Term definable classes of Boolean functions and frame definability in modal logic

M. COUCEIRO, L. HELLA, J. KIVELÄ

ABSTRACT. We establish a connection between term definability of Boolean functions and definability of finite modal frames. We introduce a bijective translation between functional terms and uniform degree-1 formulas and show that a class of Boolean functions is defined by functional terms if and only if the corresponding class of Scott-Montague frames is defined by the translations of these functional terms, and vice versa. As a special case, we get that the clone Λ_1 of all conjunctions corresponds to the class of all Kripke frames. We also characterize some classes of Scott-Montague frames corresponding to subclones of Λ_1 by restricting the class of Kripke frames in a natural way. Furthermore, by modifying Kripke semantics, we extend our results to correspondences between linear clones and classes of Kripke frames equipped with modified semantics.

1. Introduction

In [16] Segerberg studied a semantics for modal logic which is more general than the well-known Kripke semantics. In literature this semantics is often called *neighbourhood semantics*. Hansson and Gärdenfors formulated the same semantics in a different, yet an equivalent, way in [9]. Following them, we call this semantics *Scott-Montague semantics* and here we also adopt their definition for the semantics.

A Scott-Montague frame is a structure $\langle W, F \rangle$ where W is a non-empty set and F is a set function $\mathcal{P}(W) \rightarrow \mathcal{P}(W)$. The truth condition for the modal operator $\Box$ in a model $\mathcal{M}$ based on a Scott-Montague frame is given by

$$\mathcal{M}, w \models \Box\varphi \text{ if and only if } w \in F(\|\varphi\|^\mathcal{M}),$$

where $\|\varphi\|^\mathcal{M}$ is the truth set of φ in $\mathcal{M}$. In the case of finite Scott-Montague frames, the set function F can be seen as a vector-valued Boolean function $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$, and the Scott-Montague frames thus obtained are called *Boolean frames*. The idea of Boolean frames was introduced by Virtanen in [19].

It is well known that certain classes of Boolean functions can be defined by means of functional terms (see e.g. [13]). This approach to definability of Boolean function classes has certain similarities to frame definability in modal logic. We achieve

Key words and phrases: Boolean functions, functional terms, term definable classes, clones, propositional logic, modal logic, modal axioms, Kripke frames, Scott-Montague frames.

The work of the first author was partially supported by the Graduate School in Mathematical Logic MALJA. The work of the first and second authors was supported in part by the grant #28139 from the Academy of Finland. The work of the third author was supported by the grant #28139 from the Academy of Finland.

a general correspondence between definability of Boolean functions by functional terms and definability of Scott-Montague frames by modal formulas of a specific form, so-called *uniform degree-1 formulas*. This correspondence is given by a bijective translation between functional terms and uniform degree-1 formulas. We prove that a class of Boolean functions is defined by functional terms if and only if the corresponding class of Scott-Montague frames is defined by the uniform degree-1 formulas which are the translations of the defining functional terms. Conversely, a class of Scott-Montague frames is defined by uniform degree-1 formulas if and only if the corresponding class of Boolean functions is defined by the translations of these formulas.

In this paper we are especially interested in classes of Boolean functions which are closed under composition and contain all projections. These classes are called *clones* and it is well known that clones are definable by functional terms. As a corollary to our general correspondence result, we obtain for each clone a corresponding class of Scott-Montague frames.

There is a known way to interpret Kripke frames as Scott-Montague frames. Conversely, if a Scott-Montague frame is *augmented* then it can be interpreted as a Kripke frame. In the case of finite frames, being augmented just means that the frame validates the modal axioms $\Box\top$ and $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$. For the definition for augmentation, see [3]. This gives rise to the question: What are the connections between classes of Kripke frames and clones. We prove that the class of all Kripke frames corresponds to the clone Λ_1 , i.e. to the clone of all conjunctions. Furthermore, by restricting Kripke frames in natural ways, we get classes of Kripke frames corresponding to some subclones of Λ_1 .

Not all the classes of Scott-Montague frames corresponding to clones can be characterized by means of the standard Kripke semantics. However, by modifying Kripke semantics, we are also able to characterize the classes of Scott-Montague frames corresponding to the linear clones L , L_0 , L_1 , L_c and LS . The modifications we consider are based on the parity of R -successors in Kripke frames. For example, one of the modified semantics is given by

$$\mathcal{M}, w \models_O \Box\varphi \text{ if and only if } |R[w] \cap \|\varphi\|_O^{\mathcal{M}}| \text{ is odd.}$$

That is, a formula $\Box\varphi$ is true at w if and only if φ is true at an odd number of R -successors of w . With respect to this semantics, we show that the class of all Kripke frames corresponds to the clone L_0 , i.e. to the clone of all 0-preserving linear functions. By varying this semantics, we get classes of Kripke frames with given semantics that correspond to the other linear clones.

Structure of the paper. In Section 2, we give the basic definitions and background information on modal logic and Boolean functions. The clones of Boolean functions and the notion of term definability is defined in Section 3. We establish a general correspondence between definability of Boolean functions by functional terms and definability of Scott-Montague frames by uniform degree-1 formulas in Section 4. In Section 5, we present some correspondences between clones and classes

of Kripke frames, with respect to the standard Kripke semantics. The correspondences between the linear clones and classes of Kripke frames are given in Section 6. In the last section we list some open problems and discuss topics for future work.

Acknowledgement. We are grateful to Ari Virtanen for introducing us to the notion of Boolean frames, and for pointing out the importance of Scott-Montague semantics. The fruitful discussions with Virtanen on his work [19] were the starting point of our research.

2. Preliminaries

2.1. Modal logic. Our results concern *basic modal logic* which is obtained from propositional logic by adding one unary modal operator $\Box$. There are several generalizations of this basic modal logic. For example, we could have many different unary modal operators and in that case we are dealing with *multi-modal logic*. Or we could have modal operators of higher arity. For background and further generalizations, see [1]. Throughout this paper, when we speak of modal logic we mean basic modal logic.

Let Φ be a countable set of proposition symbols denoted $p_1, p_2, \dots$, or $p, q, r, \dots$. The *formulas of modal logic* are defined inductively as follows:

- (1) All proposition symbols $p \in \Phi$ and constants $\perp$ and $\top$ are formulas.
- (2) If φ and ψ are formulas, then $\neg\varphi$, $(\varphi \wedge \psi)$ and $\Box\varphi$ are formulas.

We also make use of other connectives, like $\vee$, $\rightarrow$ and $\leftrightarrow$, and define them in a usual way by using $\neg$ and $\wedge$. The dual operator $\Diamond$ for $\Box$ is defined as $\Diamond = \neg\Box\neg$. As an analogue to the quantifier rank of a formula in first-order logic we have the concept of *the degree of a formula* in modal logic. The degree of a formula φ , denoted by $\deg(\varphi)$, is defined inductively as follows:

$$\begin{aligned} \deg(\perp) = \deg(\top) &= 0, \\ \deg(p) &= 0, \text{ for all } p \in \Phi, \\ \deg(\neg\varphi) &= \deg(\varphi), \\ \deg(\varphi \wedge \psi) &= \max\{\deg(\varphi), \deg(\psi)\}, \\ \deg(\Box\varphi) &= \deg(\varphi) + 1. \end{aligned}$$

Note that modal formulas of degree 0 are just *propositional formulas*.

The most popular semantics for modal logic is the relational semantics, known as *Kripke semantics*. A *Kripke frame* is a structure $\mathcal{F} = \langle W, R \rangle$ where W , called the *universe*, is a non-empty set and R is a binary relation on W . From a Kripke frame we get a *Kripke model* $\mathcal{M} = \langle \mathcal{F}, V \rangle$ (or $\mathcal{M} = \langle W, R, V \rangle$) by adding a *valuation function* $V : \Phi \rightarrow \mathcal{P}(W)$, where $\mathcal{P}(W)$ denotes the set of all subsets of W . The notion of a formula φ being *true* (or *satisfied*) in the model $\mathcal{M}$ at a state (or world)

$w \in W$, denoted $\mathcal{M}, w \models \varphi$, is defined inductively as follows:

$$\begin{aligned} \mathcal{M}, w &\models \perp && \text{never,} \\ \mathcal{M}, w &\models \top && \text{always,} \\ \mathcal{M}, w &\models p && \text{iff } w \in V(p), \text{ for all } p \in \Phi, \\ \mathcal{M}, w &\models \neg\varphi && \text{iff } \mathcal{M}, w \not\models \varphi, \\ \mathcal{M}, w &\models \varphi \wedge \psi && \text{iff } \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi, \\ \mathcal{M}, w &\models \Box\varphi && \text{iff for all } v \in W: \text{ if } wRv \text{ then } \mathcal{M}, v \models \varphi. \end{aligned}$$

It is easy to verify that $\mathcal{M}, w \models \Diamond\varphi$ if and only if $\mathcal{M}, v \models \varphi$ for some $v \in W$ such that wRv .

In this paper we are also interested in a more general semantics for modal logic, so-called *Scott-Montague semantics*. A *Scott-Montague frame* is a structure $\mathcal{F} = \langle W, F \rangle$ where W is a non-empty set and F is a *set function* $\mathcal{P}(W) \rightarrow \mathcal{P}(W)$. A *Scott-Montague model* is a structure $\mathcal{M} = \langle \mathcal{F}, V \rangle$ (or $\mathcal{M} = \langle W, F, V \rangle$) where $\mathcal{F}$ is a Scott-Montague frame and V is a valuation. The truth conditions for the formulas are the same as in Kripke semantics except for the modal operator $\Box$, whose interpretation in such a model $\mathcal{M}$ is given by

$$\mathcal{M}, w \models \Box\varphi \text{ iff } w \in F(\|\varphi\|^{\mathcal{M}}),$$

where $\|\varphi\|^{\mathcal{M}}$ denotes the *truth set* of φ in $\mathcal{M}$, i.e. $\|\varphi\|^{\mathcal{M}} = \{v \in W \mid \mathcal{M}, v \models \varphi\}$. In other words F assigns to the truth set of φ the truth set of $\Box\varphi$, i.e. $\|\Box\varphi\|^{\mathcal{M}} = F(\|\varphi\|^{\mathcal{M}})$. Sometimes when there is no danger of confusion, we also use a notation $\|\varphi\|$ without a superscript for the truth set of φ . For an equivalent model theoretic formulation for Scott-Montague semantics, see [3]. Based on that formulation, the semantics is also known as *neighbourhood semantics*.

Let $\mathcal{M}$ be a Kripke or Scott-Montague model with a universe W and let $\mathcal{F}$ be Kripke or Scott-Montague frame with a universe W . We say that a formula φ is *valid in a model* $\mathcal{M}$, denoted $\mathcal{M} \models \varphi$, if $\mathcal{M}, w \models \varphi$ for all $w \in W$. A formula φ is *valid in a frame* $\mathcal{F}$, denoted $\mathcal{F} \models \varphi$, if $\langle \mathcal{F}, V \rangle \models \varphi$ for all $V : \Phi \rightarrow \mathcal{P}(W)$. If Ψ is a set of formulas such that $\mathcal{F} \models \varphi$ for all $\varphi \in \Psi$, we use a notation $\mathcal{F} \models \Psi$. Let $\mathcal{C}$ be a class of Kripke frames or a class of Scott-Montague frames and let Ψ be a set of formulas. We say that the class $\mathcal{C}$ is *defined by* Ψ (or *axiomatized by* Ψ) if

$$\mathcal{F} \in \mathcal{C} \text{ iff } \mathcal{F} \models \Psi.$$

When we talk about definability in modal logic we mean *frame-definability* since we are interested in sets of formulas defining classes of frames, not classes of models.

Kripke models can be seen as first-order structures. Let τ be a relational signature with one binary relation symbol R_0 and unary relation symbols P_i for each $p_i \in \Phi$. Then each Kripke model $\mathcal{M} = \langle W, R, V \rangle$ can be associated with a τ -structure $\mathcal{M}_{FO}$ with universe W where R_0 is interpreted as R and each P_i is interpreted as $V(p_i)$. Similarly we can interpret Kripke frame $\mathcal{F} = \langle W, R \rangle$ as a $\{R_0\}$ -structure $\mathcal{F}_{FO}$. Moreover, it is well known that for all modal formulas φ there is a first-order formula $\psi_\varphi(x)$ such that for all Kripke models $\mathcal{M} = \langle W, R, V \rangle$ and for all $w \in W$,

$$\mathcal{M}, w \models \varphi \text{ if and only if } \mathcal{M}_{FO} \models \psi_\varphi[w].$$

The formula $\psi_\varphi(x)$ is called the *standard translation* of φ . For background, see [1]. In *correspondence theory* one is interested in knowing when does a modal formula φ have a first-order counter-part θ_φ such that

$$\mathcal{F} \models \varphi \text{ iff } \mathcal{F}_{FO} \models \theta_\varphi.$$

For example, $\mathcal{F} \models \Box p \rightarrow p$ if and only if R is reflexive, which is a first-order definable property of R . It is well known that from the standard translation of φ we get a formula θ in *monadic second-order logic* (MSO) such that $\mathcal{F} \models \varphi$ iff $\mathcal{F}_{FO} \models \theta$.

In Scott-Montague semantics the validity of a formula φ in a frame $\mathcal{F} = \langle W, F \rangle$ corresponds to a set theoretical property of the function F . Let φ be a formula such that the proposition symbols occurring in φ are among $p_1, \dots, p_k$. We translate φ into a set theoretical expression $S(\varphi)$ inductively as follows:

$$\begin{aligned} S(p_i) &= X_i, \text{ for all } i \in \{1, \dots, k\}, \\ S(\neg\varphi) &= \overline{S(\varphi)}, \\ S(\varphi \wedge \psi) &= S(\varphi) \cap S(\psi), \\ S(\Box\varphi) &= F(S(\varphi)), \end{aligned}$$

where we use a notation $\overline{X}$ for the complement of $X \subseteq W$ with respect to W . The following proposition is immediate from the translation.

