Computing the mean and the variance of the cedent's share for largest claims reinsurance covers

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Abstract: We present mathematical results allowing one to evaluate the moments of order 1 and 2 of the cedent's share in the framework of reinsurance treaties based on ordered claim sizes. These results consist of closed analytical formulas that do not involve any approximation procedure. This is illustrated by numerical examples when the claim number has the Poisson or the negative binomial distribution, and the claim cost has the exponential or the Pareto distribution.

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1 Introduction

A lot of papers have been devoted to the pricing of reinsurance treaties based on the largest claims during a given time period. Especially, since the early eighties E. Kremer has dealt with this problem in a series of papers (see the reference list). Kremer has considered various situations using several approaches. More recently, Berglund [4], Walhin [20], Bathossi [1], and Ladoucette and Teugels [15] have also contributed to this subject. Basically, the goal of these authors was the pricing of Largest Claims Reinsurance (*LCR*) and/or *ECOMOR* treaties (see Section 2 for the precise definitions). In particular, calculations or estimations of the first two moments of the reinsurer's claim expenses have been developed. One of the most precise results along this line is certainly that of Berglund [4], who has established closed and tractable formulas.

In the present paper, we consider once more these sort of treaties, but from the cedent's point of view, whose concern is the assessment of the efficiency of a reinsurance cover in order to improve its solvency level at the least or the more reasonable cost. We state and prove closed formulas allowing one to calculate the retained pure premium and, more importantly, the standard deviation of the reinsured's share in the *LCR* and *ECOMOR* treaties. This solves an open problem. Indeed, as pointed out by Silvestrov *et al* [18] in a recent paper devoted to reinsurance, no such formulas existed so far.

Clearly, the problem that we address belongs to the domain of Extreme Value Theory (EVT). This theory has been studied extensively for many years and has given rise to a huge literature. Let us mention for example the book by Embrechts *et al* [8] that contains a lot of results in this area, as well as numerous applications to Insurance and Finance. A quick introduction to Extreme Value Theory, as well as related probabilistic and statistical results can also be found in the book by Beirlant *et al* [3]. The Encyclopedia of Actuarial Science also contains results on EVT in connection with Insurance or Reinsurance Mathematics (see e.g. [2]).

The paper is organized as follows: In Section 2, we present the needed definitions and preliminaries. Section 3 contains our main results, namely general analytical formulas for the first two moments of the cedent's share in *LCR* and *ECOMOR* treaties. In Section 4 several numerical examples are given. Our general conclusions are presented in Section 5.

2 Notations, Definitions, and Preliminaries

In this section, we set our notations, and we recall some needed definitions and known facts. For basic facts on Insurance Mathematics and Extreme Value Theory, we refer the reader to [7], [9], [8] and [6]. All the random variables are assumed to be defined on an underlying probability space $(\Omega, \mathcal{A}, P)$. For a given class of risks and a given period of time, N denotes the number of claims and $C_1, C_2, \dots, C_n, \dots$ denote the claim sizes. The random variables C_n are assumed to be positive, independent and identically distributed. The number of claims and the sequence of claim sizes are assumed to be independent. For each integer $n \geq 0$, we set

$$p_n = P(N = n)$$

and we denote by ψ the *probability generating function* of N , defined by

$$\psi(t) = E(t^N) = \sum_{n \geq 0} p_n t^n \quad t \in [0, 1].$$

The k -th derivative of ψ will be denoted by $\psi^{(k)}$. Further, we denote by F the common distribution function of the claim sizes and we assume the existence of a density f (with respect to the Lebesgue measure). For every integer $n \geq 1$, the sequence

$$C_{1:n} \leq C_{2:n} \leq \dots \leq C_{n:n}$$

stands for the sequence of the first n claims sizes in the increasing order. In particular

$$C_{1:n} = \min(C_1, C_2, \dots, C_n) \quad \text{and} \quad C_{n:n} = \max(C_1, C_2, \dots, C_n)$$

are respectively the smallest and the largest claim size among $C_1, C_2, \dots, C_n$. We also set $C_{i:n} = 0$ if $i < 0$ or $i > n$. For each integer k , $F_{k:n}$ denotes the cumulative distribution function of $C_{k:n}$.

In the framework of the collective risk model, the aggregate claim amount, denoted by X , has the following expression

$$X = \sum_{i=1}^N C_i \quad \text{if } N > 0, \quad 0 \quad \text{if } N = 0.$$

As is well known, provided they exist, the first two moments of X are given by

$$(2.1) \quad E(X) = E(N) E(C)$$

and

$$(2.2) \quad V(X) = E(N) V(C) + V(N) E(C)^2.$$

In particular, when the distribution of N is Poisson with parameter λ , the above formulas reduce to

$$(2.3) \quad E(X) = \lambda E(C) \quad \text{and} \quad V(X) = \lambda E(C^2).$$

