Modeling thin curved ferromagnetic films

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Abstract

The behavior of a thin film made of a ferromagnetic material in the absence of an external magnetic field, is described by an energy depending on the magnetization of the film verifying the saturation constraint. The free energy consists of an induced magnetostatic energy and an energy term with density including the exchange energy and the anisotropic energy. We study the behavior of this energy when the thickness of the curved film goes to zero. We show that the minimizers of the free energy converge to the minimizers of a local energy depending on a two-dimensional magnetization by Γ -convergence arguments.

1 Introduction

The theory of micromagnetism developed by L.D. Landau, E.M. Lifschitz [9] and W.F. Brown [3], describes the magnetic behavior of ferromagnetic bodies. According to this theory, the equilibrium state of a ferromagnetic body is described by its magnetization m , which is a vector field defined on $\mathbb{R}^3$, vanishing outside the body. The magnetization represents the volume density of macroscopic magnetic moment. This means that m induces a magnetic field in all the space. When a ferromagnetic body occupying a domain Ω is submitted to an external magnetic field H_e , the observed states are the minimizers of an energy E_e depending on the magnetization m :

$$E_e(m) = \int_{\Omega} \{ \alpha |\nabla m|^2 + Q \varphi(m) + H_e \cdot m \} + \int_{\mathbb{R}^3} \frac{1}{2} |H|^2.$$

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The first term in the free energy represents the exchange energy. It penalizes the spatial variations of the magnetization m in order to model the tendency of a specimen to exhibit regions of uniform magnetization (magnetic domains) separated by very thin transition layers (domain walls). The constant α is a non-negative constant. The second term is the anisotropic energy term. It models the existence of preferred directions of the magnetization (easy axes), along which ϕ , which is supposed to be a nonnegative, even function exhibiting cristallographic symmetry, vanishes. In general, ϕ is supposed to be a polynomial function. The coefficient Q controls the relative importance of the anisotropic energy compared to the magnetostatic energy. The magnetostatic energy, that is the fourth term of the free energy, is induced by the magnetization m , where $H = -\nabla u$ is the magnetic field induced by m in $\mathbb{R}^3$, and u is a scalar potential verifying the magnetostatic equation

$$\operatorname{div}(-\nabla u + m) = 0 \text{ in } \mathbb{R}^3. \quad (1.1)$$

The third term composing the free energy is the external energy or the Zeeman energy. It is due to the external magnetic field in which the ferromagnetic body is placed. More details can be found in A. De Simone [6]. At constant temperature the magnetization m verifies the saturation condition

$$|m| = M \text{ in } \Omega,$$

where M is a nonnegative constant that we suppose equal to 1.

The study of ferromagnetic thin films is of great interest, because this type of material is found in several fields in industry, such as audio and video tapes containing ferromagnetic ribbons used to provide high density in audio or video recording.

The first mathematical works on ferromagnetic films concerned wide films with constant thickness, see A. De Simone [6], B. Dacorogna and I. Fonseca [4]. Then, G. Gioia and R.D. James [8] studied the behavior of a thin ferromagnetic film, with no external magnetic field, when its thickness goes to zero. Gioia and James considered a thin film of thickness h occupying the domain Ω_h . The film has an energy per unit volume with no external energy term. They show that the magnetization minimizing the free energy of the thin film converges, when the thickness goes to zero, to a magnetization minimizing a limit energy. This limit energy is local, which means that there is no magnetostatic equation in the limit model. Gioia and James also show that the limit magnetization is independent of the direction normal to the film, which means that the limit model is two-dimensional. In [2], R. Alicandro and C. Leone extended the study of Gioia and James considering a more general density W , depending on the magnetization and its gradient, verifying certain growth hypotheses and including the exchange and anisotropic energies. Alicandro and Leone used the concept of tangential

quasiconvexity introduced by B. Dacorogna, I. Fonseca, J. Maly and K. Trivisa [5] in order to compute the relaxation of a class of energy functionals where the admissible fields are constrained to remain in a C^1 , q -dimensional manifold of $\mathbb{R}^d$.

In this paper, we follow [2] and we use Γ -convergence arguments to study the behavior of the energy of a curved ferromagnetic thin film and its minimizers when the thickness of the film goes to zero. After setting the problem and rescaling the energy, we study the magnetostatic energy term by examining the magnetostatic potential solution of the equation (1.1). We get the behavior of the magnetostatic energy setting a new minimization problem. Then, we recall the notion of tangential quasiconvexity before computing the Γ -limit of the sequence of energies that gives the behavior of almost minimizing sequences. Next, we rewrite the limit model on the curved surface following [11]. Finally, we apply our results to the Gioia and James model and to the model with external magnetic field. These results were announced in [14]

2 Notation and geometrical preliminaries

Throughout this article, we assume the summation convention unless otherwise specified. Greek indices take their values in the set $\{1, 2\}$ and Latin indices in the set $\{1, 2, 3\}$.

Let (e_1, e_2, e_3) be the canonical orthonormal basis of the Euclidean space $\mathbb{R}^3$. We denote by $|v|$ the norm of a vector v in $\mathbb{R}^3$, by $u \cdot v$ the scalar product of two vectors in $\mathbb{R}^3$, by $u \wedge v$ their vector product and by $u \otimes v$ their tensor product. Let $\mathbb{M}_{33}$ be the space of 3×3 real matrices endowed with the usual norm $|F| = \sqrt{\text{tr}(F^T F)}$. This is a matrix norm in the sense that $|AB| \leq |A||B|$. We denote by $A = (a_1|a_2|a_3)$ the matrix in $\mathbb{M}_{33}$ whose i th column is a_i . We consider a thin curved film of thickness $h > 0$ occupying at rest an open domain $\tilde{\Omega}_h$. This reference configuration of the film is described as follows. We are thus given a surface $\tilde{S}$, called the midsurface of the film. This surface is a bounded two-dimensional C^1 -submanifold of $\mathbb{R}^3$ and we assume for simplicity that it admits an atlas consisting of one chart. Let ψ be this chart, *i.e.* a C^1 -diffeomorphism from a bounded open subset $\bar{\omega}$ of $\mathbb{R}^2$ onto $\tilde{S}$.

Let $a_\alpha(x) = \psi_{,\alpha}(x)$ be the vectors of the covariant basis of the tangent plane $T_{\psi(x)}\tilde{S}$ associated with the chart ψ , where $\psi_{,\alpha}$ denotes the partial derivative of ψ with respect to x_α . We assume that there exists $\delta > 0$ such that $|a_1(x) \wedge a_2(x)| \geq \delta$ on $\bar{\omega}$ and we define the unit normal vector $a_3(x) = \frac{a_1(x) \wedge a_2(x)}{|a_1(x) \wedge a_2(x)|}$, which belongs to $C^1(\bar{\omega}, \mathbb{R}^3)$. The vectors $a_1(x)$, $a_2(x)$ and $a_3(x)$ constitute the covariant basis at the point x . We define the contravariant basis by the relations $a^i(x) \cdot a_j(x) = \delta_j^i$, so that $a^\alpha(x) \in T_{\psi(x)}\tilde{S}$ and $a^3(x) = a_3(x)$.

Next, we define a mapping $\Psi: \omega \times \mathbb{R} \rightarrow \mathbb{R}^3$ by

$$\Psi(x_1, x_2, x_3) = \psi(x_1, x_2) + x_3 a_3(x_1, x_2).$$

It is well known that there exists $h^* > 0$ such that for all $0 < h < h^*$, the restriction of Ψ to $\Omega_h = \omega \times]-h/2, h/2[$ is a C^1 -diffeomorphism on its image by the tubular neighborhood theorem. For such values of h , we set $\tilde{\Omega}_h = \Psi(\Omega_h)$. Alternatively, we can write

$$\tilde{\Omega}_h = \left\{ \tilde{x} \in \mathbb{R}^3, \exists \tilde{\pi}(\tilde{x}) \in \tilde{S}, \tilde{x} = \tilde{\pi}(\tilde{x}) + \eta a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))) \text{ with } \frac{-h}{2} < \eta < \frac{h}{2} \right\},$$

where $\tilde{\pi}$ denotes the orthogonal projection from $\tilde{\Omega}_h$ onto $\tilde{S}$, which is well defined and of class C^1 for $h < h^*$. Equivalently, every $\tilde{x} \in \tilde{\Omega}_h$ can be written as

$$\tilde{x} = \tilde{\pi}(\tilde{x}) + \left[(\tilde{x} - \tilde{\pi}(\tilde{x})) \cdot a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))) \right] \cdot a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))).$$

Thus, we have a curvilinear coordinate system in $\tilde{\Omega}_h$ naturally associated with the chart ψ by

$$(x_1, x_2) = \psi^{-1}(\tilde{\pi}(\tilde{x})) \text{ and } x_3 = (\tilde{x} - \tilde{\pi}(\tilde{x})) \cdot a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))).$$

Since the scalar potential $\tilde{u}_h$ is defined on $\mathbb{R}^3$ we suppose that the middle surface $\tilde{\Omega}_h$ is the image of an open domain $\omega \subset \mathbb{R}^2$ through a C^1 -diffeomorphism $\tilde{\Psi}: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ extending ψ to $\mathbb{R}^2$. More generally, we suppose that Ψ is the restriction to $\mathbb{R}^2 \times (-\frac{h}{2}, \frac{h}{2})$ of a C^1 -diffeomorphism $\tilde{\Psi}$ of $\mathbb{R}^3$. We can also suppose that Ψ is equal to the identity outside a compact set containing Ω_h . In what follows, we will keep the notation Ψ to mean $\tilde{\Psi}$.

For all $x \in \bar{\omega}$, we let $A(x) = (a_1(x)|a_2(x)|a_3(x))$. We note that $A(x)$ is an invertible matrix on $\bar{\omega}$, and that its inverse is given by $A(x)^{-1} = (a^1(x)|a^2(x)|a^3(x))^T$. We also note that $\det A(x) = |\text{cof} A(x) \cdot e_3| = |a_1(x) \wedge a_2(x)| \geq \delta > 0$ on $\bar{\omega}$. We clearly have

$$\nabla \Psi(x_1, x_2, x_3) = A(x_1, x_2) + x_3(a_{3,1}(x_1, x_2)|a_{3,2}(x_1, x_2)|0).$$

The matrix $\nabla \Psi(x_1, x_2, x_3)$ is thus everywhere invertible in $\tilde{\Omega}_h$ and its determinant is strictly positive, and therefore equal to the Jacobian of the change of variables, for h small enough.

In the following, h denotes a generic sequence of real numbers in $]0, h^*[$ that tends to zero. The next convergences are easily established.

Lemma 2.1 *We have*

$$\begin{cases} \nabla \Psi^{-1} \circ \Psi(x_1, x_2, hx_3) \rightarrow A(x)^{-1}, \\ \det \nabla \Psi(x_1, x_2, hx_3) \rightarrow \det A(x), \end{cases}$$

uniformly on $\tilde{\Omega}_1$ when $h \rightarrow 0$. In particular, $\inf_{\tilde{\Omega}_1} \det \nabla \Psi(x_1, x_2, hx_3) \geq \delta/2 > 0$ for h small enough.