Proposition 1. *Let $\mathcal{M} = \langle W, F, V \rangle$ be a Scott-Montague model and let $w \in W$. Let φ be a formula with proposition symbols $p_1, \dots, p_k$ and let $X_i = V(p_i)$ for all $i \in \{1, \dots, k\}$. Then*

$$\mathcal{M}, w \models \varphi \text{ iff } w \in S(\varphi). \quad \square$$

Corollary 1. *Let $\mathcal{F} = \langle W, F \rangle$ be a Scott-Montague frame and let φ be a formula with proposition symbols $p_1, \dots, p_k$. Then*

$$\mathcal{F} \models \varphi \text{ iff } S(\varphi) = W \text{ for all } X_1, \dots, X_k \subseteq W. \quad \square$$

Example. Let $\mathcal{F} = \langle W, F \rangle$ be a Scott-Montague frame and let φ be a formula $\Box p \rightarrow p$. The formula φ is equivalent to a formula $\neg(\Box p \wedge \neg p)$. Thus, by Corollary 1, $\mathcal{F} \models \varphi$ if and only if $F(X) \cap \overline{X} = \emptyset$ for all $X \subseteq W$. In this case φ has only a single occurrence of $\rightarrow$ and we get also a bit more intuitive correspondence result. Namely,

$$\mathcal{F} \models \Box p \rightarrow p \text{ iff } F(X) \subseteq X \text{ for all } X \subseteq W.$$

2.2. Boolean functions. Let $\mathbb{B} = \{0, 1\}$. A *Boolean function* is a map $f : \mathbb{B}^n \rightarrow \mathbb{B}$, for some positive integer n called the *arity* of f . To each sequence $(f_1, \dots, f_m)$ of n -ary Boolean functions we associate a *Boolean vector-valued map* $(f_1, \dots, f_m) : \mathbb{B}^n \rightarrow \mathbb{B}^m$ defined by $(f_1, \dots, f_m)(\mathbf{a}) = (f_1(\mathbf{a}), \dots, f_m(\mathbf{a}))$, for every $\mathbf{a} \in \mathbb{B}^n$.

By a *class* of Boolean functions we simply mean a set $K \subseteq \bigcup_{n \geq 1} \mathbb{B}^{\mathbb{B}^n}$, where $\mathbb{B}^{\mathbb{B}^n}$ denotes the class of all Boolean functions $f : \mathbb{B}^n \rightarrow \mathbb{B}$. In the sequel, we shall say that a vector-valued map $g = (f_1, \dots, f_m)$ *belongs* to a class $K \subseteq \bigcup_{n \geq 1} \mathbb{B}^{\mathbb{B}^n}$ if for every $1 \leq i \leq m$ we have $f_i \in K$.

For each $n \geq 1$, the n different *projection maps* $(a_1, \dots, a_n) \mapsto a_i$, $1 \leq i \leq n$, are also called *variables*, and denoted $x_1, \dots, x_n$, where the arity is clear from the context.

Let f be a Boolean function $\mathbb{B}^n \rightarrow \mathbb{B}$. For each $1 \leq i \leq n$, x_i is said to be an *essential variable of f* if there are $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n$ in $\mathbb{B}$ such that

$$f(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) \neq f(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n).$$

Otherwise, x_i is called a *dummy variable of f* . Note that constant functions are the only Boolean functions whose variables are all dummy.

The set $\mathbb{B}^n$ is a Boolean lattice (distributive and complemented) of 2^n elements under the component-wise order of vectors given by

$$(a_1, \dots, a_n) \preceq (b_1, \dots, b_n) \text{ if and only if } a_i \leq b_i, \text{ for all } 1 \leq i \leq n.$$

The *complement* of a vector $\mathbf{a} = (a_1, \dots, a_n)$ is also defined component-wise, i.e. $\bar{\mathbf{a}} = (1 - a_1, \dots, 1 - a_n)$. We denote the *all-zero-vector* and the *all-one-vector* by $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$, respectively. The set $\mathbb{B}^n$ is also a Boolean lattice of 2^{2^n} elements under the point-wise ordering of functions given by

$$f \leq g \text{ if and only if } f(\mathbf{a}) \leq g(\mathbf{a}), \text{ for all } \mathbf{a} \in \mathbb{B}^n.$$

In $\mathbb{B}^n$, the *complement* of a Boolean function f is the function $\bar{f}$ defined by $\bar{f}(\mathbf{a}) = 1 - f(\mathbf{a})$, for all $\mathbf{a} \in \mathbb{B}^n$. The dual of f , denoted f^d , is given by $f^d(\mathbf{a}) = \bar{f}(\bar{\mathbf{a}})$, for all $\mathbf{a} \in \mathbb{B}^n$. The functions (of any arity) having constant value 0 and 1 on $\mathbb{B}^n$ are denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively.

Boolean functions are many times defined as maps which can be “specified” by means of “Boolean formulas” (see e.g. [2]). Let X be a countable set of *vector-variable symbols* $\mathbf{x}_1, \mathbf{x}_2, \dots$. A *Boolean formula* is a formal expression defined inductively as follows:

- (1) All vector-variable symbols $\mathbf{x}_i \in X$ and constant functions $\mathbf{0}$ and $\mathbf{1}$ are Boolean formulas.
- (2) If G_1 and G_2 are Boolean formulas, then $\neg G_1$ and $(G_1 \wedge G_2)$ are Boolean formulas.

We also make use of other connectives, like $\vee$, $\rightarrow$ and $\leftrightarrow$, as usually defined. We write $G(\mathbf{x}_1, \dots, \mathbf{x}_n)$ if the vector-variable symbols are among $\mathbf{x}_1, \dots, \mathbf{x}_n$, and we say that G has arity n .

Each Boolean formula G specifies a Boolean function f by interpreting the expression G in the Boolean lattice $\mathbb{B}$, and in this case G is said to be a *formal representation* of f . It is well known that every Boolean function f can be represented by a DNF (*disjunctive normal form*), i.e. by an expression of the form

$$\bigvee_{i \in I} \left(\bigwedge_{j \in P_i} x_j \wedge \bigwedge_{j \in N_i} \bar{x}_j \right),$$

where I is a finite set of indices, and in each conjunct each variable appears at most once. We regard empty disjunctions and empty conjunctions as having value 0 and

1, respectively. It is easy to verify that if

$$f = \bigvee_{i \in I} \left(\bigwedge_{j \in P_i} x_j \wedge \bigwedge_{j \in N_i} \bar{x}_j \right),$$

then the dual f^d of f is represented by

$$f^d = \bigwedge_{i \in I} \left(\bigvee_{j \in P_i} x_j \vee \bigvee_{j \in N_i} \bar{x}_j \right) \quad (1)$$

Expressions of the form (1) are called CNF (*conjunctive normal form*) representations. Note that DNF and CNF representations are not unique.

Since Stone [17], it is well-known that any Boolean lattice can be viewed as a Boolean ring (in which every element is idempotent under product) by defining multiplication and addition (mod 2) by

$$x \cdot y = x \wedge y \text{ and } x \oplus y = (\bar{x} \wedge y) \vee (x \wedge \bar{y}).$$

Thus both $\mathbb{B}^n$ and $\mathbb{B}^{\mathbb{B}^n}$ can also be treated as Boolean rings using the above algebraic translations. It is not difficult to see that each n -ary Boolean function f can be represented in this Boolean ring language by a multilinear polynomial in n indeterminates over $\mathbb{B}$, called its *Zhegalkin polynomial* or *Reed-Muller polynomial*

$$f = \sum_{j \in I} (c_j \cdot \prod_{i \in I_j} x_i)$$

Moreover, unlike DNF and CNF representations, the Zhegalkin polynomial representation of a Boolean function is unique (up to permutation of terms and permutation of variables in the terms). For further developments on Boolean normal forms, see [6].

If f is an n -ary Boolean function and $g_1, \dots, g_n$ are m -ary Boolean functions, then their *composition* is the m -ary Boolean function $f(g_1, \dots, g_n)$, whose value on every $\mathbf{a} \in \mathbb{B}^m$ is $f(g_1(\mathbf{a}), \dots, g_n(\mathbf{a}))$. Let I and J be classes of Boolean functions of various arities. The *class composition* $I \circ J$, is defined as the set of all composites of functions in I with functions in J , i.e.

$$I \circ J = \{f(g_1, \dots, g_n) \mid n, m \geq 1, f \text{ } n\text{-ary in } I, g_1, \dots, g_n \text{ } m\text{-ary in } J\}.$$

When $I = \{f\}$, we write $f \circ J$ instead of $\{f\} \circ J$. (For background see [4] and [5].)

3. Clones and Term definable classes

A (*Boolean*) *clone* is a class $C \subseteq \bigcup_{n \geq 1} \mathbb{B}^{\mathbb{B}^n}$ containing all projections and idempotent under class composition, that is, satisfying $C \circ C = C$. The clones of Boolean functions form an algebraic lattice by defining the meet as the intersection of clones, and the join as the smallest clone containing the union. This lattice is known as Post Lattice (see Figure 1), named after Emil Post who first described and classified in [14] the set of all Boolean clones (for recent and shorter proofs of Post's classification see [15], [18], [20]; for general background see [10] and [11]).

In this paper we shall be especially interested in the following clones:

- Ω : the class of all Boolean functions;

FIGURE 1. Post Lattice.

- T_0 and T_1 : the classes of 0- and 1-preserving functions, respectively, i.e.,
 $T_0 = \{f \in \Omega \mid f(0, \dots, 0) = 0\}$, $T_1 = \{f \in \Omega \mid f(1, \dots, 1) = 1\}$;
- T_c : the class of constant-preserving functions, i.e., $T_c = T_0 \cap T_1$.
- M : the class of all monotone functions, i.e.,
 $M = \{f \in \Omega \mid f(\mathbf{a}) \leq f(\mathbf{b}), \text{ whenever } \mathbf{a} \preceq \mathbf{b}\}$;
- S : the class of all self-dual functions, i.e.,
 $S = \{f \in \Omega \mid f^d = f\}$;
- $S_c = S \cap T_c$, $SM = S \cap M$;
- L : the class of all linear functions, i.e.,
 $L = \{f \in \Omega \mid f = c_0 \mathbf{1} \oplus c_1 x_1 \oplus \dots \oplus c_n x_n \text{ for some } n \text{ and } c_0, \dots, c_n \in \mathbb{B}\}$;
- $L_0 = L \cap T_0$, $L_1 = L \cap T_1$, $LS = L \cap S$, $L_c = L \cap T_c$;
- Λ : the class of all conjunctions and constants, i.e.,
 $\Lambda = \{f \in \Omega \mid f = \mathbf{0}, \mathbf{1}, x_{i_1} \wedge \dots \wedge x_{i_n} \text{ for some } n \geq 1 \text{ and } i_j \text{'s}\}$;
- $\Lambda_0 = \Lambda \cap T_0$, $\Lambda_1 = \Lambda \cap T_1$, $\Lambda_c = \Lambda \cap T_c$;
- V : the class of all disjunctions and constants, i.e.,
 $V = \{f \in \Omega \mid f = \mathbf{0}, \mathbf{1}, x_{i_1} \vee \dots \vee x_{i_n} \text{ for some } n \geq 1 \text{ and } i_j \text{'s}\}$;
- $V_0 = V \cap T_0$, $V_1 = V \cap T_1$, $V_c = V \cap T_c$;
- $\Omega(1)$: the class of all variables, negated variables, and constants;
- I^* : the class of all variables and negated variables;
- I : the class of all variables and constants;
- $I_0 = I \cap T_0$, $I_1 = I \cap T_1$;
- I_c : the smallest clone containing only variables, i.e., $I_c = I \cap T_c$.

Remark 1. Using the above definitions, it is easy to verify that every function in:

- L_0 is of the form $f = c(x_1 \oplus \dots \oplus x_n)$, for $n \geq 1$ and $c \in \mathbb{B}$,
- L_1 is either the sum of an odd number of variables, or the negation of the sum of an even number of variables,
- LS is either the sum of an odd number of variables, or the negation of the sum of an even number of variables,
- L_c is the sum of an odd number of variables.

Observe also that the only functions in Λ with essential variables are conjunctions of variables, and since $\mathbf{0} \notin \Lambda_1$, for every $f \in \mathbb{B}^{\mathbb{B}^n} \cap \Lambda_1$, we have that

$f(a_1, \dots, a_n) = 1$ if and only if $a_i = 1$, whenever x_i is an essential variable of f .

A *functional term* is a formal expression

$$T = G(\mathbf{f}(G_1(\mathbf{x}_1, \dots, \mathbf{x}_r)), \dots, \mathbf{f}(G_m(\mathbf{x}_1, \dots, \mathbf{x}_r))) \quad (2)$$

where $m, r \geq 1$, G is an m -ary Boolean formula, $G_1, \dots, G_m$ are r -ary Boolean formulas, $\mathbf{x}_1, \dots, \mathbf{x}_r$ are r distinct vector-variable symbols, and $\mathbf{f}$ is a *function symbol*. Note that, for each positive integer n , the functional term T defines a map $T : \mathbb{B}^{\mathbb{B}^n} \times \mathbb{B}^{nr} \rightarrow \mathbb{B}$ given by

$$T(f, \mathbf{a}_1, \dots, \mathbf{a}_r) = G(f(G_1(\mathbf{a}_1, \dots, \mathbf{a}_r)), \dots, f(G_m(\mathbf{a}_1, \dots, \mathbf{a}_r)))$$

by interpreting the Boolean formula G in $\mathbb{B}$ and the Boolean formulas G_i in $\mathbb{B}^n$.

An n -ary Boolean function f is said to *satisfy* the functional term T , denoted $T(f) \equiv 1$, if for all $\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathbb{B}^n$ we have

$$T(f, \mathbf{a}_1, \dots, \mathbf{a}_r) = 1$$

For background, see [13, 8]. See also [7, 12] for further variants, and [5] for further extensions. For our purposes, we extend this notion of satisfaction to Boolean vector-valued maps $g = (f_1, \dots, f_n)$, by saying that g *satisfies* (2) if for each $1 \leq i \leq n$, f_i satisfies (2). In this setting, a class $K \subseteq \bigcup_{n \geq 1} \mathbb{B}^{\mathbb{B}^n}$ is said to be defined by a set $\mathcal{T}$ of functional terms, denoted $K = \langle \mathcal{T} \rangle$, if K is the set of all those functions satisfying every member of $\mathcal{T}$. If $\mathcal{T} = \{T\}$, then we write $K = \langle T \rangle$ instead of $K = \langle \{T\} \rangle$. A class K is said to be *term definable* if there is a set $\mathcal{T}$ of functional terms such that $K = \langle \mathcal{T} \rangle$.

An m -ary Boolean function g is said to be obtained from an n -ary Boolean function f by *simple variable substitution*, if there are m -ary projections $p_1, \dots, p_n \in I_c$ such that $g = f(p_1, \dots, p_n)$. A class $K \subseteq \Omega$ is said to be *closed under simple variable substitutions* if each Boolean function obtained from a function f in K by simple variable substitution is also in K . In other words, the class K is closed under simple variable substitutions if and only if $K \circ I_c = K$. See [4, 5] for generalizations of the notion of simple variable substitution. The relevance of simple variable substitutions for definability of Boolean function classes by means of functional terms is made apparent by following result which was first obtained, in a somewhat variant form, by Ekin, Foldes, Hammer and Hellerstein in [7].

