FOURIER'S LAW FOR A MICROSCOPIC MODEL OF HEAT CONDUCTION.

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ABSTRACT. We consider a chain of N harmonic oscillators perturbed by a conservative stochastic dynamics and coupled at the boundaries to two gaussian thermostat at different temperatures. The stochastic perturbation is given by a diffusion process that exchange momentum between nearest neighbor oscillators conserving the total kinetic energy. The resulting total dynamics is a degenerate hypoelliptic diffusion with a smooth stationary state. We prove for the stationary state, in the limit as $N \rightarrow \infty$, the Fourier's law and the linear profile for the energy average.

1. INTRODUCTION

In insulating crystals heat is transported by lattice vibrations, and since the pioneering work of Debye, one-dimensional chains of anharmonic oscillators have been used as microscopic models for heat conduction (for a review cf. [13] and [5]). These chains are then connected at the extremities to two thermostats at different temperatures. Non-linear effects are extremely important in order to obtain finite conductivity. Enough strong non-linearity causes scattering between phonons and should imply a sufficiently fast decay of correlations for heat currents. In fact it is well known that harmonic chains, because of their infinitely many conserved quantities, have infinite conductivity and do not obey Fourier's law (cf. [15]). On the other hand a rigorous treatment of a nonlinear chain, even the proof of the existence of the conductivity coefficient, is out of reach of current mathematical techniques.

In the present paper we study a model of a chain of *harmonic* oscillators where the hamiltonian dynamic is perturbed by a random continuous exchange of kinetic energy between nearest neighbors oscillators. This random exchange conserves the total kinetic energy and destroy all other conservation laws. In this sense it simulates the long time effect of the non-linearities in the deterministic model. This random exchange on kinetic energy is realized by a diffusion on the circle of constant kinetic energy of the nearest neighbor oscillators. We expect the same macroscopic behavior and results if this diffusions are replaced by jump processes.

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The interaction with the reservoirs are modeled by Ornstein-Uhlenbeck processes at the corresponding temperatures. It results that the total dynamics of system is a degenerate hypoelliptic diffusion on the phase space. By Hörmander theorem this process has a smooth stationary state. This stationary state is product and gaussian only if the temperatures of the thermostats are equal (equilibrium).

We prove that in the stationary state a Fourier's law is valid for the energy flow and that the total energy of the system is proportional to the size of it. Then we prove a linear profile for the energy. A corresponding law of large number (hydrodynamic limit) should be valid for this system, but at the moment we have not been able to prove this.

The macroscopic evolution of the *dynamical* fluctuation in equilibrium for the corresponding infinite model, have been proven in a companion paper (cf. [10]).

With similar motivations other stochastic models have been proposed before. In 1970, Bosterly, Rich and Visscher (cf. [3]) considered a chain of harmonic oscillators where each oscillator is also connected to an interior bath, modeled, like the boundary terms, by Ornstein-Uhlenbeck processes. The temperature of each bath is then chosen in a self-consistent way. The Fourier's law and the linear profile of temperature for this model in the steady state have been proven recently by Bonetto, Lebowitz and Lukkarinen (cf. [4]). There are two main difference between this model and ours. In the Bosterly, Rich and Visscher model, energy is not conserved by the bulk dynamics, event though the temperatures of the internal baths are regulated so that the *average* flow of energy between the oscillators and the internal baths is null. In our system the bulk dynamic conserves energy, and only the boundary reservoirs can change the total energy. The second difference is that the dynamic of the Bosterly, Rich and Visscher model is linear, and consequently the stationary state is fully gaussian. Fourier's law, linear profile of temperatures and other result can be then obtained by computing the limit of the 2-point correlations of the stationary state. The stochastic perturbation we consider is intrinsically non-linear and the stationary state is non-gaussian (except in the equilibrium case).