Assume that a reinsurance treaty has been concluded between a reinsurer and a reinsured (or cedent). We denote by X' (resp. X'') the total aggregate claim amount paid by the reinsured (resp. the reinsurer). These random variables obviously satisfy $X = X' + X''$. Basically, a reinsurance treaty based on ordered claim sizes can be given by a sequence of functions R_n of the following type

$$(2.4) \quad R_n(c_1, \dots, c_n) = \sum_{i=1}^n h_i(c_{i:n}) \quad n \geq 1$$

where $c_{i:n}$ denotes a possible value of the random variable $C_{i:n}$ ($1 \leq i \leq n$) and $(h_i)_{i \geq 1}$ denotes a given sequence of measurable functions

$$h_i : [0, +\infty) \rightarrow \mathbf{R}$$

such that

$$0 \leq \sum_{i=1}^n h_i(c_{i:n}) \leq \sum_{i=1}^n c_i$$

for all $n \geq 1$ and all $(c_1, \dots, c_n) \in \mathbf{R}^n$. In such a case, X' and X'' read as follows

$$(2.5) \quad X'' = R_N(C_1, \dots, C_N) = \sum_{i=1}^N h_i(c_{i:N}) \quad \text{and} \quad X' = X - X''.$$

More precisely, if $N = n$ and $C_i = c_i$ for $i = 1, \dots, n$, then X'' takes on the value $R_n(c_1, \dots, c_n)$. We also set $R_0 = 0$. Let us mention the following three examples, where p denotes a positive integer.

(i) The *largest claims reinsurance treaty of order p*, denoted by $LCR(p)$, is defined by

$$(2.6) \quad X'' = \sum_{j=1}^p C_{N-j+1:N}$$

where, for each $j \geq 1$, the random variable $C_{N-j+1:N}$ is given by

$$C_{N-j+1:N} = \begin{cases} \text{the } j^{\text{th}} \text{ largest claim amount} & \text{if } N \geq j \\ 0 & \text{otherwise.} \end{cases}$$

Here, formula (2.4) holds with

$$h_i(x) = x \quad \text{if } i = N - p + 1, \dots, N \quad \text{and} \quad h_i(x) = 0 \quad \text{if } i \leq N - p$$

so that the reinsurer will pay the p largest claims that have occurred during the reference time period. In particular, when $N < p$ one has $X'' = X$ and $X' = 0$.

(ii) The *ECOMOR*(p) treaty is defined by

$$(2.7) \quad X'' = \sum_{j=1}^{p-1} (C_{N-j+1:N} - C_{N-p+1:N}) = \sum_{j=1}^{p-1} C_{N-j+1:N} - (p-1)C_{N-p+1:N}$$

for $p \geq 2$ (for $p = 1$ or $N \leq p$, one has $X'' = 0$ and $X' = X$). In this case, formula (2.5) is valid with

$$h_i(x) = \begin{cases} x & \text{if } i = N - p + 2, \dots, N \\ (1-p)x & \text{if } i = N - p + 1 \\ 0 & \text{if } i \geq N - p \end{cases}$$

Clearly, the p -th largest claim size plays the role of a random deductible (or priority). This treaty was introduced by the French actuary Thépaut [19]. The term *ECOMOR* is an abbreviation for the French “Excédent de coût moyen relatif”.

3 The first two moments of the reinsured share in the *LCR* and *ECOMOR* treaties

Since the aggregate claim amount X satisfies $X = X' + X''$ where X' (resp. X'') is the cedent's (resp. reinsurer's) share, we immediately get

$$(3.1) \quad E(X) = E(X') + E(X'')$$

This formula allows us to deduce the expectation of the reinsured's share once the expectation of the reinsurer's share is known. Formulas (2.6) and (2.7) show that in the $LCR(p)$ treaty one has

$$(3.2) \quad E(X'') = \sum_{j=1}^p E(C_{N-j+1:N})$$

and in the $ECOMOR(p)$ treaty

$$(3.3) \quad E(X'') = \sum_{j=1}^{p-1} E(C_{N-j+1:N}) - (p-1)E(C_{N-p+1:N}).$$

In order to simplify the presentation, we first consider the case where $N \geq p$ for the LCR treaty and $N > p$ for the $ECOMOR$ treaty. In practice, the probability of these events are often very close to 1. An indication on how to deal with the general case is given in Remark 3.1 (ii).

The variance of the reinsurer's share in the $LCR(p)$ treaty reads as follows

$$(3.4) \quad V(X'') = \sum_{j=1}^p V(C_{N-j+1:N}) + 2 \sum_{j=2}^p \sum_{i=1}^{j-1} Cov(C_{N-i+1:N}, C_{N-j+1:N}).$$

In the $ECOMOR(p)$ treaty, the following formula holds

$$(3.5) \quad \begin{aligned} V(X'') &= \sum_{j=1}^{p-1} V(C_{N-j+1:N}) + (p-1)^2 V(C_{N-p+1:N}) \\ &+ 2 \sum_{j=2}^{p-1} \sum_{i=1}^{j-1} Cov(C_{N-i+1:N}, C_{N-j+1:N}) - 2(p-1) \sum_{i=1}^{p-1} Cov(C_{N-i+1:N}, C_{N-p+1:N}). \end{aligned}$$