Theorem 1. *A class K of Boolean functions is definable by a set of functional terms if and only if K is closed under simple variable substitutions.*

Note that every clone is in particular closed under simple variable substitutions, and thus every clone is definable by some set functional terms. In the sequel, we shall make use of some of the examples listed below. For a complete list of functional terms defining every clone of Boolean functions, see [8].

$$\begin{aligned} T_0 &= \langle \neg \mathbf{f}(\mathbf{0}) \rangle, \\ T_1 &= \langle \mathbf{f}(\mathbf{1}) \rangle, \\ T_c &= \langle \{ \neg \mathbf{f}(\mathbf{0}), \mathbf{f}(\mathbf{1}) \} \rangle, \\ M &= \langle \mathbf{f}(\mathbf{x}_1) \rightarrow \mathbf{f}(\mathbf{x}_2 \vee \mathbf{x}_1) \rangle, \\ S &= \langle \neg \mathbf{f}(\mathbf{x}_1) \leftrightarrow \mathbf{f}(\neg \mathbf{x}_1) \rangle, \\ L &= \langle \mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}_1 \oplus \mathbf{x}_2) \oplus \mathbf{f}(\mathbf{x}_1) \oplus \mathbf{f}(\mathbf{x}_2) \rangle, \\ L_0 &= \langle \mathbf{1} \oplus \mathbf{f}(\mathbf{x}_1 \oplus \mathbf{x}_2) \oplus \mathbf{f}(\mathbf{x}_1) \oplus \mathbf{f}(\mathbf{x}_2) \rangle, \\ L_1 &= \langle \{ \mathbf{f}(\mathbf{1}), \mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}_1 \oplus \mathbf{x}_2) \oplus \mathbf{f}(\mathbf{x}_1) \oplus \mathbf{f}(\mathbf{x}_2) \} \rangle, \\ LS &= \langle \{ \mathbf{f}(\mathbf{1}) \oplus \mathbf{f}(\mathbf{0}), \mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}_1 \oplus \mathbf{x}_2) \oplus \mathbf{f}(\mathbf{x}_1) \oplus \mathbf{f}(\mathbf{x}_2) \} \rangle, \\ L_c &= \langle \{ \mathbf{f}(\mathbf{1}), \mathbf{1} \oplus \mathbf{f}(\mathbf{x}_1 \oplus \mathbf{x}_2) \oplus \mathbf{f}(\mathbf{x}_1) \oplus \mathbf{f}(\mathbf{x}_2) \} \rangle, \\ \Lambda &= \langle \mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2) \leftrightarrow \mathbf{f}(\mathbf{x}_1) \wedge \mathbf{f}(\mathbf{x}_2) \rangle, \\ \Lambda_1 &= \langle \{ \mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2) \leftrightarrow \mathbf{f}(\mathbf{x}_1) \wedge \mathbf{f}(\mathbf{x}_2) \} \rangle, \\ \Lambda_c &= \langle \{ \neg \mathbf{f}(\mathbf{0}), \mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2) \leftrightarrow \mathbf{f}(\mathbf{x}_1) \wedge \mathbf{f}(\mathbf{x}_2) \} \rangle, \\ I &= \langle \{ \mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2) \leftrightarrow \mathbf{f}(\mathbf{x}_1) \wedge \mathbf{f}(\mathbf{x}_2), \mathbf{f}(\mathbf{x}_1 \vee \mathbf{x}_2) \leftrightarrow \mathbf{f}(\mathbf{x}_1) \vee \mathbf{f}(\mathbf{x}_2) \} \rangle, \\ I_1 &= \langle \{ \mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2) \leftrightarrow \mathbf{f}(\mathbf{x}_1) \wedge \mathbf{f}(\mathbf{x}_2), \neg \mathbf{f}(\mathbf{x}_1) \rightarrow \mathbf{f}(\neg \mathbf{x}_1) \} \rangle, \\ I_c &= \langle \{ \mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2) \leftrightarrow \mathbf{f}(\mathbf{x}_1) \wedge \mathbf{f}(\mathbf{x}_2), \neg \mathbf{f}(\mathbf{x}_1) \leftrightarrow \mathbf{f}(\neg \mathbf{x}_1) \} \rangle. \end{aligned}$$

Note that if a class K is defined by a finite set $\mathcal{T}$ of functional terms, then it is defined by a single functional term which is obtained by taking the conjunction of the functional terms in $\mathcal{T}$. For example,

$$\Lambda_1 = \langle \mathbf{f}(\mathbf{1}) \wedge (\mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2) \leftrightarrow \mathbf{f}(\mathbf{x}_1) \wedge \mathbf{f}(\mathbf{x}_2)) \rangle.$$

4. General correspondence

In this section we will show that definability of Boolean functions by functional terms corresponds exactly to definability of finite Scott-Montague frames by formulas of a specific form. We call these formulas *uniform degree-1 formulas*. We will make use of the natural bijection between vector-valued Boolean functions $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$ and set functions $F : \mathcal{P}(W) \rightarrow \mathcal{P}(W)$, where W is a set with n elements. By using this natural bijection, finite Scott-Montague frames can be interpreted by Boolean functions and this correspondence can be used to define certain classes of frames by using known theories of Boolean function classes. We will only consider finite frames and throughout this section we fix W to be the n -element set $\{w_1, \dots, w_n\}$.

Let $X \subseteq W$ and $\mathbf{x} \in \mathbb{B}^n$. By $\mathbf{a}_X$ we denote the characteristic vector of X , i.e. for all $1 \leq i \leq n$,

$$\mathbf{a}_X[i] = 1 \text{ if and only if } w_i \in X,$$

where $\mathbf{a}_X[i]$ denotes the i th component of the vector $\mathbf{a}_X$. Conversely, by $A_{\mathbf{x}}$ we denote the set whose characteristic vector is $\mathbf{x}$. It is immediate that $\mathbf{a}_{A_{\mathbf{x}}} = \mathbf{x}$ and $A_{\mathbf{a}_X} = X$.

Let $f = (f_1, \dots, f_n)$ be a vector-valued Boolean function $\mathbb{B}^n \rightarrow \mathbb{B}^n$. The functions f_i are called the *component functions*, or simply the *components*, of f . We define F_f to be the function $\mathcal{P}(W) \rightarrow \mathcal{P}(W)$ such that $F_f(X) = A_{f(\mathbf{a}_X)}$ for all $X \subseteq W$. That is, for all $X \subseteq W$ and $w_i \in W$,

$$w_i \in F_f(X) \text{ if and only if } f_i(\mathbf{a}_X) = 1.$$

We denote by $\mathcal{F}_f$ the Scott-Montague frame $\langle W, F_f \rangle$.

Let $\mathcal{F} = \langle W, F \rangle$ be a Scott-Montague frame. We denote by f_F the vector-valued Boolean function $\mathbb{B}^n \rightarrow \mathbb{B}^n$ for which $f_F(\mathbf{x}) = \mathbf{a}_{F(A_{\mathbf{x}})}$ for all $\mathbf{x} \in \mathbb{B}^n$. In other words, for all $\mathbf{x} \in \mathbb{B}^n$ and for all $1 \leq i \leq n$,

$$f_i(\mathbf{x}) = 1 \text{ if and only if } w_i \in F(A_{\mathbf{x}}),$$

where f_i is the i th component of f_F . We also use a notation $f_{\mathcal{F}} = f_F$.

It is immediate from these definitions that $f_{F_f} = f$ and $F_{f_F} = F$.

Let T be a functional term, let $f = (f_1, \dots, f_n)$ be a vector-valued Boolean function $\mathbb{B}^n \rightarrow \mathbb{B}^n$ and let $\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathbb{B}^n$. By $T(f, \mathbf{a}_1, \dots, \mathbf{a}_r)$ we denote the n -vector whose i th component is $T(f_i, \mathbf{a}_1, \dots, \mathbf{a}_r)$. We write $T(f) \equiv \mathbf{1}$ when $T(f, \mathbf{a}_1, \dots, \mathbf{a}_r) = \mathbf{1}$ for all $\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathbb{B}^n$.

In order to establish the desired connection between term definable classes of Boolean functions and definable classes of Scott-Montague frames, we need a translation of functional terms into modal formulas and, conversely, of modal formulas

into functional terms. We will first define a translation of functional terms T into modal formulas ψ_T such that for every vector-valued Boolean function f , $T(f) \equiv \mathbf{1}$ if and only if $\mathcal{F}_f \models \psi_T$. The idea of the translation is very simple; we just replace vector-variable symbols $\mathbf{x}_i$ by proposition symbols p_i and the function symbol $\mathbf{f}$ by the modal operator $\Box$, and all the connectives remain the same. The exact definition of the translation is done in two parts. First we define a translation for Boolean formulas and then, using this, we define a translation for functional terms.

Let $G = G(\mathbf{x}_1, \dots, \mathbf{x}_r)$ be an r -ary Boolean formula. The *translation* of G , denoted by ψ_G , is defined inductively as follows.

- If $G = \mathbf{0}$ (or $\mathbf{1}$), then $\psi_G = \perp$ (or $\top$ respectively).
- If $G(\mathbf{x}_1, \dots, \mathbf{x}_r) = \mathbf{x}_i$ for some $1 \leq i \leq r$, then $\psi_G = p_i$.
- If $G = \neg H$, then $\psi_G = \neg \psi_H$.
- If $G = H_1 \wedge H_2$, then $\psi_G = \psi_{H_1} \wedge \psi_{H_2}$.

Let $T = G(\mathbf{f}(G_1(\mathbf{x}_1, \dots, \mathbf{x}_r)), \dots, \mathbf{f}(G_m(\mathbf{x}_1, \dots, \mathbf{x}_r)))$ be a functional term as defined in Section 3, let $\psi_G(p_1, \dots, p_m)$ be the translation of G and let ψ_{G_i} be the translation of G_i for each $1 \leq i \leq m$. The *translation* of T , denoted by ψ_T , is defined as

$$\psi_T = \psi_G(\Box \psi_{G_1}, \dots, \Box \psi_{G_m}).$$

For example, let $T = \mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2) \rightarrow (\mathbf{f}(\mathbf{x}_1) \wedge \mathbf{f}(\mathbf{x}_2))$. Then

$$\psi_T = \Box(p_1 \wedge p_2) \rightarrow (\Box p_1 \wedge \Box p_2).$$

Let $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_r) \in \mathbb{B}^{nr}$. We denote by $V_{\mathbf{a}}$ the valuation function $\Phi \rightarrow \mathcal{P}(W)$ such that $V_{\mathbf{a}}(p_j) = A_{\mathbf{a}_j}$ for all $1 \leq j \leq r$.

Lemma 1. *Let G be an r -ary Boolean formula and let $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_r) \in \mathbb{B}^{nr}$. Let $\mathcal{M} = \langle W, F, V_{\mathbf{a}} \rangle$ be a Scott-Montague model. Then for all $1 \leq i \leq n$,*

$$G(\mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1 \text{ if and only if } \mathcal{M}, w_i \models \psi_G,$$

i.e. $\|\psi_G\|^{\mathcal{M}} = A_{\mathbf{b}}$, where $\mathbf{b} = G(\mathbf{a}_1, \dots, \mathbf{a}_r)$.

Proof. The proof is by induction on the construction of G . If G is a constant $\mathbf{0}$ (or $\mathbf{1}$) then ψ_G is $\perp$ (or $\top$ respectively) and the result is trivial. If $G(\mathbf{x}_1, \dots, \mathbf{x}_r) = \mathbf{x}_j$ for some $1 \leq j \leq r$ then $\psi_G = p_j$ and

$$G(\mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1 \Leftrightarrow \mathbf{a}_j[i] = 1 \Leftrightarrow w_i \in A_{\mathbf{a}_j} \Leftrightarrow \mathcal{M}, w_i \models p_j \Leftrightarrow \mathcal{M}, w_i \models \psi_G.$$

Let G be of the form $\neg H$ and suppose that the claim holds for H . Then

$$\begin{aligned} G(\mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1 &\Leftrightarrow H(\mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 0 \\ &\stackrel{(*)}{\Leftrightarrow} \mathcal{M}, w_i \not\models \psi_H \\ &\Leftrightarrow \mathcal{M}, w_i \models \neg \psi_H \\ &\Leftrightarrow \mathcal{M}, w_i \models \psi_G, \end{aligned}$$

where the equivalence $(*)$ holds by the induction hypothesis. In the case where G is of the form $H_1 \wedge H_2$, the result also follows straightforwardly from the induction hypothesis. $\square$

Theorem 2. *Let T be a functional term*

$$G(\mathbf{f}(G_1(\mathbf{x}_1, \dots, \mathbf{x}_r), \dots, \mathbf{f}(G_m(\mathbf{x}_1, \dots, \mathbf{x}_r)))$$

and let $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_r) \in \mathbb{B}^{nr}$. Let f be a vector-valued Boolean function $\mathbb{B}^n \rightarrow \mathbb{B}^n$. Then for all $1 \leq i \leq n$,

$$T(f, \mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1 \text{ if and only if } \langle \mathcal{F}_f, V_{\mathbf{a}} \rangle, w_i \models \psi_T.$$

Proof. The proof is by induction on the construction of G . The cases where G is a constant, or of the form $\neg H$ or $H_1 \wedge H_2$, are handled similarly as in the previous lemma. If $G(\mathbf{x}_1, \dots, \mathbf{x}_m) = \mathbf{x}_j$ for some $1 \leq j \leq m$, then $T = \mathbf{f}(G_j(\mathbf{x}_1, \dots, \mathbf{x}_r))$ and $\psi_T = \Box \psi_{G_j}$. Thus

$$\begin{aligned} T(f, \mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1 &\Leftrightarrow f(G_j(\mathbf{a}_1, \dots, \mathbf{a}_r))[i] = 1 \\ &\Leftrightarrow w_i \in A_{f(\mathbf{b})}, \text{ where } \mathbf{b} = G_j(\mathbf{a}_1, \dots, \mathbf{a}_r) \\ &\Leftrightarrow w_i \in F_f(A_{\mathbf{b}}), \text{ where } \mathbf{b} = G_j(\mathbf{a}_1, \dots, \mathbf{a}_r) \\ &\stackrel{(*)}{\Leftrightarrow} w_i \in F_f(\|\psi_{G_j}\|^{\mathcal{M}}), \text{ where } \mathcal{M} = \langle \mathcal{F}_f, V_{\mathbf{a}} \rangle \\ &\Leftrightarrow \langle \mathcal{F}_f, V_{\mathbf{a}} \rangle, w_i \models \Box \psi_{G_j} \\ &\Leftrightarrow \langle \mathcal{F}_f, V_{\mathbf{a}} \rangle, w_i \models \psi_T, \end{aligned}$$

where the equivalence $(*)$ holds by Lemma 1. $\square$

As an immediate corollary to Theorem 2 we get that the translation ψ_T works as desired.