Another model has been introduced in 1982 by Kipnis, Marchioro and Presutti (cf. [11]) where the energy is microscopically conserved but the hamiltonian part of the dynamics is removed. The dynamics consist here only in a random exchange of energy between nearest-neighbor *oscillators*, given by properly defined jump processes. The striking duality properties of this process make it explicitly solvable, and in [11] Fourier's law and linear profile of temperature are proven. Recently a deterministic hamiltonian model has been proposed in [8] where, in a proper high temperature limit and under a chaoticity assumption, the model of Kipnis, Marchioro and Presutti can be recovered.

The main tool we use in our proof is a bound of the entropy production of the bulk dynamics. This tool has been successful in the analogous problem of Fick's law in some lattice dynamics (cf. [7], [12]).

One of the main difficulties in proving Fourier's law and hydrodynamic limit is to establish a *fluctuation-dissipation* relation, i.e. a decomposition of the current of the conserved quantity (here the energy) in a dissipative part (a spatial *gradient*) and a fluctuating part (a *time derivative*). Thanks to the stochastic perturbation one can write here an exact fluctuation-dissipation relation (cf. equation (27)). Then, in order to obtain the Fourier's law, we have to bound (uniformly in the size of the system) the second moment of the positions and velocity at the boundary. It results that we can bound the second moments of all the coordinates, that gives a bound of the expectation of the total energy proportional to the size of the system.

2. THE MODEL

Atoms are labeled by $x \in \{1, \dots, N-1\}$. Atom 1 and $N-1$ are in contact with two separate heat reservoirs at two different temperatures T_l and T_r . The interaction between the reservoirs is modeled by two Ornstein-Uhlenbeck processes at the corresponding temperatures. The moments of the atoms are denoted by $p_1, \dots, p_{N-1}$ and the positions by $q_1, \dots, q_{N-1}$. The distances between the positions are denoted by $r_1, \dots, r_{N-2}$, where $r_x = q_{x+1} - q_x$. The hamiltonian of the system that represents the total energy inside the system is given by

(1)

$$\mathcal{H}_N = \sum_{x=1}^{N-1} e_x, \quad e_x = \frac{(p_x^2 + (r_x - \rho)^2)}{2} \quad x = 1, \dots, N-2; \quad e_{N-1} = \frac{p_{N-1}^2}{2}.$$

The dynamics is described by the following system of stochastic differential equations:

(2)

$$\begin{aligned} dr_x &= (p_{x+1} - p_x)dt, \quad x = 1, \dots, N-2 \\ dp_x &= (r_x - r_{x-1})dt - \gamma p_x dt + \sqrt{\gamma} (p_{x-1} dw_{x-1,x} - p_{x+1} dw_{x,x+1}), \quad x = 2, \dots, N-2 \\ dp_1 &= (r_1 - \rho)dt - \frac{1+\gamma}{2} p_1 dt - \sqrt{\gamma} p_2 dw_{1,2} + \sqrt{T_l} dw_{0,1}, \\ dp_{N-1} &= -(r_{N-2} - \rho)dt - \frac{1+\gamma}{2} p_{N-1} dt + \sqrt{\gamma} p_{N-2} dw_{N-2,N-1} + \sqrt{T_r} dw_{N-1,N}, \end{aligned}$$

where $\gamma > 0$ is a parameter that regulates the strength of the random exchange of momenta between the nearest neighbor particles. Observe that by translating r_x in $r_x - \rho$ one has the same equations for the new coordinate but with $\rho = 0$. So we set $\rho = 0$ without any loss of generality.

The generator writes as

$$(3) \quad \begin{aligned} L_N = & \sum_{x=1}^{N-2} (p_{x+1} - p_x) \partial_{r_x} + \sum_{x=2}^{N-2} (r_x - r_{x-1}) \partial_{p_x} + r_1 \partial_{p_1} - r_{N-2} \partial_{p_{N-1}} \\ & + \frac{\gamma}{2} \sum_{x=1}^{N-2} X_{x,x+1}^2 + \frac{1}{2} (T_l \partial_{p_1}^2 - p_1 \partial_{p_1}) + \frac{1}{2} (T_r \partial_{p_{N-1}}^2 - p_{N-1} \partial_{p_{N-1}}) \end{aligned}$$

where

$$(4) \quad X_{x,x+1} = p_{x+1} \partial_{p_x} - p_x \partial_{p_{x+1}}$$

One can check easily that the Lie algebra generated by these vector fields and the hamiltonian part of L_N has full rank at every point of the state space $\mathbb{R}^{N-1} \times \mathbb{R}^{N-2}$. By Hörmander theorem it follows that this operator is hypoelliptic (cf. thm 22.2.1 in [9]), so the stationary measure has a smooth density. We denote with $\langle \cdot \rangle$ the expectation with respect to the stationary measure. The existence of a unique stationary measure can be proved similarly as in [16] or in [6].