As to the variance of X' , the cedent's share, it can be derived from the following relationship

$$(3.6) \quad V(X') = V(X - X'') = V(X) + V(X'') - 2Cov(X, X'')$$

where

$$(3.7) \quad Cov(X, X'') = E(XX'') - E(X)E(X'').$$

Formulas for $E((C_{N-i+1:N})^k)$ and $E(C_{N-i+1:N} C_{N-j+1:N})$ have been proposed in several works, especially in that of Berglund [4]. These formulas are recalled hereafter, because they are used in Section 4 to work out our numerical applications and because our own formulas for the cedent's share, although more complex, have similar features. The k -th moment of $C_{i:N}$ is given by

$$(3.8) \quad E((C_{N-i+1:N})^k) = \frac{1}{(i-1)!} \int_0^1 F^{-1}(u)^k (1-u)^{i-1} \psi^{(i)}(u) du$$

where $1 \leq i \leq n$ and $k \geq 1$. As already mentioned, ψ stands for the probability generating function of N . As to the expectation of the cross product of the i -th and the j -th ordered claim sizes, where $1 \leq i < j \leq n$, we have

$$(3.9) \quad \begin{aligned} E(C_{N-i+1:N} C_{N-j+1:N}) &= \\ &\frac{1}{(i-1)!(j-i-1)!} \int_0^1 F^{-1}(v)(1-v)^{j-1} \psi^{(j)}(v) \left(\int_0^1 F^{-1}(1-u(1-v)) u^{i-1} (1-u)^{j-i-1} du \right) dv. \end{aligned}$$

Clearly, the above two equalities and formulas (3.2) to (3.5) allow us to calculate $E(X'')$ and $V(X'')$ for the LCR and $ECOMOR$ treaties. Then, $E(X')$ immediately follows from (3.1). As to $V(X')$, in view of (3.6) and (3.7), it only remains to calculate $E(XX'')$. The following result provides formulas that allow for calculating this expectation in the $LCR(p)$ and the $ECOMOR(p)$ treaties.

Proposition 3.1 *If $E(N)$ and $E(C^2)$ are finite, the following two formulas hold.*

(a) *In the $LCR(p)$ treaty the expectation of XX'' is given by*

$$(3.10) \quad E(XX'') = \sum_{j=1}^p E(NE(C_1 C_{N-j+1:N}/N)).$$

(b) *In the $ECOMOR(p)$ treaty, we have*

$$(3.11) \quad E(XX'') = \sum_{j=1}^{p-1} E(NE(C_1 C_{N-j+1:N}/N)) - (p-1)E(NE(C_1 C_{N-p+1:N}/N)).$$

Proof. Let us recall first that, for each $j = 1, \dots, p$, the conditional expectation $E(C_1 C_{N-j+1}/N)$ is the random variable taking on the value

$$E(C_1 C_{N-j+1} / N = n) = E(C_1 C_{n-j+1})$$

with probability $p_n = P(N = n)$. In the $LCR(p)$ treaty we have

$$(3.12) \quad E(XX'') = E\left\{ \left(\sum_{i=1}^N C_i \right) \left(\sum_{j=1}^p C_{N-j+1:N} \right) \right\} = \sum_{j=1}^p E(C_{N-j+1:N} \sum_{i=1}^N C_i).$$

For each integer $j = 1, \dots, p$ and each $n \geq 1$, the hypotheses of the collective risk model allow us to write

$$E(C_{N-j+1:N} \sum_{i=1}^N C_i / N = n) = E(C_{n-j+1:n} \sum_{i=1}^n C_i) = \sum_{i=1}^n E(C_{n-j+1:n} C_i).$$

The important point is that, due to the exchangeability of the sequence $(C_i)_{i \geq 1}$ we have

$$E(C_{N-j+1:N} \sum_{i=1}^N C_i / N = n) = n E(C_{n-j+1:n} C_1)$$

which entails almost surely

$$E(C_{N-j+1:N} \sum_{i=1}^N C_i / N) = N E(C_{N-j+1:N} C_1 / N)$$

where, for each $j = 1, \dots, p$, $E(C_1 C_{N-j+1}/N)$ is defined as above. It follows

$$E(C_{N-j+1:N} \sum_{i=1}^N C_i) = E\left(E(C_{N-j+1:N} \sum_{i=1}^N C_i / N)\right) = E\left(N E(C_{N-j+1:N} C_1 / N)\right)$$

which in view of (3.12) yields the desired result. The formula for the $ECOMOR(p)$ treaty is derived similarly. Q.E.D.

