Two-Sided Matching with One-Sided Preferences*

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Abstract

A stylized fact of most centralized matching markets (for example school choice problems) is that participants tend to submit preference lists over a small number of potential matches. This paper is based on the premise that this fact provides additional information that can be used to match agents. In a school choice context we show that considering only schools' priorities and the set of acceptable schools for each student (but not their submitted preferences), we can restrict the set of possible stable matchings that can arise for *any* preference profile of the students that leaves the set of acceptable schools unchanged. Our approach consists of linking Hall's marriage condition to stable matchings and offers a methodology that can be used for data analysis when one has access to the preferences of only one side of the market. We use our technique to propose a new mechanism for school choice problems that Pareto dominates the Student Optimal Stable Mechanism, and show that in this mechanism it

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is a dominant strategy to always rank the schools that have been revealed as acceptable in the same order as in the true preferences.

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1. INTRODUCTION

This paper contributes to the further understanding of stability in two-sided matching markets. There is now a large consensus in the matching literature that stability, which is an important equilibrium notion for matching markets, is one of the most desirable properties we should seek when designing new market institutions.¹ Roth (1990), and later Kagel and Roth (2000), indeed suggest that the ability of a market institution to produce stable matchings is key determinant of its viability. Stability in matching theory is a conjunction of two requirements: individual rationality (no agent who would be better off by dropping some of his current partners), and the absence of *blocking pairs* (that is, there is no pair of agents who would prefer to be matched to one another rather than to their current mate). Stability is thus a concept that requires taking into account the preferences of *both* sides of the market.² We show in this paper that in many cases we may not need to know the preferences of both sides of the market to identify potential stable matchings. In some cases, we can even identify a stable matching (or a part thereof) by just considering just the preferences of one side of the market.

Our goal here is not to identify stable matchings, but rather to say whether a pair of agents can be matched at *a* stable matching for *some* preference profile. While this methodology obviously has some limitations (e.g., we may not always be able to pinpoint a stable matching), it provides additional tools for data analysis when one has access to partial data, namely, the preferences of one side of the market. We argue that even though stability in two-sided matching markets refers to the preferences of both side of the market, it is still possible to make some predictions when the available data is incomplete.

From a broader perspective, this paper addresses the question of extracting information from “partial” matching data. One of the earliest results in this direction is the so-called *Rural Hospital Theorem* in the context of matching students to hospitals, which states that if a hospital does not fill its capacity at some stable matching, then it is matched to the same set of students at any stable matching.³ In other words, this result allows us to extract

¹See Azevedo and Leshno (2012) for the link between market competitive equilibrium and stable matchings.

²In one-to-one matching markets or in many-to-one matching markets when there are no complementarities between agents individual rationality is not very demanding.

³The original Rural Hospital Theorem (Roth, 1986) is stated for a many-to-one matching problem where hospitals have responsive preferences. Another celebrated result is that in marriage problems, the set of

information about *all* stable matchings given the observation of *one* stable matching. More recently, Echenique *et al.* (2012) went one step further and showed for aggregate matchings how the observation of stable matchings reveals agents' preferences.⁴ Our contribution takes the opposite approach: we show how partial observation of preferences allows us to deduce (partial) information on stable matchings. In another vein, Martínez *et al.* (2012) show how to identify incomplete preferences so that any extension that completes those preferences yields the same core. A work closely related to ours is that of Rastegeri *et al.* (2012), who consider a one-to-one matching market where agents from both sides do not know their preferences and must conduct interviews to uncover them. They show that as preferences become more complete, some potential interviews may be discarded because the parties involved in such interviews can never be matched at any stable matching.

To streamline the analysis, we consider the school choice model introduced by Abdulkadiroğlu and Sönmez (1999), but our results are not confined to this class of models; they can be applied to any many-to-one matching model with responsive preferences. The first objective of this paper is to characterize what we call *impossible matches*, i.e., pairs of student-school that cannot be matched at any stable matching. Given a profile of school priorities over students and student lists of acceptable schools (but not their preferences over those schools), a student i is an *impossible match for a school s* if there is no profile of students' preferences that leaves unchanged the set of acceptable schools for each student such that student i is matched to school s at a stable matching. Crucial in our analysis is the premise that agents on at least one side of the market do not submit complete preference lists over their potential partners from the other side of the market. This hypothesis is motivated by the fact that most (if not all) real-life applications of stable matching mechanisms involve a large number of participants and thus it is virtually impossible to expect all agents to submit a preference list over all possible partners. While this is obviously satisfied in markets that formally constrain agents with respect to the length of their preference lists (e.g., school choice in New York City, university admission in Spain), unconstrained markets (e.g., school choice in Boston, Medical Resident matching in the US) also exhibit this property.⁵

agents matched at a stable matching is the same for all stable matchings (Roth (1984) and Gale and Sotomayor (1985)).

⁴In the same line, Roth and Sotomayor (1989) show how the observation of different stable matchings for a College Admissions problem gives information about colleges' preferences over group of students.

⁵See for instance Abdulkadiroğlu, Pathak and Roth (2005), Haeringer and Klijn (2009), Calsamiglia,

We show that for a student i to be an impossible match for a school s is the existence of a set of students who can exhaust the seats made available by a set of schools (that includes school s), and those schools are the only acceptable schools for those students. This condition is linked to one of the very first result of matching theory, namely Hall’s (1935) marriage theorem, which provides a necessary and sufficient condition for the existence of a matching where no individuals from a given set remain unmatched. Hall’s condition is not sufficient for our purposes, however, as it provides no link to stability. Part of our contribution consists precisely of carefully selecting the set of individuals for which we check Hall’s condition. As we shall see, this selection can be entirely by considering only the preferences of one side of the matching market.

We use this paper to call attention to another problem related to the use of stable mechanisms in school choice problems (i.e., market mechanisms that produce stable matchings). School priorities are usually constructed considering several criteria, such as the presence of siblings or proximity, in order to give some students a higher priority at a given school. Our approach allows us to show that some correlation in the set of acceptable schools (but not necessarily correlation in the way schools are ordered in the preferences) is sufficient to impede students from other districts to have access to these schools, thus limiting substantially district mobility in a school district.

In school choice problems the Student Optimal Stable Mechanism (SOSM) is perhaps the most natural mechanism. This mechanism consists of simply asking students their preferences over schools and running Gale and Shapley’s (1962) Deferred Acceptance algorithm with students proposing. SOSM’s success comes mainly from the fact that for each student it is a dominant strategy to submit his true preferences and the matching that obtains is not only a stable matching but is the student-optimal stable matching, i.e., the “best” stable matching from the students’ point of view. But SOSM is not exempt of drawbacks, the major one being lack of efficiency. That is, there might exist a matching that makes all students better off when compared to the student-optimal matching. The matching literature has studied to what extent we can improve upon SOSM while maintaining as much as possible the non-manipulation and stability as much as possible.⁶ Building on our charac-

Haeringer and Klijn (2010), Kojima, Pathak and Roth (2010), or Pathak and Sönmez (2012) for accounts on the length of submitted preference lists in centralized matching markets.

⁶Another source of inefficiency may come from the fact that schools may have weak priority orderings — see Erdil and Ergin (2006) or Abdulkadiroğlu, Pathak and Roth (2009).

terization of impossible matches, we propose a new mechanism that can alleviate the lack of efficiency of SOSM. Our mechanism adopts the framework of Kesten’s (2010) Efficiency-Adjusted Deferred Acceptance Mechanism (EADAM), which (roughly) consists of deleting some students from some schools’ priorities before running the Deferred Acceptance algorithm. In EADAM, a student is deleted from a school’s priority if by doing so all students will be (weakly) better off at the new student-optimal matching. In our mechanism we delete a student from a school’s priority if the student is an impossible match for that school. Like in EADAM, once all deletions are made we compute the student-optimal matching. A major drawback of Kesten’s mechanism, however, is that it is not strategyproof, i.e., some students may gain by misrepresenting their preferences. For our mechanism we are able to show that it is a dominant strategy for all students to truthfully rank all schools that are declared as acceptable. In other words, while both Kesten’s and our mechanism are variations of the SOSM, only our mechanism retains some of SOSM’s strategyproofness.⁷

The paper is structured as follows. In Section 2 we outline the basic model we shall work with. Section 3 contains the first main contribution of this paper, i.e., the characterization of impossible matches. In Section 5 we propose a new algorithm that improves upon Gale and Shapley’s Deferred Acceptance algorithm and study some of its properties. Most of the proofs are relegated to the Appendix.

2. PRELIMINARIES

2.1. Matchings

We consider throughout this paper a (finite) set I of students and a (finite) set S of schools and each school s is endowed with a positive capacity q_s . The problem studied here is that of matching students to schools in the limit of their capacities. A **matching** is a mapping $\mu : I \cup S \rightarrow 2^I \cup S$. A matching is **feasible** if for any $i \in I$ and any $s \in S$

- $\mu(i) \in S \cup i$,
- $\mu(s) \in 2^I$,
- $\mu(i) = s$ if and only if $i \in \mu(s)$, and

⁷Our mechanism is not fully strategyproof; some students may be better off by changing the set of schools declared as acceptable.

- $|\mu(s)| \leq q_s$.

For $v \in S \cup I$, we call $\mu(v)$ agent v 's assignment. For $i \in I$, if $\mu(i) = s \in S$ then student i is matched a seat at school s under μ . If $\mu(i) = i$ then student i is unmatched under μ .

2.2. School choice problem

A **school choice problem** is given by a 5-tuple $(I, S, q, \succ_I, \succ_S)$, where I is a set of students, S a set of schools, q a vector of schools' capacities, $\succ_I$ the students' preferences over schools and $\succ_S$ are schools' priorities over students.

Each student $i \in I$ is assumed to have a strict preference $\succ_i$ relation over the schools and the option of remaining unassigned, i.e., $\succ_i$ is a linear order over $S \cup \{i\}$, where i denotes his outside option (e.g., going to a private school). The notation $s \succ_i s'$ means that student i prefers to go to school s than school s' . For a set $I' \subseteq I$, we denote by $\succ_{I'}$ the profile $(\succ_i)_{i \in I'}$. Given a student's preferences $\succ_i$, we denote by $A(\succ_i)$ the set of schools that are acceptable to i and for a set I' of students $A(\succ_{I'})$ denotes the set of all schools that are acceptable to students in I' , i.e., $A(\succ_{I'}) = \cup_{i \in I'} A(\succ_i)$.