3 The three-dimensional and rescaled problems

The behavior of a curved ferromagnetic thin film of thickness h occupying the domain $\tilde{\Omega}_h$ is described by a micromagnetic energy $\tilde{e}_h$ depending on its magnetization $\tilde{m}_h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. The vector field $\tilde{m}_h$ vanishes outside the film. We refer to [6] and the references therein for more information about ferromagnetic materials. The magnetization of the film minimizes an energy $\tilde{e}_h$ of the form

$$\tilde{e}_h(\tilde{m}_h) = \frac{1}{h} \int_{\tilde{\Omega}_h} \left[W(\tilde{m}_h, \nabla \tilde{m}_h) + \frac{1}{2} \nabla \tilde{u}_h \cdot \tilde{m}_h \right] d\tilde{x}, \quad (3.1)$$

under the saturation constraint

$$|\tilde{m}_h| = 1 \text{ in } \tilde{\Omega}_h.$$

The free energy is composed of an energy term with density W depending on the magnetization and its gradient, that includes the exchange and anisotropic energies. The density verifies the following growth, coercivity and locally lipschitz dependence assumptions

$$\begin{cases} \exists c > 0, \exists p \in]1, +\infty[, \forall y \in \mathbb{R}^3 \text{ and } \forall F \in M^3, |W(y, F)| \leq c(1 + |F|^p), \\ \exists \gamma > 0, \exists \beta \geq 0, \forall y \in \mathbb{R}^3 \text{ and } \forall F \in M^3, W(y, F) \geq \gamma|F|^p - \beta, \\ \forall y \in \mathbb{R}^3 \text{ and } \forall F, F' \in M^3, |W(y, F) - W(y, F')| \leq c(1 + |F|^{p-1} + |F'|^{p-1})|F - F'|, \end{cases} \quad (3.2)$$

In the case of Gioia and James we have $W(y, F) = \varphi(y) + \alpha|F|^2$ and $p = 2$.

The second term in (3.1) depends on the magnetization $\tilde{m}_h$ and a scalar potential $\tilde{u}_h$ for the induced magnetic field $H_h = -\nabla \tilde{u}_h$, solution of the magnetostatic equation

$$\operatorname{div}(-\nabla \tilde{u}_h + \tilde{m}_h) = 0 \text{ in } \mathbb{R}^3,$$

which we can also write $\operatorname{div} B_h = 0$, where $B_h = H_h + \tilde{m}_h$ in $\mathbb{R}^3$.

The observed states of the ferromagnetic film are the solutions of the minimization problem

$$\tilde{e}_h(\tilde{m}_h) = \min_{\tilde{m} \in \tilde{V}_h} \tilde{e}_h(\tilde{m}), \quad (3.3)$$

where

$$\begin{aligned} \tilde{V}_h = \{ \tilde{m} \in L^\infty(\mathbb{R}^3; \mathbb{R}^3), \text{ such that } \tilde{m}|_{\tilde{\Omega}_h} \in W^{1,p}(\tilde{\Omega}_h; \mathbb{R}^3), \\ |\tilde{m}| = 1, \text{ a.e. in } \tilde{\Omega}_h \text{ and } \tilde{m} = 0, \text{ a.e. in } \tilde{\Omega}_h^c \}, \end{aligned}$$

with $p > 1$. In this article we do not make any convexity assumption on W , consequently problem (3.3) may well not possess any solutions. Thus, we consider

a diagonal minimizing sequence $\tilde{m}_h$ for the sequence of energies $\tilde{e}_h$ over the sets $\tilde{V}_h$. More specifically, we assume that

$$\tilde{m}_h \in \tilde{V}_h \text{ and } \tilde{e}_h(\tilde{m}_h) = \inf_{\tilde{m} \in \tilde{V}_h} \tilde{e}_h(\tilde{m}) + h\varepsilon(h),$$

where ε is a positive function such that $\varepsilon(h) \rightarrow 0$ when $h \rightarrow 0$.

Using Γ -convergence arguments, we study the behavior of the energy $\tilde{e}_h$ and its almost minimizers in the above sense when the thickness of the curved film goes to zero. We begin by flattening and rescaling the minimizing problem in order to work on a fixed cylindrical domain.

In order to flatten the domain, we define $m_h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $u_h : \mathbb{R}^3 \rightarrow \mathbb{R}$ by setting for all $x \in \mathbb{R}^3$, $m_h(x) = \tilde{m}_h(\Psi(x))$ and $u_h(x) = \tilde{u}_h(\Psi(x))$. This implies that $\nabla \tilde{m}_h(\tilde{x}) = \nabla m_h(x) (\nabla \Psi^{-1}(\Psi(x)))$ and $\nabla \tilde{u}_h(\tilde{x}) = (\nabla \Psi^{-1}(\Psi(x)))^T \nabla u_h(x)$ with $\tilde{x} = \Psi(x)$. We thus set $e^h(m_h) = \tilde{e}_h(\tilde{m}_h)$ and we obtain

$$e^h(m_h) = \frac{1}{h} \int_{\Omega_h} \left[W(m_h, \nabla m_h (\nabla \Psi^{-1} \circ \Psi)) + \frac{1}{2} (\nabla \Psi^{-1} \circ \Psi)^T \nabla u_h \cdot m_h \right] \det \nabla \Psi \, dx_h.$$

Then, we rescale the problem by setting $m(h)(x) = m_h(x_1, x_2, hx_3)$ and $u(h)(x) = u_h(x_1, x_2, hx_3)$ for all $x \in \mathbb{R}^3$. This implies that

$$\nabla \tilde{m}_h(\tilde{x}) = (m(h)_{,1} | m(h)_{,2} | \frac{1}{h} m(h)_{,3}) A_h \text{ and } \nabla \tilde{u}_h(\tilde{x}) = A_h^T \nabla u(h),$$

with $A_h(x) = \nabla \Psi^{-1} \circ \Psi(x_1, x_2, hx_3)$ and $\nabla_h u(h) = \begin{pmatrix} u(h)_{,1} \\ u(h)_{,2} \\ h^{-1} u(h)_{,3} \end{pmatrix}$. These relations

may also be written $\nabla \tilde{m}_h = \nabla m(h) I_h A_h$ and $\nabla \tilde{u}_h = A_h^T I_h \nabla u(h)$ with $I_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{h} \end{pmatrix}$.

We note that $A_h(x) = \left(A(x_1, x_2) + hx_3 (a_{3,1} | a_{3,2} | 0) \right)^{-1}$, for all $x \in \bar{\omega} \times [-1, 1]$ (but not necessarily outside of this domain).

Setting $e(h)(m(h)) = e^h(m_h)$, we obtain

$$e(h)(m(h)) = \int_{\Omega_1} \left[W(m(h), (m(h)_{,1} | m(h)_{,2} | h^{-1} m(h)_{,3}) A_h) + \frac{1}{2} A_h^T \nabla_h u(h) \cdot m(h) \right] d_h \, dx,$$

with $d_h(x) = \det \nabla \Psi(x_1, x_2, hx_3)$.

The magnetostatic equation then reads

$$\left(\nabla \left(-A_h^T(x) I_h \nabla u(h)(x) + m(h)(x) \right) \right) : (A_h^T(x) I_h) = 0 \text{ in } \mathbb{R}^3. \quad (3.4)$$

and the saturation constraint reads

$$|m(h)(x)| = 1 \text{ for all } x \in \Omega_1. \quad (3.5)$$

4 Magnetostatic energy behavior

In this section we study the behavior of the magnetostatic energy term

$$E_{\text{mag}}(h)(m) = \int_{\Omega_1} \frac{1}{2} A_h^T \nabla_h u \cdot m \, d_h \, dx,$$

when the thickness of the curved film goes to zero. Let us thus be given $m : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $m|_{\Omega_1} \in L^\infty(\Omega_1; S^2)$ and $m = 0$ on Ω_1^c . We consider the minimization problem: Find $u(h, m) \in U$ such that

$$I_m(h)(u(h, m)) = \inf_{v \in U} I_m(h)(v), \quad (4.1)$$

with

$$I_m(h)(v) = \frac{1}{2} \int_{\mathbb{R}^3} \left| A_h^T \nabla_h v - m \right|^2 d_h \, dx$$

and

$$U = \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^3), \nabla v \in L^2(\mathbb{R}^3; \mathbb{R}^3), \int_B v \, dx = 0 \right\},$$

where B is the unit ball of $\mathbb{R}^3$. The condition $\int_B v \, dx = 0$ excludes the trivial translations $v \rightarrow v + c$. We endow U with the scalar product

$$(u, v)_{U, h} = \int_{\mathbb{R}^3} A_h^T \nabla_h u \cdot A_h^T \nabla_h v \, d_h \, dx.$$

When endowed with this scalar product, U is a Hilbert space (see [13]). Thus, problem (4.1) has a unique solution $u(h, m) \in U$ verifying the following Euler-Lagrange equation

$$\int_{\mathbb{R}^3} (A_h^T \nabla_h u(h, m) - m) \cdot A_h^T \nabla_h v \, d_h \, dx = 0, \forall v \in U, \quad (4.2)$$

that is the weak form of (3.4). Setting $v = u(h, m)$ in (4.2), we obtain

$$\int_{\mathbb{R}^3} \left| A_h^T \nabla_h u(h, m) \right|^2 d_h \, dx = \int_{\mathbb{R}^3} A_h^T \nabla_h u(h, m) \cdot m \, d_h \, dx, \quad (4.3)$$

where the right-hand side is equal to twice the magnetostatic energy term $E_{\text{mag}}(h)$ corresponding to a given magnetization m . The following proposition gives the behavior of the magnetostatic energy term under convergence of the magnetization.

Proposition 4.1 *Let $\bar{m}(h)$ be a sequence of functions in $L^\infty(\mathbb{R}^3; \mathbb{R}^3)$ such that $\bar{m}(h) = 0$ on Ω_1^c and $|\bar{m}(h)| = 1$ on Ω_1 , verifying $\bar{m}(h) \rightarrow \bar{m}(0)$ strongly in $L^2(\mathbb{R}^3; \mathbb{R}^3)$. Let $\bar{u}(h, \bar{m}(h))$ be the solution of the minimizing problem (4.1) associated to $\bar{m}(h)$. Then, we have*

$$\nabla \bar{u}(h, \bar{m}(h)) \rightarrow 0 \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^3) \text{ and } \frac{1}{h} \bar{u}(h, \bar{m}(h))_{,3} \rightarrow w \text{ in } L^2(\mathbb{R}^3),$$

where $w \in L^2(\mathbb{R}^3)$ verifies

$$\int_{\mathbb{R}^3} |w \tilde{a}^3|^2 d_0 dx = \int_{\Omega_1} |a_3 \cdot \bar{m}(0)|^2 d_0 dx,$$

and $\tilde{a}^3$ represents the third column vector of $A_0^T(x) = \nabla \Psi^{-1}(\Psi(x_1, x_2, 0))$ which is equal to a_3 on Ω_1 . We have also,

$$E_{\text{mag}}(h)(\bar{m}(h)) \rightarrow E_{\text{mag}}(0)(\bar{m}(0)) = \frac{1}{2} \int_{\Omega_1} |(a_3, \bar{m}(0))|^2 d_0 dx.$$

We recall that a_3 is the third vector of the covariant basis associated to the diffeomorphism Ψ on Ω_1 .