Corollary 2. *Let T be a functional term and let $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_r) \in \mathbb{B}^{nr}$. Let f be a vector-valued Boolean function $\mathbb{B}^n \rightarrow \mathbb{B}^n$. Then*

$$T(f, \mathbf{a}_1, \dots, \mathbf{a}_r) = \mathbf{1} \text{ if and only if } \langle \mathcal{F}_f, V_{\mathbf{a}} \rangle \models \psi_T,$$

and furthermore

$$T(f) \equiv \mathbf{1} \text{ if and only if } \mathcal{F}_f \models \psi_T. \quad \square$$

The translation of a functional term T yields a modal formula of a specific form. We say that ψ is a *uniform degree-1 formula* if it is of the form

$$\psi = \varphi(\Box \varphi_1, \dots, \Box \varphi_m),$$

where $\varphi, \varphi_1, \dots, \varphi_m$ are propositional formulas. Here φ is a formula with proposition symbols $p_1, \dots, p_m$ and ψ is a formula obtained from φ by replacing every occurrence of p_i by $\Box \varphi_i$ for each $1 \leq i \leq m$. Thus ψ is of degree 1 and uniform in the sense that ψ has no subformula in which a binary Boolean connective combines a formula of the form $\Box \theta$ and a propositional formula other than a constant. For example, $\top \wedge \Box \top \wedge \Box p_1$ is a uniform degree-1 formula, whereas $\Box p_1 \vee p_2$ is not.

Our next aim is to translate uniform degree-1 formulas ψ into functional terms T_ψ such that for every Scott-Montague frame $\mathcal{F}$, $\mathcal{F} \models \psi$ if and only if $T_\psi(f_{\mathcal{F}}) \equiv \mathbf{1}$. The idea of the translation is similar to the one introduced above. We just replace the modal operator $\Box$ in ψ by the function symbol $\mathbf{f}$ and proposition symbols p_i by vector-variable symbols $\mathbf{x}_i$. As we will observe later in Theorem 4, the translations $T \mapsto \psi_T$ and $\psi \mapsto T_\psi$ are inverses of each other. We will first define a translation

of a propositional formula φ into a Boolean formula G_φ , and then using this we define a translation of a uniform degree-1 formula into a functional term.

If φ is a modal formula in which there occur r different proposition symbols, we always assume for simplicity that those proposition symbols are $p_1, \dots, p_r$. Let φ be a propositional formula. The *translation* of φ , denoted by G_φ , is defined inductively as follows.

- If $\varphi = \perp$ (or $\top$), then $G_\varphi = \mathbf{0}$ (or $\mathbf{1}$ respectively).
- If $\varphi(p_1, \dots, p_r) = p_i$ for some $1 \leq i \leq r$, then $G_\varphi(\mathbf{x}_1, \dots, \mathbf{x}_r) = \mathbf{x}_i$.
- If $\varphi = \neg\theta$, then $G_\varphi = \neg G_\theta$.
- If $\varphi = \theta_1 \wedge \theta_2$, then $G_\varphi = G_{\theta_1} \wedge G_{\theta_2}$.

Let $\psi = \varphi(\Box\varphi_1, \dots, \Box\varphi_m)$ be a uniform degree-1 formula. The *translation* of ψ , denoted by T_ψ , is defined as

$$T_\psi = G_\varphi(\mathbf{f}(G_{\varphi_1}(\mathbf{x}_1, \dots, \mathbf{x}_r)), \dots, \mathbf{f}(G_{\varphi_m}(\mathbf{x}_1, \dots, \mathbf{x}_r))).$$

If, for example, $\psi = \Box\top \wedge (\Box p_1 \wedge \Box p_2 \leftrightarrow \Box(p_1 \wedge p_2))$, then

$$T_\psi = \mathbf{f}(\mathbf{1}) \wedge (\mathbf{f}(\mathbf{x}_1) \wedge \mathbf{f}(\mathbf{x}_2) \leftrightarrow \mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2)).$$

Lemma 2. *Let φ be a propositional formula with proposition symbols $p_1, \dots, p_r$ and let $\mathcal{M} = \langle W, F, V \rangle$ be a Scott-Montague model. Let $\mathbf{a}_j = \mathbf{a}_{V(p_j)}$ for all $1 \leq j \leq r$. Then for all $1 \leq i \leq n$,*

$$\mathcal{M}, w_i \models \varphi \text{ if and only if } G_\varphi(\mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1,$$

i.e. $G_\varphi(\mathbf{a}_1, \dots, \mathbf{a}_r) = \mathbf{a}_{\|\varphi\|^\mathcal{M}}$.

Proof. The lemma is easily proved by induction on the construction of φ , similarly to the proof of Lemma 1. $\square$

Theorem 3. *Let ψ be a uniform degree-1 formula $\varphi(\Box\varphi_1, \dots, \Box\varphi_m)$ and let $p_1, \dots, p_r$ be the proposition symbols which occur in ψ . Let $\mathcal{M} = \langle W, F, V \rangle$ be a Scott-Montague model and let $\mathbf{a}_j = \mathbf{a}_{V(p_j)}$ for all $1 \leq j \leq r$. Then for all $1 \leq i \leq n$,*

$$\mathcal{M}, w_i \models \psi \text{ if and only if } T_\psi(f_F, \mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1.$$

Proof. The proof is by induction on the construction of φ . The cases where φ is a constant $\perp$ or $\top$, or of the form $\neg\theta$ or $\theta_1 \wedge \theta_2$, are straightforward. If $\varphi(p_1, \dots, p_m) = p_j$ for some $1 \leq j \leq m$, then $\psi = \Box\varphi_j$ and therefore $T_\psi = \mathbf{f}(G_{\varphi_j}(\mathbf{x}_1, \dots, \mathbf{x}_r))$. Hence

$$\begin{aligned} \mathcal{M}, w_i \models \psi &\Leftrightarrow \mathcal{M}, w_i \models \Box\varphi_j \\ &\Leftrightarrow w_i \in F(\|\varphi_j\|^\mathcal{M}) \\ &\Leftrightarrow f_F(\mathbf{a}_{\|\varphi_j\|^\mathcal{M}})[i] = 1 \\ &\stackrel{(*)}{\Leftrightarrow} f_F(G_{\varphi_j}(\mathbf{a}_1, \dots, \mathbf{a}_r))[i] = 1 \\ &\Leftrightarrow T_\psi(f_F, \mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1, \end{aligned}$$

where the equivalence $(*)$ holds by Lemma 2. $\square$

From Theorem 3 we get immediately the intended corollary:

Corollary 3. *Let ψ be a uniform degree-1 formula. Let $\mathcal{F} = \langle W, F \rangle$ be a Scott-Montague frame and let V be a function $\Phi \rightarrow \mathcal{P}(W)$. Let $p_1, \dots, p_r$ be the proposition symbols which occur in ψ and let $\mathbf{a}_j = \mathbf{a}_{V(p_j)}$ for all $1 \leq j \leq r$. Then*

$$\langle \mathcal{F}, V \rangle \models \psi \text{ if and only if } T_\psi(f_F, \mathbf{a}_1, \dots, \mathbf{a}_r) = \mathbf{1},$$

and furthermore

$$\mathcal{F} \models \psi \text{ if and only if } T_\psi(f_F) \equiv \mathbf{1}. \quad \square$$

The following lemma and theorem show that the translations $T \mapsto \psi_T$ and $\psi \mapsto T_\psi$ are inverses of each other. Thus there is a one-to-one correspondence between functional terms and uniform degree-1 formulas.

Lemma 3. *Let G be a propositional term and let φ be a modal formula of degree 0. Then $G_{\psi_G} = G$ and $\psi_{G_\varphi} = \varphi$.*

Proof. Both claims are proved by a straightforward induction. $\square$

Theorem 4. *Let T be a functional term*

$$G(\mathbf{f}(G_1(\mathbf{x}_1, \dots, \mathbf{x}_r), \dots, \mathbf{f}(G_m(\mathbf{x}_1, \dots, \mathbf{x}_r)))$$

and let ψ be a uniform degree-1 formula

$$\varphi(\Box\varphi_1, \dots, \Box\varphi_m).$$

Then $T_{\psi_T} = T$ and $\psi_{T_\psi} = \psi$.

Proof. The first claim is proved by induction on the construction of G . The cases where G is a constant, or of the form $\neg H$ or $H_1 \wedge H_2$, are straightforward. If $G(\mathbf{x}_1, \dots, \mathbf{x}_m) = \mathbf{x}_j$ for some $1 \leq j \leq m$, then $T = \mathbf{f}(G_j(\mathbf{x}_1, \dots, \mathbf{x}_r))$ and hence $\psi_T = \Box\psi_{G_j}$. Therefore

$$\begin{aligned} T_{\psi_T} &= \mathbf{f}(G_{\psi_{G_j}}(\mathbf{x}_1, \dots, \mathbf{x}_r)) \\ &\stackrel{(*)}{=} \mathbf{f}(G_j(\mathbf{x}_1, \dots, \mathbf{x}_r)) \\ &= T, \end{aligned}$$

where the equality $(*)$ holds by Lemma 3.

The second claim of the theorem is proved similarly by induction on the construction of φ . $\square$

Our main result is that definability of Boolean functions by functional terms corresponds exactly to definability of Scott-Montague frames by uniform degree-1 formulas. With this purpose in mind, we define a class of Scott-Montague frames *corresponding* to a class of Boolean functions and a class of Boolean functions *corresponding* to a class of Scott-Montague frames. Let K be a class of Boolean functions and let $\mathcal{C}$ be a class of Scott-Montague frames. We denote by $\mathcal{C}_K$ the class of Scott-Montague frames

$$\{\mathcal{F}_f \mid f \text{ is a vector-valued Boolean function such that } f \in K\},$$

and $K_{\mathcal{C}}$ denotes the class of Boolean functions

$$\{g \in \Omega \mid g \text{ is a component of } f_{\mathcal{F}}, \text{ where } \mathcal{F} \in \mathcal{C}\}.$$

Theorem 5. (a) Let K be the class of Boolean functions defined by a set $\mathcal{T}$ of functional terms. Then the class $\mathcal{C}_K$ is defined by $\Psi_{\mathcal{T}}$, where $\Psi_{\mathcal{T}} = \{\psi_T \mid T \in \mathcal{T}\}$.

(b) Let $\mathcal{C}$ be the class of Scott-Montague frames defined by a set Ψ of uniform degree-1 formulas. Then the class $K_{\mathcal{C}}$ is defined by $\mathcal{T}_{\Psi}$, where $\mathcal{T}_{\Psi} = \{T_{\psi} \mid \psi \in \Psi\}$.

Proof. For (a), let $\mathcal{F} = \langle W, F \rangle$. If $\mathcal{F} \in \mathcal{C}_K$ then $\mathcal{F} = \mathcal{F}_f$ for some vector-valued Boolean function $f \in K$. Now $T(f) \equiv \mathbf{1}$ for all $T \in \mathcal{T}$, and it follows from Corollary 2 that $\mathcal{F} \models \psi_T$ for all $T \in \mathcal{T}$. Therefore $\mathcal{F} \models \Psi_{\mathcal{T}}$. Suppose that $\mathcal{F} \models \Psi_{\mathcal{T}}$. Then $\mathcal{F} \models \psi_T$ for all $T \in \mathcal{T}$ and we get from Corollary 3 that $T_{\psi_T}(f_F) \equiv \mathbf{1}$ for all $T \in \mathcal{T}$. Since $T_{\psi_T} = T$ by Theorem 4, we have that $T(f_F) \equiv \mathbf{1}$ for all $T \in \mathcal{T}$. Therefore $f_F \in K$, and since $F = F_{f_F}$ we conclude that $\mathcal{F} = \mathcal{F}_{f_F} \in \mathcal{C}_K$.

For (b), suppose first that $g \in K_{\mathcal{C}}$. Now g is a component of a vector-valued Boolean function $f_{\mathcal{F}}$ where $\mathcal{F} \in \mathcal{C}$. Since $\mathcal{F} \models \psi$ for all $\psi \in \Psi$, it follows from Corollary 3 that $T_{\psi}(f_{\mathcal{F}}) \equiv \mathbf{1}$ for all $\psi \in \Psi$. Since g is a component of $f_{\mathcal{F}}$, we get that $T_{\psi}(g) \equiv \mathbf{1}$ for all $\psi \in \Psi$. Suppose then that $T_{\psi}(g) \equiv \mathbf{1}$ for all $\psi \in \Psi$. Let $\psi \in \Psi$ and let g' be a vector-valued Boolean function all of which components are g . Then $T_{\psi}(g') \equiv \mathbf{1}$ and it follows from Corollary 2 that $\mathcal{F}_{g'} \models \psi_{T_{\psi}}$. By Theorem 4, we know that $\psi_{T_{\psi}} = \psi$, and therefore $\mathcal{F}_{g'} \models \psi$. So $\mathcal{F}_{g'} \models \Psi$ and hence $\mathcal{F}_{g'} \in \mathcal{C}$. Since g is a component of g' and $g' = f_{\mathcal{F}_{g'}}$, we conclude that $g \in K_{\mathcal{C}}$. $\square$

Theorem 6. (a) Let K be a class of Boolean functions defined by a set $\mathcal{T}$ of functional terms. Then $K_{\mathcal{C}_K} = K$.

(b) Let $\mathcal{C}$ be a class of Scott-Montague frames defined by a set Ψ of uniform degree-1 formulas. Then $\mathcal{C}_{K_{\mathcal{C}}} = \mathcal{C}$.

Proof. To prove (a), let g be a Boolean function. Since the class K is defined by $\mathcal{T}$, it follows from Theorem 5 (a) that the class $\mathcal{C}_K$ is defined by the set $\Psi_{\mathcal{T}}$. Thus

$$\begin{aligned} g \in K_{\mathcal{C}_K} &\stackrel{(1)}{\Leftrightarrow} g \models T_{\psi} \text{ for all } \psi \in \Psi_{\mathcal{T}} \\ &\Leftrightarrow g \models T_{\psi_T} \text{ for all } T \in \mathcal{T} \\ &\stackrel{(2)}{\Leftrightarrow} g \models T \text{ for all } T \in \mathcal{T} \\ &\Leftrightarrow g \in K, \end{aligned}$$

where the equivalence (1) holds by Theorem 5 (b) and (2) by Theorem 4. Hence $K_{\mathcal{C}_K} = K$.

The proof of (b) is similar. $\square$

To summarize, Theorems 5 and 6 together with Theorem 4 tell us that a class $\mathcal{C}$ of Scott-Montague frames is definable by uniform degree-1 formulas if and only if the corresponding class $K_{\mathcal{C}}$ of Boolean functions is defined by functional terms. Moreover, the defining functional terms are simply the translations of the defining formulas. Similarly a class K of Boolean functions is definable by functional terms if and only if the corresponding class $\mathcal{C}_K$ of Scott-Montague frames is definable by uniform degree-1 formulas, and the defining formulas are the translations of the defining functional terms.