Each school $s \in S$ is endowed with a strict priority ordering $\succ_s$ over the students and a fixed capacity of seats q_s . The priority ordering $\succ_s$ is a linear ordering over I . Without loss of generality, we assume that $q_s > 0$ for each school in S . We denote by $\succ_S = (\succ_s)_{s \in S}$ and $q = (q_s)_{s \in S}$ the profile of priorities and capacities, respectively.

When there is no risk of confusion we shall simply use $\succ$ to denote a school choice problem.

Students' preferences over schools can be straightforwardly extended to preferences over matchings. We some abuse of notation, we say that student i prefers the matching μ to the matching μ' , which we denote $\mu \succ_i \mu'$ if, and only if, $\mu(i) \succ_i \mu'(i)$. When considering preferences over matchings it is necessary to consider the weak preference relation $\succeq_i$, where $\mu \succeq_i \mu'$ if $\mu(i) = \mu'(i)$ or $\mu(i) \succ_i \mu'(i)$.

A key property of matchings in the two-sided matching literature is stability. A matching is stable if, for any student, all the schools he prefers to the one he is assigned to have exhausted their capacity with students that have higher priority. Formally, let P be a school choice problem. A matching μ for a school choice problem $\succ$ is **stable** if

- it is **individually rational**, i.e., for all $i \in I$, $\mu(i) \succeq_i i$,

- it is **non wasteful** (Balinski and Sönmez, 1999), *i.e.*, for all $i \in I$ and all $s \in S$, $s \succ_i \mu(i)$ implies $|\mu(s)| = q_s$, and
- there is no **justified envy**, *i.e.*, for all $i, j \in I$ with $\mu(j) = s \in S$, $s \succ_i \mu(i)$ implies $j \succ_s i$.

Given a profile $\succ$, we denote the set of stable matchings by $S(\succ)$.

Another desirable property for a matching is Pareto-efficiency. We follow the traditional approach here by restricting welfare analysis to the students.⁸ A matching μ' Pareto dominates a matching μ if all students prefer μ' to μ and there is at least one student that strictly prefers μ' to μ . Formally, μ' **Pareto dominates** μ if $\mu'(i) \succeq_i \mu(i)$ for all $i \in I$, and $\mu'(i') \succ_{i'} \mu(i')$ for some $i' \in I$. A matching is **Pareto efficient** if it is not Pareto dominated by any other matching.

2.3. Pre-matching problem

A **pre-matching problem** is similar to a school choice problem except that for students it only specifies the schools that are acceptable to them and schools' priorities are restricted to the set of students that view them as acceptable. Formally, a pre-matching problem is given by a 4-tuple (I, S, P_S, q_s) where I is the set of students, S is the set of schools, P_S is a profile of schools' **pre-priorities** and q_s is a vector of schools' capacities. In a pre-matching problem, the pre-priority of school s , P_s , is a linear order over a set $A_s \subseteq I$. For each student $i \in I$, let $A_i = \{s \in S : i \in A_s\}$. The collection of schools' pre-priorities P_S is a **pre-profile**. Whenever there is no risk of confusion we shall use the shorthand P to denote a pre-matching problem. For any $I' \subseteq I$, let $A_{I'} = \cup_{i \in I'} A_i$. Similarly, for any $S' \subseteq S$, $A_{S'} = \cup_{s \in S'} A_s$. Given a pre-matching problem (I, S, P_S, q_s) a matching μ is **feasible** if it is a matching (see Section 2.1) and for each $i \in I$, $\mu(i) = s$ implies $i \in A_s$.

Let P be a pre-profile, i and s such that $i \in A_s$, and let $B \subseteq I$. The upper-counter set of i at s restricted to B , denoted by $U_i^B(s)$, is the set of students in B that have a higher priority than i at s , $U_i^B(s) = \{j \in B : j P_s i\}$.

Given a school choice problem $\succ$ it is easy to construct an associated pre-matching problem P by simply deleting from each school's priority ordering the students who view that school as unacceptable. Abusing vocabulary, we shall say that if for a school s we have

⁸See Abdulkadiroğlu and Sönmez (2003), Haeringer and Klijn (2009) or Kesten (2010).

$i \in A_s$ then student i is **acceptable** for school s , and similarly if $s \in A_i$ then school s is **acceptable** for student i . One can also consider the reverse operation, i.e., constructing a school choice problem from a pre-matching problem. Note however that the same pre-matching problem can generate different school-choice problems. Formally, given a pre-matching problem (I, S, P_S, q) , a school choice problem $(I', S', \succ_I, \succ_S, q')$ is **P_S -compatible** (or **P -compatible**) if

- (a) $I = I', S = S', q = q'$;
- (b) for each student i and each school s , $s \succ_i i$ if, and only if, $i \in A_s$,
- (c) for each school $s \in S$, and each pair of students $i, i' \in A_s$, $i P_s i'$ if, and only if, $i \succ_s i'$.

Given a pre-profile P_S we denote by $\Theta(P_S)$ the set of school choice problems that are P_S -compatible. So a school choice problem is P -compatible if (a) it involves the same sets of students and schools and schools' capacities are the same; (b) a student considers a school as acceptable if, and only if, he appears in the schools' pre-priority; and (c) each school's pre-priority ordering is compatible with its priority ordering.

3. IMPOSSIBLE MATCHES

The purpose of this section is to define and characterize the students that are **impossible matches** (for some specific school). The question addressed by the concept of impossible match is the following: upon observing one side of the markets' preferences/priorities (i.e. a pre-matching problem), is it possible to say whether there exists a stable matching with respect to some realization of the other side's preferences (i.e. a school choice problem) where a particular student is matched to a particular school at a stable matching? If the answer is "no" then that student is said to be an impossible match for that school.

Definition 1 A student i is an **impossible match for school s** at the pre-matching problem P if for any school choice problem $\succ \in \Theta(P)$, there is no matching $\mu \in S(\succ)$ such that $\mu(i) = s$.

To understand how the preferences of only one-side of the market may be sufficient to check the stability of a matching consider the following school choice problem. Let i_2, i_3 and i_4 be students who applied only to schools s_2 and s_3 , s_3 and s_4 , and s_2 and s_4 , respectively.

Let i_1 be a student who applied only to schools s_1 and s_2 . Table 1 below describes the priority orderings for those four schools, where we deleted from a school's priority ordering all the students among students i_1, i_2, i_3 and i_4 who did not put that school in their submitted preference list.⁹

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}
i_1	.	.	.
.	i_2	i_3	i_4
.	.	.	.
.	i_1	i_2	i_3
.	.	.	.
.	i_4	.	.

Table 1: A simple example

For instance, at school s_2 student i_2 has a higher priority than student i_1 , who has higher priority than student i_4 . Dots in the priority orderings indicate potential students (different from students $i_1, \dots, i_4$) who also applied to the corresponding school.

We claim that for any preference profile of the students that does not affect the set of acceptable schools (e.g., s_2 and s_3 are always acceptable for student i_2), it is impossible to obtain a stable matching where i_1 is matched to s_2 . To see this, note that if this is the case then i_2 must be matched to s_3 , which in turn implies that i_3 is matched to s_4 , which is only possible if i_4 is matched to s_2 , a contradiction. So, there is no preference profile (that does leaves unchanged the set of acceptable schools for each student) and a stable matching for that preference profile where i_1 is matched to s_2 .

Without entering into much details, it is important to note that the key property of schools s_2, s_3 and s_4 's priorities to ensure that i_1 is an impossible match for school s_2 is not, as it may appear, that there is some sort of cycle with the relative positions of students i_2, i_3 and i_4 . As we shall see, i_1 being an impossible match for s_1 is rather due to the fact that there are 3 students competing for three schools, and those students have a higher priority than i_1 .

⁹Note that as long as we consider stable matchings (and thus individually rational matchings), "cleaning" schools' priority orderings this way does not affect our arguments.

Part of our contribution consists of showing that we do not need to consider all possible profiles of students' preferences (and compute all the corresponding stable matchings) to check whether a student is an impossible match for a school. The key intuition is the following. Recall first that by definition any feasible matching will also be individually rational in any compatible matching problem. So, if a matching is not stable for a particular instance of a matching problem it must be that there exists a blocking pair involving a student i and a school s . However, recall that in a pre-matching problem we do have access only to students' set of acceptable schools (but not their preferences). The only way to accommodate this constraint is that i is matched to himself and school s either has an empty seat available for i or is matched to a student with a lower priority than i . We call such a matching unstable.

Definition 2 A matching μ is **stable** for a pre-matching problem P if

- (a) it is **feasible**, i.e., for each $i \in I$, $\mu(i) = s$ implies $i \in A_s$;
- (b) it is **non-wasteful**, i.e., $|\mu(s)| < q_s$ implies $\mu(i) \in S$ for each $i \in A_s$;
- (c) there is no **justified envy**, i.e., for each $i \in I$, $\mu(i) \in S$ implies $\mu(i') \in S$ for each $i' \in I$ such that $i' P_{\mu(i)} i$.

So, a feasible matching is stable if whenever a student is matched a school all the students that have higher priority at that school are matched to a school. Furthermore, if a school does not fill its capacity then all the students that are acceptable to it are matched to a school.

Given the definition of a stability of matching for a pre-profile P we can reformulate the definition of an impossible match student in the following way.

Proposition 1 *A student i_0 is an impossible match for s_0 in the pre-matching problem P if, and only if, there is no stable and maximum matching μ for the pre-matching problem P^{i_0, s_0} such that $\mu(i_0) = s_0$.*

Proof Let μ be a stable matching for a matching problem $(I, S, \succ_I, \succ_S, q_S)$ such that $\mu(i_0) = s_0$. Let P be the corresponding pre-matching problem. It is straightforward to see that μ is feasible, non-wasteful and that there is no justified envy at P . Conversely, let μ be stable for a pre-matching problem (I, S, P, q_S) . Let $\succ_I$ be the preference profile such that

for each $i \in I$ with $\mu(i) \neq i$ school $\mu(i)$ is the most preferred school under $\succ_i$. If $\mu(i) = i$ then $\succ_i$ be any preference profile that is P -compatible. It is easy to see that $\succ \in \Theta(P)$ and that μ is stable for the P -compatible matching problem $(I, S, \succ_I, \succ_S, q_S)$. ■

In order to check whether a student i_0 is an impossible match student for a school s_0 at a pre-profile P we need to show that there is no matching μ stable for P such that $\mu(i_0) = s_0$. That is, we need to show that there is no feasible matching μ that is non-wasteful and without justified envy. The non-wastefulness condition implicitly calls for a matching that maximizes the number of students being matched. The existence of maximum matching is one of the oldest in matching theory.

