Cost allocation protocols for network formation on connection situations

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Summary

Model and objectives

Game and properties

An optimal protocol

Building a network in a strategic setting

Situation: a set of agents building a network in order to be connected to a given source.

Assumptions:

- making a link $e = (i, j)$ has a cost/consumes energy $w(e)$;
- non cooperative game: agents do not make binding agreements on the design of the network. Each agent wants to minimize its own cost.

Question and objectives

How should we **design cost allocation protocols** to minimize the efficiency loss caused by rational players that are only willing to perform update leading to an immediate reduction of their individual cost shares?

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→ address the problem of the design of cost allocation protocols to **coordinate players** placed on the nodes of a graph in such a way that:

- ▶ **convergence** under **Better Response Dynamics** (BRD) holds
- ▶ an **efficient** (minimum cost) communication network is built.

The strategic game

- $G = (N', E, w)$ is an **undirected**, **connected** and **weighted graph**, where $N' = N \cup \{0\}$ and $N = \{1, 2, \dots, n\}$, and $w(e)$ is the **cost/power** needed to build/use the link e .

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- The cost of **remaining disconnected** from the source is **infinite**.

An example of game form with two players

1 \ 2		0		1	
		0		1	
0	0				
2	0				

Some properties for protocols and games

- A cost allocation protocol c such that $\sum_{i \in S} c_i(w, S) = w(T_S)$ for every strategy profile S is said **budget-balanced**.
(for connected players if some players are not connected)
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A protocol is said **state-dependent** iff for every state S and every weight functions w, w' , with $w(e) = w'(e)$ for every $e \in T_S$, then $c_i(w, S) = c_i(w', S)$ for every $i \in S$.

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Nash equilibrium (0, 1) is **optimal** (i.e. $T_{(0,1)}$ is a minimum cost spanning tree (**mcs**t) connecting all nodes in N'), but (2, 0) is not.

Better Response Dynamic and Optimality

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- A **Better Response Dynamic** (*BRD*, also called Nash dynamics) (associated with a protocol c) is a sequence of states $S^0, S^1, \dots$, such that each state S^k (except S^0) is resulted by a better response of some player from the state S^{k-1} .

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Theorem If a protocol is budget-balanced and optimal, then it is not state-dependent.

A budget-balanced and optimal protocol

Idea: If the network is not optimal (extra cost Δ), charge this cost Δ to a player (a set of players) to create for them an incentive to change.

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How to find the victims: based on a set of players which

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Two protocols:

- ▶ One fairly (equally) shares cost between players in a optimal situation;
- ▶ One is more like Bird's protocol (players pay one link).

Ordinal potential game

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Thanks to a potential function $\Phi(S)$, ie Φ only decreases during a BRD.

$$\Phi(S) = (|N \setminus \text{con}(S)|, |\hat{V}(S)|, \sum_{i \in \hat{V}(S)} |E_S(i)|)$$

where $E_S(i) = \{j \in N' : w(i, j) < w(i, S_i)\}$.

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- we have analyzed properties for protocols in relation the the convergence of the best reply dynamics to efficient Nash equilibria.
- the inherent limitations of the optimal protocols proposed in this paper is that it depends on the choice of an *a priori* selected mcst.