Energy is conserved by the bulk part of the dynamics and we have

$$(5) \quad L_N e_x = j_{x-1,x} - j_{x,x+1}$$

with

$$(6) \quad \begin{aligned} j_{x,x+1} &= -r_x p_{x+1} - \frac{\gamma}{2} (p_{x+1}^2 - p_x^2), \quad x = 1, \dots, N-2 \\ j_{0,1} &= \frac{1}{2} (T_l - p_1^2), \quad j_{N-1,N} = -\frac{1}{2} (T_r - p_{N-1}^2) \end{aligned}$$

Consequently $j_{x,x+1}$ is called instantaneous current of energy. Because of stationarity, for any $x = 1, N-1$ we have

$$(7) \quad \langle j_{x,x+1} \rangle = \langle j_{0,1} \rangle = \langle j_{N-1,N} \rangle$$

The following theorems are the main results of this paper.

Theorem 1. *For any $\gamma > 0$*

$$(8) \quad \lim_{N \rightarrow \infty} N \langle j_{x,x+1} \rangle = \frac{1}{2} (\gamma + \gamma^{-1}) (T_l - T_r).$$

Furthermore there exists a constant C depending only on γ , T_l and T_r such that

$$(9) \quad \langle \mathcal{H}_N \rangle \leq CN$$

Theorem 2. *For $\gamma = 1$ and any bounded function $G : [0, 1] \rightarrow \mathbb{R}$, we have*

$$(10) \quad \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \sum_{x=1}^{N-1} G(x/N) e_x \right\rangle = \int_0^1 G(q) T(q) dq$$

where $T(q) = T_l + (T_r - T_l)q$ is the linear profile interpolating T_l and T_r .

3. ENTROPY PRODUCTION

Denote by $g_{T_r}(p_1, r_1, \dots, p_{N-2}, r_{N-2}, p_{N-1})$ the density of the product on gaussians with mean 0 and variance T_r . We denote by f_N the density of the stationary measure with respect to g_{T_r} . By hypoellipticity this density is smooth.

By stationarity we have

$$(11) \quad 0 = -2 \langle L_N \log f_N \rangle = \gamma \sum_{x=1}^{N-2} \int \frac{(X_{x,x+1} f_N)^2}{f_N} g_{T_r} d\bar{p} d\bar{r} + T_r \int \frac{(\partial_{p_{N-1}} f_N)^2}{f_N} g_{T_r} d\bar{p} d\bar{r} - 2 \langle L_l \log f_N \rangle$$

where $L_l = (T_l \partial_{p_1}^2 - p_1 \partial_{p_1})$. Define $h = g_{T_l}/g_{T_r}$, then we can rewrite the last term as

$$(12) \quad \begin{aligned} -2 \langle L_l \log f_N \rangle &= -2 \int \frac{f_N}{h} L_l \log \left(\frac{f_N}{h} \right) g_{T_l} d\bar{p} d\bar{r} - 2 \int f_N L_l (\log h) g_{T_r} d\bar{p} d\bar{r} \\ &= T_l \int \frac{[\partial_{p_1} (f_N/h)]^2}{f_N/h} g_{T_l} d\bar{p} d\bar{r} + (T_l^{-1} - T_r^{-1}) (T_l - \langle p_1^2 \rangle). \end{aligned}$$