Proof Since $\bar{u}(h, \bar{m}(h))$ minimizes $I_{\bar{m}(h)}^h$ over U and $0 \in U$, we have

$$\begin{aligned} I_{\bar{m}(h)}(h)(\bar{u}(h, \bar{m}(h))) &\leq I_{\bar{m}(h)}(0) = \frac{1}{2} \int_{\mathbb{R}^3} |\bar{m}(h)|^2 d_h dx \\ &= \frac{1}{2} \text{vol}(\tilde{\Omega}_h) \leq c. \end{aligned}$$

Using the triangle inequality, we obtain that

$$\left(\int_{\mathbb{R}^3} \left| A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \right|^2 d_h dx \right)^{\frac{1}{2}} \leq c + \left(\int_{\Omega_1} d_h dx \right)^{\frac{1}{2}} \leq c'.$$

Since

$$\left| \nabla_h \bar{u}(h, \bar{m}(h)) \right| \leq \|A_h^{-T}\|_{L^\infty(\mathbb{R}^3; M^3)} \left| A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \right|,$$

we have

$$\|\nabla \bar{u}(h, \bar{m}(h))\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)} \leq c \quad \text{and} \quad \frac{1}{h} \|\bar{u}(h, \bar{m}(h))_{,3}\|_{L^2(\mathbb{R}^3)} \leq c.$$

Thus, for a subsequence still denoted h , there exists $H \in L^2(\mathbb{R}^3; \mathbb{R}^3)$ and $w \in L^2(\mathbb{R}^3)$ such that

$$\nabla \bar{u}(h, \bar{m}(h)) \rightharpoonup H \text{ weakly in } L^2(\mathbb{R}^3; \mathbb{R}^3) \text{ and } \frac{1}{h} \bar{u}(h, \bar{m}(h))_{,3} \rightharpoonup w \text{ weakly in } L^2(\mathbb{R}^3). \quad (4.4)$$

The L^p version of the Poincaré Lemma implies the existence of $u(0) \in H_{loc}^1(\mathbb{R}^3; \mathbb{R}^3)$ such that $H = \nabla u(0)$. Thus,

$$\nabla \bar{u}(h, \bar{m}(h)) \rightharpoonup \nabla u(0) \text{ weakly in } L^2(\mathbb{R}^3; \mathbb{R}^3), \quad (4.5)$$

with $u(0)_{,3} = 0$, which implies that $\nabla u(0) = 0$ in $L^2(\mathbb{R}^3)$ since $\nabla u(0) \in L^2(\mathbb{R}^3; \mathbb{R}^3)$. Consequently,

$$u(0) = 0, \quad (4.6)$$

since $u(0) \in U$. Let us show that the convergences in (4.4) and (4.5) are actually strong. Equation (4.3) reads in this case

$$\int_{\mathbb{R}^3} |A_h^T \nabla_h \bar{u}(h, \bar{m}(h))|^2 d_h dx = \int_{\mathbb{R}^3} A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \cdot \bar{m}(h) d_h dx. \quad (4.7)$$

The convergences (4.4), (4.5) and the strong convergence of $\bar{m}(h)$ to $\bar{m}(0)$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$ imply that

$$\int_{\mathbb{R}^3} A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \cdot \bar{m}(h) d_h dx \rightarrow \int_{\Omega_1} A_0^T (\nabla_p u(0) + w e_3) \cdot \bar{m}(0) d_0 dx, \quad (4.8)$$

where $\nabla_p u = \begin{pmatrix} u_{,1} \\ u_{,2} \\ 0 \end{pmatrix}$. Thus, the $L^2(\mathbb{R}^3; \mathbb{R}^3)$ norm of $A_h^T \nabla_h \bar{u}(h, \bar{m}(h))$ is conver-

gent. In the next step, we identify the limit of this norm using some test functions constructed on the model of the one used by Gioia and James in [8]. We consider a sequence of functions $w^\varepsilon \in C_c^\infty(\mathbb{R}^3)$ that converges strongly in $L^2(\mathbb{R}^3)$ to w when $\varepsilon \rightarrow 0$. Let $a_\varepsilon > 0$ be such that the projection of the support of w^ε on the x_3 axis belongs to $[-a_\varepsilon, a_\varepsilon]$ and we suppose that $a_\varepsilon > 1$. We set for $\lambda > 0$

$$v_h^{\varepsilon, \lambda}(x_1, x_2, x_3) = h \int_{-a_\varepsilon}^{x_3} w^\varepsilon(x_1, x_2, s) ds - \frac{h}{\lambda} \int_{a_\varepsilon}^{x_3} \chi_{[a_\varepsilon, a_\varepsilon + \lambda]}(r) dr \int_{-a_\varepsilon}^{a_\varepsilon} w^\varepsilon(x_1, x_2, s) ds + c^\varepsilon.$$

The constant c^ε is chosen in such a way as to satisfy the condition on the ball B and the second term guarantees that $\nabla v_h^{\varepsilon, \lambda} \in L^2(\mathbb{R}^3; \mathbb{R}^3)$. We have that $\lim_{h \rightarrow 0} v_h^{\varepsilon, \lambda} = \lim_{h \rightarrow 0} v_h^{\varepsilon, \lambda} = 0$ in $L^2(\mathbb{R}^3)$ for ε, λ fixed and

$$\frac{1}{h} v_h^{\varepsilon, \lambda} = w^\varepsilon - \frac{1}{\lambda} \chi_{[a_\varepsilon, a_\varepsilon + \lambda]}(x_3) \int_{-a_\varepsilon}^{a_\varepsilon} w^\varepsilon(x_1, x_2, s) ds.$$

Since $\bar{u}(h, \bar{m}(h))$ is the solution of the minimization problem (4.1) we have that

$$\begin{aligned} \int_{\mathbb{R}^3} |A_h^T \nabla_h \bar{u}(h, \bar{m}(h))|^2 d_h dx - 2 A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \cdot \bar{m}(h) d_h dx \\ \leq \int_{\mathbb{R}^3} |A_h^T \nabla_h v_h^{\varepsilon, \lambda}|^2 d_h dx - 2 A_h^T \nabla_h v_h^{\varepsilon, \lambda} \cdot \bar{m}(h) d_h dx. \end{aligned} \quad (4.9)$$

Using (4.8), (4.6) and the fact that $A_0^T = (\nabla \Psi^{-1} \circ \Psi(x_1, x_2, 0))^T = (a^1 | a^2 | a^3)$ in Ω_1 , where a^i represents the vectors of the contravariant basis associated with the diffeomorphism ψ with $|a^3| = 1$, we obtain

$$\int_{\mathbb{R}^3} A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \cdot \bar{m}(h) d_h dx \rightarrow \int_{\Omega_1} A_0^T (w e_3) \cdot \bar{m}(0) d_0 dx = \int_{\Omega_1} w a^3 \cdot \bar{m}(0) d_0 dx. \quad (4.10)$$

On the other hand, we have

$$A_h(x) = (\nabla \Psi^{-1} \circ \Psi(x_1, x_2, h x_3))^T \rightarrow (\nabla \Psi^{-1} \circ \Psi(x_1, x_2, 0))^T = A_0^T(x),$$

in $C^0(\mathbb{R}^3)$. Let $\tilde{a}^i$ be the column vectors of A_0^T which are defined in $\mathbb{R}^3$. We know that $\tilde{a}^i = a^i$, in Ω_1 . Passing to the limit in (4.9) when $h \rightarrow 0$ and using (4.10), we obtain that

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_{\mathbb{R}^3} \left| A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \right|^2 d_h dx \\ & \leq \int_{\mathbb{R}^3} \left| \left(w^\varepsilon - \frac{1}{\lambda} \chi_{[a_\varepsilon, a_\varepsilon + \lambda]}(x_3) \int_{-a_\varepsilon}^{a_\varepsilon} w^\varepsilon(x_1, x_2, s) ds \right) \tilde{a}^3 \right|^2 d_0 dx \\ & - 2 \int_{\Omega_1} \left\{ \left(w^\varepsilon - \frac{1}{\lambda} \left(\chi_{[a_\varepsilon, a_\varepsilon + \lambda]}(x_3) \int_{-a_\varepsilon}^{a_\varepsilon} w^\varepsilon(x_1, x_2, s) ds \right) \right) a^3 \cdot \bar{m}(0) + 2 w a^3 \cdot \bar{m}(0) \right\} d_0 dx. \end{aligned}$$

Expanding this expression, we obtain

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_{\mathbb{R}^3} \left| A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \right|^2 d_h dx \\ & \leq \int_{\mathbb{R}^3} \left(|w^\varepsilon|^2 + \frac{1}{\lambda^2} \left| \chi_{[a_\varepsilon, a_\varepsilon + \lambda]}(x_3) \int_{-a_\varepsilon}^{a_\varepsilon} w^\varepsilon(x_1, x_2, s) ds \right|^2 \right. \\ & \quad \left. - \frac{2}{\lambda} \left(w^\varepsilon \chi_{[a_\varepsilon, a_\varepsilon + \lambda]}(x_3) \int_{-a_\varepsilon}^{a_\varepsilon} w^\varepsilon(x_1, x_2, s) ds \right) \right| \tilde{a}^3|^2 d_0 dx \\ & \quad + \int_{\Omega_1} \left\{ \frac{2}{\lambda} \left(\chi_{[a_\varepsilon, a_\varepsilon + \lambda]}(x_3) \int_0^{a_\varepsilon} w^\varepsilon(x_1, x_2, s) ds \right) a^3 \cdot \bar{m}(0) \right. \\ & \quad \left. - 2 w^\varepsilon a^3 \cdot \bar{m}(0) + 2 w a^3 \cdot \bar{m}(0) \right\} d_0 dx. \quad (4.11) \end{aligned}$$