Theorem 1 gave a characterization for definability of classes of Boolean functions by functional terms. Using Theorems 1 and 5, one can easily derive a similar characterization for definability of classes of Scott-Montague frames by uniform degree-1 formulas. That is, a class $\mathcal{C}$ of Scott-Montague frames $\mathcal{F}$ is definable by a set of uniform degree-1 formulas if and only if $\mathcal{C}$ satisfies the following closure condition:

If every component f_i of $f_{\mathcal{F}}$ can be represented as $f_i = g(x_{j_1}, \dots, x_{j_n})$ where g is a component of $f_{\mathcal{F}'}$ for some $\mathcal{F}' \in \mathcal{C}$, then $\mathcal{F} \in \mathcal{C}$.

This condition can be formulated in terms of the set function F , without using the component functions, but in that case the formulation is very technical and therefore more difficult to understand.

5. Kripke correspondence

In addition to the general correspondence, there are also connections between classes of Kripke frames and Boolean clones. This is not so surprising since Kripke frames are just special cases of Scott-Montague frames. In this section we show that several classes of Kripke frames correspond to classes of Scott-Montague frames which are defined by the translations ψ_T of functional terms T defining some Boolean clones.

Let $\mathcal{M}$ and $\mathcal{M}'$ be Kripke or Scott-Montague models with a universe W and a valuation V . We say that the models $\mathcal{M}$ and $\mathcal{M}'$ are *pointwise equivalent* if for all $w \in W$ and for all formulas φ ,

$$\mathcal{M}, w \models \varphi \text{ if and only if } \mathcal{M}', w \models \varphi.$$

It is well known that there is a one-to-one correspondence between Kripke frames and so-called *augmented* Scott-Montague frames such that for all valuations the models based on these frames are pointwise equivalent. We will give this correspondence on finite frames in the next two lemmas and Proposition 2. In this section we consider only finite frames, in which case augmented Scott-Montague frames are just frames in which the axioms $\Box \top$ and $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$ are valid. If we would also consider infinite frames, then we would need an additional condition on Scott-Montague frames. For the correspondence and more information on augmented Scott-Montague frames, see [3].

Lemma 4. *Let $\langle W, R \rangle$ be a Kripke frame and let $\langle W, F_R \rangle$ be a Scott-Montague frame where F_R is given by*

$$F_R(X) = \{w \in W \mid \forall v \in W : wRv \Rightarrow v \in X\} \text{ for all } X \subseteq W.$$

Then for all valuations V , the models $\langle W, R, V \rangle$ and $\langle W, F_R, V \rangle$ are pointwise equivalent. Furthermore the axioms $\Box \top$ and $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$ are valid in the frame $\langle W, F_R \rangle$. $\square$

Let $\langle W, R \rangle$ be a Kripke frame and let $w \in W$. By $R[w]$ we denote the set of R -successors of w , i.e. $R[w] = \{v \in W \mid wRv\}$.

Lemma 5. *Let $\langle W, F \rangle$ be a Scott-Montague frame which validates the axioms $\Box\top$ and $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$ and let $f = f_F$. Let $\langle W, R_F \rangle$ be a Kripke frame where R_F is given by*

$$R_F[w_i] = \{w_j \in W \mid x_j \text{ is an essential variable of } f_i\} \text{ for all } w_i \in W,$$

where f_i is the i th component function of f . Then for all valuations V , the models $\langle W, F, V \rangle$ and $\langle W, R_F, V \rangle$ are pointwise equivalent.

Proof. The clone Λ_1 is defined by the functional terms

$$\mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}),$$

and so, by Theorem 5, the class $\mathcal{C}_{\Lambda_1}$ of Scott-Montague frames is defined by the axioms $\Box\top$ and $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$. Thus $\langle W, F \rangle \in \mathcal{C}_{\Lambda_1}$. Let f_i be a component function of f . Then $f_i \in K_{\mathcal{C}_{\Lambda_1}}$ and it follows from Theorem 6 that $f_i \in \Lambda_1$. Hence, as observed in Remark 1,

$$f_i(\mathbf{a}) = 1 \text{ if and only if } \mathbf{a}[j] = 1 \text{ for every essential variable } x_j \text{ of } f_i.$$

Let V be a valuation and let $\mathcal{M} = \langle W, F, V \rangle$ and $\mathcal{M}' = \langle W, R_F, V \rangle$. We have to show that for every formula φ and for every $w_i \in W$,

$$\mathcal{M}, w_i \models \varphi \text{ if and only if } \mathcal{M}', w_i \models \varphi.$$

We proceed by induction on the construction of φ . Clearly, the claim holds for constants and proposition symbols. Using the induction hypothesis, it is easy to verify that the claim also holds for $\varphi = \neg\psi$ and $\varphi = \psi_1 \wedge \psi_2$. Suppose that $\varphi = \Box\psi$. By the induction hypothesis, we have that $\|\psi\|^{\mathcal{M}} = \|\psi\|^{\mathcal{M}'}$ and therefore

$$\begin{aligned} \mathcal{M}, w_i \models \Box\psi &\Leftrightarrow w_i \in F(\|\psi\|^{\mathcal{M}}) \\ &\Leftrightarrow f_i(\mathbf{a}_{\|\psi\|^{\mathcal{M}}}) = 1 \\ &\Leftrightarrow \mathbf{a}_{\|\psi\|^{\mathcal{M}}}[j] = 1 \text{ for every essential variable } x_j \text{ of } f_i \\ &\Leftrightarrow w_j \in \|\psi\|^{\mathcal{M}} \text{ for every essential variable } x_j \text{ of } f_i \\ &\Leftrightarrow w_j \in \|\psi\|^{\mathcal{M}'} \text{ for every } w_j \in R_F[w_i] \\ &\Leftrightarrow \mathcal{M}', w_i \models \Box\psi. \end{aligned}$$

By the induction principle it follows that the claim holds for every formula φ , which completes the proof of the lemma. $\square$

Lemma 6. *Let $\langle W, R \rangle$ be a Kripke frame and let F_R be defined as in Lemma 4. Let f_i be the i th component of f_{F_R} . Then*

$$x_j \text{ is an essential variable of } f_i \text{ if and only if } w_j \in R[w_i].$$

Proof. By Lemma 4 axioms $\Box\top$ and $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$ are valid in $\langle W, F_R \rangle$. Thus $\langle W, F_R \rangle \in \mathcal{C}_{\Lambda_1}$ and hence $f_i \in \Lambda_1$ as shown in the proof of Lemma 5. Let $W = \{w_1, \dots, w_n\}$. From the definition of the function F_R we get that for every $1 \leq i \leq n$ and $\mathbf{a} \in \mathbb{B}^n$,

$$f_i(\mathbf{a}) = 1 \Leftrightarrow (\forall k : w_i R w_k \Rightarrow \mathbf{a}[k] = 1).$$

From this fact, one can see that if $w_j \notin R[w_i]$ then for all $\mathbf{a} \in \mathbb{B}^n$, the value of $f_i(\mathbf{a})$ does not depend on the j th component of $\mathbf{a}$. Hence, if $w_j \notin R[w_i]$ then x_j is not an essential variable of f_i .

Suppose then that $w_i R w_j$. Since $f_i \in \Lambda_1$, we have that $f_i(\mathbf{1}) = 1$. Let $\mathbf{a} \in \mathbb{B}^n$ such that $\mathbf{a}[j] = 0$ and $\mathbf{a}[k] = 1$ for every $k \neq j$. Since $w_i R w_j$ and $\mathbf{a}[j] = 0$ we get that $f_i(\mathbf{a}) = 0$. Thus $f_i(\mathbf{1}) \neq f_i(\mathbf{a})$ and so x_j is an essential variable of f_i . $\square$

Proposition 2. *Let $\langle W, R \rangle$ be a Kripke frame, let $\langle W, F \rangle \in \mathcal{C}_{\Lambda_1}$ and let F_R and R_F be defined as in Lemmas 4 and 5. Then the translations $R \mapsto F_R$ and $F \mapsto R_F$ are inverses of each other, i.e. $R_{F_R} = R$ and $F_{R_F} = F$.*

Proof. We show first that $R_{F_R} = R$. Let $w_i, w_j \in W$ and let f_i be the i th component of f_{F_R} . Then

$$\begin{aligned} w_j \in R_{F_R}[w_i] &\Leftrightarrow x_j \text{ is an essential variable of } f_i \\ &\Leftrightarrow w_j \in R[w_i], \end{aligned}$$

where the first equivalence holds by the definition of R_{F_R} and the second equivalence holds by Lemma 6. Therefore $R_{F_R} = R$.

To prove the second claim of the proposition, let $w_i \in W$ and $X \subseteq W$. Let $g = (g_1, \dots, g_n)$ be the vector-valued Boolean function determined by F_{R_F} , and let f_i be the i th component of f_F . Since $\langle W, F \rangle \in \mathcal{C}_{\Lambda_1}$, we know that $f_i \in \Lambda_1$. Thus

$$\begin{aligned} w_i \in F_{R_F}(X) &\Leftrightarrow g_i(\mathbf{a}_X) = 1 \\ &\stackrel{(1)}{\Leftrightarrow} \forall k : w_i R_F w_k \Rightarrow \mathbf{a}_X[k] = 1 \\ &\stackrel{(2)}{\Leftrightarrow} \mathbf{a}_X[k] = 1 \text{ for every essential variable } x_k \text{ of } f_i \\ &\stackrel{(3)}{\Leftrightarrow} f_i(\mathbf{a}_X) = 1 \\ &\Leftrightarrow w_i \in F(X). \end{aligned}$$

The equivalence (1) is explained in the proof of Lemma 6, the equivalence (2) holds by the definition of R_F and the equivalence (3) follows from Remark 1. Hence, we conclude that $F_{R_F} = F$. $\square$

Let $\mathcal{K}$ be a class of Kripke frames and let $\mathcal{C}$ be a class of Scott-Montague frames. We say that the class $\mathcal{K}$ *corresponds* to the class $\mathcal{C}$ if there are translations $R \mapsto F_R$ and $F \mapsto R_F$ such that the following conditions hold.

- If $\langle W, R \rangle \in \mathcal{K}$, then $\langle W, F_R \rangle \in \mathcal{C}$, and for all valuations V , the models $\langle W, R, V \rangle$ and $\langle W, F_R, V \rangle$ are pointwise equivalent.
- If $\langle W, F \rangle \in \mathcal{C}$, then $\langle W, R_F \rangle \in \mathcal{K}$, and for all valuations V , the models $\langle W, F, V \rangle$ and $\langle W, R_F, V \rangle$ are pointwise equivalent.
- The translations $R \mapsto F_R$ and $F \mapsto R_F$ are inverses of each other.

Furthermore, we say that the class $\mathcal{K}$ *corresponds* to a Boolean clone C if $\mathcal{K}$ corresponds to the class $\mathcal{C}_C$ of Scott-Montague frames. From Lemmas 4 and 5 and Proposition 2 we obtain the following proposition.

Proposition 3. *Let $\mathcal{K}$ be the class of all Kripke frames. Then $\mathcal{K}$ corresponds to the clone Λ_1 .* $\square$

Remark 2. Let $\langle W, F \rangle$ be a Scott-Montague frame. Since the class $\mathcal{C}_{\Lambda_1}$ is defined by the axioms $\Box \top$ and $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$, it is easy to see that $\langle W, F \rangle \in \mathcal{C}_{\Lambda_1}$ if and only if F satisfies the conditions

$$F(W) = W \text{ and}$$

$$F(X) \cap F(Y) = F(X \cap Y)$$

hold for all $X, Y \subseteq W$.

We also get interesting correspondence results between some subclasses of $\mathcal{K}$ and subclones of Λ_1 . For example, there is a class of Kripke frames which corresponds to the minimal clone I_c . This clone plays a key role when characterizing definability of classes of Boolean functions by functional terms, see Theorem 1 and the discussion before it. For this correspondence result, we need to consider Kripke frames $\mathcal{F} = \langle W, R \rangle$ where R is a *serial* relation, i.e. every $w \in W$ has an R -successor. In the following propositions we use the same translations $R \mapsto F_R$ and $F \mapsto R_F$ as in Lemmas 4 and 5. While Lemmas 4 and 5 and Proposition 2 do the most of the work of the proofs, we need to show that the frames obtained from the translations belong to the right classes of Kripke and Scott-Montague frames in question.

Proposition 4. *Let $\mathcal{K}_s$ be the class of all Kripke frames $\mathcal{F} = \langle W, R \rangle$ such that R is a serial relation. Then $\mathcal{K}_s$ corresponds to the clone Λ_c .*

Proof. The clone Λ_c is defined by the functional terms

$$\neg \mathbf{f}(\mathbf{0}), \mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}),$$

and so the class $\mathcal{C}_{\Lambda_c}$ is defined by the axioms $\neg \Box \perp$, $\Box \top$ and $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$. It is easy to verify that a Scott-Montague frame $\mathcal{F} = \langle W, F \rangle$ validates these axioms if and only if the conditions

- (1) $F(\emptyset) = \emptyset$,
- (2) $F(W) = W$ and
- (3) $F(X) \cap F(Y) = F(X \cap Y)$

hold for every $X, Y \subseteq W$.

Let $\langle W, R \rangle \in \mathcal{K}_s$. Now $R[w] \neq \emptyset$ for every $w \in W$ and it is immediate from the definition of F_R that $F_R(\emptyset) = \emptyset$. It follows from Lemma 4 that $\langle W, F_R \rangle \in \mathcal{C}_{\Lambda_1}$ and thus, by Remark 2, F_R satisfies conditions (2) and (3). Hence $\langle W, F_R \rangle \in \mathcal{C}_{\Lambda_c}$.

Let $\langle W, F \rangle \in \mathcal{C}_{\Lambda_c}$. Now $\langle W, F \rangle \models \neg \Box \perp$ and it follows from Lemma 5 that also $\langle W, R_F \rangle \models \neg \Box \perp$. From this fact, one can easily conclude that R_F has to be a serial relation and hence $\langle W, R_F \rangle \in \mathcal{K}_s$. $\square$

Proposition 5. *Let $\mathcal{K}_f$ be the class of all Kripke frames $\mathcal{F} = \langle W, R \rangle$ such that R is a function. Then $\mathcal{K}_f$ corresponds to the clone I_c .*

Proof. The clone I_c is defined by the functional terms

$$\mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}), \neg \mathbf{f}(\mathbf{x}) \leftrightarrow \mathbf{f}(\neg \mathbf{x}).$$

Hence the class $\mathcal{C}_{I_c}$ is defined by the axioms $\Box \top$, $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$ and $\neg \Box p \leftrightarrow \Box \neg p$. It is easy to check that these axioms are valid in a Scott-Montague frame $\mathcal{F} = \langle W, F \rangle$ if and only if the conditions

- (1) $F(W) = W$,
- (2) $F(X) \cap F(Y) = F(X \cap Y)$ and
- (3) $\overline{F(X)} = F(\overline{X})$

hold for every $X, Y \subseteq W$.

