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► To cite this version:

Justin Kruger, M. Remzi Sanver. The relationship between Arrow's and Wilson's theorems on restricted domains. Mathematical Social Sciences, Elsevier, 2021, 114, pp.95-97. 10.1016/j.mathsocsci.2021.08.002 . hal-03464586

HAL Id: hal-03464586

<https://hal.archives-ouvertes.fr/hal-03464586>

Submitted on 3 Dec 2021

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The relationship between Arrow's and Wilson's theorems on restricted domains*

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August 7, 2021

Abstract

We consider Arrow's and Wilson's theorems as conditions on domains, as well as a third condition due to Malawski and Zhou (1994). We describe when domains satisfying each combination of these conditions exist.

JEL Classifications: D71.

Keywords: social welfare function, restricted domains, Arrow's impossibility, Wilson's impossibility.

*We are grateful to two referees and the associate editor whose anonymous reports contributed to the improvement of our paper.

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1 Introduction

Wilson’s (1972) theorem is sometimes considered a strengthening of Arrow’s (1950) theorem. Indeed, on the full domain, Arrow’s theorem is a ready corollary of Wilson’s result. Although the reverse implication is less intuitively obvious, Malawski and Zhou (1994) give an intermediate result which can be used to fashion a proof of Wilson’s theorem using Arrow’s theorem. They also give domain conditions under which their intermediate result applies. Thus, for domains satisfying these conditions, Wilson’s theorem can be obtained through Arrow’s theorem. Campbell and Kelly (2003) further explore the two theorems under domain restrictions by giving example domains where Arrow’s theorem fails and Wilson’s holds, and vice-versa.

Following Campbell and Kelly (2003), we consider these theorems as conditions on domains. To see how we do this, first consider Arrow’s and Wilson’s theorems in the traditional manner, that is, as implications between properties of social welfare functions. One such version of Arrow’s theorem is the following: if a social welfare function

- (1) is defined on a domain containing free triples,
 - (2) satisfies binary independence, and
 - (3) is Pareto optimal,
- then
- (4) this social welfare function is dictatorial.

We remove the property concerning domains—here, (1) the free triple property—from the antecedent of the implication, and designate those domains over which the remaining antecedents (2) and (3) imply the original consequent (4) as Arrow domains. Similarly, we define Wilson domains as those where the implication of Wilson’s theorem—minus any domain properties— holds; and similarly Malawski-Zhou domains for the result of Malawski and Zhou (1994).

Obviously, each of these three conditions -namely Arrow, Wilson and Malawski-Zhou - will be satisfied by some domains and failed by others. An interesting question concerns what logical combinations of the three conditions are possible. This paper gathers the findings that are dispersed in the literature; adds new ones, and provides an almost complete map with respect to the logical (in)compatibilities between these three conditions. To this end, the remainder of the paper is split into three further sections. Immediately after this introductory section, we provide formal definitions in Section 2. Section 3 contains the results that are appraised and catalogued in a Venn diagram in Section 4.

2 Definitions

Consider a finite set of *alternatives* A with $|A| \geq 3$ and a set of *voters* $N = \{1, \dots, n\}$ with $n \geq 2$. We denote $W(A)$ for the set of complete preorders over A . We write $R_i \in W(A)$ for the *preference* of $i \in N$ over A . We use P_i to refer to the

strict part of R_i . We let $R = (R_1, \dots, R_n) \in W(A)^N$ stand for a *(preference) profile*. Any non-empty $D \subseteq W(A)^N$ is called a *domain*.