Theorem 1 (Hall's Marriage Theorem, 1935) *There exists a matching where all students are matched to some school if, and only if,*

$$\text{For each } I' \subseteq I, ; |I'| \leq \sum_{s \in A_{I'}} q_s . \quad (1)$$

In words, there exists a matching where all students can be matched to some school if, and only if, for each subset I' of students the number of students in I' is at most equal to the total number of seats among the schools that consider some of the students in I' as acceptable. Eq. (1) is known as *Hall's marriage condition*.¹⁰ To incorporate the no-justified envy condition we shall not apply Hall's marriage condition to the original pre-matching problem but rather to a collection of truncated pre-matching problems.

Definition 3 For a pre-matching problem (I, S, P_S, q_S) , and some $i \in A_s$ for some $s \in S$, the **truncated pre-matching problem at i** is the pre-matching problem (I, S, P_S^i, q_S) such that

(a) $P_s^i = P_s$ for each $s \notin A_i$,

(b) for each $s \in A_i$, P_s^i is an ordering of U_i^s and P_s^i and P_s coincide for U_i^s .

¹⁰Hall's marriage condition was originally stated for the one-to-one matching problem. It is easy to see that this condition can be easily extended to the case of school choice problems when schools have responsive preferences by making for each school as many copies as its capacity (where each copy has only one seat to offer) and a student acceptable for a school will be acceptable for each of its copy.

In other words, given a pre-matching problem P , a truncated pre-matching problem at i , P^i , is obtained from P by declaring unacceptable, for each school acceptable to i , the students ranked below i and i are now unacceptable. Given a pre-matching problem P , we denote by A_s^i and A_j^i the set of acceptable students of school s and set of acceptable schools of student i in the problem P^i , respectively.

We can now state our first main result.

Theorem 2 *Let P be a pre-matching problem and $i_0 \in A_{s_0}$. The following conditions are equivalent:*

1. *Student i_0 is an impossible match of s_0 in the pre-matching problem P .*
2. *There exists a nonempty subset of students $\mathbf{J} \subseteq I \setminus \{i_0\}$ with $\mathbf{J}_0 := \mathbf{J} \cap \{i : iP_{s_0}i_0\} \neq \emptyset$ such that for any (possibly empty) $\bar{\mathbf{J}} \subseteq \mathbf{J} \setminus \mathbf{J}_0$, there exists $T \subseteq \mathbf{J} \setminus \bar{\mathbf{J}}$ such that*

$$|T| > \sum_{s \in A_T^{\bar{\mathbf{J}}}} \bar{q}_s \quad (2)$$

where $\bar{q}_{s_0} = q_{s_0} - 1$ and $\bar{q}_s = q_s$ if $s \neq s_0$.

Before proving Theorem 2 a few remarks are in order

Remark 1 *Note that $A_{\mathbf{J} \setminus \bar{\mathbf{J}}}^{\bar{\mathbf{J}}} \neq \emptyset$ for every $\bar{\mathbf{J}} \subseteq \mathbf{J} \setminus \mathbf{J}_0$ since $\mathbf{J}_0 := \mathbf{J} \cap \{i : iP_{s_0}i_0\} \neq \emptyset$. If $\bar{\mathbf{J}} = \mathbf{J} \setminus \mathbf{J}_0$ then $A_{\mathbf{J} \setminus \bar{\mathbf{J}}}^{\bar{\mathbf{J}}} = A_{\mathbf{J}_0} \ni s_0$. Applying the condition to $\bar{\mathbf{J}} = \mathbf{J} \setminus \mathbf{J}_0$ we also get that $|\mathbf{J}_0| \geq |T| > s \in A_T^{\bar{\mathbf{J}}} \bar{q}_s \geq q_{s_0} - 1$. If $\bar{\mathbf{J}} = \emptyset$ then we are back to the pre-matching problem (I, S, P_S, q_S) and the condition states that there is no matching such that every student in $\mathbf{J}$ is matched to a school in the pre-matching (I, S, P_S, q_S) .*

Remark 2 *Note that in condition (2) of Theorem 2 the set $\mathbf{J}$ can be chosen among the subsets of $\mathbf{J}(i_0, s_0)$, instead of $I \setminus \{i_0\}$, where $\mathbf{J}(i_0, s_0)$ is the minimal "connected component" of students emanating from (i_0, s_0) which can be defined recursively as follows: the minimal set of students containing $\{i : iP_{s_0}i_0\}$ such that for every $i \in \mathbf{J}(i_0, s_0)$, $s \in A_i$ implies that every $j \neq i_0$ such that jP_si belongs to $\mathbf{J}(i_0, s_0)$. The set is uniquely defined (and does not contain i_0 , otherwise it would not be minimal).*

Remark 3 *Note that, in condition (2) of Theorem 2, if $\mathbf{J}(i_0, s_0) = \{i : iP_{s_0}i_0\}$, then $\mathbf{J} = \mathbf{J}_0$ and $A_{\mathbf{J}} = s_0$. Hence the theorem establishes that i_0 can be an impossible match only if i_0 is not among the q_{s_0} best ranked students.*

The next lemma clarifies the role of the cardinality condition in Theorem 2. We omit the proof as it is a direct application of Hall's Marriage Theorem (with capacities).

Lemma 1 *Let P be a pre-matching problem and $i_0 \in A_{s_0}$. Let $\mathbf{J} \subseteq I \setminus \{i_0\}$ be a nonempty subset of students with $\mathbf{J}_0 := \mathbf{J} \cap \{i : iP_{s_0}i_0\} \neq \emptyset$. The following conditions are equivalent:*

1. *For any (possibly empty) $\bar{\mathbf{J}} \subseteq \mathbf{J} \setminus \mathbf{J}_0$, there exists $T \subseteq \mathbf{J} \setminus \bar{\mathbf{J}}$ such that $|T| > \sum_{s \in A_T^{\bar{\mathbf{J}}}} \bar{q}_s$.*
2. *For any (possibly empty) $\bar{\mathbf{J}} \subseteq \mathbf{J} \setminus \mathbf{J}_0$, there is no matching in the pre-matching problem $P^{\bar{\mathbf{J}}}$ such that every student in $\bar{\mathbf{J}}$ is matched to a school.*

We prove now Theorem 2.

Proof (1) $\Rightarrow$ (2) Let i_0 be an impossible match for s_0 . So, there is no stable matching μ for P such that $\mu(i_0) = s_0$.

Set $\mathbf{J} := \mathbf{J}(i_0, s_0)$ as defined in Remark 2. By construction $\mathbf{J}_0 = \{i : iP_{s_0}i_0\}$ and obviously $\mathbf{J}_0 \neq \emptyset$ otherwise i_0 would not be an impossible match of s_0 .

Let $\bar{\mathbf{J}} \subseteq \mathbf{J} \setminus \mathbf{J}_0$ and suppose by way of contradiction that for every $T \subseteq \mathbf{J}$, $|T| \leq \sum_{s \in A_T^{\bar{\mathbf{J}}}} \bar{q}_s$. From Lemma 1 there exists a matching μ in the pre-matching problem $P^{\bar{\mathbf{J}}}$ such that each student in $\mathbf{J}$ is matched to a school. The matching is stable for the pre-matching $P^{\bar{\mathbf{J}}}$. Note also that by construction $P_{s_0}^{\bar{\mathbf{J}}} = P_{s_0}$.

We get stability for the initial pre-matching problem P by filling the wasteful schools (taken in any particular order) with the best ranked and unmatched remaining students (if any), except for i_0 who is matched to s_0 (he is the best ranked and unmatched student remaining in s_0 , where is there is still an available seat by construction). This leads to a contradiction.

2. $\Rightarrow$ 1. Suppose now that there exists a nonempty subset $\mathbf{J} \subseteq I \setminus \{i_0\}$ with $\mathbf{J}_0 := \mathbf{J} \cap \{i : iP_{s_0}i_0\} \neq \emptyset$ such that for any (possibly empty) $\bar{\mathbf{J}} \subseteq \mathbf{J} \setminus \mathbf{J}_0$, there exists $T \subseteq \mathbf{J}$ such that $|T| > \sum_{s \in A_T^{\bar{\mathbf{J}}}} \bar{q}_s$. Let μ be any stable matching for P and let $\bar{\mathbf{J}} := \{i \in I : \mu(i) = i\} \cap \mathbf{J}$. We show next that $\mu(i_0) \neq s_0$.

Suppose there exists $j \in \bar{\mathbf{J}} \cap \mathbf{J}_0$. So $\mu(i_0) = s_0$ cannot be stable for P , a contradiction. Hence, $\bar{\mathbf{J}} \subseteq \mathbf{J} \setminus \mathbf{J}_0$ (or $\bar{\mathbf{J}} = \emptyset$). Applying the previous inequalities given by assumption to the

set $\bar{\mathbf{J}}$, we obtain from Lemma 1 that there is no matching such that all students in $\mathbf{J} \setminus \bar{\mathbf{J}}$ are matched to a school in the pre-matching $(I, S, P^{\bar{\mathbf{J}}}, \bar{q}_S)$.¹¹

From the definition of $\bar{\mathbf{J}}$ we have that at μ all students in $\mathbf{J} \setminus \bar{\mathbf{J}}$ are matched in the pre-matching problem $(I, S, P^{\bar{\mathbf{J}}}, q_S)$. Now remark that if $\mu(i_0) = s_0$ then μ' such that $\mu'(i) = \mu(i)$ for each $i \in \mathbf{J} \setminus \bar{\mathbf{J}}$ and $\mu'(i) = i$ otherwise (especially $\mu'(i_0) = i_0$) is a feasible matching in the pre-matching problem $(I, S, P^{\bar{\mathbf{J}}}, \bar{q}_S)$ that matches all students in $\mathbf{J} \setminus \bar{\mathbf{J}}$, a contradiction. ■

Theorem 2 suggests that identifying impossible matches can be computationally hard. Condition (2) in the theorem indeed states that one has not only have to identify a set $\mathbf{J}$ but also check Eq. (2) for all possible subsets $\bar{\mathbf{J}}$ of $\mathbf{J}$. We now show that in fact the question whether a student i_0 is an impossible match for a school s_0 can be solved in polynomial time. To this end we introduce the following algorithm.

Matching algorithm at (i_0, s_0)

Step 1 Let $\tilde{P}$ to be the pre-profile such that $\tilde{P}_{s_0} = P_{s_0}$, and for each school $s \neq s_0$, $\tilde{P}_s$ is a priority ordering over $A_s \setminus \{i_0\}$ and $i\tilde{P}_s i'$ if, and only if, $iP_s students i'$.

Step 2 Let μ^0 be a maximal and stable matching for the pre-profile $\tilde{P}$.

If $\mu^0(i_0) = s_0$ then STOP. Otherwise go to step 3.