Let $\langle W, R \rangle \in \mathcal{K}_f$. We show that $\overline{F_R(X)} = F_R(\overline{X})$ for all $X \subseteq W$. Let $w \in W$ and $X \subseteq W$. Since $\langle W, R \rangle \in \mathcal{K}_f$, the relation R is a function and we have that

$$\begin{aligned} w \in \overline{F_R(X)} &\Leftrightarrow w \notin F_R(X) \\ &\Leftrightarrow \exists v \in W : wRv \wedge v \in \overline{X} \\ &\Leftrightarrow \forall v \in W : wRv \Rightarrow v \in \overline{X} \\ &\Leftrightarrow w \in F_R(\overline{X}). \end{aligned}$$

Hence F_R satisfies the condition (3). As before, F_R satisfies also the conditions (1) and (2), and so $\langle W, F_R \rangle \in \mathcal{C}_{I_c}$.

Let $\langle W, F \rangle \in \mathcal{C}_{I_c}$. Since $\langle W, F \rangle \models \neg \Box p \leftrightarrow \Box \neg p$, it follows from Lemma 5 that $\langle W, R_F \rangle \models \neg \Box p \leftrightarrow \Box \neg p$. We have to show that R_F is a function. Suppose on the contrary that R_F is not a function. Then either some element of W does not have an R_F -successor or some element of W has at least two of them. Assume first that there is $w \in W$ which has no R_F -successor. Let $\mathcal{M} = \langle W, R_F, V \rangle$ be a Kripke model. Then trivially $\mathcal{M}, w \models \Box \neg p$ and $\mathcal{M}, w \not\models \neg \Box p$. Thus the axiom $\neg \Box p \leftrightarrow \Box \neg p$ is not valid in the frame $\langle W, R_F \rangle$. Assume then that there is $w \in W$ which has two R_F -successors w_1 and w_2 . Consider a Kripke model $\mathcal{M} = \langle W, R_F, V \rangle$ where V is a valuation such that $V(p) = \{w_1\}$. Then $\mathcal{M}, w \models \neg \Box p$ and $\mathcal{M}, w \not\models \Box \neg p$. Hence the axiom $\neg \Box p \leftrightarrow \Box \neg p$ is not valid in the frame $\langle W, R_F \rangle$. So, both cases lead up to a contradiction and therefore R_F is a function. $\square$

Proposition 6. *Let $\mathcal{K}_p$ be the class of all Kripke frames $\mathcal{F} = \langle W, R \rangle$ such that R is a partial function. Then $\mathcal{K}_p$ corresponds to the clone I_1 .*

Proof. The clone I_1 is defined by the functional terms

$$\mathbf{f}(1), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}), \neg \mathbf{f}(\mathbf{x}) \rightarrow \mathbf{f}(\neg \mathbf{x}).$$

Thus the class $\mathcal{C}_{I_1}$ of Scott-Montague frames is defined by the axioms $\Box \top$, $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$ and $\neg \Box p \rightarrow \Box \neg p$. These axioms are valid in a Scott-Montague frame $\mathcal{F} = \langle W, F \rangle$ if and only if the conditions

- (1) $F(W) = W$,
- (2) $F(X) \cap F(Y) = F(X \cap Y)$ and
- (3) $\overline{F(X)} \subseteq F(\overline{X})$

hold for every $X, Y \subseteq W$.

Let $\langle W, F \rangle \in \mathcal{K}_p$. Again F_R satisfies the conditions (1) and (2). We will show that $\overline{F_R(X)} \subseteq F_R(\overline{X})$. Let $X \subseteq W$ and let $w \in \overline{F_R(X)}$. Then $w \notin F_R(X)$ and thus there exists $v \in W$ such that wRv and $v \notin X$. Since R is a partial function, there is only one v such that wRv . Hence we have that for all $v \in W$, wRv implies $v \in \overline{X}$, and therefore $w \in F_R(\overline{X})$. Thus $\overline{F_R(X)} \subseteq F_R(\overline{X})$ and we have proved that $\langle W, F_R \rangle \in \mathcal{C}_{I_1}$.

Let then $\langle W, F \rangle \in \mathcal{C}_{I_1}$. We know that $\langle W, F \rangle \models \neg \Box p \rightarrow \Box \neg p$ and hence, by Lemma 5, $\langle W, R_F \rangle \models \neg \Box p \rightarrow \Box \neg p$. We have to show that $\langle W, R_F \rangle \in \mathcal{K}_p$, i.e. R_F is a partial function. If R_F is not a partial function then there is $w \in W$ which has two R_F -successors. By proceeding as in the proof of Proposition 5, one can

show that the axiom $\neg\Box p \rightarrow \Box\neg p$ is not valid in the frame $\langle W, R_F \rangle$. This is a contradiction and thus R_F has to be a partial function. $\square$

Note that if we had chosen $\Diamond$ as the basic operator in modal logic and given the truth condition of $\Diamond$ in Scott-Montague semantics by the set function, then the class of all Kripke frames would correspond to the clone V_0 , which is the dual of Λ_1 . Similarly, the classes $\mathcal{K}_s$ and $\mathcal{K}_p$ would correspond to the duals of Λ_c and I_1 , respectively, i.e. to the clones V_c and I_0 . Since the clone I_c is self-dual, the class $\mathcal{K}_f$ would still correspond to I_c .

6. Linear classes

In this section we establish further correspondences between classes of Kripke frames and clones of Boolean functions by modifying the standard Kripke semantics. These modifications are based on the parity of the sets of R -successors. As in the previous section, we consider only finite frames.

Throughout this section we make use of the exclusive-or, denoted by $\oplus$ and defined by

$$\varphi \oplus \psi = (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi),$$

as a propositional connective in formulas of modal logic. One can easily verify that $\oplus$ is associative and commutative. Furthermore, for every Kripke or Scott-Montague model $\mathcal{M}$ with universe W and for every $w \in W$, $\mathcal{M}, w \models \varphi_1 \oplus \dots \oplus \varphi_m$ if and only if $\mathcal{M}, w \models \varphi_i$ for an odd number of indices $1 \leq i \leq m$.

Let $X, Y \subseteq W$. By $\overline{X}$ we denote the complement of X with respect to W . We use the symbol $\oplus$ also for the symmetric difference

$$X \oplus Y = (X \setminus Y) \cup (Y \setminus X) = (X \cap \overline{Y}) \cup (Y \cap \overline{X}).$$

Similarly to the exclusive-or, the symmetric difference is associative and commutative.

The frames and models that we shall consider are essentially the same as those used in Kripke semantics. In fact, the semantics that we will introduce differ from Kripke semantics only in the account for the modal operator $\Box$. However, the notion of correspondence between classes of Kripke and Scott-Montague frames must be also modified accordingly. In the sequel we will define three new *truth relations*, $\models_O$, $\models_S$ and $\models_M$, which give the modifications of Kripke semantics that we are going to consider.

Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model and let $\mathcal{M}' = \langle W, F, V \rangle$ be a Scott-Montague model. We say that the models $\mathcal{M}$ and $\mathcal{M}'$ are *pointwise equivalent with respect to $\models_\eta$* , where $\eta \in \{O, S, M\}$, if for all modal formulas φ and for all $w \in W$,

$$\mathcal{M}, w \models_\eta \varphi \text{ if and only if } \mathcal{M}', w \models \varphi.$$

Let $(\mathcal{K}, \models_\eta)$ be a pair where $\mathcal{K}$ is a class of Kripke frames and $\models_\eta$ is the truth relation referring to the semantics which is being used, and let $\mathcal{C}$ be a class of Scott-Montague frames. We say that the pair $(\mathcal{K}, \models_\eta)$ *corresponds* to the class $\mathcal{C}$ if there are translations $R \mapsto F_R$ and $F \mapsto R_F$ such that the following conditions hold.

- If $\langle W, R \rangle \in \mathcal{K}$, then $\langle W, F_R \rangle \in \mathcal{C}$, and for all valuations V , the models $\langle W, R, V \rangle$ and $\langle W, F_R, V \rangle$ are pointwise equivalent with respect to $\models_\eta$.
- If $\langle W, F \rangle \in \mathcal{C}$, then $\langle W, R_F \rangle \in \mathcal{K}$, and for all valuations V , the models $\langle W, R_F, V \rangle$ and $\langle W, F, V \rangle$ are pointwise equivalent with respect to $\models_\eta$.
- The translations $R \mapsto F_R$ and $F \mapsto R_F$ are inverses of each other.

As in the previous section, we say that the pair $(\mathcal{K}, \models_\eta)$ *corresponds* to a clone $\mathcal{C}$ if the pair $(\mathcal{K}, \models_\eta)$ corresponds to the class $\mathcal{C}_C$.

We will provide pairs $(\mathcal{K}, \models_\eta)$ for each of the linear clones L, L_0, L_1, L_c and LS . Consider first the clone L_0 which is defined by the functional term

$$1 \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}).$$

It follows from Theorem 5 that the class $\mathcal{C}_{L_0}$ is defined by the axiom $\top \oplus \Box p \oplus \Box q \oplus \Box(p \oplus q)$. It is easy to verify that this axiom is valid in a Scott-Montague frame $\mathcal{F} = \langle W, F \rangle$ if and only if the condition

$$(1) \quad F(X) \oplus F(Y) = F(X \oplus Y)$$

holds for all $X, Y \subseteq W$.

The following lemma gives rise to one modification of Kripke semantics which will, together with the class of all Kripke frames, correspond to the clone L_0 .

Lemma 7. *Let $F : \mathcal{P}(W) \rightarrow \mathcal{P}(W)$ be a set function satisfying the condition (1). Let R_F be the relation defined by*

$$R_F[w] = \{v \in W \mid w \in F(\{v\})\}$$

for every $w \in W$. Then for all $X \subseteq W$ and $w \in W$,

$$w \in F(X) \text{ if and only if } |R_F[w] \cap X| \text{ is odd.}$$

Proof. Let $X \subseteq W$. First, note that if $X = X_1 \oplus \dots \oplus X_k$, then $w \in X$ if and only if $w \in X_i$ for an odd number of indices i . Since F satisfies the condition (1) and $\oplus$ is associative, it is easy to see that $F(X_1 \oplus \dots \oplus X_k) = F(X_1) \oplus \dots \oplus F(X_k)$. From the condition (1) we also get that $F(\emptyset) = F(\emptyset \oplus \emptyset) = F(\emptyset) \oplus F(\emptyset) = \emptyset$.

Let $w \in W$. Let X' denote the set $R_F[w] \cap X$, and suppose that $|X'|$ is even. We show first that $w \notin F(X')$. If $X' = \emptyset$ then $w \notin F(X')$, since $F(\emptyset) = \emptyset$. Let $X' = \{w_1, \dots, w_{2m}\}$ for some $m \geq 1$. In other words, $X' = \{w_1\} \oplus \dots \oplus \{w_{2m}\}$ and it follows from the definitions of $R_F[w]$ and X' that $w \in F(\{w_i\})$ for all $1 \leq i \leq 2m$. From this fact and the observations above, we get that

$$w \notin F(\{w_1\}) \oplus \dots \oplus F(\{w_{2m}\}) = F(\{w_1\} \oplus \dots \oplus \{w_{2m}\}) = F(X').$$

Now we show that $w \notin F(X \setminus X')$. The case $(X \setminus X') = \emptyset$ is clear. Suppose that $(X \setminus X') = \{w_1, \dots, w_k\}$ for some $k \geq 1$. By the definitions of $R_F[w]$ and X' we know that $w \notin F(\{w_i\})$ for all $1 \leq i \leq k$. Thus

$$w \notin F(\{w_1\}) \oplus \dots \oplus F(\{w_k\}) = F(\{w_1\} \oplus \dots \oplus \{w_k\}) = F(X \setminus X').$$

So $w \notin F(X')$ and $w \notin F(X \setminus X')$, and hence $w \notin F(X') \oplus F(X \setminus X')$. Since $X = X' \oplus (X \setminus X')$, it follows from the condition (1) that

$$w \notin F(X') \oplus F(X \setminus X') = F(X' \oplus (X \setminus X')) = F(X).$$

Suppose now that $|X'|$ is odd. Let $X' = \{w_1, \dots, w_{2m-1}\}$ for some $m \geq 1$. Then $X' = \{w_1\} \oplus \dots \oplus \{w_{2m-1}\}$ and since $w \in F(\{w_i\})$ for all $1 \leq i \leq 2m-1$, we have that

$$w \in F(\{w_1\}) \oplus \dots \oplus F(\{w_{2m-1}\}) = F(\{w_1\} \oplus \dots \oplus \{w_{2m-1}\}) = F(X').$$

Furthermore $w \notin F(X \setminus X')$ and thus

$$w \in F(X') \oplus F(X \setminus X') = F(X' \oplus (X \setminus X')) = F(X).$$

Hence, we have proved the lemma. $\square$

The previous lemma motivates the following semantics. Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model and let $w \in W$. We define the truth relation $\models_O$ inductively for modal formulas. For constants, proposition symbols and Boolean connectives $\models_O$ is defined as $\models$ in Kripke semantics. Let φ be a modal formula. For the modal operator $\Box$, the truth relation is defined by

$$\mathcal{M}, w \models_O \Box \varphi \text{ iff } |R[w] \cap \|\varphi\|_O^{\mathcal{M}}| \text{ is odd}$$

where $\|\varphi\|_O^{\mathcal{M}}$ denotes the truth set of φ in the model $\mathcal{M}$ with respect to $\models_O$.

We denote by $\mathcal{L}_0$ the pair $(\mathcal{K}, \models_O)$ where $\mathcal{K}$ is the class of all Kripke frames. We denote by $\mathcal{L}_c$ the pair $(\mathcal{K}_O, \models_O)$ where $\mathcal{K}_O$ denotes the class of all Kripke frames $\langle W, R \rangle$ such that every $w \in W$ has an odd number of R -successors.

Proposition 7. *The pair $\mathcal{L}_0$ corresponds to the clone L_0 .*

Proof. As mentioned before Lemma 7, the class $\mathcal{C}_{L_0}$ consists of exactly those Scott-Montague frames $\langle W, F \rangle$ for which

$$(1) \quad F(X) \oplus F(Y) = F(X \oplus Y)$$

holds for all $X, Y \subseteq W$.

The translation $R \mapsto F_R$ is given by

$$F_R(X) = \{w \in W \mid |R[w] \cap X| \text{ is odd}\}$$

for all $X \subseteq W$ and the translation $F \mapsto F_F$ is defined as in Lemma 7.