2.1 Social welfare functions and their properties

A *social welfare function* (SWF) is a mapping $f : D \longrightarrow W(A)$. For any $R \in D$, we write $f^*(R)$ for the strict part of $f(R)$. A SWF f is

- *Pareto optimal (PO)* if for all $x, y \in A$ and for all $R \in D$, $x P_i y$ for all $i \in N$ implies $x f^*(R) y$.
- *anti-Pareto optimal (A-PO)* if for all $x, y \in A$ and for all $R \in D$, $x P_i y$ for all $i \in N$ implies $y f^*(R) x$.
- *non-imposed (NI)* if for all $x, y \in A$, there exists $R \in D$ such that $x f(R) y$.
- *dictatorial (DICT)* if there exists $i \in N$ such that for all $x, y \in A$ and for all $R \in D$, $x P_i y$ implies $x f^*(R) y$.
- *anti-dictatorial (A-DICT)* if there exists $i \in N$ such that for all $x, y \in A$ and for all $R \in D$, $x P_i y$ implies $y f^*(R) x$.
- *null (NULL)* if for all $x, y \in A$ and for all $R \in D$, $x f(R) y$.
- *binary independent (IND)* if for all $x, y \in A$ and for all $R, R' \in D$, $x R_i y \iff x R'_i y$ for all $i \in N$ implies $x f(R) y \iff x f(R') y$.

2.2 Properties of domains

We say that a domain D is

- **ARROW** if for every SWF $f : D \longrightarrow W(A)$, f is IND and PO implies f is DICT.
- **WILSON** if for every SWF $f : D \longrightarrow W(A)$, f is IND and NI implies f is DICT or A-DICT or NULL.
- **MALAWSKI-ZHOU** if for every SWF $f : D \longrightarrow W(A)$, f is IND and NI implies f is PO or A-PO or NULL.

Although we are centrally concerned with the above three conditions on domains, we also consider two other conditions. A domain D is *Cartesian* if there is some $E \subseteq W(A)$ such that $D = E^N$. A domain D is *Paretian complete* if for every pair $x, y \in A$, there exists $R \in D$ such that $x P_i y$ for all $i \in N$. A domain that is both Cartesian and Paretian complete is qualified as *basic*.

3 Results

Given three possible conditions on domains, ARROW, WILSON and MALAWSKI-ZHOU, there are eight possible logical combinations. We give below seven propositions (Proposition 4 covers two of these logical combinations) that together determine for which combinations domains exist. In case of existence, we discuss the possibility of ensuring the domain to be basic.

PROPOSITION 1: There are basic domains that are not ARROW, not WILSON and not MALAWSKI-ZHOU.

PROOF: Let $A = \{a_1, \dots, a_m\}$. A preference $R_i \in W(A)$ is single-peaked (with respect to order of natural numbers) if for all $j, k, l \in \{1, \dots, m\}$ with $j > k > l$, $a_j R_i a_k$ implies $a_k R_i a_l$ and $a_l R_i a_k$ implies $a_k R_i a_j$. Let $S(A) \subseteq W(A)$ be the set of single-peaked preferences. The pairwise majority rule is the SWF $f^\mu : S(A)^N \rightarrow W(A)$ where for all $R \in S(A)^N$, for all $x, y \in A$, we have $x f^\mu(R) y \iff |\{i \in N : x R_i y\}| \geq |\{i \in N : y R_i x\}|$. Over $S(A)^N$, f^μ satisfies IND, PO (hence NI) but fails each of DICT, A-DICT and NULL. Thus $S(A)^N$ exemplifies a domain which is neither ARROW nor WILSON.

Now, consider the SWF $f : S(A)^N \rightarrow W(A)$ that is identical to f^μ except at profiles $R \in S(A)^N$ with $a_1 P_i a_2$ for all $i \in N$, in which case a_1 and a_2 are indifferent in the output. To see that the output of f is always transitive, take any $R \in S(A)^N$ and any $x, y, z \in A$ with $x f(R) y$ and $y f(R) z$. Clearly, $x f(R) z$ when R is such that $f(R) = f^\mu(R)$. So let $a_1 P_i a_2$ for all $i \in N$. Unless $(a_1 = x \text{ and } a_2 \in \{y, z\})$ or $(a_1 = y \text{ and } a_2 = z)$ we have $x f^\mu(R) y$ and $y f^\mu(R) z$, implying $x f^\mu(R) z$, which in turn implies $x f(R) z$. Now let $a_1 = x$ and $a_2 = y$, without loss of generality. Again, we have $x f^\mu(R) y$ and $y f^\mu(R) z$, implying $x f^\mu(R) z$, which in turn implies $x f(R) z$. Thus, the output of f is always transitive. Moreover, f is IND and NI but does not satisfy any of PO, A-PO or NULL, hence showing that $S(A)^N$ is not MALAWSKI-ZHOU. Q.E.D.