Since the projection of the support of w^ε on the x_3 -axis belongs to $[-a_\varepsilon, a_\varepsilon]$, we have that $w^\varepsilon(x) \chi_{[a_\varepsilon, a_\varepsilon + \lambda]}(x_3) = 0$, a.e. in $\mathbb{R}^3$, thus the third term of the right hand side of inequality (4.11) vanishes. We also have that $\chi_{[a_\varepsilon, a_\varepsilon + \lambda]}(x_3) = 0$ in Ω_1 since $a_\varepsilon > 1$. Thus, the fourth term of the right hand side of inequality (4.11) also

vanishes. Considering all this, inequality (4.11) becomes

$$\begin{aligned}
& \lim_{h \rightarrow 0} \int_{\mathbb{R}^3} \left| A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \right|^2 d_h dx \\
& \leq \int_{\mathbb{R}^3} \left(\left| w^\varepsilon \right|^2 + \frac{1}{\lambda^2} \left| \chi_{[a_\varepsilon, a_\varepsilon + \lambda]}(x_3) \int_{-a_\varepsilon}^{a_\varepsilon} w^\varepsilon(x_1, x_2, s) ds \right|^2 \right) |\tilde{a}^3|^2 d_0 dx \\
& \quad - 2 \int_{\Omega_1} [w^\varepsilon a^3 \cdot \bar{m}(0) - w a^3 \cdot \bar{m}(0)] d_0 dx \\
& = \int_{\mathbb{R}^3} \left| w^\varepsilon \tilde{a}^3 \right|^2 d_0 dx - 2 \int_{\Omega_1} [w^\varepsilon a^3 \cdot \bar{m}(0) - w a^3 \cdot \bar{m}(0)] d_0 dx \\
& \quad + \frac{1}{\lambda^2} \int_{\mathbb{R}^2} \left| \int_{-a_\varepsilon}^{a_\varepsilon} w^\varepsilon(x_1, x_2, s) ds \right|^2 |\tilde{a}^3|^2 d_0 d\sigma \left(\int_{\mathbb{R}} \chi_{[a_\varepsilon, a_\varepsilon + \lambda]}^2(x_3) dx_3 \right).
\end{aligned}$$

Noting that $\int_{\mathbb{R}} \chi_{[a_\varepsilon, a_\varepsilon + \lambda]}^2(x) dx = \lambda$, and passing to the limit when $\lambda \rightarrow +\infty$ we obtain

$$\begin{aligned}
& \lim_{h \rightarrow 0} \int_{\mathbb{R}^3} \left| A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \right|^2 d_h dx \\
& \leq \int_{\mathbb{R}^3} \left| w^\varepsilon \tilde{a}^3 \right|^2 d_0 dx - 2 \int_{\Omega_1} w^\varepsilon a^3 \cdot \bar{m}(0) - w a^3 \cdot \bar{m}(0) d_0 dx.
\end{aligned}$$

Then we take the limit as $\varepsilon \rightarrow 0$ and obtain

$$\lim_{h \rightarrow 0} \int_{\mathbb{R}^3} \left| A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \right|^2 d_h dx \leq \int_{\mathbb{R}^3} \left| w \tilde{a}^3 \right|^2 d_0 dx. \quad (4.12)$$

Due to the weak convergences in (4.4), (4.5) and since $A_0^T(\nabla_p u(0) + w e_3) = A_0^T(w e_3) = w \tilde{a}^3$ in $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} \left| w \tilde{a}^3 \right|^2 d_0 dx \leq \liminf \int_{\mathbb{R}^3} \left| A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \right|^2 d_h dx. \quad (4.13)$$

Then, we deduce the convergence of the norm thanks to (4.12) and (4.13), therefore

$$A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \rightarrow w \tilde{a}^3,$$

in $L^2(\mathbb{R}^3; \mathbb{R}^3)$. It follows that

$$\nabla_h \bar{u}(h, \bar{m}(h)) = A_h^{-T} A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) \rightarrow A_0^{-T} A_0^T(w e_3) = w e_3,$$

that gives the strong convergence in (4.4) and (4.5). Let us now identify the norm of $w \tilde{a}^3$ in $L^2(\mathbb{R}^3)$. Letting $h \rightarrow 0$ in (4.7), we get

$$\int_{\mathbb{R}^3} \left| w \tilde{a}^3 \right|^2 d_0 dx = \int_{\Omega_1} w a_3 \cdot \bar{m}(0) d_0 dx. \quad (4.14)$$

We set

$$\xi_h^\varepsilon = h \int_{-1}^{x_3} (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))(x_1, x_2, s) ds - \frac{h}{\lambda} \int_1^{x_3} \chi_{[1, 1+\lambda]}(r) dr \int_{-1}^1 (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))(x_1, x_2, s) ds + c^\varepsilon,$$

where $\bar{m}^\varepsilon(0)$ is a sequence in $C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$ that converges to $\bar{m}(0)$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$ when ε goes to zero with $\text{supp } \bar{m}^\varepsilon(0) \subset \Omega_1$ and c^ε a constant depending on ε . We have that $\xi_h^\varepsilon \in U$, with

$$\frac{1}{h} \xi_{h,3}^\varepsilon = \tilde{a}^3 \cdot \bar{m}^\varepsilon(0) - \frac{1}{\lambda} \chi_{[1, 1+\lambda]}(x_3) \int_{-1}^1 (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))(x_1, x_2, s) ds.$$

The minimization problem (4.1) implies that

$$\int_{\mathbb{R}^3} \left| A_h^T \nabla_h \bar{u}(h, \bar{m}(h)) - \bar{m}(h) \right|^2 d_h dx \leq \int_{\mathbb{R}^3} \left| A_h^T \nabla_h \xi_h^\varepsilon - \bar{m}(h) \right|^2 d_h dx.$$

Letting h go to zero, we see that

$$\begin{aligned} \int_{\mathbb{R}^3} \left| A_0^T (we_3) - \bar{m}(0) \right|^2 d_0 dx &\leq \int_{\mathbb{R}^3} \left| A_0^T \left((\tilde{a}^3 \cdot \bar{m}^\varepsilon(0) \right. \right. \\ &\quad \left. \left. - \frac{1}{\lambda} \chi_{[1, 1+\lambda]}(x_3) \int_{-1}^1 (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))(x_1, x_2, s) ds \right) e_3 \right) - \bar{m}(0) \right|^2 d_0 dx. \end{aligned} \quad (4.15)$$

Let us estimate the right-hand side of this inequality, we have that

$$\begin{aligned} &\int_{\mathbb{R}^3} \left| A_0^T \left((\tilde{a}^3 \cdot \bar{m}^\varepsilon(0) - \frac{1}{\lambda} \chi_{[1, 1+\lambda]}(x_3) \int_{-1}^1 (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))(x_1, x_2, s) ds) e_3 \right) - \bar{m}(0) \right|^2 d_0 dx \\ &= \int_{\mathbb{R}^3} \left| \left(\tilde{a}^3 \cdot \bar{m}^\varepsilon(0) - \frac{1}{\lambda} \chi_{[1, 1+\lambda]}(x_3) \int_{-1}^1 (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))(x_1, x_2, s) ds \right) \tilde{a}^3 - \bar{m}(0) \right|^2 d_0 dx \\ &= \int_{\mathbb{R}^3} \left| \tilde{a}^3 \cdot \bar{m}^\varepsilon(0) \right|^2 + \frac{1}{\lambda^2} \chi_{[1, 1+\lambda]}^2(x_3) \left| \int_{-1}^1 (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))(x_1, x_2, s) ds \right|^2 |\tilde{a}^3|^2 \\ &\quad + \left| \bar{m}(0) \right|^2 - \frac{2}{\lambda} \left(\tilde{a}^3 \cdot \bar{m}^\varepsilon(0) \chi_{[1, 1+\lambda]}(x_3) \int_{-1}^1 (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))(x_1, x_2, s) ds \right) \\ &\quad - 2 \left((\tilde{a}^3 \cdot \bar{m}^\varepsilon(0)) \tilde{a}^3 \cdot \bar{m}(0) \right) - \frac{2}{\lambda} \left(\chi_{[1, 1+\lambda]}(x_3) \int_{-1}^1 (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))(x_1, x_2, s) ds \tilde{a}^3 \cdot \bar{m}(0) \right) d_0 dx \end{aligned} \quad (4.16)$$

and since the projection of the support of $\bar{m}^\varepsilon(0)$ and of $\bar{m}(0)$ on the x_3 axis stays in $[0, 1]$, it follows that

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{1}{\lambda} \left((\tilde{a}^3 \cdot \bar{m}^\varepsilon(0)) \chi_{[1, 1+\lambda]}(x_3) \int_{-1}^1 (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))(x_1, x_2, s) ds \right) d_0 dx \\ &= \int_{\mathbb{R}^3} \frac{1}{\lambda} \left((\chi_{[1, 1+\lambda]}(x_3) \int_{-1}^1 (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))(x_1, x_2, s) ds) \tilde{a}^3 \cdot \bar{m}(0) \right) d_0 dx = 0. \end{aligned}$$

We also have that

$$\begin{aligned} \frac{1}{\lambda^2} \int_{\mathbb{R}^3} \chi_{[1,1+\lambda]}^2(x_3) \left| \int_{-1}^1 (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))(x_1, x_2, s) ds \right|^2 |\tilde{a}^3|^2 d_0 dx \\ = \frac{1}{\lambda} \int_{\mathbb{R}^2} \left| \int_{-1}^1 (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))(x_1, x_2, s) ds \right|^2 |\tilde{a}^3|^2 d_0 d\sigma \\ \xrightarrow{\lambda \rightarrow +\infty} 0 \end{aligned}$$

Applying all this in (4.15) we get, letting $\lambda \rightarrow \infty$

$$\int_{\mathbb{R}^3} \left| w\tilde{a}^3 - \bar{m}(0) \right|^2 d_0 dx \leq \int_{\mathbb{R}^3} \left| (\tilde{a}^3 \cdot \bar{m}^\varepsilon(0))\tilde{a}^3 - \bar{m}(0) \right|^2 d_0 dx.$$

Then, we let $\varepsilon \rightarrow 0$ and obtain

$$\int_{\mathbb{R}^3} \left[|w\tilde{a}^3|^2 - 2w(\tilde{a}^3 \cdot \bar{m}(0)) \right] d_0 dx \leq \int_{\mathbb{R}^3} \left[|(\tilde{a}^3 \cdot \bar{m}(0))|^2 - 2|(\tilde{a}^3 \cdot \bar{m}(0))|^2 \right] d_0 dx,$$

and using (4.14) we get

$$- \int_{\mathbb{R}^3} w(\tilde{a}^3 \cdot \bar{m}(0)) d_0 dx \leq - \int_{\mathbb{R}^3} |(\tilde{a}^3 \cdot \bar{m}(0))|^2 d_0 dx.$$

Consequently, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |(\tilde{a}^3 \cdot \bar{m}(0))|^2 d_0 dx &\leq \int_{\mathbb{R}^3} w(\tilde{a}^3 \cdot \bar{m}(0)) d_0 dx \\ &= \int_{\mathbb{R}^3} |w\tilde{a}^3|^2 d_0 dx. \end{aligned} \tag{4.17}$$

Thanks to the Cauchy-Schwartz inequality, equation (4.14) gives

$$\begin{aligned} \int_{\mathbb{R}^3} |w\tilde{a}^3|^2 d_0 dx &= \int_{\mathbb{R}^3} (w\tilde{a}^3 \cdot \bar{m}(0)) d_0 dx \\ &\leq \left(\int_{\mathbb{R}^3} |w|^2 d_0 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |(\tilde{a}^3 \cdot \bar{m}(0))|^2 d_0 dx \right)^{\frac{1}{2}}, \end{aligned}$$

thus,

$$\left(\int_{\mathbb{R}^3} |w\tilde{a}^3|^2 d_0 dx \right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}^3} |(\tilde{a}^3 \cdot \bar{m}(0))|^2 d_0 dx \right)^{\frac{1}{2}},$$

and using (4.17) we obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^3} |w\tilde{a}^3|^2 d_0 dx \right)^{\frac{1}{2}} &= \left(\int_{\mathbb{R}^3} |(\tilde{a}^3 \cdot \bar{m}(0))|^2 d_0 dx \right)^{\frac{1}{2}} \\ &= \left(\int_{\Omega_1} |(a_3 \cdot \bar{m}(0))|^2 d_0 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Finally, we have

$$E_{\text{mag}}(h)(\bar{m}(h)) \rightarrow E_{\text{mag}}(0)(\bar{m}(0)) = \frac{1}{2} \int_{\Omega_1} |(a_3 \cdot \bar{m}(0))|^2 d_0 dx.$$

□

Now that we know the behavior of the magnetostatic energy term, we can compute the Γ -limit of the free energy $e(h)$.