So by (29) we have the following bound

$$(13) \quad \begin{aligned} \gamma \sum_{x=1}^{N-2} \int \frac{(X_{x,x+1} f_N)^2}{f_N} g_{T_r} d\bar{p} d\bar{r} + T_r \int \frac{(\partial_{p_{N-1}} f_N)^2}{f_N} g_{T_r} d\bar{p} d\bar{r} + T_l \int \frac{[\partial_{p_1} (f_N/h)]^2}{f_N/h} g_{T_l} d\bar{p} d\bar{r} \\ = (T_r^{-1} - T_l^{-1}) (T_l - \langle p_1^2 \rangle) \end{aligned}$$

In section 4, we prove that this last expression is bounded by CN^{-1} for some constant C (cf. (29) and (6)). This relation also gives us the right sign for the energy current, i.e. if $T_l < T_r$ we have $\langle j_{x,x+1} \rangle < 0$.

4. SOME BOUNDS

From (6) and (7) we have

$$(14) \quad \langle p_1^2 \rangle + \langle p_{N-1}^2 \rangle = T_l + T_r$$

Observe that, since $L_N r_1^2 = 2(r_1 p_2 - r_1 p_1)$, we have

$$(15) \quad \langle r_1 p_2 \rangle = \langle r_1 p_1 \rangle$$

Equation (7) for $x = 1$ gives

$$(16) \quad \langle j_{1,2} \rangle = - \langle r_1 p_2 \rangle - \frac{\gamma}{2} (\langle p_2^2 \rangle - \langle p_1^2 \rangle)$$

Since this last is equal to $\langle j_{0,1} \rangle$, using (15), we obtain

$$(17) \quad \frac{\gamma}{2} \langle p_2^2 \rangle = - \langle r_1 p_1 \rangle + \frac{1}{2}(\gamma + 1) \langle p_1^2 \rangle - \frac{1}{2} T_l$$

Then by Schwarz inequality there exists a constant C , depending only on γ , such that

$$(18) \quad \langle p_2^2 \rangle \leq C (\langle r_1^2 \rangle + \langle p_1^2 \rangle)$$

Analogous computation for the index $x = N - 2$ gives

$$(19) \quad \langle p_{N-2}^2 \rangle \leq C (\langle p_{N-1}^2 \rangle + \langle r_{N-2}^2 \rangle).$$

Observe now that

$$(20) \quad L_N(r_1 p_1) = p_1(p_2 - p_1) + r_1^2 - \frac{\gamma+1}{2} p_1 r_1$$

so we have the relation

$$(21) \quad \langle r_1^2 \rangle = \langle p_1^2 \rangle - \langle p_1 p_2 \rangle + \frac{\gamma+1}{2} \langle p_1 r_1 \rangle$$

and by use of (17)

$$(22) \quad \langle r_1^2 \rangle = \langle p_1^2 \rangle - \langle p_1 p_2 \rangle + \left(\frac{\gamma+1}{2} \right)^2 \langle p_1^2 \rangle - \frac{\gamma(\gamma+1)}{4} \langle p_2^2 \rangle - \frac{\gamma+1}{4} T_l$$

and by Schwarz inequality, for any $\alpha > 0$

$$(23) \quad \langle r_1^2 \rangle \leq \left(1 + \left(\frac{\gamma+1}{2} \right)^2 + \frac{1}{2\alpha} \right) \langle p_1^2 \rangle + \left(\frac{\alpha}{2} - \frac{\gamma(\gamma+1)}{4} \right) \langle p_2^2 \rangle$$

choosing properly α one obtains a constant C depending only on γ , such that

$$(24) \quad \langle r_1^2 \rangle \leq C \langle p_1^2 \rangle$$

and an analogous bound is obtained for $\langle r_{N-2}^2 \rangle$.