Step 3 Define $Z = \{i : iR_{s_0} i_0 \text{ and } \mu^0(i) = i\}$. Let $Z = \{i_1, \dots, i_k\}$ such that $i_h P_{s_0} i_{h+1}$ for each $h = 1, \dots, k-1$ (so $i_k = i_0$).

Step 3.h.1 Let $\hat{I} = \{i : \mu^{h-1}(i) \neq i\}$ and $I_0 = \{i \in \hat{I} : iP_{s_0} i_h\}$.

Step 3.h.2 Let $\hat{P}$ and $\hat{q}$ such that

- (a) $\hat{P}_{s_0}$ is a truncation of $\tilde{P}_{s_0}$ at i_h .
- (b) For each $s \neq s_0$, $\hat{P}_s$ is a truncation of $\tilde{P}_s$ at $I \setminus \hat{I}$.
- (c) $\hat{q}_{s_0} = q_{s_0} - 1$ and $\hat{q}_s = q_s$ for each $s \neq s_0$.
- (d) for each school $s \neq s_0$, $\hat{P}_s$ is a priority ordering over $A_s \setminus \{i_h\}$ and $i\hat{P}_s i'$ if, and only if, $iP_s i'$.

¹¹Note that by Remark 1 we have $(\mathbf{J} \setminus \bar{\mathbf{J}}) \neq \emptyset$.

Step 3.h.3 For each $i \in \widehat{I} \setminus I_0$, find a maximum matching $\widehat{\mu}$ for the pre-profile $\widehat{P}^i$, defined as the pre-profile $\widehat{P}$ truncated at i .

If there exists $i \in \widehat{I}$ such that for each student $j \in \widehat{I} \setminus \{i\}$, $\widehat{\mu}(j) \neq j$ then let $\mu^h = \widehat{\mu}$ for each $j \neq i_h$ and $\mu^h(i_h) = s_0$. If $h < k$ set $h = h + 1$ and go to Step 3.h.1, otherwise go to Step 4. If there no no student $i \in \widehat{I}$ such that for each student $j \in \widehat{I} \setminus \{i\}$, $\widehat{\mu}(j) \neq j$ then STOP and let $\mu = \mu^h$.

Step 4 Finalize the matching with $\mu = \mu^k$.

The most demanding part of the algorithm is Step 3. Notice however that each iteration in that step consists of finding maximum matchings for a particular pre-profile and set of students. It is well known that finding a maximum matching can be done in polynomial time —Lovász and Plummer (1986)— so since the number of students is finite our algorithm is polynomial. More interesting is the following result.

Proposition 2 *Let (I, S, P, q_S) be a pre-matching problem and let μ be the matching produced by the matching algorithm at (i_0, s_0) . Then i_0 is an impossible match for s_0 if, and only if, $\mu(i_0) \neq s_0$.*

Proposition 2 and the fact that our algorithm runs in polynomial time contrast with some previous results in the computer science literature by Manlove *et al.* (2002) and more recently by Rastegeri *et al.* (2012). They consider a two-sided matching problem with short preference lists and indifferences and address several questions related to stable matchings. It is straightforward to see a matching is stable for a pre-matching problem if it is stable for a matching problem where each student is indifferent across all the schools that are acceptable to him. So, pre-matching problems are a special case of the problems studied in Manlove *et al.* or Rastegeri *et al.*. These two papers show that the question whether a pair is an impossible match is NP-complete, i.e., that is cannot be solved in polynomial time.

4. THE ILLUSION OF CHOICE IN A STABLE SCHOOL CHOICE MECHANISM

School choice programs usually endow students with a high priority to some school close to their homes.¹² Our characterization result shows an additional adverse effect that such a

¹²See Abdulkadiroğlu, Pathak and Roth (2005), and Abdulkadiroğlu, Pathak, Roth and Sönmez (2005).

policy may induce. We assume here that the school choice mechanism in used is a stable mechanism (e.g., SOSM).

To begin with, we will consider a special case of block configuration. For ease of exposition, assume that in the following school choice problem each school has only one seat to propose. Let $\{i_1, \dots, i_n\}$ and $\{s_1, \dots, s_n\}$ be the set of students and schools, respectively. Each student is assigned a *district school* and schools' priority give to each student top priority at his district school.

We now consider a profile of submitted preference lists $\succ$ by the students that satisfies the following property. There exists an ordering of schools' qualities and each student only put in his preference lists schools that have a higher or equal quality as that of his district school. Formally, there is a partition $\{I_1, I_2, \dots, I_k\}$ such that for each set I_h of students, $A_{I_h} \subseteq \cup_{h \leq k} S_{h'}$, where $S_{h'}$ is the set of district schools of the students in $I_{h'}$. In other words, a student belonging to the set I_h of the partition only puts in his submitted preference lists schools that are the district school of some student in the first h elements of the partition.¹³ Also, we assume that each student's preference list contains his district school.¹⁴

We then have the following result.¹⁵

Proposition 3 *If μ is a stable matching for some submitted preference list then for each I_h , $i \in I_h$ implies $\mu(i) \in S_h$.*

Proof The proof is straightforward, it suffices to observe that for each $h < h'$, each student $i \in I_{h'}$ is an impossible match for each school $s \in S_h$. To see this, observe that students in $\cup_{h'' \leq h} I_{h''}$ are necessarily matched at any maximum, stable matching and they can only be matched to schools in $\cup_{h'' < h'} S_{h''}$ (i.e., they belong to the set $\mathbf{J}^*$). So by Proposition 7 in the Appendix student i is an impossible match for each school $s \in \cup_{h'' < h'} S_{h''}$.¹⁶ ■

We used here a particular (albeit not unrealistic) preference profile, but in fact it is sufficient to have correlation in terms of acceptable schools (but not necessarily correlation in terms of preferences over those schools) to generate blocks.

¹³Note that the ordering of schools' qualities may not need be complete nor strict.

¹⁴See Calsamiglia, Haeringer and Klijn (2010) for experimental evidence.

¹⁵When there is a continuum of students and a finite number of schools Calsamiglia and Miralles (2012) show that, under the assumption that there is one school that is the least preferred by all students, all students end up being matched to their district school for any preference profile.

¹⁶In fact, one easily checks that the students $\cup_{h'' \leq h} I_{h''}$ form a block at (i, s) .

5. A WELFARE IMPROVEMENT MECHANISM

The classic mechanism used to obtain a stable matching is the so-called Student-Optimal Stable Mechanism. This mechanism consists of asking students (or their parents) a preference list over schools and then compute a matching using schools' priority rankings with the deferred acceptance (DA) algorithm introduced by Gale and Shapley (1962). Let $\succ_I$ be a profile of ordered lists submitted by the students and $\succ_S$ be schools' priority rankings over students. The DA algorithm finds a matching through the following steps.

STEP 1: Each student i proposes to the school that is ranked first in $\succ_i$ (if there is no such school then i remains unassigned). Each school s tentatively assigns up to q_s seats to its proposers one at a time following the priority order $\succ_s$. Remaining students are rejected.

STEP l , $l \geq 2$: Each student i that is rejected in Step $l - 1$ proposes to the next school in the ordered list $\succ_i$ (if there is no such school then i remains unassigned). Each school s considers the new proposers and the students that have a (tentative) seat at s . School s tentatively assigns up to q_s seats to these students one at a time following the priority order $\succ_s$. Remaining students are rejected.

The algorithm stops when no student is rejected. Each student is assigned to his final tentative school. Let $\varphi(\succ)$ denote the matching. The mechanism φ is the *Student-Optimal Stable mechanism* and the matching $\varphi(\succ)$ is called the *student-optimal matching*. It is well known that SOSM is a stable mechanism that is Pareto superior to any other stable matching mechanism (Gale and Shapley, 1962).

A crucial property of SOSM is that it is strategy-proof (Dubins and Freedman, 1981; Roth, 1982). That is, for each student it is a dominant strategy to submit his true preferences,

$$\varphi(\succ_i, \succ_{-i}) \succeq_i \varphi(\succ'_i, \succ_{-i}), \quad \text{for each } \succ'_i, \succ_{-i}.$$

It turns out, however, that SOSM is not a Pareto-efficient mechanism. As Ergin (2002) shows, one needs very restrictive conditions on the priority structure to ensure that SOSM is Pareto-efficient for any preference profile. In fact, there is no hope to find a more efficient matching while keeping with stability and strategyproofness.

Proposition 4 (Kesten, 2010) *There is no Pareto-efficient and strategy-proof mechanism that selects the Pareto-efficient and stable matching whenever it exists.*

When computing the student-optimal matching there might be a student, say, i , who (for a given preference profile) will not end up being matched to some school, say, s , although that school s is preferred to the one they are matched with. Kesten's starting observation is that if i has now the least priority at school s his final matching will not be changed but he might make some other students better off. The main part of his contribution consists of identifying precisely those students. He then proposes a mechanism where students are asked whether they agree to waive their priority at those schools where they might make other students better off and then run the DA algorithm. However, a major drawback of Kesten's mechanism is that it is not strategyproof. Some students may gain by misrepresenting their preferences.

The structure of the mechanism we propose is the same as that of Kesten's. The main difference is in the identification of the students for which we shall modify their priority: In our mechanism, a student will wave his priority at a school if he is an impossible match for that school. As we shall see, since the qualification of being an impossible match for a school does not depend on the exact order of the schools in the submitted preference profile our mechanism retains some of the incentives of the SOSM.

We now introduce a new mechanism, the **Student Optimal Stable without Dummies mechanism** (SOSDM), whose purpose is to allow for Pareto improvements with respect to the SOSM but still maintaining part of the incentives carried by SOSM. Our mechanism simply consists of running the Deferred Acceptance algorithm with the reduced form of the submitted preference profile. Given a school choice problem $(I, S, q, \succ_I, \succ_S)$, we denote by $\varphi^d(\succ)$ the matching obtained with the DA algorithm.

An immediate result is that no student can be worse off under the SOSDM mechanism.

Proposition 5 *Let (I, S, q, P, f) be a school choice problem. Then $\varphi^d(\succ) \succeq_i \varphi(\succ)$ for each $i \in I$.*

Proof It is easy to verify that if $\mu \in S(\succ)$, then $\mu \in S(\succ^d)$. Since $\varphi^d(\succ)$ is the student-optimal matching with the preference profile $\succ^d$, $\varphi^d(\succ) \succeq_i^d \varphi(\succ)$. Note that $v \succ_i^d v' \succeq_i^d i$ implies $v \succ_i v' \succeq_i i$. Since $\varphi^d(\succ)(i) \succeq_i^d i$ for each $i \in I$, we have $\varphi^d(\succ) \succeq_i \varphi(\succ)$, the desired result. ■

By Proposition 4 SOSDM is not strategyproof. We show however that is the type of profitable manipulations is reduced with our mechanism.