Our task is now to calculate $E\left(N E(C_{N-j+1:N} C_1 / N)\right)$, which is involved in formulas (3.10) and (3.11). Observe first that for each $j = 1, \dots, p$

$$(3.13) \quad E\left(N E(C_{N-j+1:N} C_1 / N)\right) = \sum_{n \geq j} n p_n E(C_1 C_{n-j+1:n}).$$

In order to calculate

$$I_{k:n} := E(C_1 C_{k:n}),$$

we need to know the probability measure, say $\mu_{k:n}$, that defines the distribution of the pair $(C_1, C_{k:n})$, where $k = n - j + 1$ ranges from 1 to n . At this step, it is important to observe that $\mu_{k:n}$ is not absolutely continuous with respect to the two-fold Lebesgue measure λ_2 on the measurable space $(\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2))$, because the event $\{C_1 = C_{k:n}\}$ has a non zero probability (namely $1/n$). More precisely, the measure $\mu_{k:n}$ admits the following decomposition

$$(3.14) \quad \mu_{k:n} = \mathbf{1}_{\Delta}(x, y) a(x, y) \lambda_2 + \mathbf{1}_{\Delta'}(x, y) b(x, y) \lambda_2 + \mathbf{1}_D(x) c(x) \lambda_D$$

where λ_2 denotes the Lebesgue measure on $(\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2))$ and λ_D the Lebesgue measure on the line

$$D = \{(x, y) \in \mathbf{R}^2 : x = y\}.$$

The subsets Δ and Δ' are defined by

$$\Delta = \{(x, y) \in \mathbf{R}^2 : x < y\} \quad \text{and} \quad \Delta' = \{(x, y) \in \mathbf{R}^2 : x > y\}.$$

Formula (3.14) provides us with three pieces of information the distribution of $(C_1, C_{k:n})$, namely:

(i) the restriction of $\mu_{k:n}$ to Δ admits a density $a(x, y)$ with respect to the two-dimensional Lebesgue measure λ_2 ,

(ii) the restriction of $\mu_{k:n}$ to Δ' admits a density $b(x, y)$ with respect to λ_2

(iii) the restriction of $\mu_{k:n}$ to D , which is a one dimensional subset of $\mathbf{R}^2$, admits a density c with respect to the one-dimensional Lebesgue measure λ_D .

On Δ , namely when $x < y$, standard differential and combinatorial arguments (see e.g. [DN] p. 11) allow us to derive the formulas

$$a(x, y) = \frac{(n-1)!}{(k-2)!(n-k)!} f(x)f(y)F(y)^{k-2}(1-F(y))^{n-k} \quad (x, y) \in \Delta$$

valid for $k = 2, 3, \dots, n$. On Δ' , namely when $x > y$, we get similarly

$$b(x, y) = \frac{(n-1)!}{(k-1)!(n-k-1)!} f(x)f(y)F(y)^{k-1}(1-F(y))^{n-k-1} \quad (x, y) \in \Delta'$$

valid for $k = 1, 2, \dots, n-1$. On the line D , the density function $c(\cdot)$ is given by

$$c(x) = \frac{(n-1)!}{(k-1)!(n-k)!} f(x)F(x)^{k-1}(1-F(x))^{n-k} \quad x \in D.$$

When $k = 1$, one has $a(x, y) = 0$ because the event $\{C_1 < C_{1:n}\}$ is impossible. Similarly, when $k = n$ one has $b(x, y) = 0$, because the event $\{C_1 > C_{n:n}\}$ is impossible. Using the above formulas, we get

$$I_{k:n} = E(C_1 C_{k:n}) = \int \int_{\Delta} xy a(x, y) dx dy + \int \int_{\Delta'} xy b(x, y) dx dy + \int_D x^2 c(x) dx$$

It is also convenient to introduce the function H defined by

$$(3.15) \quad H(z) = \int_0^z t f(t) dt \quad z \geq 0$$

which satisfies $H(+\infty) = E(C) = m$, where m is assumed to be finite. From the above formulas, we infer

$$\begin{aligned} I_{k:n} &= \frac{(n-1)!}{(k-2)!(n-k)!} \int_0^{+\infty} y f(y) F(y)^{k-2} (1-F(y))^{n-k} H(y) dy \\ &+ \frac{(n-1)!}{(k-1)!(n-k-1)!} \int_0^{+\infty} y f(y) F(y)^{k-1} (1-F(y))^{n-k-1} (m - H(y)) dy \\ &+ \frac{(n-1)!}{(k-1)!(n-k)!} \int_0^{+\infty} x^2 f(x) F(x)^{k-1} (1-F(x))^{n-k} dy. \end{aligned}$$

After substituting $v = F(y)$ in the first two integrals and $u = F(x)$ in the third one, we get

$$I_{k:n} = \frac{(n-1)!}{(k-2)!(n-k)!} \int_0^1 F^{-1}(v) v^{k-2} (1-v)^{n-k} H(F^{-1}(v)) dv$$

$$\begin{aligned}
& + \frac{(n-1)!}{(k-1)!(n-k-1)!} \int_0^1 F^{-1}(v) v^{k-1} (1-v)^{n-k-1} (m - H(F^{-1}(v))) dv \\
& + \frac{(n-1)!}{(k-1)!(n-k)!} \int_0^1 F^{-1}(u)^2 u^{k-1} (1-u)^{n-k} du.
\end{aligned}$$