REMARK: f^μ is IND, NI and PO, thus it does not establish the failure of MALAWSKI-ZHOU over $S(A)^N$ and to our knowledge, the literature admits no consideration of domains that fail all three conditions.

PROPOSITION 2: No domain is ARROW, MALAWSKI-ZHOU and not WILSON.

PROOF: Suppose a domain D is ARROW and MALAWSKI-ZHOU. We will show that D is WILSON. Take any SWF $f : D \rightarrow W(A)$ that is IND and NI. As D is MALAWSKI-ZHOU, f is PO or A-PO or NULL. If f is PO, then by D being ARROW, f is DICT. Now let f be A-PO. Consider the SWF $g : D \rightarrow W(A)$ that outputs the reverse of the output of f at every preference profile. By construction, g is PO and satisfies IND. As D is ARROW, g is DICT, implying that f is A-DICT, establishing that D is WILSON. Q.E.D.

REMARK: An interpretation of Proposition 2 is that under MALAWSKI-ZHOU, ARROW implies WILSON. This may be attributed to Malawski and Zhou (1994), though they do not give a formal proof nor a formal statement.¹

PROPOSITION 3: No Paretian complete domain is WILSON, MALAWSKI-ZHOU and not ARROW. On the other hand, there exist Cartesian domains that are WILSON, MALAWSKI-ZHOU and not ARROW.

PROOF: The set of Paretian complete domains is a subset of what Campbell and Kelly (2003) call “Type 1 domains”, for which WILSON implies ARROW, hence the negative result. To establish the positive result consider the domain E^N where E consists of all preferences where a_m is ranked last while the ranking of $\{a_1, \dots, a_{m-1}\}$ is single-peaked with respect to the order of natural numbers. Over E^N , the pairwise majority rule f^μ (defined in the proof of Proposition 1) satisfies IND and PO but fails DICT. Thus, E^N is not ARROW. Now we claim that over E^N , any SWF f that satisfies NI must always output indifference between all alternatives, which implies that E^N is WILSON and MALAWSKI-ZHOU. We complete the proof by showing our claim. Take any $a_i \in \{a_1, \dots, a_{m-1}\}$. By NI, there exists $R, R' \in W(A)^N$ with $a_i f(R) a_m$ and $a_m f(R') a_i$. Moreover, IND implies $a_i f(R') a_m$ and $a_m f(R) a_i$. Noting that this argument is valid for any $a_i \in \{a_1, \dots, a_{m-1}\}$ shows the claim. Q.E.D.

REMARK: The positive result is interesting because it shows that ARROW and WILSON are not equivalent under MALAWSKI-ZHOU, a point that is not discussed by Malawski and Zhou (1994).

PROPOSITION 4: Every domain that is WILSON is also MALAWSKI-ZHOU.

PROOF: Take any domain D that is domain is WILSON and any SWF $f : D \rightarrow W(A)$ that is IND and NI. By WILSON, f is DICT or A-DICT or NULL. If f is DICT then f is PO. If f is A-DICT then f is A-PO. Thus, D is MALAWSKI-ZHOU. Q.E.D.

REMARK: Proposition 4 establishes that there is no domain that is ARROW, WILSON and not MALAWSKI-ZHOU; as well as no domain that is not ARROW, WILSON and not MALAWSKI-ZHOU.

PROPOSITION 5: There are Cartesian domains that are MALAWSKI-ZHOU, not ARROW and not WILSON.

PROOF: Malawski and Zhou (1994, p.106) exemplify a domain that is MALAWSKI-ZHOU, not ARROW and not WILSON.