Let $(I, S, q, \succ_I, \succ_S)$ be a school choice problem. Let $\succ'_i$ denote any preference ordering over $S \cup \{i\}$. We say that $\succ'_i$ is **semi-truthful** if for each $s \in S$ and $s' \in S \cup \{i\}$ such that $s \succ'_i s' \succeq_i i$, then $s \succ_i s' \succeq_i i$. That is, a preference $\succ'_i$ is semi-truthful if all the schools that are declared as acceptable in $\succ'_i$ are ranked in the same order as in the true preferences $\succ_i$. Given a preference ordering $\succ_i$ we write $A(\succ_i)$ to denote the set of preference orderings whose set of acceptable schools is the same as that of $\succ_i$. Finally, if $\succ_i$ is the true preference ordering and $\succ'_i$ another preference ordering, $\tilde{\succ}_i \in A(\succ'_i)$ denotes the semi-truthful preference ordering.

We say that a mechanism ϕ is **semi-strategyproof** if for each $i \in I$, and each $\succ_i, \succ'_i$ and $\succ_{-i}$,

$$\phi(\tilde{\succ}_i, \succ_{-i}) \succeq_i \phi(\succ''_i, \succ_{-i}) \quad \text{for each } \succ''_i \in A(\succ'_i)$$

where $\tilde{\succ}_i$ is the semi-truthful preference ordering in $A(\succ'_i)$. That is, a mechanism is semi-strategyproof if for each student, once the student has chosen which school to put in his revealed preferences, it is a dominant strategy to put those schools in the true order.

Proposition 6 *SOSDM semi-strategyproof.*

Proof Consider a school choice problem $(I, S, q, \succ_I, \succ_S)$ and let $\succ'_I$ be a profile of submitted preferences and let $\tilde{\succ}_i$ the semi-truthful preference ordering in $A(\succ'_i)$. Suppose there exists $\succ''_i \in A(\succ_i)$ such that

$$\phi^d(\succ''_i, \succ_{-i}) \succ_i \phi^d(\tilde{\succ}_i, \succ_{-i}) .$$

for some $\succ_{-i}$. Since $\tilde{\succ}_i, \succ''_i \in A(\succ'_i)$, any student j dummy for a school s in the school choice problem $(I, S, q, (\tilde{\succ}_i, \succ_{-i}), \succ_S)$ is also an impossible match for school s in the school choice problem $(I, S, q, (\succ''_i, \succ_{-i}), \succ_S)$. Clearly, $\phi^d(\tilde{\succ}_i, \succ_{-i}) \succeq_i i$. So, $\phi^d(\succ''_i, \succ_{-i}) \in S$. Let $s = \phi^d(\succ''_i, \succ_{-i})$. It follows that i is not an impossible match for s (otherwise s would be unacceptable for i in $\succ''_i$). So, $\phi^d(\succ''_i, \succ_{-i}) \succ_i \phi^d(\tilde{\succ}_i, \succ_{-i})$ implies $\phi^d(\succ''_i, \succ_{-i}) \tilde{\succ}_i \phi^d(\tilde{\succ}_i, \succ_{-i})$, which in turn implies

$$\phi^d(\succ''_i, \succ_{-i}) \tilde{\succ}_i^d \phi^d(\tilde{\succ}_i, \succ_{-i}) .$$

By definition of φ^d ,

$$\begin{aligned} \phi^d(\succ''_i, \succ_{-i}) &= \phi(\succ''_i^d, \succ_{-i}^d) \\ \phi^d(\tilde{\succ}_i, \succ_{-i}) &= \phi(\tilde{\succ}_i^d, \succ_{-i}^d) . \end{aligned}$$

So we have $\phi(\succ_i^d, \succ_{-i}^d) \widetilde{\succ}_i^d \phi(\widetilde{\succ}_i, \succ_{-i}^d)$, which contradicts the strategyproofness of φ . ■

In an SOSDM, the only ways to have a successful manipulation consist of altering the set of schools declared as acceptable.

Example 1 Consider the following school choice problem with three schools, s_1, s_2 , and s_3 , and three students, i_1, i_2, i_3 . Each school has one seat to offer. Let the students (true) preferences and schools' priorities be the following,

$\succ_{i_1}$	$\succ_{i_2}$	$\succ_{i_3}$	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
s_1	s_2	s_1	i_2	i_1	i_1
s_2	s_1	s_2	i_3	i_2	i_2
s_3	s_3	s_3	i_1	i_3	i_3

Suppose that i_1 and i_3 submit their true preferences, and i_2 submits the preferences $\succ'_{i_2} = s_2, s_1, i_2$ (i.e., s_3 is declared as unacceptable, and s_2 declared preferred to s_1). It is easy to see that no student is an impossible match for any school. So SOSDM coincides with SOSM and we obtain the matching $\mu = \{(i_1, s_2), (i_2, s_1), (i_3, s_3)\}$.

Suppose now that student i_1 submits the preferences $\succ'_{i_1} = s_1, s_2, i_1$. Note that i_1 only altered the set of acceptable schools. The two schools that are declared acceptable are ranked as in his true preferences. Students i_2 and i_3 still submit $\succ'_{i_2}$ and $\succ_{i_3}$, respectively. With this new profile it is easy to see that i_3 is now an impossible match for school s_1 . So, running SOSDM we obtain that i_1 is now matched to s_1 , i_2 matched to s_2 and i_3 matched to s_3 . That is, student i_1 is strictly better off declaring i_3 as unacceptable.

Note that when all but one student submit their true (i.e., complete) preferences, it is always a best-response for the remaining student to also submit his true preferences. This is so because in this case the last student has no possibility to create a block by eliminating some schools from his set of acceptable schools.

APPENDIX

5.1. Preliminaries

Let (I, S, P_S, q_S) be a pre-matching problem. For our purposes it is convenient to describe it by a non-directed graph $G(V, E)$ where the set of vertices is $V = I \cup S$ and the set of edges is $E = \{(i, s) : i \in A_s\}$. A **path** π in the graph $G(V, E)$ is a (finite) sequence of vertices $(v_1, v_2, \dots, v_k)$ such that for each $h = 1, \dots, k-1$ we have $(v_h, v_{h+1}) \in E$. Given a matching μ , an edge (v, v') is **free** if v is not matched to v' under the matching μ . Otherwise the edge is **matched**. A path $\pi \in G(V, E)$ is an **alternating path for a matching μ** if it alternates matched and free edges. A path $\pi = (v_1, \dots, v_k)$ in $G(I', S', \mu)$ is an **augmenting path** if it is an alternating path where both $(v_1 v_2)$ and $(v_{k-1} v_k)$ are free edges. A matching μ is **maximum for a set of students $I' \subseteq I$** if there does not exist a matching μ' that matches more students in I' than μ , i.e., $|\{i \in I' : \mu(i) \neq i\}| \geq |\{i \in I' : \mu'(i) \neq i\}|$ for any matching $\mu' \in \mathcal{M}$.¹⁷

Theorem 3 (Berge, 1957) *A matching μ is maximum for I' and S' if, and only if, there is no augmenting path in $G(I', S', \mu)$.*

Let μ be a matching and $\pi = (i_1, s_1, i_2, s_2, \dots, i_k, s_k)$ and augmenting path. The matching $\mu' = \mu \oplus \pi$ is the matching obtained when updating μ with the path π , i.e., $\mu'(i) = \mu(i)$ for each $i \notin \pi$ and $\mu'(i_h) = s_h$ for each $i_h \in \pi$, $h = 1, \dots, k$.

A path is **simple** if no student appears in more than one edge in π .¹⁸

Lemma 2 *Let π be an augmenting path for a matching μ and i be the first student in the path (so $\mu(i) = i$). Then there exists a simple augmenting path π' for μ such that i is the first student in π' .*

Proof Let μ be a matching and π be an augmenting in $G(I, S, \mu)$. assume that there are some students appearing several times. Without loss of generality, index the students in π in the order of the augmenting path, i.e., $\pi = (i_1, s_1, i_2, s_2, \dots, i_\ell, s_\ell)$. So, there is at least

¹⁷A related concept (albeit different) is that of a *maximal* matching: a matching μ is **maximal for the students** if there does not exist a matching μ' such that $\mu(i) \neq i \Rightarrow \mu'(i) \neq i$ and there exists i such that $\mu(i) = i$ and $\mu'(i) \neq i$.

¹⁸Note that since schools may be matched to two or more students the same school can appear in different edges of a simple path.

one student i and two indices h and h' such that $j < j'$ and $i_j = i_{j'}$. Let h' be the highest index in π such that there is no $h' < k < k' \leq \ell$ such that $i_k = i_{k'}$, and let h be the lowest index such that $i_h = i_{h'}$ (i.e., there is no $h'' < h$ such that $i_{h''} = i_h$). Let π' be the following path,

$$\pi' = (i_1, s_1, \dots, s_{h-1}, i_h, s_{h'}, i_{h'+1}, \dots, i_\ell, s_\ell).$$

Since π is an augmenting path in $G(I, S, \mu)$, $\mu(i_h) = s_{h-1}$, $\mu(i_{h'+1}) = s_{h'}$ and similarly for the rest of the students in π' except student i_1 for which $\mu(i_1) = i_1$. So, π' is an augmenting path. Note that by construction, i_h appears only once in π . If there is no student in π' appearing more than once then we are done. Otherwise, it suffices to repeat the procedure until we end up with a simple augmenting path. $\blacksquare$

Lemma 3 *A student i_0 is an impossible match for s_0 in the pre-matching problem P if, and only if, there is no stable and maximum matching μ for the pre-matching problem P^{i_0, s_0} such that $\mu(i_0) = s_0$.*

Proof The “if” direction is straightforward. Let i_0 be an impossible match for s_0 and suppose there exists a stable (and maximum) pre-matching for the pre-matching problem P^{i_0, s_0} such that $\mu(i_0) = s_0$. In this case it is easy to see that μ is also stable for P .¹⁹ By Proposition 1 one can easily construct a profile $\succ \in \Theta(P)$ such that $\mu \in S(\succ)$, a contradiction.

Let i_0 be such that i_0 is not an impossible match for s_0 . So, there exists $\succ \in \Theta(P_S)$ and μ such that $\mu(i_0) = s_0$ and $\mu \in S(\succ)$. Clearly, μ is also stable for the pre-matching problem P , and thus it is stable for P^{i_0, s_0} . If μ is a maximum matching for P then μ is also a maximum matching for P^{i_0, s_0} and μ is also obviously stable for P^{i_0, s_0} . Suppose then that μ is not a maximum matching, so that by Berge’s theorem there exists an augmenting path π . We present the following procedure that will yield a maximum matching.