Let $\langle W, R \rangle \in \mathcal{K}$. We show that $\langle W, F_R \rangle \in \mathcal{C}_{L_0}$, i.e. F_R satisfies the condition (1). Let $X, Y \subseteq W$ and let $w \in W$. We have to show that $w \in F_R(X) \oplus F_R(Y)$ if and only if $w \in F_R(X \oplus Y)$. It is obvious that the parities of the sets $R[w] \cap X$, $R[w] \cap Y$ and $R[w] \cap (X \oplus Y)$ depend only on the parities of the sets $R[w] \cap (X \setminus Y)$, $R[w] \cap (Y \setminus X)$ and $R[w] \cap (X \cap Y)$. So there are eight different cases. We consider here one of them, the other cases follow similarly. So suppose that $|R[w] \cap (X \setminus Y)|$ and $|R[w] \cap (Y \setminus X)|$ are odd and $|R[w] \cap (X \cap Y)|$ is even. Then $|R[w] \cap X|$ and $|R[w] \cap Y|$ are odd and $|R[w] \cap (X \oplus Y)|$ is even. So $w \in F_R(X)$, $w \in F_R(Y)$ and $w \notin F_R(X \oplus Y)$. Thus $w \notin F_R(X) \oplus F_R(Y)$ and $w \notin F_R(X \oplus Y)$, and hence $w \in F_R(X) \oplus F_R(Y)$ if and only if $w \in F_R(X \oplus Y)$. The rest of the cases can be proved similarly, and therefore $\langle W, F_R \rangle \in \mathcal{C}_{L_0}$.

Let V be a valuation and let $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W, F_R, V \rangle$. We show that the claim

$$\mathcal{M}, w \models_O \varphi \text{ if and only if } \mathcal{M}', w \models \varphi$$

holds for every modal formula φ and $w \in W$. This is done by induction on the construction of φ . The base cases, $\varphi = \perp$, $\varphi = \top$ or $\varphi = p$, follow immediately from the definition of $\models_O$, and the cases, where $\varphi = \neg\psi$ or $\varphi = \psi_1 \wedge \psi_2$, can be easily verified by making use of the induction hypothesis. So let $\varphi = \Box\psi$ and suppose that ψ satisfies the claim. Then $\|\psi\|_O^{\mathcal{M}} = \|\psi\|^{\mathcal{M}'}$ and

$$\begin{aligned} \mathcal{M}, w \models_O \Box\psi &\Leftrightarrow |R[w] \cap \|\psi\|_O^{\mathcal{M}}| \text{ is odd} \\ &\Leftrightarrow |R[w] \cap \|\psi\|^{\mathcal{M}'}| \text{ is odd} \\ &\Leftrightarrow w \in F_R(\|\psi\|^{\mathcal{M}'}) \\ &\Leftrightarrow \mathcal{M}', w \models \Box\psi. \end{aligned}$$

Thus the models $\mathcal{M}$ and $\mathcal{M}'$ are pointwise equivalent with respect to $\models_O$.

Let then $\langle W, F \rangle \in \mathcal{C}_{L_0}$. Since R_F is a binary relation on W , we have that $\langle W, R_F \rangle \in \mathcal{K}$.

Let V be a valuation and let $\mathcal{M} = \langle W, F, V \rangle$ and $\mathcal{M}' = \langle W, R_F, V \rangle$. We show that for all formulas φ and for all $w \in W$,

$$\mathcal{M}, w \models \varphi \text{ if and only if } \mathcal{M}', w \models_O \varphi.$$

This is again done by induction. We consider only the case $\varphi = \Box\psi$, the other cases are straightforward. So let $\varphi = \Box\psi$ where ψ satisfies the claim above. Then $\|\psi\|^{\mathcal{M}} = \|\psi\|_O^{\mathcal{M}'}$ and

$$\begin{aligned} \mathcal{M}, w \models \Box\psi &\Leftrightarrow w \in F(\|\psi\|^{\mathcal{M}}) \\ &\stackrel{(*)}{\Leftrightarrow} |R_F[w] \cap \|\psi\|^{\mathcal{M}}| \text{ is odd} \\ &\Leftrightarrow |R_F[w] \cap \|\psi\|_O^{\mathcal{M}'}| \text{ is odd} \\ &\Leftrightarrow \mathcal{M}', w \models_O \Box\psi, \end{aligned}$$

where the equivalence $(*)$ holds by Lemma 7. Hence the models $\mathcal{M}'$ and $\mathcal{M}$ are pointwise equivalent with respect to $\models_O$.

Let $\langle W, R \rangle \in \mathcal{K}$ and let $\langle W, F \rangle \in \mathcal{C}_{L_0}$. We still have to prove that $R_{F_R} = R$ and $F_{R_F} = F$. Let $w \in W$ and $X \subseteq W$. Since for all $v \in W$,

$$\begin{aligned} v \in R_{F_R}[w] &\Leftrightarrow w \in F_R(\{v\}) \\ &\Leftrightarrow |R[w] \cap \{v\}| \text{ is odd} \\ &\Leftrightarrow v \in R[w], \end{aligned}$$

and

$$\begin{aligned} w \in F_{R_F}(X) &\Leftrightarrow |R_F[w] \cap X| \text{ is odd} \\ &\stackrel{(*)}{\Leftrightarrow} w \in F(X), \end{aligned}$$

where the equivalence $(*)$ holds again by Lemma 7, we can conclude that $R_{F_R} = R$ and $F_{R_F} = F$. $\square$

Proposition 8. *The pair $\mathcal{L}_c$ corresponds to the clone L_c .*

Proof. The proof of Proposition 8 follows exactly the same steps as in the proof of Proposition 7. The clone L_c is defined by the functional terms

$$\mathbf{f}(\mathbf{1}), \mathbf{1} \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}),$$

and so the class $\mathcal{C}_{L_c}$ is defined by the axioms $\Box\top$ and $\top \oplus \Box p \oplus \Box q \oplus \Box(p \oplus q)$. These axioms are valid in a Scott-Montague frame $\mathcal{F} = \langle W, F \rangle$ if and only if the conditions

- (i) $F(W) = W$ and
- (ii) $F(X) \oplus F(Y) = F(X \oplus Y)$

hold for all $X, Y \subseteq W$.

The translations $R \mapsto F_R$ and $F \mapsto R_F$ are the same as in Proposition 7.

Let $\langle W, R \rangle \in \mathcal{K}_O$. We show that $\langle W, F_R \rangle \in \mathcal{C}_{L_c}$. Since $|R[w]|$ is odd for every $w \in W$ it is clear that $F_R(W) = W$. By Lemma 7, the function F_R also satisfies the condition (ii). Thus $\langle W, F_R \rangle \in \mathcal{C}_{L_c}$.

Let $\langle W, F \rangle \in \mathcal{C}_{L_c}$. We have to show that $|R_F[w]|$ is odd for every $w \in W$. Let $w \in W$. Since $F(W) = W$, it follows from Lemma 7 that $R_F[w] \cap W$ has an odd number of elements. Since $R_F[w] \cap W = R_F[w]$, we have that $|R_F[w]|$ is odd, and therefore $\langle W, R_F \rangle \in \mathcal{K}_O$.

The remainder of the proof is similar to the proof of Proposition 7. $\square$

Let F be a set function $\mathcal{P}(W) \rightarrow \mathcal{P}(W)$. For the rest of the section we define the translation $F \mapsto R_F$ by

$$R_F[w] = \{v \in W \mid w \in F(\{v\}) \Leftrightarrow w \notin F(\emptyset)\}$$

for all $w \in W$. Note that this definition coincides with that in Lemma 7 because in that case $F(\emptyset) = \emptyset$.

The following lemma will be useful in the sequel.

Lemma 8. *Let F be a set function $\mathcal{P}(W) \rightarrow \mathcal{P}(W)$ which satisfies the condition*

$$(2) \quad F(\emptyset) \oplus F(X) \oplus F(Y) = F(X \oplus Y)$$

for all $X, Y \subseteq W$. Let $X \subseteq W$ and let $w \in W$.

- (a) *If $w \notin F(\emptyset)$, then $w \in F(X)$ if and only if $|R_F[w] \cap X|$ is odd.*
- (b) *If $w \in F(\emptyset)$, then $w \in F(X)$ if and only if $|R_F[w] \cap X|$ is even.*

Proof. As mentioned before, if $X = X_1 \oplus \dots \oplus X_k$ then $w \in X$ if and only if $w \in X_i$ for odd number of indices i . If $w \notin F(\emptyset)$ then $R_F[w] = \{v \in W \mid w \in F(\{v\})\}$. We denote by X' the set $X \cap \{v \in W \mid w \in F(\{v\})\}$. And if $w \in F(\emptyset)$ then $R_F[w] = \{v \in W \mid w \notin F(\{v\})\}$. For the set $X \cap \{v \in W \mid w \notin F(\{v\})\}$ we use a notation X'' .

To prove (a), suppose that $w \notin F(\emptyset)$. It follows from the condition (2) that

$$w \in F(X) \oplus F(Y) \text{ if and only if } w \in F(X \oplus Y)$$

and from this, it is easy to see that $w \in F(X_1 \oplus \dots \oplus X_k)$ if and only if $w \in F(X_1) \oplus \dots \oplus F(X_k)$. Now the rest of the proof of (a) is essentially the same as the proof of Lemma 7.

For the part (b), suppose that $w \in F(\emptyset)$. By using the condition (2), it is easy to see that $w \in F(X')$. In addition, an easy induction on the cardinality of X'' shows that $w \in F(X'')$ if and only if $|X''|$ is even. Since $w \in F(\emptyset)$ and $w \in F(X')$, we have that $w \in F(\emptyset) \oplus F(X') \oplus F(X'')$ if and only if $w \in F(X' \oplus X'')$. Since $X' \oplus X'' = X$,

we get by the condition (2) that

$$\begin{aligned} w \in F(X) &\Leftrightarrow w \in F(X' \oplus X'') \\ &\Leftrightarrow w \in F(\emptyset) \oplus F(X') \oplus F(X'') \\ &\Leftrightarrow w \in F(X'') \\ &\Leftrightarrow |X''| \text{ is even.} \end{aligned}$$

This proves (b). $\square$

Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model and let $w \in W$. The truth relation $\models_S$ is defined as $\models_O$ except for the modal operator $\Box$, for which we define

$$\mathcal{M}, w \models_S \Box \varphi \text{ iff the sets } R[w] \text{ and } R[w] \cap \|\varphi\|_S^{\mathcal{M}} \text{ have the same parity,}$$

where $\|\varphi\|_S^{\mathcal{M}}$ denotes the truth set of φ in the model $\mathcal{M}$ with respect to $\models_S$.

We denote by $\mathcal{L}_1$ the pair $(\mathcal{K}, \models_S)$ where $\mathcal{K}$ is the class of all Kripke frames.

Proposition 9. *The pair $\mathcal{L}_1$ corresponds to the clone L_1 .*

Proof. The clone L_1 is defined by the functional terms

$$\mathbf{f}(\mathbf{1}), \mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}).$$

Thus the class $\mathcal{C}_{L_1}$ is defined by the axioms $\Box \top$ and $\top \oplus \Box \perp \oplus \Box p \oplus \Box q \oplus \Box(p \oplus q)$. It is easy to verify that these axioms are valid in a Scott-Montague frame $\mathcal{F} = \langle W, F \rangle$ if and only if the conditions

- (i) $F(W) = W$ and
- (ii) $F(\emptyset) \oplus F(X) \oplus F(Y) = F(X \oplus Y)$

hold for all $X, Y \subseteq W$.

The translation $R \mapsto F_R$ is given by

$$F_R(X) = \{w \in W \mid |R[w] \cap X| \text{ is odd} \Leftrightarrow |R[w]| \text{ is odd}\}$$

for all $X \subseteq W$.

Let $\langle W, R \rangle \in \mathcal{K}$. We have to show that $\langle W, F_R \rangle \in \mathcal{C}_{L_1}$, i.e. F_R satisfies the conditions (i) and (ii). It is immediate from the definition of F_R that $F_R(W) = W$. It also follows straightforwardly from the definition of F_R that for all $w \in W$, $w \in F_R(\emptyset)$ if and only if $|R[w]|$ is even. Using this fact one can prove, similarly as in the proof of Proposition 7, that for all $X, Y \subseteq W$ and for all $w \in W$,

$$w \in F_R(\emptyset) \oplus F_R(X) \oplus F_R(Y) \text{ if and only if } w \in F_R(X \oplus Y).$$

Thus F_R satisfies also the condition (ii) and therefore $\langle W, F_R \rangle \in \mathcal{C}_{L_1}$.

Let V be a valuation function and let $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W, F_R, V \rangle$. Similarly, as in the proof of Proposition 7, it can be proved that the models $\mathcal{M}$ and $\mathcal{M}'$ are pointwise equivalent with respect to $\models_S$.

Let $\langle W, F \rangle \in \mathcal{C}_{L_1}$. Then R_F is a relation on W and $\langle W, R_F \rangle \in \mathcal{K}$.

Let V be a valuation and let $\mathcal{M} = \langle W, F, V \rangle$ and $\mathcal{M}' = \langle W, R_F, V \rangle$. By using Lemma 8, one can easily show that the models $\mathcal{M}'$ and $\mathcal{M}$ are pointwise equivalent with respect to $\models_S$.

Let $\langle W, R \rangle \in \mathcal{K}$ and $\langle W, F \rangle \in \mathcal{C}_{L_1}$. Our final task is to prove that $R_{F_R} = R$ and $F_{R_F} = F$. Let $w, v \in W$. Suppose that $v \in R_{F_R}[w]$ and $w \in F_R(\emptyset)$. Now it

follows from the definition of R_{F_R} that $w \notin F_R(\{v\})$. From the definition of F_R we get that $|R[w] \cap \{v\}|$ is odd if and only if $|R[w]|$ is even. Since $w \in F_R(\emptyset)$ and $|R[w] \cap \emptyset|$ is even, it has to be the case that $|R[w]|$ is even. Thus $|R[w] \cap \{v\}|$ is odd and hence $v \in R[w]$. A similar proof shows that $v \in R[w]$ if $w \notin F_R(\emptyset)$.

Now suppose that $v \in R[w]$, i.e. $|R[w] \cap \{v\}|$ is odd. If $|R[w]|$ is also odd then $w \in F_R(\{v\})$ and $w \notin F_R(\emptyset)$ and thus $v \in R_{F_R}[w]$ by the definition of R_{F_R} . Also, if $|R[w]|$ is even, it is easy to see that $v \in R_{F_R}[w]$. Hence we have proved that $R_{F_R} = R$.