Equivalently, since $k = n - j + 1$ we have

$$I_{n-j+1:n} = T_1(n, j) + T_2(n, j) + T_3(n, j)$$

where

$$\begin{aligned}
T_1(n, j) &= \frac{(n-1)!}{(n-j-1)!(j-1)!} \int_0^1 F^{-1}(v) v^{n-j-1} (1-v)^{j-1} H(F^{-1}(v)) dv \\
T_2(n, j) &= \frac{(n-1)!}{(n-j)!(j-2)!} \int_0^1 F^{-1}(v) v^{n-j} (1-v)^{j-2} (m - H(F^{-1}(v))) dv \\
T_3(n, j) &= \frac{(n-1)!}{(n-j)!(j-1)!} \int_0^1 F^{-1}(u)^2 u^{n-j} (1-u)^{j-1} du.
\end{aligned}$$

According to a previous observation, the first term is absent if $j = n$ and the second term is absent if $j = 1$. More precisely, we have $T_1(n, n) = 0$ and $T_2(n, 1) = 0$ for all $n \geq 1$. Now, we return to (3.13) and to the computation of

$$(3.16) \quad E(N E(C_1 C_{N-j+1:N}/N)) = \sum_{n \geq j+1} n p_n T_1(n, j) + \sum_{n \geq j} n p_n T_2(n, j) + \sum_{n \geq j} n p_n T_3(n, j).$$

As to the first summation in (3.16), we get for all $j \geq 1$

$$\begin{aligned}
\sum_{n \geq j+1} n p_n T_1(n, j) &= \frac{1}{(j-1)!} \int_0^1 F^{-1}(v) (1-v)^{j-1} H(F^{-1}(v)) \left(\sum_{n \geq j+1} \frac{n!}{(n-j-1)!} v^{n-j-1} \right) dv \\
&= \frac{1}{(j-1)!} \int_0^1 F^{-1}(v) (1-v)^{j-1} H(F^{-1}(v)) \psi^{(j+1)}(v) dv
\end{aligned}$$

where $\psi^{(j+1)}$ denotes the $(j+1)$ -th derivative of ψ , the probability generating function of N . As to the second summation in (3.16), we get similarly for all $j \geq 2$

$$\sum_{n \geq j} n p_n T_2(n, j) = \frac{1}{(j-2)!} \int_0^1 F^{-1}(v) (1-v)^{j-2} (m - H(F^{-1}(v))) \psi^{(j)}(v) dv,$$

whereas for $j = 1$, we have

$$\sum_{n \geq 1} n p_n T_2(n, 1) = 0.$$

As to the third summation in (3.16), it is readily checked that

$$\sum_{n \geq j} n p_n T_3(n, j) = \frac{1}{(j-1)!} \int_0^1 F^{-1}(u)^2 (1-u)^{j-1} \psi^{(j)}(u) du.$$

Consequently, we have proven the following result.

Proposition 3.2 *Assume that $E(N)$ and $E(C^2)$ are finite. For each $j = 2, \dots, p$ one has*

$$\begin{aligned}
E(N E(C_1 C_{N-j+1:N}/N)) &= \frac{1}{(j-1)!} \int_0^1 F^{-1}(v) (1-v)^{j-1} H(F^{-1}(v)) \psi^{(j+1)}(v) dv \\
&+ \frac{1}{(j-2)!} \int_0^1 F^{-1}(v) (1-v)^{j-2} (m - H(F^{-1}(v))) \psi^{(j)}(v) dv + \frac{1}{(j-1)!} \int_0^1 F^{-1}(u)^2 (1-u)^{j-1} \psi^{(j)}(u) du
\end{aligned}$$

For $j = 1$ the above formula become

$$E(N E(C_1 C_{N:N}/N)) = \int_0^1 F^{-1}(v) H(F^{-1}(v)) \psi^{(2)}(v) dv + \int_0^1 F^{-1}(u)^2 \psi^{(1)}(u) du$$

As already mentioned, ψ denotes the probability generating function of N and $\psi^{(j)}$ its j^{th} derivative.

Using Proposition 3.2 together with formulas (3.10) and (3.11) we can derive formulas for $E(XX'')$ in the *LCR* and in the *ECOMOR* treaties. Since $V(X)$ and $V(X'')$ are already known (see formulas (2.2) and (3.4)), it suffices to return to (3.6), which allows for deriving formulas for $V(X')$ in both treaties. More precisely, we proceed recursively starting from $p = 1$ in the *LCR* treaty and from $p = 2$ in the *ECOMOR* treaty. First, we calculate $E(X'')$ and $V(X'')$ using Berglund's formulas, namely equalities (3.8) and (3.9). Then, we determine $E(X')$ and $V(X')$ with the help of the following two equalities

$$E(X') = E(X) - E(X'')$$

and

$$V(X') = V(X) + V(X'') - 2 \left(E(XX'') - E(X)E(X'') \right)$$

where $E(XX'')$ is given by (3.10) or (3.11). These formulas are used in Section 4 for performing numerical calculations.