¹We thank Lin Zhou for an exchange we had on this matter.

REMARK: By Proposition 2, under MALAWSKI-ZHOU, ARROW implies WILSON. Thus, Proposition 5 can be read as MALAWSKI-ZHOU does not imply WILSON, i.e., the converse of Proposition 4 does not hold. Nevertheless, the domain in Malawski and Zhou (1994) is not Paretian complete. It remains an open question whether there are basic domains that are MALAWSKI-ZHOU, not ARROW and not WILSON.

PROPOSITION 6: There are basic domains that are ARROW, not WILSON and not MALAWSKI-ZHOU.

PROOF: For any $X \subseteq A$, let $L(X)$ be the set of linear orders on X . Fix some $a \in A$. For any $Q_i \in L(A \setminus \{a\})$, define $R_i(Q_i, a) \in W(A)$ for all $x, y \in A \setminus \{a\}$ as, (i) $x R_i(Q_i, a) y \iff x Q_i y$; (ii) $x R_i(Q_i, a) a$; (iii) $a R_i(Q_i, a) x \iff z Q_i x$ for all $z \in A \setminus \{a, x\}$. So the bottom indifference class of $R_i(Q_i, a)$ is a doubleton that consists of a and the (unique) last ranked alternative according to Q_i , while $R_i(Q_i, a)$ linearly ranks all alternatives in $A \setminus \{a, x\}$ as Q_i does. Let $D_a = L(A) \cup \{R_i(Q_i, a) : Q_i \in L(A \setminus \{a\})\}$. By construction D_a^N is basic. Using Theorem 1 of Kelly (1994), it can be seen to be ARROW as well. We now show that D_a^N is not MALAWSKI-ZHOU (thus also not WILSON by Proposition 4). Fix some $d \in N$ and define the SWF $f : D_a^N \rightarrow W(A)$ such that for all $R \in D_a^N$, for all $x, y \in A \setminus \{a\}$, (i) $x f(R) y \iff x R_d y$; (ii) $x f(R) a$; (iii) $a f(R) x \iff x R_d a$ and $a R_d x$. So f copies the preference of d when it comes from $\{R_i(Q_i, a) : Q_i \in L(A \setminus \{a\})\}$ and otherwise, copies it except by placing a at the bottom. Observe that f is NI and IND but not PO, A-PO or A-DICT, showing that D_a^N is not MALAWSKI-ZHOU.

REMARK: Campbell and Kelly (2003) give several examples of domains that are ARROW but not WILSON. By Proposition 2 we know that these are not MALAWSKI-ZHOU either. However, their examples are not Cartesian.

PROPOSITION 7: There are basic domains that are ARROW, WILSON and MALAWSKI-ZHOU.

PROOF: The full domain is well-known to satisfy all three conditions. Q.E.D.

4 Final remarks

Our analysis is a continuation of the work of Malawski and Zhou (1994) and Campbell and Kelly (2003).

From Malawski and Zhou (1994), we isolate MALAWSKI-ZHOU as a condition on domains in its own right, and consider its logical relationship with ARROW and WILSON. Part of our contribution is an explicit proof of the known result that on MALAWSKI-ZHOU domains ARROW implies WILSON.

We also show that, technically speaking, the reverse implication does not hold: on MALAWSKI-ZHOU domains, WILSON does not generally imply ARROW; although adding the further condition of Paretian completeness does get the equivalence of ARROW and WILSON.

The fact that on Paretian complete domains WILSON implies ARROW was observed by Campbell and Kelly (2003), who actually prove a stronger statement. Indeed, these authors provide a wealth of technical results, many of which overlap significantly with our contributions.

Throughout our work, we provide supplementary remarks giving attributions to previous versions—partial or complete—of our results. In fact, many of our existence results are stronger than previous versions as they illustrate domains that are Cartesian.

In brief, we combine our findings with those that already exist in the literature to present a perspective to the relationship between Arrow’s and Wilson’s theorems on restricted domains. Figure 1 summarizes our results.

Figure 1: Summary of the findings

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