Similarly we can show that $F_{R_F} = F$. $\square$

For the remaining two linear clones, L and LS , we consider Kripke frames with an additional unary relation $U \subseteq W$. Let $\mathcal{K}^U$ be the class of all frames $\mathcal{F} = \langle W, R, U \rangle$ where $U \subseteq W$. For the models based on these frames, we define a truth relation $\models_M$ inductively as follows. Let $\mathcal{M} = \langle \mathcal{F}, V \rangle$ be a model where $\mathcal{F} \in \mathcal{K}^U$ and V is a valuation function, and let $w \in W$. For constants, proposition symbols and Boolean connectives $\models_M$ is defined as $\models$ in Kripke semantics. Let φ be a modal formula. If $w \in U$ then

$$\mathcal{M}, w \models_M \Box \varphi \text{ if and only if } |R[w] \cap \|\varphi\|_M^{\mathcal{M}}| \text{ is odd,}$$

and if $w \notin U$ then

$$\mathcal{M}, w \models_M \Box \varphi \text{ if and only if } |R[w] \cap \|\varphi\|_M^{\mathcal{M}}| \text{ is even.}$$

Here $\|\varphi\|_M^{\mathcal{M}}$ denotes the truth set of φ in the model $\mathcal{M}$ with respect to $\models_M$.

We denote by $\mathcal{L}$ the pair $(\mathcal{K}^U, \models_M)$. Let $\mathcal{K}_O^U$ be the class of all frames $\langle W, R, U \rangle$ in $\mathcal{K}^U$ such that every $w \in W$ has an odd number of R -successors. We denote by $\mathcal{L}_S$ the pair $(\mathcal{K}_O^U, \models_M)$.

The addition of a unary relation also affects the definition of corresponding classes. Instead of translating a binary relation to a set function and vice versa, we also have to take into account the unary relation defined on the universe. Let $\mathcal{K}'$ be a subclass of $\mathcal{K}^U$ and let $\mathcal{C}$ be a class of Scott-Montague frames. We say that the pair $(\mathcal{K}', \models_M)$ *corresponds* to the class $\mathcal{C}$ if there are translations $(R, U) \mapsto F_{(R, U)}$ and $F \mapsto (R_F, U_F)$ such that the following conditions hold.

- If $\langle W, R, U \rangle \in \mathcal{K}'$, then $\langle W, F_{(R, U)} \rangle \in \mathcal{C}$, and for all valuations V , the models $\langle W, R, U, V \rangle$ and $\langle W, F_{(R, U)}, V \rangle$ are pointwise equivalent with respect to $\models_M$.
- If $\langle W, F \rangle \in \mathcal{C}$, then $\langle W, R_F, U_F \rangle \in \mathcal{K}'$, and for all valuations V , the models $\langle W, R_F, U_F, V \rangle$ and $\langle W, F, V \rangle$ are pointwise equivalent with respect to $\models_M$.
- $R_{F_{(R, U)}} = R$, $U_{F_{(R, U)}} = U$ and $F_{(R_F, U_F)} = F$.

Furthermore, we say that the pair $(\mathcal{K}', \models_M)$ *corresponds* to a clone C if the pair $(\mathcal{K}', \models_M)$ corresponds to the class $\mathcal{C}_C$.

Proposition 10. *The pair $\mathcal{L}$ corresponds to the clone L .*

Proof. The clone L is defined by the functional term

$$\mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}).$$

Thus the class $\mathcal{C}_L$ of Scott-Montague frames is definable by the axiom $\top \oplus \Box \perp \oplus \Box p \oplus \Box q \oplus \Box(p \oplus q)$. This axiom is valid in a Scott-Montague frame $\mathcal{F} = \langle W, F \rangle$ if and only if the condition

$$(i) \quad F(\emptyset) \oplus F(X) \oplus F(Y) = F(X \oplus Y)$$

holds for all $X, Y \subseteq W$.

The translations $(R, U) \mapsto F_{(R,U)}$ and $F \mapsto (R_F, U_F)$ are defined as follows. For all $X \subseteq W$,

$$F_{(R,U)}(X) = \{w \in W \mid w \in U \text{ and } |R[w] \cap X| \text{ is odd}\} \cup \{w \in W \mid w \notin U \text{ and } |R[w] \cap X| \text{ is even}\},$$

R_F is defined as before Lemma 8 and $U_F = \{w \in W \mid w \notin F(\emptyset)\}$.

Let $\langle W, R, U \rangle \in \mathcal{K}^U$. As in the proof of Proposition 7, we can show that for all $X, Y \subseteq W$ and for all $w \in W$,

$$w \in F_{(R,U)}(\emptyset) \oplus F_{(R,U)}(X) \oplus F_{(R,U)}(Y) \text{ if and only if } w \in F_{(R,U)}(X \oplus Y).$$

Thus $F_{(R,U)}$ satisfies the condition (i) and $\langle W, F_{(R,U)} \rangle \in \mathcal{C}_L$.

Let V be a valuation function and let $\mathcal{M} = \langle W, R, U, V \rangle$ and $\mathcal{M}' = \langle W, F_{(R,U)}, V \rangle$. We show that for all modal formulas φ and for all $w \in W$,

$$\mathcal{M}, w \models_M \varphi \text{ if and only if } \mathcal{M}', w \models \varphi.$$

The proof is by induction on the construction of φ . We consider only the case where φ is of the form $\Box\psi$, the other ones are straightforward. So let $\varphi = \Box\psi$ and suppose that the claim holds for ψ , whence $\|\psi\|_M^{\mathcal{M}} = \|\psi\|^{\mathcal{M}'}$. Let $w \in U$. Then

$$\begin{aligned} \mathcal{M}, w \models_M \Box\psi &\Leftrightarrow |R[w] \cap \|\psi\|_M^{\mathcal{M}}| \text{ is odd} \\ &\Leftrightarrow |R[w] \cap \|\psi\|^{\mathcal{M}'}| \text{ is odd} \\ &\Leftrightarrow w \in F_{(R,U)}(\|\psi\|^{\mathcal{M}'}) \\ &\Leftrightarrow \mathcal{M}', w \models \Box\psi. \end{aligned}$$

Similarly, $\mathcal{M}, w \models_M \Box\psi$ if and only if $\mathcal{M}', w \models \Box\psi$, whenever $w \notin U$. Thus the models $\mathcal{M}$ and $\mathcal{M}'$ are pointwise equivalent with respect to $\models_M$.

Let $\langle W, F \rangle \in \mathcal{C}_L$. Since R_F is a binary relation on W and U_F is a subset of W , we have that $\langle W, R_F, U_F \rangle \in \mathcal{K}^U$.

Let V be a valuation and let $\mathcal{M} = \langle W, F \rangle$ and $\mathcal{M}' = \langle W, R_F, U_F \rangle$. As above, it can be shown that the models $\mathcal{M}'$ and $\mathcal{M}$ are pointwise equivalent with respect to $\models_M$.

Let $\langle W, R, U \rangle \in \mathcal{K}^U$ and $\langle W, F \rangle \in \mathcal{C}_L$. We still have show that $R_{F_{(R,U)}} = R$, $U_{F_{(R,U)}} = U$ and $F_{(R_F, U_F)} = F$. Let $w, v \in W$. It is easy to see that

$$w \in U \Leftrightarrow w \notin F_{(R,U)}(\emptyset) \Leftrightarrow w \in U_{F_{(R,U)}}.$$

Also, it is not difficult to see that

$$\begin{aligned} v \in R[w] &\Leftrightarrow w \in \overline{U} \oplus F_{(R,U)}(\{v\}) \\ &\Leftrightarrow w \in F_{(R,U)}(\emptyset) \oplus F_{(R,U)}(\{v\}) \\ &\Leftrightarrow v \in R_{F_{(R,U)}}[w]. \end{aligned}$$

Thus $R_{F_{(R,U)}} = R$ and $U_{F_{(R,U)}} = U$.

Let $X \subseteq W$ and $w \in W$. Suppose that $w \in F_{(R_F, U_F)}(X)$. Let $w \in F_{(R_F, U_F)}(\emptyset)$. Then $w \notin U_F$ and furthermore $w \in F(\emptyset)$. Since $w \notin U_F$ and $w \in F_{(R_F, U_F)}(X)$, we can conclude that $|R_F[w] \cap X|$ is even. Since $w \in F(\emptyset)$, it follows from Lemma 8 that $w \in F(X)$. Similarly we can prove that $w \in F(X)$ if $w \notin F_{(R_F, U_F)}(\emptyset)$.

Let $w \in F(X)$. Assume first that $w \in F(\emptyset)$. Then $w \notin U_F$. Since $w \in F(\emptyset)$ and $w \in F(X)$, it follows from Lemma 8 that $|R_F[w] \cap X|$ is even, and hence we get that $w \in F_{(R_F, U_F)}(X)$. A similar proof shows that $w \in F_{(R_F, U_F)}(X)$ when $w \notin F(\emptyset)$. Thus $F_{(R_F, U_F)} = F$. $\square$

With similar methods, using the same translations $(R, U) \mapsto F_{(R, U)}$ and $F \mapsto (R_F, U_F)$ as in the proof of Proposition 10, we can also prove the following proposition which completes the cases of linear clones.

Proposition 11. *The pair $\mathcal{L}_S$ corresponds to the clone LS .* $\square$

7. Conclusion

We have established a complete correspondence between definability of Boolean functions by functional terms and definability of finite Scott-Montague frames by uniform degree-1 formulas. We introduced a bijective translation between functional terms and uniform degree-1 formulas and, based on this, we showed that a class of Boolean functions is defined by functional terms if and only if the corresponding class of Scott-Montague frames is defined by the translations of these functional terms, and vice versa. Furthermore, we characterized classes of Scott-Montague frames corresponding to the clone Λ_1 and to some particular subclones of Λ_1 by classes of Kripke frames. By modifying Kripke semantics, we were also able to characterize the classes of Scott-Montague frames corresponding to the linear clones L , L_0 , L_1 , L_c and LS .

At this point the obvious question is: For which of the remaining clones the corresponding class of Scott-Montague frames can be characterized by further modifications of Kripke semantics? In the bottom of Post Lattice, we still lack characterizations for the classes of frames corresponding to the clones V , V_1 , Λ , Λ_0 , $\Omega(1)$, I and I^* . Our intention is to study these cases in a subsequent paper. It seems that some of them can be obtained from the results in this paper by using some minor technical tricks.

However, some interesting clones, such as S , M and SM , seem to be more difficult to handle. In the case of the bottom clones and linear clones, the associativity of the generating functions of the clones seems to be crucial. There is a qualitative difference when associativity does not hold. For example, the clone SM is generated by the majority of three variables, which is not an associative function. Nevertheless, we are going to study in a follow-up paper whether the classes of Scott-Montague frames corresponding to the clones in the top of Post Lattice can also be characterized by modifying Kripke semantics in a different way.

REFERENCES

- [1] P. Blackburn, M. de Rijke, Y. Venema. *Modal Logic*, Cambridge University Press, Cambridge, 2001.
- [2] F. M. Brown. *Boolean Reasoning: The Logic of Boolean Equations*, 2nd ed., Dover, Mineola, New York, 2003.
- [3] B. F. Chellas. *Modal Logic, an Introduction*, Cambridge University Press, 1980.
- [4] M. Couceiro, S. Foldes. “Function Class Composition, Relational Constraints and Stability under Compositions with Clones”, Rutcor Research Report 22-2004, Rutgers University, <http://rutcor.rutgers.edu/~rrr/>.
- [5] M. Couceiro, S. Foldes. “Constraints, Functional Equations, Definability of Function Classes, and Functions of Boolean Variables”, Rutcor Research Report 36-2004, Rutgers University, <http://rutcor.rutgers.edu/~rrr/>.
- [6] M. Couceiro, S. Foldes, E. Lehtonen. “Composition of Post Classes and Normal Forms of Boolean Functions”, Rutcor Research Report 05-2005, Rutgers University, <http://rutcor.rutgers.edu/~rrr/>.
- [7] O. Ekin, S. Foldes, P. L. Hammer, L. Hellerstein. “Equational Characterizations of Boolean Functions Classes”, *Discrete Mathematics*, **211** (2000) 27–51.
- [8] S. Foldes, G. Pogosyan. “Post Classes Characterized by Functional Terms”, *Discrete Applied Mathematics* **142** (2004) 35–51.
- [9] B. Hansson, P. Gärdenfors. “A Guide to Intensional Semantics”, *Modality, Morality and Other Problems of Sense and Nonsense: Essays Dedicated to Sören Halldén*, ss. 151-67, Gleerup, Lund, 1973.
- [10] S. W. Jablonski, G. P. Gawrilow, W. B. Kudrjawzew. *Boolesche Funktionen und Postsche Klassen*, Vieweg, Braunschweig, 1970.
- [11] N. Pippenger. *Theories of Computability*, Cambridge University Press, Cambridge, 1997.
- [12] N. Pippenger. “Galois Theory for Minors of Finite Functions”, *Discrete Mathematics*, **254** (2002) 405–419.
- [13] G. Pogosyan. “Classes of Boolean Functions Defined by Functional Terms”, *Multiple Valued Logic*, **7** (2002) 417–448.
- [14] E. L. Post. “The Two-Valued Iterative Systems of Mathematical Logic”, *Annals of Mathematical Studies* **5** (1941) 1–122.
- [15] M. Reschke, K. Denecke. “Ein neuer Beweis für die Ergebnisse von E. L. Post über abgeschlossene Klassen Boolescher Funktionen”, *Elektronische Informationsverarbeitung und Kybernetik* **25**(7) (1989) 361–380.
- [16] K. Segerberg. “An Essay in Classical Modal Logic”, *Filosofiska Studier utgivna av Filosofiska Föreningen och Filosofiska Institutionen vid Uppsala Universitet*, nr 13, Uppsala, 1971.
- [17] M. H. Stone. “Subsumption of the Theory of Boolean Algebras under the Theory of Rings”, *Proceedings of the National Academy of Sciences USA*, **31** (1935) 103–105.
- [18] A. B. Ugolnikov. “Closed Post Classes” (in Russian), *Izvestiya VUZ. Matematika* **32** (1988) 79–88.
- [19] A. Virtanen. “Essee Deskriptiivisistä Modaalilogiikasta”, manuscript, Tampere, 1994.
- [20] I. E. Zverovich. “Characterizations of Closed Classes of Boolean Functions in Terms of Forbidden Subfunctions and Post Classes”, Rutcor Research Report 17-2000, Rutgers University, <http://rutcor.rutgers.edu/~rrr/>. To appear in *Discrete Applied Mathematics*.

DEPARTMENT OF MATHEMATICS, STATISTICS AND PHILOSOPHY, UNIVERSITY OF TAMPERE, KALEVANTIE 4, 33014 TAMPERE, FINLAND