On a labeled vehicle routing problem

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Introduction

Typical combinatorial optimization problems

Given a set of **weighted** facilities (e.g. edges or vertices of a graph), find a *feasible* subset of facilities with optimum (i.e. minimum or maximum) total weight

e.g. minimum weight spanning tree, TSP, etc

Labeled optimization problems

Given a set of **labeled** facilities, find a *feasible* subset of facilities with optimum **number** of labels

assumption: either **all** facilities having the same label are purchased or **none** of them

Weights often represent a cost/distance/duration while a label (or color) can represent

- a mode of transport (plane, bus, metro, etc)
- a technology (optical fiber, encryption)
- name of the facility's owner (a company)

Sometimes the *quality* of a solution **is not** dominated by its total distance

- number of changings in the metro
- direct flights are sometimes more expensive than trips with one stop in the opposite direction

The vehicle routing problem (VRP)

Classical VRP

A complete graph with

- a depot denoted by 0
- n customers denoted by $1, \dots, n$
- a capacity k
- a distance on the edges

Find a collection of cycles such that

- every vertex/customer is visited exactly once
- every cycle contains the depot plus at most k customers
- the total distance is minimized

The LABELED vehicle routing problem (LVRP)

VRP with labels

A complete graph with

- a depot denoted by 0
- n customers denoted by $1, \dots, n$
- a capacity k
- a **color** on each edge

color=label

Find a collection of cycles such that

- every vertex/customer is visited exactly once
- every cycle contains the depot plus at most k customers
- **the number of colors** used is minimized

- identify polynomial/NP-hard cases of the problem
- poly-time approximation algorithms for NP-hard cases
 - approximation ratio: $\frac{SOL}{OPT}$ in the worst case

bad news

Unless $P = NP$, and taking $\epsilon \in (0, 1)$, there is no $n^{1-\epsilon}$ -approximation algorithm for $LVRP(k)$, where $k \geq n^\epsilon$

A trivial algorithm gives an $(n + 1)$ -approximation

frequencies

The frequency of a color is the number of times it appears in the graph

f denotes the maximum frequency in the graph

For a set of edges E' , $f(E')$ is the maximum frequency for the colors appearing in E'

Simple observations

observation 1

one can visit a customer using a color and return back to the depot at “no cost”

observation 2

every cycle of length 3 can be replaced by two cycles of length one (the number of colors cannot increase)

assumption: no cycle of length 3

Simple observations

observation 3

The blue color must appear twice. Otherwise, replace by:

LVRP is polynomial if $k = 1, 2$ or $f = 1$. What about the case $k = 3$ and $f = 2$?

k -partition problem

decide whether a simple graph on kq vertices can be partitioned into q paths of length $k - 1$

NP-complete problem when $k \geq 3$

NP-hardness of LVRP when $k = 3$ and $f = 2$

reduction of the 3-partition problem

- keep all vertices of G , add a depot 0 and add one “edge-vertex” v_e per edge e of G
- one color per edge e in G and edge $(0, v_e)$
- any other edge receives a **unique** color

NP-hardness of LVRP when $k = 3$ and $f = 2$

G admits a 3-partition iff there is a solution using less than $m + 2q$ colors

- every “edge-vertex” is covered by a cycle of length 2
- the other vertices are in cycles of length $k + 1$

Approximation

We suppose that we cover every customer with a cycle of length 2, i.e. the approximate solution is a star with the depot at the center

Notations

$E_0 = \{[0, i] : 1 \leq i \leq n\}$ is the star (approximate solution)

E^* is the optimal solution

s_i is the number of cycles of length i in the star

s_i^* is the number of cycles of length i in the optimal solution

Two cases

- $f \leq \frac{k+1}{2}$ approx. ratio $\frac{f(k-2)+1}{k-1}$ *presented*
- $f \geq \frac{k+1}{2}$ approx. ratio $\frac{f(k-2)+k+1}{k+1}$ *skipped*

Case $f \leq (k+1)/2$

For a set of edges E' , $R(E')$ is defined as $|E' \cap E_0|$ minus the number of colors in $E' \cap E_0$ (colors saved for the edges adjacent to the depot)

$R(E') = 4 - 3 = 1$ in the example

$$OPT \geq s_2^* + 2 \sum_{i=4}^{k+1} s_i^* - R(E^*)$$

$$OPT \geq s_2^* + 2 \sum_{i=4}^{k+1} s_i^* - R(E^*) \quad \times \beta$$

$$OPT \geq (s_2^* + \sum_{i=4}^{k+1} i s_i^*)/f \quad \times \alpha$$

$$OPT \geq s_2^* + 2 \sum_{i=4}^{k+1} s_i^* - R(E^*) \quad \times \beta$$

$$OPT \geq (s_2^* + \sum_{i=4}^{k+1} i s_i^*) / f \quad \times \alpha$$

$$(\alpha + \beta)OPT \geq (\frac{\alpha}{f} + \beta)s_2^* + \sum_{i=4}^{k+1} (\frac{\alpha}{f} + 2\beta)s_i^* - \beta R(E^*)$$

Take $\alpha = f \frac{k-2}{k-1}$ and $\beta = \frac{1}{k-1}$

We have $\frac{\alpha}{f} + \beta \geq 1$, $\frac{\alpha}{f} + 2\beta \geq i - 1$ and $\beta \leq 1$

$$\begin{aligned} \frac{f(k-2) + 1}{k-1} OPT &\geq s_2^* + \sum_{i=4}^{k+1} (i-1)s_i^* - R(E^*) \\ &\geq n - R(E^*) \geq SOL \end{aligned}$$

Tightness

One can show examples where the star is $\frac{f(k-2)+1}{k-1}$ -approximate for every k and f satisfying $f \leq \frac{k+1}{2}$

$k = 4$ and $f = 2$

instance

optimum = 3

sol = 5

LVRP when $k = 3, 4$, $f = 2$ is APX-hard

LVRP when $k = 3$, $f = 2$ is polynomial if every color of the star has frequency 1

LVRP when $k = 3$, $f = 3$ is NP-hard even if every color of the star has frequency 1

With local improvements performed on the star, one can improve the previous approximation ratio if $R(E_0) = 0$

Thank You