Putting all together we have obtained the following lemma:

Lemma 1. *There exists a constant C depending only on γ and linearly on T_l and T_r such that*

$$(25) \quad \langle r_1^2 \rangle + \langle p_1^2 \rangle + \langle p_2^2 \rangle + \langle r_{N-2}^2 \rangle + \langle p_{N-1}^2 \rangle + \langle p_{N-2}^2 \rangle \leq C(T_l + T_r)$$

The bulk dynamics is only apparently non-gradient since defining

$$(26) \quad h_x = \frac{1}{2\gamma} p_{x+1}(r_x + r_{x+1}) + \frac{1}{4} p_{x+1}^2, \quad x = 1, \dots, N-3$$

permits to rewrite

$$(27) \quad j_{x,x+1} = -\nabla \left(\frac{1}{2\gamma} r_x^2 + \frac{\gamma}{2} p_x^2 + \frac{1}{2\gamma} p_x p_{x+1} + \frac{\gamma}{4} \nabla(p_x^2) \right) + L h_x, \quad x = 1, \dots, N-3.$$

where the discrete gradient ∇ of a discrete function w is defined by $(\nabla w)(x) = w(x+1) - w(x)$. Using again (7) we have

(28)

$$\begin{aligned} \langle j_{0,1} \rangle &= \frac{1}{N-3} \sum_{x=1}^{N-3} \langle j_{x,x+1} \rangle \\ &= -\frac{1}{N-3} \left(\frac{1}{2\gamma} \langle r_{N-2}^2 \rangle + \frac{\gamma}{2} \langle p_{N-2}^2 \rangle + \frac{1}{2\gamma} \langle p_{N-2} p_{N-1} \rangle \right. \\ &\quad \left. + \frac{\gamma}{4} (\langle p_{N-1}^2 \rangle - \langle p_{N-2}^2 \rangle) - \frac{1}{2\gamma} \langle r_1^2 \rangle - \frac{\gamma}{2} \langle p_1^2 \rangle - \frac{1}{2\gamma} \langle p_2 p_1 \rangle - \frac{\gamma}{4} (\langle p_2^2 \rangle - \langle p_1^2 \rangle) \right) \end{aligned}$$

and by (25) we obtain that there exists a constant C depending only on T_l , T_r and γ such that

$$(29) \quad |\langle j_{x,x+1} \rangle| \leq \frac{C}{N}, \quad x = 0, \dots, N-1.$$

5. FOURIER'S LAW

Proposition 1. *For $x = 1$ and $N - 2$ we have*

$$(30a) \quad \lim_{N \rightarrow \infty} \langle p_x p_{x+1} \rangle = 0$$

$$(30b) \quad \lim_{N \rightarrow \infty} \langle r_x p_{x+1} \rangle = 0$$

$$(30c) \quad \lim_{N \rightarrow \infty} \langle (p_x^2 - p_{x+1}^2) \rangle = 0$$

Proof. Let us prove the case $x = 1$, for $x = N - 2$ the proof is similar. By (13), (29) and (25)

(31)

$$\begin{aligned} \langle r_1 p_2 \rangle &= \langle r_1 p_1 \rangle = \int r_1 p_1 (f_N/h) g_{T_l} d\bar{p} d\bar{r} = T_l \int r_1 \partial_{p_1} (f_N/h) g_{T_l} d\bar{p} d\bar{r} \\ &\leq T_l \langle r_1^2 \rangle^{1/2} \left(\int \frac{[\partial_{p_1} (f_N/h)]^2}{f_N/h} g_{T_l} \right)^{1/2} d\bar{p} d\bar{r} \leq \frac{C}{\sqrt{N}} \end{aligned}$$

The proof for $\langle p_1 p_2 \rangle$ is similar.

Now by (29) for $x = 1$ we have

$$(32) \quad \lim_{N \rightarrow \infty} \langle (p_1^2 - p_2^2) \rangle = 0$$

□

Then by (21) we have

$$(33) \quad \lim_{N \rightarrow \infty} \langle r_1^2 \rangle = \lim_{N \rightarrow \infty} \langle p_1^2 \rangle = T_l$$

and similarly

$$(34) \quad \lim_{N \rightarrow \infty} \langle r_{N-2}^2 \rangle = \lim_{N \rightarrow \infty} \langle p_{N-1}^2 \rangle = T_r$$

By (28) it follows that

$$(35) \quad \lim_{N \rightarrow \infty} N \langle j_{x,x+1} \rangle = \frac{1}{2} (\gamma + \gamma^{-1}) (T_l - T_r)$$

i.e. the law of Fourier.