Remark 3.1 (i) *As we can see, the formulas that appear in Proposition 3.2 involve the inverse of the distribution function F of the claim cost, the probability generating function ψ (and its derivatives) of the claim number and the function H defined by*

$$H(x) = \int_0^x t f(t) dt \quad x > 0$$

where f stands for the density function of the claim cost distribution. Apart from function H , the same elements were already present in Berglund's formulas.

(ii) *As already mentioned, the formulas for the moments of order 1 and 2 of the cedent's share and the reinsurer's share in the LCR treaty are only valid if the event $A = \{N \geq p\}$ is realized. However, general formulas can be derived easily in a standard way. For example, for the cedent's share we would have*

$$E(X') = P(A) E(X'/A) + P(A^c) E(X'/A^c)$$

and

$$V(X') = P(A) V(X'/A) + P(A^c) V(X'/A^c) + 2P(A)P(A^c)(E(X'/A) - E(X'/A^c))^2.$$

In the numerical examples that are presented in Section 4 the values of $P(A^c) = P(N < p)$ ($p = 1, \dots, 10$) are very small, so that the corresponding terms can be neglected.

(iii) *Formulas for the distribution of C_{N-i+1} and the pair (C_{N-i+1}, C_{N-j+1}) ($1 \leq i < j \leq N$) had been already presented by Ciminelli [5] in the middle seventies and later by Kremer [12]. These formulas were used by Berglund [4] to determine the moments of order 1 and 2 of the above random variables (and the moments of the cross products).*

4 Numerical Examples

Numerical examples are given below. They provide an illustration of the formulas of the previous section for the cedent's share. We also present results concerning the reinsurer's share for sake of completeness. We consider the case where the number of claims has either the Poisson distribution or the negative binomial distribution. We recall that N has the Poisson distribution of parameter λ ($\lambda > 0$) if the probabilities are given by

$$P(N = n) = e^{-\lambda} \frac{\lambda^n}{n!} \quad n \in \mathbf{N}.$$

The expectation, the variance and the probability generating function are given by

$$E(N) = V(N) = \lambda \quad \text{and} \quad \psi(t) = \exp(\lambda(t - 1)) \quad 0 \leq t \leq 1.$$

The negative binomial distribution with parameters r and p ($r > 0$ and $0 < p < 1$) is defined by

$$P(N = 0) = p^r \quad \text{and} \quad P(N = n) = \frac{r(r+1)\dots(r+n-1)}{n!} p^r (1-p)^n \quad n \geq 1.$$

As to the expectation, the variance and the probability generating function, one has

$$E(N) = \frac{r(1-p)}{p} \quad V(N) = \frac{r(1-p)}{p^2} \quad \text{and} \quad \psi(t) = \left(\frac{p}{1-(1-p)t} \right)^r \quad 0 \leq t \leq 1.$$

The individual claim cost is assumed to have either the exponential distribution or the Pareto distribution. These claim cost distributions are strongly different in that the exponential distribution has a light tail, whereas the Pareto has a heavy tail. If C is exponentially distributed, its cumulative distribution function F is given by

$$F(x) = 1 - \exp(-\alpha x) \quad \text{if } x \geq 0, \quad 0 \quad \text{if } x < 0.$$

where α is a positive parameter. The expectation and the variance are given by

$$E(C) = 1/\alpha \quad \text{and} \quad V(C) = 1/\alpha^2.$$

The following equality provides the expression of the inverse of F .

$$F^{-1}(u) = -\frac{1}{\alpha} \ln(1-u) \quad 0 \leq u < 1.$$

As to the (shifted) Pareto distribution, the cumulative distribution function is given by

$$F(x) = 1 - \left(\frac{b}{x+b} \right)^\beta \quad \text{if } x \geq 0, \quad 0 \quad \text{otherwise,}$$

where the parameters satisfy $\beta > 0$ and $b > 0$. Here the expectation is given by $E(C) = b/(\beta - 1)$, provided $\beta > 1$. If $\beta > 2$, the variance is finite and given by

$$V(C) = \frac{b^2\beta}{(\beta-1)^2(\beta-2)}.$$

The inverse of F reads as follows

$$F^{-1}(u) = \frac{b}{(1-u)^{1/\beta}} - b \quad 0 \leq u < 1.$$

In the following examples, we have chosen $\lambda = 40$ for the Poisson distribution, and $r = 40$ and $p = 0.5$ for the negative binomial distribution. Consequently, the expectations of the claim number are both equal to 40. The standard deviations are given by $\sigma(N) = 6.32$ for the Poisson distribution and $\sigma(N) = 8.94$ for the negative binomial.