5 Γ -convergence and behavior of minimizers

We are interested in the behavior of the diagonal minimizing sequence $m(h)$ of the energy $e(h)$, that verifies

$$m(h) \in V \text{ and } e(h)(m(h)) = \inf_{m \in V} e(h)(m) + \varepsilon(h),$$

with $\varepsilon(h) \rightarrow 0$ when $h \rightarrow 0$ and

$$V = \{m \in L^\infty(\mathbb{R}^3; \mathbb{R}^3), \text{ such that } m|_{\Omega_1} \in W^{1,p}(\Omega_1; \mathbb{R}^3), \\ |m| = 1, \text{ a.e. in } \Omega_1 \text{ and } m = 0, \text{ a.e. in } \Omega_1^c\}.$$

In the following, we will identify functions in V with functions in $W^{1,p}(\Omega_1; \mathbb{R}^2)$ while keeping the same notation.

Let us begin by a lemma on deformations with bounded energies.

Lemma 5.1 *Let $m(h)$ be a sequence of magnetizations in V verifying $e(h)(m(h)) \leq c$. Then, we have*

$$\|\nabla m(h)\|_{L^p(\Omega_1; \mathbb{R}^3)} \leq c \text{ and } \frac{1}{h} \|m(h)_{,3}\|_{L^p(\Omega_1; \mathbb{R}^3)} \leq c. \quad (5.1)$$

Moreover, the limit points of this sequence for the weak topology of $W^{1,p}$ belong to the set

$$V_M = \{m \in V \text{ et } m_{,3} = 0 \text{ in } \Omega_1\}.$$

Proof Since the magnetostatic energy term is positive, we have

$$\int_{\Omega_1} W(m(h), (m(h)_{,1}, m(h)_{,2}, \frac{1}{h}m(h)_{,3})A_h) d_h dx \leq c,$$

and (3.2) implies

$$\|(m(h)_{,1}, m(h)_{,2}, \frac{1}{h}m(h)_{,3})A_h\|_{L^p(\Omega_1; \mathbb{R}^3)} \leq c.$$

Thus, we obtain (5.1). We deduce the existence of a subsequence, still denoted h and $m(0) \in W^{1,p}(\Omega_1; \mathbb{R}^3)$ satisfying

$$m(h) \rightharpoonup m(0) \text{ in } W^{1,p}(\Omega_1; \mathbb{R}^3)$$

with $m(0)_{,3} = 0$. The weak convergence in $W^{1,p}(\Omega_1; \mathbb{R}^3)$ implies that for a subsequence still denoted h we have $m(h) \rightarrow m(0)$ in $L^p(\Omega_1; \mathbb{R}^3)$, and thus, for a subsequence, we have $m(h) \rightarrow m(0)$ a.e. in Ω_1 , which implies that for almost every $x \in \Omega_1$ we have $|m(0)| = 1$. $\square$

Before announcing the main result of this section, let us recall the definition of tangential quasiconvexity, a notion introduced by B. Dacorogna, I. Fonseca, J. Maly and K. Trivisa in [5] that gives the relaxation of a class of functionals where admissible deformations are constrained to remain in a C^1 submanifold of dimension q of $\mathbb{R}^d$. This definition was generalized by R. Alicandro and C. Leone in [2] in the case of functionals depending on the gradient of the deformation and the deformation itself. We denote by $M^{d \times N}$ the set of matrices $d \times N$. Let $f : M^{d \times N} \rightarrow [0, +\infty]$ be a Borel measurable function and M a C^1 submanifold of dimension q of $\mathbb{R}^d$. We denote by $T_y(M)$ the tangent space to M at $y \in M$.

Definition 5.1 *Let $y \in M$ and $\xi \in [T_y(M)]^N$. The tangential quasiconvexification of f in ξ relative to y is defined by*

$$Q_T^{N,d} f(y, \xi) := \inf \left\{ \int_Q f(y, \xi + \nabla \varphi(x)) dx : \varphi \in W_0^{1,\infty}(Q; T_y(M)) \right\}, \quad (5.2)$$

with Q a cube in $\mathbb{R}^N$.

In our case M is the unit sphere S^2 of $\mathbb{R}^3$ and $T_y(S^2) = y^\perp$, the plane orthogonal to y . We recall two results proved in [5]

Proposition 5.1 *Let $f : M^{d \times N} \rightarrow [0, +\infty)$ be a continuous function such that there exists $c > 0$ and $p \geq 1$ verifying $0 \leq f(\xi) \leq c(1 + |\xi|^p)$, for every $\xi \in M^{d \times N}$. We define the functional J by*

$$J(u) := \int_{\Omega_1} Q_T^{N,d} f(u, \nabla u) dx, \quad u \in W^{1,p}(\Omega_1; M).$$

Then, $J(\cdot)$ is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega_1; M)$.

Theorem 5.1 *Let $f : M^{d \times N} \rightarrow [0, +\infty)$ a continuous function verifying $0 \leq f(\xi) \leq c(1 + |\xi|^p)$, for $p \geq 1$, $c > 0$ and every $\xi \in M^{d \times N}$. If M is a C^1 submanifold of dimension q , then*

$$F(u) = \int_{\Omega_1} Q_T^{N,d} f(u, \nabla u) dx,$$

where

$$F(u) := \inf_{u_n} \left\{ \liminf_{n \rightarrow +\infty} \int_{\Omega_1} f(\nabla u_n) dx : u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega_1; M) \right\}.$$

In our case, the magnetization m is constrained to remain in the sphere S^2 . Consequently, m_i will be in $m^\perp$ almost everywhere. As in Acerbi, Buttazzo and Percivale [1] for nonlinearly elastic strings and Le Dret and Raoult [10] for membranes,

we define $W_0 : \Omega_1 \times \mathbb{R}^3 \times M^{3 \times 2} \rightarrow \mathbb{R}$ by setting

$$\begin{aligned} W_0(x, y, \bar{F}) &:= \inf_{z \in y^\perp} W(y, (\bar{F}|z)A_0(x)) \\ &= \inf_{z \in \mathbb{R}^3} \bar{W}(y, (\bar{F}|z)A_0(x)), \end{aligned}$$

with $A_0(x) = \nabla \Psi^{-1} \circ \Psi(x_1, x_2, 0)$ and

$$\bar{W}(y, (\bar{F}|z)A_0(x)) = \begin{cases} W(y, (\bar{F}|z)A_0(x)), & \text{si } z \in y^\perp, \\ +\infty, & \text{otherwise.} \end{cases}$$

We show as in Le Dret and Raoult [10], that the function W_0 is continuous and satisfies properties analogous to those verified by W , that is to say

$$\begin{cases} \exists c' > 0, \forall \bar{F} \in M^{3,2}, \forall y \in \mathbb{R}^3, \forall x \in \bar{\omega}, |W_0(x, y, \bar{F})| \leq c' (1 + |\bar{F}|^p), \\ \exists \gamma' > 0, \exists \beta' \geq 0, \forall \bar{F} \in M^{3,2}, \forall y \in \mathbb{R}^3, \forall x \in \bar{\omega}, W_0(x, y, \bar{F}) \geq \gamma' |\bar{F}|^p - \beta'. \end{cases}$$

In order to study the behavior of the free energy of the ferromagnetic curved film and its possible minimizers when the thickness of the film goes to zero, we are going to compute its Γ -limit. The natural space in which we would like to compute the Γ -limit is $W^{1,p}(\Omega_1; \mathbb{R}^3)$. However, $W^{1,p}(\Omega_1; \mathbb{R}^3)$ endowed with the weak topology is not metrizable. Thus, we extend the energy $e(h)$ to all $L^p(\Omega_1; \mathbb{R}^3)$ by setting

$$\text{for all } m \in L^p(\Omega_1; \mathbb{R}^3), e^*(h)(m) = \begin{cases} e(h)(m) & \text{if } m \in V, \\ +\infty & \text{otherwise,} \end{cases}$$

We now are in a position to compute the Γ -limit of the free energy.

Theorem 5.2 *The sequence of energies $e^*(h)$ Γ -converges for the strong topology of $L^p(\Omega_1; \mathbb{R}^3)$ to*

$$e^*(0)(m) = \begin{cases} \int_{\omega} Q_T^{2,3} W_0(x, m(x), (m_{,1}|m_{,2})) + |(a_3(x), m)|^2 d_0 dx & \text{if } m \in V_M \\ +\infty & \text{otherwise.} \end{cases}$$

The proof of the theorem follows from the following propositions.

Proposition 5.2 *We have*

$$\Gamma - \liminf e^*(h) \geq e^*(0).$$

Preuve We obtain the lower bound of the Γ -limit by showing that for any sequence $m^h \in L^p(\Omega_1; \mathbb{R}^3)$ converging strongly to m^0 in $L^p(\Omega_1; \mathbb{R}^3)$, we have

$$\liminf e^*(h)(m^h) \geq e^*(0)(m^0).$$

First, if $m^h \notin V$, then $e^*(h)(m^h) = +\infty$ and the result is trivial. In the same way, if $m^h \in V$ and $\liminf e^*(h)(m^h) = +\infty$, the result is obvious.