6. ENERGY BOUND

We claim now there exists a constant $C > 0$ independent of N such that

$$(36) \quad \left\langle \frac{\mathcal{H}_N}{N} \right\rangle \leq C$$

Proof. Define

$$(37) \quad \phi(x) = \frac{1}{2\gamma} \langle r_x^2 \rangle + \frac{\gamma}{4} (\langle p_x^2 \rangle + \langle p_{x+1}^2 \rangle) + \frac{1}{2\gamma} \langle p_x p_{x+1} \rangle.$$

By (5) and (27), we have

$$(38) \quad \Delta \phi(x) = 0, \quad x = 1, \dots, N-3$$

Here, $(\Delta w)(x) = w(x+1) + w(x-1) - 2w(x)$ is the usual discrete Laplacian of the function $w(x)$. By (25) and the maximum principle, it follows that there exists a constant C independent of N such that

$$(39) \quad |\phi(x)| \leq C, \quad x = 1, \dots, N-2$$

For $x \in \{2, \dots, N-3\}$, a simple computation shows

$$(40) \quad L_N(p_x p_{x+1}) = -3p_x p_{x+1} + (r_{x+1} - r_x)p_x + (r_x - r_{x-1})p_{x+1}$$

Hence, by Schwarz inequality and (39), we get

$$(41) \quad |\langle p_x p_{x+1} \rangle| \leq 6C^{1/2} (\langle p_x^2 \rangle^{1/2} + \langle p_{x+1}^2 \rangle^{1/2}), \quad x = 2, \dots, N-3$$

In fact, by (25), we can extend this inequality for $x \in \{1, \dots, N-2\}$ (with a slight modification of the constant C):

$$(42) \quad |\langle p_x p_{x+1} \rangle| \leq C (\langle p_x^2 \rangle^{1/2} + \langle p_{x+1}^2 \rangle^{1/2}), \quad x = 1, \dots, N-2$$

Reporting (42) in (39), we get

$$(43) \quad \frac{1}{2\gamma} \langle r_x^2 \rangle + \frac{\gamma}{4} (\langle p_x^2 \rangle + \langle p_{x+1}^2 \rangle) \leq C \left\{ 1 + (\langle p_x^2 \rangle^{1/2} + \langle p_{x+1}^2 \rangle^{1/2}) \right\}$$

Using the trivial inequality $\sqrt{z} \leq z/4 + 1$ valid for any z , we obtain

$$(44) \quad \langle e_x \rangle \leq C, \quad x \in \{1, \dots, N-1\}$$

for some positive constant C independent of N . The equation (36) is a trivial consequence of (44). $\square$

7. ENERGY PROFILE FOR $\gamma = 1$

We prove here Theorem 2 for $\gamma = 1$. In this case we have $\phi(x) = \langle r_x^2 \rangle + \frac{1}{2} \langle (p_{x+1} + p_x)^2 \rangle = 2 \langle e_x \rangle + \langle p_x p_{x+1} \rangle$ for $x = 1, \dots, N-2$.

By (30) and (33), $\lim_{N \rightarrow \infty} \phi(1) = 2T_l$ and $\lim_{N \rightarrow \infty} \phi(N-2) = 2T_r$. This gives the boundary conditions for the Laplace equation (38). It follows that for any $q \in [0, 1]$

$$(45) \quad \lim_{N \rightarrow \infty} \phi([Nq]) = 2T(q)$$

Then, in order to prove (10), we are left to prove

$$(46) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=1}^{N-2} G(x/N) \langle p_x p_{x+1} \rangle = 0$$

Since we have the uniform bound on the energy (44), we can assume that G has a compact support in $(0, 1)$ (meaning we can forget the boundary terms in the following discussion). By equation (29), we have

$$(47) \quad \frac{1}{N} \sum_{x=1}^N G(x/N) \langle p_{x+1} r_x \rangle = \frac{1}{2N} [G(\frac{x}{N}) - G(\frac{x-1}{N})] \langle p_x^2 \rangle + O(N^{-1})$$