Suppose first that there is a school s such that $|\mu(s)| < q_s$ and there is $i \in A_s$ such that $\mu(i) = i$. Without loss of generality, let i such that for any i' such that $i' P_s i$, $\mu(i') \in S$. So $\pi = (i, s)$ is an augmenting path, and the matching $\mu' = \mu \oplus \pi$ is obviously stable for the pre-matching problem P^{i_0, s_0} .

Suppose now that there exists i such that $\mu(i) \in S$ and $i \in \pi$. By Lemma 2 we can assume that π is a simple. Let $\mu' = \mu \oplus \pi$, and suppose that μ is not a stable matching. So,

¹⁹If $\mu(i_0) = i_0$ and μ is stable for the pre-matching problem P^{i_0, s_0} it may not be the case that μ is also stable for the pre-matching problem P . For instance, we could have $s \neq s_0$ such that $i_0 \in A_s$ and $|\mu(s)| < q_s$.

there exist $i \in \pi$ and i' such that $i'P_{\mu'(s)}i$ such that $\mu'(i') = i'$. Without loss of generality, let $\pi = (i_1, s_1, i_2, s_2, \dots, i, \mu'(i), i_h, s_h, \dots, i_k, s_k)$ such that for $h' \geq h$ there is no i'' such that $\mu'(i'') = i''$ and $i''P_{s_h}i_h$.²⁰ Let i' be such that for each student i'' with higher priority at school $\mu'(i), \mu(i'') \in S$. It is easy to see that the path $\pi' = (i', \mu'(i), i_h, s_h, \dots, i_k, s_k)$ is also a simple augmenting path. Let $\mu'' = \mu \oplus \pi'$. Given our choices of i and i' there is no student that is matched to a school but has a lower priority of an unmatched student at that school, i.e., μ'' is a stable matching. Since $\mu(i_0) = s_0$ and i_0 is only acceptable to school s_0 he cannot be part of any augmenting path. Hence, $\mu''(i_0) = s_0$. If μ'' is a maximum matching then we are done, otherwise it suffices to repeat the procedure until we get a stable maximum matching. ■

5.2. Missing proofs

To prove Proposition 2 we shall use the following alternative characterization of an impossible match student.

Definition 4 Given a pre-matching problem P , a **block at** (i_0, s_0) with $i_0 \in A_{s_0}$ is a pair of disjoint sets $(\mathbf{J}, \bar{\mathbf{J}}) \in I \setminus \{i_0\} \times I \setminus \{i_0\}$ with $|\mathbf{J} \cap \{i : iP_{s_0}i_0\}| \geq q_{s_0}$, $|\mathbf{J}| = \sum_{s \in A_{\mathbf{J}}} q_s$ such that

$$|T| \leq \sum_{s \in A_T^{\bar{\mathbf{J}}}} q_s, \quad \text{for each } T \subseteq \mathbf{J} \quad (3)$$

and for each $i \in \mathbf{J} \setminus \mathbf{J}_0$, for the pre-matching problem P^i ,

$$\exists T \subseteq \mathbf{J} \setminus \{i\} \text{ such that } |T| > \sum_{s \in A_T^{\{i\} \cup \bar{\mathbf{J}}}} \bar{q}_s \quad (\star)$$

where $\bar{q}_s = q_s$ if $s \neq s_0$, $\bar{q}_{s_0} = q_{s_0} - 1$ and $\mathbf{J}_0 = \mathbf{J} \cap \{i : iP_{s_0}i_0\}$.

Proposition 7 Let $(\mathbf{J}, \bar{\mathbf{J}})$ be a block at (i_0, s_0) for a pre-matching problem (I, S, P_S, q_S) . Then i_0 is an impossible match student for s_0 .

²⁰That is, student i is the last student in the path that will make the matching μ' not stable.

If $\mu(i_0) = s_0$ for some stable matching μ , then we must have $\bar{H} := \{j \in \mathbf{J} : \mu(j) = j\} \neq \emptyset$. The proof roughly consists of showing that we can construct a matching $\hat{\mu}$ such that for some $j \in H$ all students in $\mathbf{J} \setminus \{j\}$ are matched to a school and the matching $\hat{\mu}$ is feasible for the pre-profile $P^{\bar{\mathbf{J}} \cup \{j\}}$ with capacity vector $\bar{q}_S$. Invoking the condition $(\star)$ of the definition of a block will then yields a contradiction as it implies that there is another student that is not in H that is not matched at μ . The key observation to yield to this contradiction is that since μ is stable it must be that all students in $\mathbf{J}_0$ must be matched at μ .

Proof Let P be a profile with a block $(\mathbf{J}, \bar{\mathbf{J}})$ at (i_0, s_0) . We assume that $(\mathbf{J}, \bar{\mathbf{J}})$ is a minimal block at (i_0, s_0) in the sense that there is no block $(\mathbf{J}', \bar{\mathbf{J}}')$ at (i_0, s_0) such that $\mathbf{J} \subset \mathbf{J}'$. Suppose that i_0 is not an impossible match for i_0 . So, by Proposition 1 there exists a matching μ stable for P such that $\mu(i_0) = s_0$. Note that, μ is also stable for the pre-profile $\tilde{P}$, where $\tilde{P}_{s_0} = P_{s_0}$ and for each $s \neq s_0$, $\tilde{P}_s$ is a priority ordering over $A_s \setminus \{i_0\}$ with $i\tilde{P}_s i'$ if, and only if, $iP_s i'$. Also, if $(\mathbf{J}, \bar{\mathbf{J}})$ is a block at (i_0, s_0) for the pre-profile P it is also a block at (i_0, s_0) for the profile $\tilde{P}$. We shall therefore assume throughout the proof that P is such that $A_{i_0} = s_0$. Let $H = \{j \in \mathbf{J} : \mu(j) \neq j\}$ and $\bar{H} = \mathbf{J} \setminus H$. Since $\mu(i_0) = s_0$ and $|\mathbf{J}| = \sum_{s \in A_{\mathbf{J}}} q_s$ we have $\bar{H} \neq \emptyset$.

Claim 1 *There exists a matching $\hat{\mu}$ such that $\hat{\mu}(j) \in A_{\mathbf{J}}$ for each $j \in H$ and $\hat{\mu}$ is a feasible matching at $(I, S, P^{\bar{\mathbf{J}}}, \bar{q}_S)$.²¹*

Proof. From the definition of a block there exists a matching μ^* such that $\mu^*(j) \in A_{\mathbf{J}}$ for each $j \in \mathbf{J}$ and that is feasible for $P^{\bar{\mathbf{J}}}$. Let j_1 be any student in $\mu^*(s_0)$ and let $\hat{\mu}$ be such that

$$\hat{\mu}(j) = \begin{cases} j & \text{if } j = j_1 \\ \mu^*(j) & \text{if } j \in H \setminus \{j_1\} \\ j & \text{if } j \notin H \end{cases} \quad (4)$$

If $\hat{\mu}(j) \neq j$ for each $j \in \mathbf{J} \setminus \bar{H}$ then we are done.²² Assume then that $j_1 \in H$ and suppose by way of contradiction that we cannot match all students in H at $(I, S, P^{\bar{\mathbf{J}}}, \bar{q}_S)$, i.e., there is no augmenting path π at $\hat{\mu}$ starting with student j_1 . Suppose there exists a school $s_1 \in A_{\mathbf{J} \setminus \bar{H}}$ such that $|\hat{\mu}(s_1)| < \bar{q}_{s_1}$. Since $s_0 \in A_{j_1}$ it must be that $|\hat{\mu}(s_0)| = \bar{q}_{s_0}$. So, $s_1 \neq s_0$. Observe

²¹Note that the claim is trivial if there is no $j \notin \mathbf{J}$ such that $\mu(j) \in A_{\mathbf{J}}$.

²²Note that the existence of μ^* for the pre-profile $P^{\bar{\mathbf{J}}}$ does not imply that $jP_{s_0}i_0$ for each $j \in \mu^*(s_0)$. So it might be that there is $j_1 \in \mu^*(s_0)$ such that $i_0P_{s_0}j_1$ and $j_1 \notin H$.

that if there is a path $\pi = (j_1, \dots, s_1)$ then there is an augmenting path $\pi'(j_1, \dots, s_1)$. Define $H_1 = \{j \in H : s_1 \notin A_j\}$. We then have $A_{H_1} \cap A_{H \setminus H_1} = \emptyset$. Note that $s_0 \in A_{j_1}$ implies $j_1 \in H_1$. If there is another school s_2 such that $s_2 \in A_{H_1}$ and $|\widehat{\mu}(s_2)| < \bar{q}_{s_2}$, then let $H_2 = \{j \in H_1 : s_2 \notin A_j\}$. Again $s_0 \in A_{j_1}$ implies $j_1 \in H_2$ and $A_{H_2} \cap A_{H \setminus H_2} = \emptyset$. Continuing this way we eventually end up with a set H_k such that $|\widehat{\mu}(s)| = \bar{q}_s$ for each $s \in A_{H_k}$ and $A_{H_k} \cap A_{H \setminus H_k} = \emptyset$. Since $\widehat{\mu}(j_1) = j_1$, we then have $|H_k| = \sum_{s \in A_{H_k}} q_s$. We then have

$$\mu^* \text{ is a perfect match between } H_k \text{ and } A_{H_k} \text{ at } P^{\bar{\mathbf{J}}}. \quad (5)$$