As for the claim cost, we have chosen $\alpha = 0.01$ for the exponential distribution, and $b = 150$ and $\beta = 2.5$ for the Pareto distribution. This leads to $E(C) = 100$ for both distributions. The standard deviations are given by $\sigma(C) = 100$ for the exponential distribution and $\sigma(C) = 223.61$ for the Pareto. As to the aggregate claim amount, we have $E(X) = 4000$ for all pairs of distributions. The standard deviations are also shown in the following table. The results immediately follow from equations (2.1) and (2.2).

TABLE 1. Expectation and standard deviation of the aggregate claim amount
in the absence or reinsurance

	$E(X)$	$\sigma(X)$
Poisson/exponential	4 000	894.43
Poisson/Pareto	4 000	1 549.19
NB/exponential	4 000	1 095.45
NB/Pareto	4 000	1 673.32

The following four tables show the expectations and the standard deviation of both the reinsurer's share and the cedent's share for $LCR(p)$ and $ECOMOR(p)$ treaties when p goes from 1 to 10. Following Remark 3.1 (ii), also note that for the Poisson distribution, one has $P(N \leq 10) = 1,62 \times 10^{-8}$ and for the negative binomial distribution $P(N \leq 10) = 1,19 \times 10^{-5}$, which is very small. The reinsurer's share is denoted by X'_p and the cedent's share by X''_p . The results have been obtained by the formulas of the previous section.

TABLE 2. The reinsurer's share X'_p and the cedent's share X''_p for the $LCR(p)$ and $ECOMOR(p)$ treaties.
Claim number: Poisson distribution - Claim cost: Exponential distribution.

p	Reinsurer's share				Cedent's share			
	LCR Treaty		ECOMOR Treaty		LCR Treaty		ECOMOR Treaty	
	$E(X''_p)$	$\sigma(X''_p)$	$E(X''_p)$	$\sigma(X''_p)$	$E(X'_p)$	$\sigma(X'_p)$	$E(X'_p)$	$\sigma(X'_p)$
1	427	128	0	0	3573	843	4000	894
2	753	189	100	100	3247	803	3900	889
3	1030	236	200	141	2970	768	3800	883
4	1273	275	300	173	2727	735	3700	877
5	1491	309	400	200	2509	705	3600	872
6	1690	340	500	224	2310	676	3500	866
7	1871	368	600	245	2129	649	3400	860
8	2039	394	700	265	1961	622	3300	854
9	2193	419	800	283	1807	597	3200	849
10	2337	442	900	300	1663	572	3100	843

TABLE 3. The reinsurer's share X'_p and the cedent's share X''_p for the $LCR(p)$ and $ECOMOR(p)$ treaties.
Claim number: Poisson distribution - Claim cost: Pareto distribution.

p	Reinsurer's share				Cedent's share			
	LCR Treaty		ECOMOR Treaty		LCR Treaty		ECOMOR Treaty	
	$E(X''_p)$	$\sigma(X''_p)$	$E(X''_p)$	$\sigma(X''_p)$	$E(X'_p)$	$\sigma(X'_p)$	$E(X'_p)$	$\sigma(X'_p)$
1	827	1011	0	0	3173	951	4000	1549
2	1263	1116	391	949	2737	819	3609	1114
3	1582	1173	625	1041	2418	737	3375	1015
4	1838	1213	813	1093	2162	675	3187	954
5	2054	1244	975	1129	1946	625	3025	908
6	2241	1270	1122	1157	1759	582	2878	870
7	2405	1291	1256	1180	1595	544	2744	838
8	2551	1310	1382	1199	1449	510	2618	810
9	2682	1326	1500	1216	1318	480	2500	785
10	2801	1341	1613	1232	1199	452	2387	762

TABLE 4. The reinsurer's share X'_p and the cedent's share X''_p for the $LCR(p)$ and $ECOMOR(p)$ treaties.
Claim number: Negative binomial distribution - Claim cost: Exponential distribution.

p	Reinsurer's share				Cedent's share			
	LCR Treaty		ECOMOR Treaty		LCR Treaty		ECOMOR Treaty	
	$E(X''_p)$	$\sigma(X''_p)$	$E(X''_p)$	$\sigma(X''_p)$	$E(X'_p)$	$\sigma(X'_p)$	$E(X'_p)$	$\sigma(X'_p)$
1	425	129	0	0	3575	1045	4000	1095
2	751	192	100	100	3249	1003	3900	1091
3	1026	240	200	141	2974	966	3800	1086
4	1268	282	300	173	2732	930	3700	1082
5	1485	319	400	200	2515	897	3600	1077
6	1682	353	500	224	2318	865	3500	1072
7	1862	384	600	245	2138	834	3400	1068
8	2029	414	700	265	1971	804	3300	1063
9	2182	442	800	283	1818	774	3200	1058
10	2325	470	900	300	1675	745	3100	1054

TABLE 5. The reinsurer's share X'_p and the cedent's share X''_p for the $LCR(p)$ and $ECOMOR(p)$ treaties.
Claim number: Negative binomial distribution - Claim cost: Pareto distribution.