Remark that we have forget the boundary terms in the discrete integration by parts since G vanishes at the boundaries. Since G is continuously differentiable and because of (44), we have hence

$$(48) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=1}^{N-1} G(x/N) \langle p_{x+1} r_x \rangle = 0$$

We recall now the following trivial computations (valid in the bulk)

$$(49) \quad \begin{aligned} L_N(p_x p_{x+1}) &= -3p_x p_{x+1} + (r_{x+1} - r_x)p_x + (r_x - r_{x-1})p_{x+1} \\ L_N(r_x^2/2) &= p_{x+1} r_x - p_x r_x \\ L_N(r_x r_{x-1}) &= (p_{x+1} - p_x)r_{x-1} + (p_x - p_{x-1})r_x \end{aligned}$$

Using the second equation of (49) and the equation (48), we have:

$$(50) \quad \frac{1}{N} \sum_{x=1}^{N-2} G(x/N) \langle p_x r_x \rangle = 0$$

Hence, by the first equation of (49), (48) and (50), we have just to prove

$$(51) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=1}^{N-2} G(x/N) \langle r_{x+1} p_x - r_{x-1} p_{x+1} \rangle = 0$$

In the same way, the last equation of (49) says that

$$(52) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=2}^{N-2} G(x/N) (\langle p_{x+1} r_{x-1} - p_{x-1} r_x \rangle) = 0$$

But we have

$$(53) \quad \begin{aligned} \langle r_{x+1}p_x - r_{x-1}p_{x+1} \rangle &= - \langle p_{x+1}r_{x-1} - p_{x-1}r_x \rangle + \langle r_{x+1}p_x - r_xp_{x-1} \rangle \\ &= - \langle p_{x+1}r_{x-1} - p_{x-1}r_x \rangle + \nabla(\langle r_xp_{x-1} \rangle) \end{aligned}$$

It follows by (52) that

$$(54) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=1}^{N-2} G(x/N) \langle r_{x+1}p_x - r_{x-1}p_{x+1} \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=2}^{N-2} G(x/N) \nabla(\langle r_xp_{x-1} \rangle) = 0$$

the last equality is the consequence of the discrete integration by parts and the differentiability of G . We have hence proved (51) and we are done.

8. OPEN PROBLEMS AND OTHER MODELS

We have proven for our stochastic model the Fourier law for any value of the coupling γ and the linear profile of the *energy* for the case $\gamma = 1$. The essential tool used has been a bound on the entropy production. This bound on the entropy production together with a uniform bound on $\langle p_x^{2+\delta} \rangle$ will provide a proof of the linear *temperature* profile and of *local equilibrium* for any value of γ . Unfortunately we have not been able to prove yet such uniform bound of the higher moments of the velocities, but we conjecture that it is certainly satisfied.

A generalization in more dimension looks like a difficult problem, since the decomposition of the current in a gradient plus a time derivative (cf. equation (27)) will be much more complex, involving non-local functions.

The proof we have exposed in the present paper can be adapted for some modification of the model. For example one can add a pinning given by *on site* harmonic potential, adding to the hamiltonian a term $\sum_{x=1}^{N-1} \nu^2 q_x^2/2$. Or adding stochastic reservoirs like in the model of Bolsterli-Rich-Visscher (cf. [3] and [4]) with self consistent temperatures, i.e. we can add to the generator a term

$$\lambda \sum_{x=2}^{N-2} (T_x \partial_{p_x}^2 - p_x \partial_{p_x})$$

where the temperatures T_x are imposed to be equal to $\langle p_x^2 \rangle$. In this case we find that the self-consistent profile T_x is asymptotically linear and the Fourier law is given by

$$(55) \quad \lim_{N \rightarrow \infty} N \langle j_{x,x+1} \rangle = \left(\frac{1}{2(\gamma + \lambda)} + \frac{\gamma}{2} \right) (T_l - T_r).$$

which, in the limit as $\gamma \rightarrow 0$ is in agreement with the results of Bonetto-Lebowitz-Lukkarinen (cf. [4]). The proof of (55) is very close to the one exposed in sections 3, 4, 5. In fact one has the decomposition of the current

in the form $\nabla\phi_x + L_N h_x$ with the function h_x given by

$$(56) \quad h_x = \frac{1}{2(\gamma + \lambda)} p_{x+1}(r_x + r_{x+1}) + \frac{1}{4} p_{x+1}^2, \quad x = 1, \dots, N-3$$

Observe that this works also in the case $\gamma = 0$ if $\lambda > 0$.