Note that μ stable and $\mu(i_0) = s_0$ imply $\mu(j) \neq j$ for each $j \in \mathbf{J}_0$. Using $j_1 \in H_k$ and $s_0 \in A_{j_1}$ we then have

$$|\{j \in H_k : j P_{s_0} i_0\}| \geq q_{s_0}. \quad (6)$$

Let $j \in H_k \setminus \mathbf{J}_0$, and suppose that there exists a matching $\tilde{\mu}$ such that for each $j' \in H_k \setminus \{j\}$ we have $\tilde{\mu}(j') \neq j'$ and that $\tilde{\mu}$ is feasible for $(I, S, P^{\bar{\mathbf{J}} \cup \{j\}}, \bar{q}_S)$. Since $A_{H_k} \cap A_{H \setminus H_k} = \emptyset$ we have $P_s^{\bar{\mathbf{J}}} = P_s^{\bar{\mathbf{J}} \cup \{j\}}$ for each $s \notin A_{\mathbf{J}_k}$ and $\mu^*(j) \notin A_{H_k}$ for each $j \in \mathbf{J} \setminus H_k$.²³ So the matching $\bar{\mu}$ defined as $\bar{\mu}(j) = \tilde{\mu}(j)$ for each $j \in H_k$ and $\bar{\mu}(j) = \mu^*(j)$ for each $j \in \mathbf{J} \setminus H_k$ is a feasible matching for $(I, S, P^{\bar{\mathbf{J}} \cup \{j\}}, \bar{q}_S)$. Observe that at $\bar{\mu}$ there are $|\mathbf{J}| - 1$ students in $\mathbf{J}$ that are matched to a school, i.e., Hall's marriage condition is satisfied each set $T \subseteq \mathbf{J} \setminus \{j\}$. So, $(\mathbf{J}, \bar{\mathbf{J}})$ is not a block at (i_0, s_0) , a contradiction. Therefore, for each $j \in H_k \setminus H_0$ it is not possible to match all students in $H_k \setminus \{j\}$ at the pre-profile $P^{\bar{\mathbf{J}} \cup \{j\}}$ with capacity vector $\bar{q}_S$. So, using (5) and (6) we deduce that $(\mathbf{J}_k, \bar{\mathbf{J}})$ is a block at (i_0, s_0) . Since $H_k \subset \mathbf{J}$ we deduce that $(\mathbf{J}, \bar{\mathbf{J}})$ is not a minimal block, a contradiction. So, there exists a matching $\widehat{\mu}$ that is feasible for $(I, S, P^{\bar{\mathbf{J}}}, \bar{q}_S)$ such that $\widehat{\mu}(j) \neq j$ for each $j \in H_k$. $\square$

Let $\widehat{\mu}$ be a matching such that $\widehat{\mu}(j) \neq j$ for each $j \in H$, $\widehat{\mu}(j) = j$ for each $j \notin H$, and $\widehat{\mu}$ is a feasible matching for $(I, S, P^{\bar{\mathbf{J}}}, \bar{q}_S)$. We now consider matching the students in $\bar{H}$. Suppose first that $|\bar{H}| = 1$, say, $\bar{H} = \{j_1\}$. Since μ is stable we must have that μ is also a feasible matching at $(I, S, P^{\bar{\mathbf{J}} \cup \{j_1\}}, q_s)$. Observe now that $|\mathbf{J} - 1| = |H - 1|$ implies that μ is a perfect match between $H \setminus \{j_1\}$ and A_H when schools capacities are $\bar{q}_S$ at the pre-profile $P^{\bar{\mathbf{J}} \cup \{j_1\}}$. This contradicts the fact that $(\mathbf{J}, \bar{\mathbf{J}})$ is a block at (i_0, s_0) . So, $|\bar{H}| > 1$. Without loss of generality let $\bar{H} = \{j_1, j_2, \dots, j_k\}$, and assume $k > 1$. So at $\widehat{\mu}$ there are $|H| = |\mathbf{J}| - k$

²³recall that μ^* is a perfect matching for $\mathbf{J}$ and $A_{\mathbf{J}}$ with capacities q_S and by construction $\bar{q}_s = q_s$ for each school $s \notin A_{H_k}$.

students in $\mathbf{J}$ that are matched to a school. Since $\hat{\mu}(i_0) = s_0$ and $|\mathbf{J}| = \sum_{s \in A_{\mathbf{J}}} q_s$, we can match at most $k - 1$ additional students from $\bar{H}$ to a school while maintaining all students already matched to a school at $\hat{\mu}$ to some school when capacities are given by $\bar{q}_S$.

Our goal now consists of adding to the set of students matched to a school the students in $\bar{H}$. Most of the arguments are similar to those used to show that we can match all students in $\mathbf{J} \setminus \bar{H}$ at $P^{\bar{\mathbf{J}}}$ with capacity vector $\bar{q}_S$.

Claim 2 *For each student $j \in \bar{H}$ there exists a matching $\hat{\mu}_1$ such that $\hat{\mu}_1(j) \neq j$ for each $j' \in H \cup \{j\}$ that is feasible for $P^{\bar{\mathbf{J}}}$ with capacity vector $\bar{q}_S$.*

Proof. Let j_1 be any student in $\bar{H}$ and suppose that there is no an augmenting path starting with student j_1 at the matching $\hat{\mu}$ with the pre-profile $P^{\bar{\mathbf{J}}}$ and capacity vector $\bar{q}_S$. Note that since $k > 1$ there is at least one school $s_1 \in A_{\mathbf{J}}$ such that $|\hat{\mu}_1(s_1)| < \bar{q}_{s_1}$. Consider then the set $H_1 = \{j \in H \cup \{j_1\} : s_1 \notin A_j\}$.²⁴ Clearly, $j_1 \in H_1$. If there is another school s_2 such that $|\hat{\mu}(s_2)| < \bar{q}_{s_2}$, then let $H_2 = \{j \in H_1 : s_2 \notin A_j\}$. Continuing this way we eventually end up with a set H_k such that $|\hat{\mu}(s)| = \bar{q}_s$ for each $s \in A_{H_k}$. Since $\hat{\mu}(j_1) = j_1$ we have $|H_k| = \sum_{s \in A_{H_k}} q_s$. With capacities q_S it is possible to match all students in $\mathbf{J}$, so there exists a path $\pi = (j_1, \dots, s_0)$. It follows that $s_0 \in A_{H_k}$. So, $\mathbf{J}_0 \subset H_k$ and thus $|\{j \in H_k : j P_{s_0} i_0\}| \geq s_0$. Note also that at μ^* all students in H_k are matched to a school and $|H_k| = \sum_{s \in A_{H_k}} q_s$.

Consider now any $j \in \{j_1\} \cup (H_k \setminus \mathbf{J}_0)$ and suppose that there exists a matching $\tilde{\mu}$ such that for each $j' \in \{j_1\} \cup H_k \setminus \mathbf{J}_0$ we have $\tilde{\mu}(j') \neq j'$ and that $\tilde{\mu}$ is feasible at $P^{\bar{\mathbf{J}} \cup \{j\}}$ with capacity vector $\bar{q}_S$. From the construction of the set $\{j_1\} \cup H_k$, for each $s \notin A_{H_k \cup \{j_1\}}$ we have $P_s^{\bar{\mathbf{J}}} = P_s^{\bar{\mathbf{J}} \cup \{j\}}$. So the matching $\bar{\mu}$ defined as $\bar{\mu}(j') = \tilde{\mu}(j')$ for each $j' \in H_k \cup \{j_1\}$ and $\bar{\mu}(j') = \mu^*(j')$ for each $j' \notin H_k \cup \{j_1\}$ is a feasible matching at $P^{\bar{\mathbf{J}} \cup \{j\}}$. Indeed, students in H_k can only be matched to a school in A_{H_k} , so at μ^* all students in $\mathbf{J} \setminus H_k$ must be matched to a school $s \notin A_{H_k}$. Observe now that we have $|\{j' \in \mathbf{J} : \bar{\mu}(j') \neq j'\}| = |\mathbf{J}| - 1$, which contradicts the fact that $(\mathbf{J}, \bar{\mathbf{J}})$ is a block at (i_0, s_0) . So, for each $j \in H_k \setminus \mathbf{J}_0$ there is no such matching $\tilde{\mu}$. It follows then that $(H_k, \bar{\mathbf{J}})$ is a block at (i_0, s_0) . Since $k > 2$ we have $H_k \cup \{j_1\} \subset \mathbf{J}$ and thus $(\mathbf{J}, \bar{\mathbf{J}})$ is not a minimal block, a contradiction. So, for each student $j \in \bar{H}$ there exists an augmenting path at $\hat{\mu}$ starting with student j in the pre-profile $P^{\bar{\mathbf{J}}}$

²⁴Like for the case of the students in $\mathbf{J} \setminus \bar{H}$ it is the case that there is no path π such that for some $i, i' \in \pi$ we have $i \in H_1$ and $i' \in A_{s_1}$.

with capacity vector $\bar{q}_S$. □

Claim 3 *There exists a student $j_1 \in \bar{H}$ and a matching $\hat{\mu}_1$ such that $\hat{\mu}_1(j) \neq j$ for each $j \in \{j_1\} \cup H$ that is feasible for $P^{\bar{\mathbf{J}} \cup \bar{H} \setminus \{j_1\}}$ with capacity vector $\bar{q}_S$.*

Let j_1 be a student $j_1 \in \bar{H}$ and let $\pi_1 = (j_1, s_1, \dots)$ be an augmenting path for $\hat{\mu}$ at $P^{\bar{\mathbf{J}}}$ with capacity vector $\bar{q}_S$ such that there is no other student $j_2 \in \bar{H}$ and an augmenting path $\pi_2 = (j_2, \dots)$ such that π_2 is strictly smaller than π_1 . If the smallest augmenting path is not unique, then choose any augmenting path $\pi = (j_1, s_1, \dots, s_\ell)$ where for each j such that $j P_{s_1} j_1$ we have $\hat{\mu}(j) \neq j$. We claim that $\hat{\mu}_1 = \hat{\mu} \oplus \pi$ is a feasible matching at $P^{\bar{\mathbf{J}} \cup \bar{H} \setminus \{j_1\}}$ with capacity vector $\bar{q}_S$.