p	Reinsurer's share				Cedent's share			
	LCR Treaty		ECOMOR Treaty		LCR Treaty		ECOMOR Treaty	
	$E(X''_p)$	$\sigma(X''_p)$	$E(X''_p)$	$\sigma(X''_p)$	$E(X'_p)$	$\sigma(X'_p)$	$E(X'_p)$	$\sigma(X'_p)$
1	824	1011	0	0	3176	1110	4000	1673
2	1258	1119	390	949	2742	980	3610	1270
3	1576	1179	623	1041	2424	895	3377	1177
4	1831	1222	810	1093	2169	830	3190	1118
5	2046	1256	972	1129	1954	776	3028	1074
6	2231	1284	1118	1158	1769	729	2882	1037
7	2394	1308	1253	1181	1606	687	2747	1006
8	2540	1330	1378	1201	1460	649	2622	978
9	2670	1349	1496	1219	1330	614	2504	953
10	2788	1367	1608	1235	1212	582	2392	930

Remark 4.1 (i) In the above tables, it can be seen that the numerical values of the reinsurer's and the cedent's shares are more sensitive to the claim cost distribution rather than to the claim number distribution. The low sensitivity to the claim number distribution was also apparent in the paper by Berglund [4], which contains

numerical values of the reinsurer's share when the claim cost has the Pareto distribution. In other words, for a given claim cost distribution the numerical results do not vary significantly when the Poisson distribution is replaced with the negative binomial distribution with the same expectation. On the contrary, for the same claim number distribution the results for the exponential and the Pareto distribution strongly differ. This is particularly visible for the standard deviation of X'_p and X''_p .

(ii) In all tables from 2 to 5, it can be checked that the quantity $V(X) - V(X'_p) - V(X''_p) = 2 \text{cov}(X'_p, X''_p)$ is non negative, which shows the interest of splitting the risk. Especially, when $\text{cov}(X'_p, X''_p)$ is positive the reinsurance arrangement is globally more secure for both parties. This property is connected to comonotonicity.

(iii) Our numerical computations have been performed using the formulas presented in Section 3, with the Maple 9.5 software (this is not the latest version). We have implemented the code on a standard desktop computer. When the claim cost has the exponential distribution, it has not been necessary to introduce the specific formulas for F^{-1} , ψ and H in the expressions of $E(C_{i:N}^k)$, $E(C_{i:N} C_{j:N})$ and $E(NE(C_1 C_{N-j+1:N}/N))$. Instead, we only had to define these functions at the beginning of the Maple sheet and then use the general formulas. This has been made possible because this software includes a symbolic computation system. The Monte Carlo method has been also used for sake of comparison.

(iv) An interesting work making an extensive use of the Monte Carlo method has been published recently by Silvestrov and al. [18]. It allows for analyzing and comparing reinsurance contracts in a quite general setting, by means of a sophisticated and efficient software.

5 Conclusion

In this work, we have derived formulas allowing one to calculate the expectation and, more importantly, the variance of the cedent's share in the framework of the *LCR* or *ECOMOR* treaty. In our opinion, these results provide a better knowledge of the *LCR* and *ECOMOR* treaties. We also comment that, even if *LCR* or *ECOMOR* covers are not popular in the reinsurance world, the formulas of Section 3 provide a useful tool for assessing the impact of very large claims on the cedent's or the reinsurer's portfolio.

As already mentioned, we have mainly used a direct approach and have derived closed formulas. Our method does not involve any approximation or asymptotic device. Nevertheless, asymptotic methods are also interesting, especially to get approximations in distributions. This approach for the *LCR* and *ECOMOR* reinsurance was developed by several authors, in particular by Bathossi [1], by Beirlant [2], and by Ladoucette and Teugels [15]. Appealing to results such as the Fisher-Tippett Theorem, especially for heavy-tailed distribution, these authors managed to derive approximations and bounds for the distribution of the reinsurer's share in *LCR* and *ECOMOR* treaties. Other kinds of approximations or bounds were considered by E. Kremer (e.g. in [11] and [14]).

The results also raise interesting problems. For example, a natural question would be the classical one of optimality of a reinsurance treaty among a given class, from the cedent's or the reinsurer's point of view. In particular, it would be worthwhile to examine whether the *LCR* or the *ECOMOR* treaty has some sort of optimality property. Of course, the optimality of the risk transfer could be also considered in a cooperative game approach from both the cedent's and the reinsurer's point of view. Apart from optimality, the formulas would allow for comparing *LCR* or *ECOMOR* treaty with other reinsurance treaties, such as the quota-share or the excess-of-loss treaty. The author has already made numerical explorations along this line and would like to consider this subject more closely in a further paper. On the other hand, we have only considered moments of order 1 and 2. However, it is known that the actual aggregate claim amount distributions, as well as the claim size distribution are seldom symmetric (see e.g. [7], [16] or [17]). Thus, the introduction of moments of higher order, especially of order 3, could be pertinent. Of course, this would be feasible only for distributions whose tail is not too heavy. Otherwise, the moments of orders 2 or 3 would not exist and other risks measures should be considered. This would be also an interesting area to explore.

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