Next, we consider $m^h \in V$, such that $\liminf e^*(h)(m^h) = \liminf e(h)(m^h) < +\infty$. This implies that for a subsequence, there exists $c > 0$ such that $e(h)(m^h) < c$. Using the lemma 5.1 we get that for a subsequence still denoted h we have that $m^h \rightharpoonup m^0$ in $W^{1,p}(\Omega_1; \mathbb{R}^3)$. Thanks to the fact that m^h is uniformly bounded in $L^\infty(\Omega_1; \mathbb{R}^3)$ we obtain that $m^h \rightarrow m^0$ in $L^s(\Omega_1; \mathbb{R}^3)$, for all $s > 1$, in particular for $s = 2$. We thus deduce from proposition 4.1 that

$$E_{\text{mag}}(h)(\bar{m}^h) \rightarrow E_{\text{mag}}(0)(\bar{m}^0) = \frac{1}{2} \int_{\Omega_1} |(a_3, \bar{m}^0)|^2 d_0 dx. \quad (5.3)$$

Let us set

$$E_{\text{ea}}(h)(m) = \int_{\Omega_1} W\left(m, (m_{,1}|m_{,2}|\frac{1}{h}m_{,3})A_h\right) d_h dx,$$

which represents the exchange and anisotropic energies. We have

$$\begin{aligned} E_{\text{ea}}(h)(m^h) &= \int_{\Omega_1} W\left(m^h, (m_{,1}^h|m_{,2}^h|\frac{1}{h}m_{,3}^h)A_h\right) d_h dx \\ &= \int_{\Omega_1} \left\{ W\left(m^h, (m_{,1}^h|m_{,2}^h|\frac{1}{h}m_{,3}^h)A_0\right) + R(x, h, m^h) \right\} d_h dx, \end{aligned}$$

where

$$R(x, h, m^h) = W\left(m^h, (m_{,1}^h|m_{,2}^h|\frac{1}{h}m_{,3}^h)A_h\right) - W\left(m^h, (m_{,1}^h|m_{,2}^h|\frac{1}{h}m_{,3}^h)A_0\right).$$

Using the properties (3.2) of W and those of the diffeomorphism Ψ , we show as in [11] that

$$\int_{\Omega_1} R(x, h, m^h) d_h dx \rightarrow 0 \text{ when } h \rightarrow 0. \quad (5.4)$$

Thus, we have

$$\begin{aligned} E_{\text{ea}}(h)(m^h) &\geq \int_{\Omega_1} \left\{ W_0\left(x, m^h, (m_{,1}^h|m_{,2}^h)\right) + R(x, h, m^h) \right\} d_h dx \\ &\geq \int_{\Omega_1} \left\{ Q_T^{2,3} W_0\left(x, m^h, (m_{,1}^h|m_{,2}^h)\right) + R(x, h, m^h) \right\} d_h dx. \end{aligned}$$

Taking the limit when $h \rightarrow 0$ we obtain using (5.4) and proposition(5.1) that

$$\liminf E_{\text{ea}}(h)(m^h) \geq \int_{\Omega_1} Q_T^{2,3} W_0(x, m^0, (m_{,1}^0 | m_{,2}^0)) d_0 dx.$$

Using (5.3) and since $m_{,3}^0 = 0$ (lemma 5.1), we obtain that

$$\liminf e(h)(m^h) \geq \int_{\omega} \left\{ Q_T^{2,3} W_0(x, m^0(x), (m_{,1}^0 | m_{,2}^0)) + |(a_3(x), m^0)|^2 \right\} d_0 dx,$$

Thus, $\liminf e^*(h)(m^h) \geq e^*(0)(m^0)$ which implies that

$$\Gamma - \liminf e^*(h) \geq e^*(0).$$

□

Then we proceed to the computation of the upper bound of the Γ -limit. We will need the following lemma (see [10])

Lemma 5.2 *Let $X \hookrightarrow Y$ be two Banach spaces such that X is reflexive and compactly embedded in Y . Let $G: X \rightarrow \mathbb{R}$ a function such that $\forall v \in X$, $G(v) \geq g(\|v\|_X)$, with g verifying $g(t) \rightarrow +\infty$ when $t \rightarrow +\infty$. Let G^* defined by $G^*(v) = G(v)$ if $v \in X$, $G^*(v) = +\infty$ otherwise. Let $\Gamma - G$ be the lower semicontinuous envelope of G for the weak topology of X and $\Gamma - G^*$ the lower semicontinuous envelope of G^* for the strong topology of Y . Then $\Gamma - G^* = (\Gamma - G)^*$.*

Proposition 5.3 *We have*

$$\Gamma - \limsup e^*(h) \leq e^*(0). \quad (5.5)$$

Preuve In order to find an upper bound for the Γ -limit, we have to construct a sequence $m^h \in L^p(\Omega_1; \mathbb{R}^3)$ that converges strongly to m^0 in $L^p(\Omega_1; \mathbb{R}^3)$ verifying

$$\lim e^*(h)(m^h) \leq e^*(0)(m^0). \quad (5.6)$$

Let $m^0 \in L^p(\Omega_1; \mathbb{R}^3)$. First, if $m^0 \notin V_M$, we have $e^*(0)(m^0) = +\infty$. Setting, for all h , $m^h = m^0$ we get (5.6). Next, we consider $m^0 \in V_M$ and we set $m^h(x) = m^0(x_1, x_2) + hx_3 \xi(x_1, x_2)$, with $\xi \in W_0^{1,p}(\omega; \mathbb{R}^3)$ to be chosen later. We temporarily do not consider the saturation constraint $|m^h| = 1$, we will treat it later. We have $m^h \rightarrow m^0$ strongly in $W^{1,p}(\Omega_1; \mathbb{R}^3)$. Using the Lebesgue convergence theorem, we have that for a subsequence still denoted h

$$\begin{aligned} E_{\text{ea}}(h)(m^h) &= \int_{\Omega_1} W(m^h, (m_{,\alpha}^0 + hx_3 \xi_{,\alpha} | \xi) A_h) d_h dx \\ &\rightarrow \int_{\Omega_1} W(m^0, (m_{,\alpha}^0 | \xi) A_0) d_0 dx. \end{aligned}$$

The function $\mu : \omega \times \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $\mu(x, z) = \bar{W}(m^0(x), (m_{,\alpha}^0(x)|z)A_0(x))$ is a Caratheodory function. Thus, the measurable selection lemma implies that, there exists $\xi : \omega \rightarrow \mathbb{R}^3$ measurable such that

$$W_0(x, m^0(x), (m_{,1}^0(x)|m_{,2}^0(x))) = \bar{W}(m^0(x), (m_{,\alpha}^0(x)|\xi(x))A_0(x)),$$

(see [7]). If $\xi \notin m^{0\perp}$, this implies that $W_0(x, m^0(x), (m_{,1}^0(x)|m_{,2}^0(x))) = +\infty$ and the result is obvious. If $\xi \in m^{0\perp}$ a.e., this implies that

$$\begin{aligned} W_0(x, m^0(x), (m_{,1}^0(x)|m_{,2}^0(x))) &= \bar{W}(m^0(x), (m_{,\alpha}^0(x)|\xi(x))A_0(x)) \\ &= W(m^0(x), (m_{,\alpha}^0(x)|\xi(x))A_0(x)). \end{aligned} \quad (5.7)$$

and thanks to the properties (3.2) of W , we obtain that $\xi \in L^p(\omega; \mathbb{R}^3)$. The density of $C_c^\infty(\omega; \mathbb{R}^3)$ in $L^p(\omega; \mathbb{R}^3)$ implies the existence of a sequence $\xi^\varepsilon \in C_c^\infty(\omega; \mathbb{R}^3)$ verifying $\xi^\varepsilon \rightarrow \xi$ strongly in $L^p(\omega; \mathbb{R}^3)$. Next, we project ξ^ε on $m^{0\perp}$ setting $\bar{\xi}^\varepsilon = (I - m^0 \otimes m^0)\xi^\varepsilon$. We have $\bar{\xi}^\varepsilon \in W_0^{1,p}(\omega; \mathbb{R}^3)$ verifying $\bar{\xi}^\varepsilon \rightarrow \xi$ strongly in $L^p(\omega; \mathbb{R}^3)$. Thus, using the Lebesgue convergence theorem, we have that for a subsequence still denoted ε

$$\int_{\omega} W(m^0(x), (m_{,\alpha}^0(x)|\bar{\xi}^\varepsilon(x))A_0(x))d_0dx \rightarrow \int_{\omega} W(m^0(x), (m_{,\alpha}^0(x)|\xi(x))A_0(x))d_0dx.$$

This means that for every $\eta > 0$, there exists an $\varepsilon(\eta) > 0$ such that $\forall \varepsilon \leq \varepsilon(\eta)$ we have

$$\int_{\omega} W(m^0, (m_{,\alpha}^0|\bar{\xi}^\varepsilon)A_0)d_0dx \leq \int_{\omega} W(m^0, (m_{,\alpha}^0|\xi)A_0)d_0dx + \eta$$

and thanks to (5.7) we obtain that

$$\int_{\omega} W(m^0, (m_{,\alpha}^0|\bar{\xi}^\varepsilon)A_0)d_0dx \leq \int_{\omega} W_0(x, m^0(x), (m_{,1}^0(x)|m_{,2}^0(x)))d_0dx + \eta \text{meas}(\tilde{S}).$$

Let us set $\bar{m}_\varepsilon^h = m^0 + hx_3\bar{\xi}^\varepsilon$. We have that $\bar{m}_\varepsilon^h \in W^{1,p}(\Omega_1; \mathbb{R}^3)$. We fix $\varepsilon > 0$, for $h < \frac{1}{2\|\bar{\xi}^\varepsilon\|_{L^\infty(\omega; \mathbb{R}^3)}}$ and since $|m^0| = 1$, we have $|\bar{m}_\varepsilon^h| > |m^0| - h\|\bar{\xi}^\varepsilon\|_{L^\infty(\omega; \mathbb{R}^3)} > \frac{1}{2}$,

a.e. Then, we set $m_\varepsilon^h = \frac{\bar{m}_\varepsilon^h}{|\bar{m}_\varepsilon^h|}$. By construction, we have $m_\varepsilon^h \in V$ thanks to the algebra property of $W^{1,p}(\omega; \mathbb{R}^3) \cap L^\infty(\omega; \mathbb{R}^3)$. We have

$$\nabla m_\varepsilon^h = \left(\frac{I}{|\bar{m}_\varepsilon^h|} - \frac{\bar{m}_\varepsilon^h \otimes \bar{m}_\varepsilon^h}{|\bar{m}_\varepsilon^h|^3} \right) \nabla \bar{m}_\varepsilon^h.$$

We also have $m_\varepsilon^h \xrightarrow{h \rightarrow 0} m^0$ strongly in $L^p(\Omega_1; \mathbb{R}^3)$, and

$$m_{\varepsilon,\alpha}^h \xrightarrow{h \rightarrow 0} (I - m^0 \otimes m^0)m_{,\alpha}^0 \text{ strongly in } L^p(\Omega_1; \mathbb{R}^3).$$

On the other hand, we have

$$((m^0 \otimes m^0) \nabla m^0)_{i,\alpha} = m_i^0(m^0, m_{,\alpha}^0) = 0,$$

since $|m^0| = 1$, and we also have

$$\frac{1}{h} m_{\varepsilon,3}^h \xrightarrow{h \rightarrow 0} (I - m^0 \otimes m^0) \bar{\xi}^\varepsilon \text{ strongly in } L^p(\Omega_1; \mathbb{R}^3).$$