In this last model one can also prove local equilibrium by proper use of Log-Sobolev inequalities and the entropy production bound, similarly as done in [14]. In the case $\gamma = 0$ and in presence of pinning, local equilibrium is proved in [4].

One can also consider different stochastic perturbations that conserve energy and also momentum (as proposed in [2] and [8]). We prove in [2] that also these momentum conserving models have finite conductivity, i.e. the average energy current decrease like $1/N$.

REFERENCES

- [1] Bernardin C. : Heat Conduction model, preprint.
- [2] Bernardin C., Basile G., Olla S.: A momentum conserving microscopic model for the Fourier law, in preparation.
- [3] Bolsterli M., Rich M., Visscher W.M.: Simulation of nonharmonic interactions in a crystal by self-consistent reservoirs, *Phys.Rev. A* **4**, 1086-1088 (1970).
- [4] Bonetto, F., Lebowitz, J.L., Lukkarinen, J., Fourier's Law for a harmonic Crystal with Self-Consistent Stochastic Reservoirs, *J.Stat.Phys.* **116**, 783-813 (2004).
- [5] Bonetto, F., Lebowitz, J.L., Rey-Bellet, Fourier's law: A challenge to theorists, in *Mathematical Physics 2000*, A. Fokas, A. Grigorian, T. Kibble, and B. Zegarlinski, eds. (Imperial College Press, London, 2000), pp. 128-150.
- [6] Eckmann J.P., Pillet C.A., Rey-Bellet L., Non-equilibrium statistical mechanics of anharmonic chains coupled to two heat baths at different temperatures, *Commun. Math. Phys.* **201**, 657-697 (1999).
- [7] Eyink G., Lebowitz J.L., Spohn H., Hydrodynamics of stationary non-equilibrium states for some lattice gas models. *Comm. Math. Phys.* **132** 253-283 (1990).
- [8] Giardinà C., Kurchan J., Fourier law in a Momentum-conserving chain, preprint 2005.
- [9] Hormander, Lars, *The Analysis of Linear Partial Differential Operators III*, Springer 1985.
- [10] Fritz J., Nagy K., Olla S., Equilibrium fluctuations for a system of harmonic oscillators with conservative noise, preprint 2005.
- [11] Kipnis C., Marchioro C., Presutti E., Heat flow in an exactly solvable model, *J. Stat.Phys.* **27**, N.1, 65-74 (1982).
- [12] Kipnis C., Landim C., Olla S., Macroscopic Properties of a Stationary Non-Equilibrium Distribution for a Non-Gradient Interacting Particles System, *Ann. Inst. H. Poincaré, probabilités et statistiques*, **31**, n.1, 191-221, (1995).
- [13] Lepri, S., Livi, R., Politi, A., Thermal Conduction in classical low-dimensional lattices, *Phys. Rep.* **377**, 1-80 (2003).
- [14] Olla S., S.R.S. Varadhan, Scaling Limits for Interacting Ornstein-Uhlenbeck Processes, *Comm. Math. Phys.* **135**, 335-378, 1991.
- [15] Rieder, Z., Lebowitz, J.L., Lieb, E., Properties of harmonic cristal in a stationary non-equilibrium state, *J. Math. Phys.* **8**, 1073-1078 (1967).
- [16] Luc Rey-Bellet, Open Classical Systems, Lecture Notes of the 2003 Grenoble Summer School on Open Quantum Systems, http://www.math.umass.edu/~lr7q/ps_files/gren2.pdf

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