To see this, suppose by way of contradiction that there exists $j \in \bar{\mathbf{J}} \setminus \{j_2, \dots, j_k\}$ and $j' \in \{j_2, \dots, j_k\}$ such that $j' P_s j$, $\hat{\mu}_1(j') = j'$ and $\hat{\mu}_1(j) = s$ for some $s \in A_j$. Suppose first that $s = s_1$, i.e., the first school involved in π . Since $\hat{\mu}_1(j') = j'$ we have $\hat{\mu}(j') = j'$ and thus $\pi' = (j', s_1, i_2, \dots, j_\ell, s_\ell)$ is also an augmenting path. It has the same length as π , but it contradicts the choice of π if the smallest augmenting path is not unique. So, $s \neq s_1$. That is, there exists $1 < h \leq \ell$ such that for $j_h, s_h \in \pi$ we have $j_h = j$ and $s_h = s$. Again from $\hat{\mu}(j') = j'$ we deduce that $\pi' = (j', s_h, j_{h+1}, \dots, j_\ell, s_\ell)$ is an augmenting path. Clearly π' is smaller than π , which contradicts the choice of π . So $\hat{\mu}_1$ is feasible for $P^{\bar{\mathbf{J}} \cup \bar{H} \setminus \{j_1\}}$ with capacity vector $\bar{q}_S$. □

It suffices now to repeat the previous claims until obtaining a matching $\hat{\mu}_{k-1}$ such all students in $\bar{H}$ but one, say, j_k are matched to a school at $\hat{\mu}_{k-1}$ and $\hat{\mu}_{k-1}$ is a feasible matching at $P^{\bar{\mathbf{J}} \cup \{j_k\}}$ with capacity vector $\bar{q}_S$. Note that at $\hat{\mu}$ there are $|\bar{\mathbf{J}}| - 1$ students in $\bar{\mathbf{J}}$ that are matched to a school. Since $(\bar{\mathbf{J}}, \bar{\mathbf{J}})$ is a block this is not possible. So μ such that $\mu(i_0) = s_0$ cannot be a stable matching. ■

The definition of a block would suggest that students in $\bar{\mathbf{J}}$ are not dummies for the schools acceptable to them. This intuition is misleading. Propositions 7 only says that if there is a block at (i_0, s_0) but i_0 is matched to s_0 at some matching then for any $(P_S$ -compatible) preference profile μ cannot be stable (the proof of the proposition will show that there is at least a student and a school in $\bar{\mathbf{J}}$ and $A_{\bar{\mathbf{J}}}$, respectively, that block μ). It may well be the case that there are students outside the block with a higher priority than the students in $\bar{\mathbf{J}}$ that may prevent some students in $\bar{\mathbf{J}}$ to be matched to a school in $A_{\bar{\mathbf{J}}}$. That is, a student

may be part of a block while also being an impossible match for all the schools for which he is acceptable. The following examples illustrates this point.

Example 2 (Multiple blocks) Consider the case where there are two schools, say s_1 and s_2 who only rank students i_1, i_2, i_3 and i_4 and those schools are the only schools ranking them. Each school has only one seat to offer. Assume that for these four students schools s_1 and s_2 are acceptable. Let the schools' priorities be as follows:

P_{s_1}	P_{s_2}
i_1	i_2
i_3	i_4
i_2	i_1
i_4	i_3

Routine checks shows that for any preference profile for students $i_h, h = 1, \dots, 4$ we have, at any stable matching, student i_1 matched to school s_1 and student i_2 matched to school s_2 . For instance, $\{i_2, i_3\}$ is a block at (i_1, s_2) , although i_3 is an impossible match for both schools s_1 and s_2 . $\square$

Proof of Proposition 2 Let i_0 and s_0 such that i_0 is an impossible match for s_0 and consider the matching algorithm. If the algorithm stopped at Step 2, then it is easy to see that if μ^0 is stable for the pre-profile $\tilde{P}$ it is also stable for the pre-profile P . So, the algorithm must have gone through Step 3.

To start, notice that if the algorithm reached Step 4 it must be that for each $h = 1, \dots, k$ it computed a matching μ^h at step 3.h.3. Consider first the case when $h = 1$, where the matching produced by the algorithm, μ^1 , was obtained at Step 3.1.3. So, there is $i \in \hat{I} \setminus I_0$ such that there exists a matching $\hat{\mu}$ where each student in $\hat{I} \setminus \{i\}$ is matched to a school in the pre-profile $\hat{P}^i$. Suppose there is justified envy at μ^1 for the pre-profile $\tilde{P}$. That is, there exists j, j' and s such that $\mu^1(j) = j$, $\mu^1(j') = s$ and $j \tilde{P}_s j'$. Suppose first that $j' = i_1$. So, $s = s_0$ and thus $j \in I_0$ and $j \neq i$. Hence $j \in \hat{I} \setminus \{i\}$ and thus $\hat{\mu}(j) \neq j$, a contradiction. So, $j' \neq i_1$ and thus $\mu^1(j') = \hat{\mu}(j')$. Therefore, student j is acceptable for s in the pre-profile $\hat{P}^i$ and thus $j \in \hat{I} \setminus \{i\}$. So $\hat{\mu}(j) \neq j$, a contradiction. Therefore, there is no justified envy at μ^1 for the pre-profile $\tilde{P}$.

A simple repetition of the above argument yields that for each $h = 1, \dots, k$, provided the algorithm went did not stop at step 3.h.3, the matching μ^h is stable for the pre-profile $\tilde{P}$.

Thus, if the algorithm stops at Step 4 with a matching μ then it is stable for the pre-profile $\tilde{P}$. Since $\mu(i_0) = s_0$, μ is also stable for the pre-profile P , a contradiction with i_0 being an impossible match for s_0 . So, the algorithm stopped at a step 3.h.3 for some $h \leq k$.

Since at each step 3.h.1 the matching μ^{h-1} is stable for the pre-profile $\tilde{P}$ it is sufficient to consider the case when the algorithm stopped at Step 3.1.3. Let $\Gamma = \{i : \mu(i) \neq i\}$. We claim that there exists a set $\Gamma^* \subseteq \Gamma$ such that $|\Gamma^*| = \sum_{s \in A_{\Gamma^*}} q_s$. To see this, suppose by way of contradiction that $|\Gamma| < \sum_{s \in A_{\Gamma}} q_s$. So there is a school, say, s_1 , such that $|\mu(s_1)| < q_{s_1}$. Since μ is stable for the pre-profile $\tilde{P}$, $\mu(i) \neq i$ for each $i \in A_{s_1}$. Since $\mu(i_1) = i_1$ and $\mu = \mu^0$ is a stable maximum matching, for each $i \in A_{s_1}$ there is no augmenting path between i and i_1 in the pre-profile $\tilde{P}$.²⁵ It follows that there exists $\Gamma^* \subset \Gamma$ and $\bar{\Gamma} \subset \Gamma$ such that $\Gamma^* \cap \bar{\Gamma} = \emptyset$, $A_{\Gamma^*} \cap A_{\bar{\Gamma}} = \emptyset$, with $i_1 \in \Gamma^*$ and $A_{s_1} \subset \bar{\Gamma}$. A simple repetition of this argument for all the schools s such that $|\mu(s)| = q_s$ yields to a minimal set Γ^* such that $A_{\Gamma^*} \cap A_{\Gamma \setminus \Gamma^*} = \emptyset$ and for each school $s \in A_{\Gamma^*}$, $|\mu(s)| = q_s$, i.e., $|\Gamma^*| = \sum_{s \in A_{\Gamma^*}} q_s$.

We claim that we have a block $(\mathbf{J}, \bar{\mathbf{J}})$ at (i_1, s_0) , where $\mathbf{J} = \Gamma^*$ and $\bar{\mathbf{J}} = \{i : \mu(i) = i\}$. By the very choice of the set Γ^* , $|\mathbf{J}| = \sum_{A_{\mathbf{J}}} q_s$. Notice also that since $s_0 \in A_{i_1}$, $A_{i_1} \subseteq A_{\Gamma^*}$, and thus $I_0 \subseteq \Gamma^*$ (where I_0 is defined at Step 3.1.1). Since μ is stable for the pre-profile $\tilde{P}$ and $\mu(i_1) = i_1$ we have by $\mu(s_0) \subseteq I_0$ and $|\mu(s_0)| = q_{s_0}$, i.e., $|\mathbf{J} \cap \{i : i P_{s_0} i_1\}| \geq q_{s_0}$. It remains then to show that condition $(\star)$ holds for the block $(\mathbf{J}, \bar{\mathbf{J}})$. That is, we need to show that for each $j \in \mathbf{J}$ we cannot find a matching $\hat{\mu}$ such that all students in $\mathbf{J} \setminus \{j\}$ are matched to a school in the pre-profile $\hat{P}^{\mathbf{J} \cup \{i\}}$.

Since the algorithm stopped at Step 3.1.3, for each $i \in \hat{I} \setminus I_0$, there is no matching $\hat{\mu}$ such that all students in $\hat{I} \setminus \{i\}$ are matched in $\hat{P}^i$, the pre-profile $\hat{P}$ truncated at i . Suppose however that there is a student $i \in \Gamma^* \setminus I_0$ such that some maximum matching $\hat{\mu}$ in the pre-profile $\hat{P}$ truncated at i , $\hat{P}^i$, we have $\hat{\mu}(j) \neq j$ for each $j \in \Gamma^* \setminus \{i\}$. Since the algorithm stopped at this step there is thus a student $j \in \Gamma \setminus \Gamma^*$ such that $\hat{\mu}(j) = j$. By the definition of Γ^* , $i \in \Gamma^*$ implies $A_i \cap A_{\Gamma \setminus \Gamma^*} = \emptyset$. So, for each school $s \in A_{\Gamma \setminus \Gamma^*}$, $\hat{P}_s = \hat{P}_s^i$.

Observe that μ^0 being stable for $\tilde{P}$ implies that a matching $\bar{\mu}$ such that $\bar{\mu}(s) = \mu^0(s)$ for each $s \in A_{\Gamma \setminus \Gamma^*}$ and $\bar{\mu}(s) = \hat{\mu}(s)$ for each $s \in A_{\Gamma^*}$ is a feasible matching for $\hat{P}^i$. Clearly, for

²⁵If μ^0 is not a stable maximum matching then there exists an augmenting path π in the pre-profile $\tilde{P}$, and perhaps $i_1 \in \pi$. Clearly, $\mu^0 \oplus \pi$ may not be a stable matching for $\tilde{P}$ but by Lemma 3 we can find a path $\pi' \subseteq \pi$ such that $\mu' = \mu^0 \oplus \pi'$ is stable. It could be the case that $\mu'(i_1) = i_1$, but μ' would be a maximum matching (if not, then we can find another path that would yield to a stable matching with more students being matched).

each $s \in A_{\Gamma \setminus \Gamma^*}$, $|\bar{\mu}(s)| \geq |\hat{\mu}(s)|$, with a strict inequality for at least one school s (because there is $j \in A_{\Gamma \setminus \Gamma^*}$ such that $\bar{\mu}(j) \neq j$ and $\hat{\mu}(j) = j$). That is, $\hat{\mu}$ cannot be a maximum matching for the pre-profile $\hat{P}^i$, a contradiction. Hence, if the algorithm stopped at Step 3.1.4 it must be that there exists $i \in \Gamma^* \setminus I_0$ such that for any maximum matching $\hat{\mu}$ for the pre-profile $\hat{P}^i$ there is a student $j \in \Gamma^* \setminus \{i\}$ such that $\hat{\mu}(j) = j$. That is, Hall's marriage condition fails for $\Gamma^* \setminus \{i\}$ in the pre-profile $\hat{P}^i$, or put differently condition $(\star)$ holds for $\Gamma^* \setminus I_0$.

Observe now that $i_0 \notin \mathbf{J}, \bar{\mathbf{J}}$. So, the pair $(\mathbf{J}, \bar{\mathbf{J}})$ is also a block at (i_0, s_0) , the desired result. ■

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