Now, $\bar{\xi}^\varepsilon \in m^{0\perp}$ so that we have $(m^0 \otimes m^0) \bar{\xi}^\varepsilon = 0$ and thus, we obtain

$$m_{\varepsilon,\alpha}^h \xrightarrow{h \rightarrow 0} m_{,\alpha}^0 \text{ and } \frac{1}{h} m_{\varepsilon,3}^h \xrightarrow{h \rightarrow 0} \bar{\xi}^\varepsilon \text{ strongly in } L^p(\Omega_1; \mathbb{R}^3).$$

Using the Lebesgue theorem, we see that

$$\begin{aligned} E_{\text{ea}}(h)(m_\varepsilon^h) &= \int_{\Omega_1} W(m_\varepsilon^h, (m_{\varepsilon,\alpha}^h | \frac{1}{h} m_{\varepsilon,3}^h) A_h) d_h dx \\ &\xrightarrow{h \rightarrow 0} \int_{\omega} W(m^0, (m_{,\alpha}^0 | \bar{\xi}^\varepsilon) A_0) d_0 dx \\ &\leq \int_{\omega} W_0(x, m^0(x), (m_{,1}^0(x) | m_{,2}^0(x))) d_0 dx + \eta. \end{aligned}$$

Thus, using proposition (4.1), we obtain

$$\lim_{h \rightarrow 0} e(h)(m_\varepsilon^h) \leq \int_{\Omega_1} \left\{ W_0(x, m^0(x), (m_{,1}^0(x) | m_{,2}^0(x))) + |(a_3, m^0)|^2 \right\} d_0 dx + \eta.$$

Let us set

$$G^*(m) = \begin{cases} \int_{\omega} W_0(x, m(x), (m_{,1} | m_{,2})) + |(a_3(x), m)|^2 d_0 dx & \text{if } m \in V_M \\ +\infty & \text{otherwise.} \end{cases}$$

We have shown that

$$\lim_{h \rightarrow 0} e^*(h)(m_\varepsilon^h) \leq G^*(m^0) + \eta,$$

which implies that $\Gamma - \limsup_{h \rightarrow 0} e^*(h)(m^0) \leq G^*(m^0) + \eta$. Since this is true for every $\eta > 0$, we obtain that

$$\Gamma - \limsup_{h \rightarrow 0} e^*(h)(m^0) \leq G^*(m^0). \quad (5.8)$$

We know that the $\Gamma - \limsup$ of a function is weakly lower semicontinuous and that the lsc envelope of the function G defined on V_M by

$$G(m) = \int_{\omega} W_0(x, m(x), (m_{,1} | m_{,2})) + |(a_3(x), m)|^2 d_0 dx$$

is $e(0)$ defined on V_M by

$$e(0)(m) = \int_{\omega} Q_T^{2,3} W_0(x, m(x), (m_{,1}|m_{,2})) + |(a_3(x), m)|^2 d_0 dx,$$

see [5]. Thus, using lemma 5.2, we obtain that the lsc envelope of G^* is $e^*(0)$, and applying the lsc envelope to both sides of (5.8), we obtain that

$$\Gamma - \limsup e^*(h) \leq e^*(0). \quad (5.9)$$

□

The conjunction of Propositions 5.2 and 5.3 gives Theorem 5.2.

Corollary 5.1 *The diagonal minimizing sequence $m(h)$ of $e(h)$ is bounded in $\bar{V}$ and its limit points for the weak topology of $W^{1,p}(\Omega_1, \mathbb{R}^3)$ belong to V_M and minimize the energy $e(0)$ defined by*

$$e(0)(m) = \int_{\omega} \left\{ Q_T^{2,3} W_0(x, m(x), (m_{,1}|m_{,2})) + |(a_3(x), m)|^2 \right\} d_0 dx.$$

Preuve The proof of the corollary follows from Lemma 5.1 and the standard Γ -convergence argument. □

6 The curved two-dimensional limit model

Since every magnetization m in V_M satisfies $m_{,3} = 0$ in Ω_1 , we will identify in the following magnetizations $m \in V_M$ with magnetizations $\bar{m} \in \bar{V}_M$ where

$$\bar{V}_M = \{\bar{m} \in L^p(\mathbb{R}^2; \mathbb{R}^3), \text{ with } \bar{m}|_{\omega} \in W^{1,p}(\omega; \mathbb{R}^3), |\bar{m}| = 1 \text{ a.e. in } \omega, \bar{m} = 0 \text{ a.e. in } \omega^c\}.$$

We consider another chart ψ' from ω' , an open set in $\mathbb{R}^2$ into $\tilde{S}$. Working with this new chart, we obtain exactly the same Γ -convergence result, but this time, through another diffeomorphism Ψ' . This means that the limit model is intrinsic and only depends on the curved surface $\tilde{S}$. Let us thus write the limit model on the curved surface $\tilde{S}$. As in [11], for any unit vector e of S^2 , we consider a bounded open set $O_e \subset e^\perp$ and denote by π_e the orthogonal projection on this set. We extend any function $\chi \in W_0^{1,\infty}(O_e, \mathbb{R}^3)$ setting $\chi_e(y) = \chi(\pi_e(y))$ and we define for any $y \in O_e$, $D_{e^\perp} \chi(y) = \nabla \chi_e(y)$. To any $m \in \bar{V}_M$ we associate a magnetization $\tilde{m} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $\tilde{m}(\tilde{x}) = m(\psi^{-1}(\tilde{x}))$. We have that $\tilde{m} \in \tilde{V}_M$ with

$$\tilde{V}_M = \{\tilde{m} \in W^{1,p}(\psi(\mathbb{R}^2); \mathbb{R}^3), |\tilde{m}| = 1 \text{ a.e. in } \tilde{S}, \tilde{m} = 0 \text{ a.e. in } \tilde{S}^c \text{ et } \nabla \tilde{m} \tilde{n}(\tilde{x}) = 0\},$$

where $\tilde{n}$ denotes the normal vector to $\tilde{S}$ (for a given orientation). All functions defined on the surface are implicitly assumed to be extended to a tubular neighborhood of the surface by being constant on each normal fiber. Partial derivatives are computed on these extensions and then restricted to the surface.

Proposition 6.1 *Every magnetization $\tilde{m}$ associated with a magnetization m minimizing the limit energy $e(0)$, minimizes on $\tilde{V}_M$ the energy $\tilde{e}(0)$ defined by*

$$\tilde{e}(0)(\tilde{m}) = \int_{\tilde{S}} \left\{ \tilde{W}(\tilde{n}(\tilde{x}), \tilde{m}, \nabla \tilde{m}(\tilde{x})) + |(\tilde{n}(\tilde{x}), \tilde{m}(\tilde{x}))|^2 \right\} d\tilde{x},$$

where $\tilde{W} : S^2 \times S^2 \times M^3$ is defined by

$$\tilde{W}(e, y, F) = \inf_{\chi \in W_0^{1,\infty}(O_e; y^\perp)} \left[\frac{1}{\text{meas}(O_e)} \int_{O_e} \left\{ \inf_{z \in y^\perp} W(y, F + z \otimes e + D_{e^\perp} \chi(s)) \right\} ds \right].$$

Preuve We proceed as in Le Dret and Raoult [11]. For every $m \in V_M$ we have

$$e(0)(m) = \int_{\omega} Q_T^{2,3} W_0(x, m(x), (m_{,1}|m_{,2})) + |(a_3(x), m)|^2 dx.$$

We set $\tilde{x} = \psi(x)$, $\tilde{m}(\tilde{x}) = m(x)$. This implies that $\nabla m(x) = \nabla \tilde{m}(\tilde{x}) \nabla \psi(x)$ and setting $\tilde{e}(0)(\tilde{m}) = e(0)(m)$ we obtain

$$\tilde{e}(0)(\tilde{m}) = \int_{\tilde{S}} \left\{ Q_T^{2,3} W_0(\psi^{-1}(\tilde{x}), \tilde{m}(\tilde{x}), \nabla \tilde{m}(\tilde{x}) \nabla \psi(\psi^{-1}(\tilde{x}))) + |(a_3(\psi^{-1}(\tilde{x})), \tilde{m}(\tilde{x}))|^2 \right\} d\tilde{x}.$$

The integral representation of the tangential quasiconvex envelope (5.2) is written in our case

$$Q_T^{2,3} W_0(x, y, \xi) := \inf \left\{ \frac{1}{\text{meas}(O)} \int_O W_0(x, y, \xi + \nabla \chi(s)) ds : \chi \in W_0^{1,\infty}(O; y^\perp) \right\},$$

with O a bounded open domain in $\mathbb{R}^3$ and the infimum does not depend on the choice of this open domain. Thus, we have for all $\tilde{x} \in \tilde{S}$

$$\begin{aligned} & Q_T^{2,3} W_0(\psi^{-1}(\tilde{x}), \tilde{m}(\tilde{x}), \nabla \tilde{m}(\tilde{x}) \nabla \psi(\psi^{-1}(\tilde{x}))) \\ &= \inf_{\chi \in W_0^{1,\infty}(O; \tilde{m}(\tilde{x})^\perp)} \frac{1}{\text{meas}(O)} \int_O W_0(\psi^{-1}(\tilde{x}), \tilde{m}(\tilde{x}), \nabla \tilde{m}(\tilde{x}) \nabla \psi(\psi^{-1}(\tilde{x})) + \nabla \chi(s)) ds. \end{aligned}$$

The definition of W_0 implies that

$$\begin{aligned} & Q_T^{2,3} W_0(\psi^{-1}(\tilde{x}), \tilde{m}(\tilde{x}), \nabla \tilde{m}(\tilde{x}) \nabla \psi(\psi^{-1}(\tilde{x}))) \\ &= \inf_{\chi \in W_0^{1,\infty}(O; \tilde{m}(\tilde{x})^\perp)} \frac{1}{\text{meas}(O)} \int_O \inf_{z \in \tilde{m}(\tilde{x})^\perp} W \left(\tilde{m}(\tilde{x}), \right. \\ & \quad \left. (\nabla \tilde{m}(\tilde{x}) \nabla \psi(\psi^{-1}(\tilde{x})) + \nabla \chi(s)|z) A_0(\psi^{-1}(\tilde{x})) \right) ds. \quad (6.1) \end{aligned}$$

Moreover, we have

$$\begin{aligned} (\nabla \tilde{m}(\tilde{x}) \nabla \Psi(\Psi^{-1}(\tilde{x})) + \nabla \chi(s)|z) A_0(\Psi^{-1}(\tilde{x})) &= (\nabla \tilde{m}(\tilde{x}) \nabla \Psi(\Psi^{-1}(\tilde{x}))|0) A_0(\Psi^{-1}(\tilde{x})) \\ &\quad + (\nabla \chi(s)|0) A_0(\Psi^{-1}(\tilde{x})) + (0|z) A_0(\Psi^{-1}(\tilde{x})). \end{aligned}$$

We also have

$$\begin{aligned} (\nabla \tilde{m}(\tilde{x}) \nabla \Psi(\Psi^{-1}(\tilde{x}))|0) &= \nabla \tilde{m}(\tilde{x}) (\nabla \Psi(\Psi^{-1}(\tilde{x}))|0) \\ &= \nabla \tilde{m}(\tilde{x}) (a_1(\Psi^{-1}(\tilde{x}))|a_2(\Psi^{-1}(\tilde{x}))|0). \end{aligned}$$

The condition $m_3 = 0$ becomes $\nabla \tilde{m}(\tilde{x}) a_3(\Psi^{-1}(\tilde{x})) = 0$ and so we have

$$\nabla \tilde{m}(\tilde{x}) (\nabla \Psi(\Psi^{-1}(\tilde{x}))|0) = \nabla \tilde{m}(\tilde{x}) (a_1(\Psi^{-1}(\tilde{x}))|a_2(\Psi^{-1}(\tilde{x}))|a_3(\Psi^{-1}(\tilde{x}))).$$

We also have

$$\begin{aligned} A_0(x) &= \nabla \Psi^{-1}(\Psi(x_1, x_2, 0)) \\ &= (\nabla \Psi(x_1, x_2, 0))^{-1} \\ &= (a_1(\Psi^{-1}(\tilde{x}))|a_2(\Psi^{-1}(\tilde{x}))|a_3(\Psi^{-1}(\tilde{x})))^{-1}, \end{aligned}$$

which implies that

$$\nabla \tilde{m}(\tilde{x}) (\nabla \Psi(\Psi^{-1}(\tilde{x}))|0) A_0(\Psi^{-1}(\tilde{x})) = \nabla \tilde{m}(\tilde{x}). \quad (6.2)$$

Next, we set $\bar{s} = \nabla \Psi(\Psi^{-1}(\tilde{x}))s$ and $\bar{\chi}(\bar{s}) = \chi(s)$. We have $\bar{\chi} \in W_0^{1,\infty}(O_{a_3(\Psi^{-1}(\tilde{x}))}; \mathbb{R}^3)$ and

$$(\nabla \chi(s)|0) A_0(\Psi^{-1}(\tilde{x})) = D_{a_3(\Psi^{-1}(\tilde{x}))^\perp} \bar{\chi}(\bar{s}). \quad (6.3)$$

Finally, for every $z \in \mathbb{R}^3$ we have

$$(0|z) A_0(\Psi^{-1}(\tilde{x})) = z \otimes a_3(\Psi^{-1}(\tilde{x})). \quad (6.4)$$

Choosing $O = (\nabla \Psi(\Psi^{-1}(\tilde{x})))^{-1} O_{a_3(\Psi^{-1}(\tilde{x}))}$ and replacing (6.2), (6.3) and (6.4) in (6.1), we obtain

$$\begin{aligned} &\mathcal{Q}_T^{2,3} W_0(\Psi^{-1}(\tilde{x}), \tilde{m}(\tilde{x}), \nabla \tilde{m}(\tilde{x}) \nabla \Psi(\Psi^{-1}(\tilde{x}))) \\ &= \inf_{\chi \in W_0^{1,\infty}(O_{a_3(\Psi^{-1}(\tilde{x}))}; \tilde{m}^\perp)} \left[\frac{1}{\text{meas}(O_{a_3(\Psi^{-1}(\tilde{x}))})} \int_{O_{a_3(\Psi^{-1}(\tilde{x}))}} \left\{ \inf_{z \in \tilde{m}^\perp} W(\tilde{m}, F + z \otimes a_3(\Psi^{-1}(\tilde{x}))) \right. \right. \\ &\quad \left. \left. + D_{a_3(\Psi^{-1}(\tilde{x}))^\perp} \chi(s) \right\} ds \right], \end{aligned}$$

and thus the result. □

6.1 Model with external magnetic field

If the curved film is placed in a uniform external magnetic field H_e , the free energy governing the behavior of the film will contain a term of external energy E_{ext} , called the Zeeman energy, depending on the magnetization $\tilde{m}_h$ of the film via

$$\tilde{E}_{\text{ext}}(\tilde{m}_h) = \frac{1}{h} \int_{\tilde{\Omega}_h} (H_e, \tilde{m}_h) d\tilde{x}.$$

Setting for all $x \in \Omega_1$, $m(h)(x) = \tilde{m}(\Psi(x_1, x_2, hx_3))$ and $E_{\text{ext}}(h)(m(h)(x)) = \tilde{E}_{\text{ext}}(\tilde{m}_h)$ we obtain

$$E_{\text{ext}}(h)(m(h)) = \int_{\Omega_1} (H_e, m(h)) d_h dx.$$

When $m(h)$ converges to $m(0)$ strongly in $L^2(\Omega_1; \mathbb{R}^3)$ with $m(0)_{,3} = 0$ in Ω_1 , we have for a subsequence still denoted h

$$\begin{aligned} E_{\text{ext}}(h)(m(h)) &\rightarrow E_{\text{ext}}(0)(m(0)) \\ &= \int_{\omega} (H_e, m(0)) d_0 dx, \end{aligned}$$

and the Γ -convergence analysis remains unchanged. The total limit energy will be

$$\begin{aligned} e(0)(m(0)) &= \int_{\omega} \left\{ Q_T^{2,3} W_0(x, m(0)(x), (m(0)_{,1} | m(0)_{,2})) + |(a_3(x), m(0))|^2 \right. \\ &\quad \left. + (H_e, m(0)) \right\} d_0 dx \end{aligned}$$

and its expression on the curved surface is

$$\tilde{e}(0)(\tilde{m}) = \int_{\tilde{S}} \left\{ \tilde{W}(\tilde{n}(\tilde{x}), \tilde{m}, \nabla \tilde{m}(\tilde{x})) + |(\tilde{n}(\tilde{x}), \tilde{m}(\tilde{x}))|^2 + (\tilde{H}_e, \tilde{m}) \right\} d\tilde{x}.$$

□

6.2 The curved Gioia and James model

We apply our results to the particular case of the Gioia and James model [8], that is with $W(y, F) = \varphi(y) + \alpha |F|^2$. The limit energy reads

$$\tilde{e}(0)(\tilde{m}) = \int_{\tilde{S}} \left\{ \tilde{W}(\tilde{n}(\tilde{x}), \tilde{m}, \nabla \tilde{m}(\tilde{x})) + |(\tilde{n}(\tilde{x}), \tilde{m}(\tilde{x}))|^2 \right\} d\tilde{x},$$

with

$$\begin{aligned} &\tilde{W}(\tilde{n}(\tilde{x}), \tilde{m}, \nabla \tilde{m}(\tilde{x})) \\ &= \inf_{\chi \in W_0^{1,\infty}(O_{\tilde{n}}; \tilde{m}^\perp)} \left[\frac{1}{\text{meas}(O_{\tilde{n}})} \int_{O_{\tilde{n}}} \left\{ \inf_{z \in \tilde{m}^\perp} [\varphi(\tilde{m}) + \alpha |\nabla \tilde{m} + z \otimes \tilde{n} + D_{\tilde{n}^\perp} \chi(s)|^2] \right\} ds \right]. \end{aligned}$$

We have that

$$\begin{aligned} |\nabla \tilde{m} + z \otimes \tilde{n} + D_{\tilde{n}^\perp} \chi(s)|^2 &= |\nabla \tilde{m}|^2 + |z \otimes \tilde{n}|^2 + |D_{\tilde{n}^\perp} \chi(s)|^2 \\ &\quad + 2\nabla \tilde{m} : z \otimes \tilde{n} + 2\nabla \tilde{m} : D_{\tilde{n}^\perp} \chi(s) + 2z \otimes \tilde{n} : D_{\tilde{n}^\perp} \chi(s). \end{aligned} \quad (6.5)$$

Since $\nabla \tilde{m} \tilde{n} = 0$, this implies that $\nabla \tilde{m} : z \otimes \tilde{n} = 0$. We also have, since $D_{\tilde{n}^\perp} \chi(s) \tilde{n} = 0$, that $z \otimes \tilde{n} : D_{\tilde{n}^\perp} \chi(s) = 0$. Thus, we have

$$|\nabla \tilde{m} + z \otimes \tilde{n} + D_{\tilde{n}^\perp} \chi(s)|^2 = |\nabla \tilde{m}|^2 + |z \otimes \tilde{n}|^2 + |D_{\tilde{n}^\perp} \chi(s)|^2 + 2\nabla \tilde{m} : D_{\tilde{n}^\perp} \chi(s).$$

This means that the infimum for $z \in \tilde{m}^\perp$ is reached when z belongs to $\tilde{n}^\perp \cap \tilde{m}^\perp$ in particular for $z = 0$. Thus, we obtain

$$\begin{aligned} \tilde{W}(\tilde{n}(\tilde{x}), \tilde{m}, \nabla \tilde{m}(\tilde{x})) &= \inf_{\chi \in W_0^{1,\infty}(O_{\tilde{n}}; \tilde{m}^\perp)} \left[\frac{1}{\text{meas}(O_{\tilde{n}})} \int_{O_{\tilde{n}}} [\varphi(\tilde{m}) + \alpha |\nabla \tilde{m} + D_{\tilde{n}^\perp} \chi(s)|^2] ds \right] \\ &= \inf_{\chi \in W_0^{1,\infty}(O; \tilde{m}^\perp)} \left[\frac{1}{\text{meas}(O)} \int_O [\varphi(\tilde{m}) + \alpha |\nabla \tilde{m} + (\nabla \chi(s)|0)A_0(\psi^{-1}(\tilde{x}))|^2] ds \right], \end{aligned}$$

with O an arbitrary bounded open domain of $\mathbb{R}^2$. The invertibility of the matrix $A_0(\psi^{-1}(\tilde{x}))$ implies that

$$\begin{aligned} \tilde{W}(\tilde{n}(\tilde{x}), \tilde{m}, \nabla \tilde{m}(\tilde{x})) &= \inf_{\chi \in W_0^{1,\infty}(D; \tilde{m}^\perp)} \left[\frac{1}{\text{meas}(D)} \int_D [\varphi(\tilde{m}) + \alpha |\nabla \tilde{m} + \nabla \chi(s)|^2] ds \right] \\ &= \varphi(\tilde{m}) + \alpha Q_T^{3,3} \Phi(\nabla \tilde{m}), \end{aligned}$$

with $\Phi : M^3 \rightarrow \mathbb{R}$ defined by $\Phi(A) = |A|^2$ and D an arbitrary bounded open domain of $\mathbb{R}^3$. Since Φ is convex, its tangential quasiconvex envelope is Φ itself and we have $\tilde{W}(\tilde{n}(\tilde{x}), \tilde{m}, \nabla \tilde{m}(\tilde{x})) = \varphi(\tilde{m}) + \alpha |\nabla \tilde{m}|^2$, therefore

$$\tilde{e}(0)(\tilde{m}) = \int_{\tilde{S}} \left\{ \varphi(\tilde{m}) + \alpha |\nabla \tilde{m}|^2 + |(\tilde{n}(\psi^{-1}(\tilde{x})), \tilde{m}(\tilde{x}))|^2 \right\} d\tilde{x}.$$

And thus we have a generalization of the results of [8].

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