Conditional viability for Impulse Differential Games

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presented by

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1957... Isaacs writes *Differential Games*

The game of two cars...
The homicidal chauffeur...

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ \psi \end{pmatrix} = \mathbf{f}(\mathbf{x}, u, d) = \begin{pmatrix} -v + v \cos \psi + uy \\ v \sin \psi - ux \\ d - u \end{pmatrix}$$

'evader'
(control)

'pursuer'
(disturbance)

March 1971

2000... accurate reachability computations

- Sets are mathematically defined
- Algorithms are proved to converge
- Numerical implementation is accurate
- Validation against analytical solutions

2002... forty five years after...

- **1957:** Isaacs links differential games to the Hamilton-Jacobi equation.
- **1982:** Crandall, Evans and Lions discover the viscosity solution of Hamilton-Jacobi equation.
- **1991:** Aubin defines the viability kernel and capture basins mathematically.
- **1994:** First numerical computations of reachable sets: Saint-Pierre, Falcone.
- **1997:** Link between the static Hamilton-Jacobi equation and viability: Frankowska, Cardaliaguet, Quincampoix, Saint-Pierre.
- **2000:** Level set methods are used to compute accurately reachable sets: Tomlin, Sastry, Lygeros, Mitchell.
- **2002:** Hyperbolic Hamilton-Jacobi equation is linked to reachability and viability: Mitchell, Bayen, Tomlin.

But in the mean time: hybrid systems...

Systems with resets...

Systems which can switch dynamics...

- **Sets are mathematically defined..... 😊**
- **Algorithms are proved to converge..... ?**
- **Numerical implementation is accurate..... ?**
- **Validation against analytical solutions..... ?**

Outline

1. A short history of differential games

2. Impulse evolutionary systems

1. Runs

2. Impulse viability kernel

3. The swimmer problem

3. Impulse evolutionary games

1. Impulse conditional viability kernel

2. Impulse discriminating kernel

3. Tax pollution

4. Conclusion

Evolutionary systems

$$x \in X \mapsto \mathcal{S}(x) \subset \mathcal{C}(0, \infty; X)$$

Example: dynamical system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ x(0) = x \end{cases}$$

$$\begin{cases} \dot{x}(t) \in F(x(t)) = \{f(x(t), u) | u \in U\} \\ x(0) = x \end{cases}$$

Space X
Point x
Set of trajectories $\mathcal{S}(x)$

Evolutionary systems

$$x \in X \hookrightarrow \mathcal{S}(x) \subset \mathcal{C}(0, \infty; X)$$

Space X

Point x

Set of trajectories $\mathcal{S}(x)$

Definition:

$x \in X \hookrightarrow \mathcal{S}(x) \subset \mathcal{C}(0, \infty; X)$
is called an evolutionary system

Runs of impulse evolutionary systems

$$\vec{x}(\cdot) := (\tau_n, x_n, x_n(\cdot))_{n \geq 0} \in (\mathbf{R}_+ \times X)^{\mathbf{N}} \times \prod_{n=0}^{\infty} C(0, \tau_n; X)$$

Pieces of trajectories, with jumps

$\Phi : X \hookrightarrow X$ **Reset function**

Impulse viability kernel

Definition 2.1 Let K and $C \subset K$ be two subsets. The impulse viability kernel of K with target $C \subset K$

$$\text{ImpViab}(K, C) := \text{ImpViab}_{(\mathcal{S}, \Phi)}(K, C)$$

of K with target $C \subset K$ under the impulse evolutionary system $(\mathcal{S}, \Phi)$ is the subset of initial states $x \in K$ from which starts at least one run viable in K forever or until it reaches the target C in finite time.

Impulse viability kernel

Impulse viability kernel

$\text{ImpViab}_{(\mathcal{S}, \Phi)}(K, C)$

Points whose all trajectories exit K

➡ $C \subset \text{ImpViab}_{(\mathcal{S}, \Phi)}(K, C) \subset K$

➡ if $C_1 \subset C_2$ and $Q_1 \subset Q_2$

then $\text{ImpViab}_{(\mathcal{S}, \Phi)}(Q_1, C_1) \subset \text{ImpViab}_{(\mathcal{S}, \Phi)}(Q_2, C_2)$

Properties of the impulse viability kernel

$\text{ImpViab}_{(\mathcal{S}, \Phi)}(K, C)$ is:

1. *the largest subset D satisfying $C \subset D \subset K$ and $D \subset \text{ImpViab}_{(\mathcal{S}, \Phi)}(D, C)$,*
2. *the smallest subset D satisfying $C \subset D \subset K$ and $\text{ImpViab}_{(\mathcal{S}, \Phi)}(K, D) \subset D$,*
3. *the unique subset D satisfying $C \subset D \subset K$ and*

$$D = \text{ImpViab}_{(\mathcal{S}, \Phi)}(K, D) = \text{ImpViab}_{(\mathcal{S}, \Phi)}(D, C)$$

Example: the Zermelo problem

Example: the Zermelo problem

In this region, the swimmer can:

- 1) swim / jump / swim / jump ... forever
- 2) swim / jump / swim ... reach the island

Here, the swimmer goes into the waterfall (and dies)

$$\text{ImpViab}_{(\mathcal{S}, \Phi)}(K, C)$$

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1. A short history of differential games

2. Impulse evolutionary systems

- 1. Runs**
- 2. Impulse viability kernel**
- 3. The swimmer problem**

3. Impulse evolutionary games

- 1. Impulse conditional viability kernel**
- 2. Impulse discriminating kernel**
- 3. Tax pollution**

4. Conclusion

Evolutionary games

$$\boxed{G} : \boxed{(x, \tilde{v})} \in \boxed{X \times \tilde{Q}} \mapsto \boxed{\mathcal{S}_{\tilde{v}}(x)} \in \mathcal{C}(0, \infty; X)$$

evolutionary
game

maps a
point and a
feedback

from the
space
and the set
of feedbacks

to a set of
trajectories

Example: differential games

Differential system:

$$\begin{cases} x'(t) \in F(x(t), u(t), v(t)) \\ u(t) \in P(x(t)) \\ v(t) \in Q(x(t)) \end{cases}$$

Evolutionary game: $(x, \tilde{v}) \mapsto \mathcal{S}_{\tilde{v}}(x)$ where

$\mathcal{S}_{\tilde{v}}(x)$ is the set of solution of:

$$x'(t) \in F(x(t), P(x(t)), \tilde{v}(x(t)))$$

Conditional Impulse Viability Kernel

$$\text{CondImpViab}_{\mathcal{G}}(K, C) := \bigcap_{\tilde{v}} \text{ImpViab}_{\tilde{v}}(K, C)$$

For all feedback $\tilde{v}$

The set of points which stay in K forever, or reach C in final time whatever $\tilde{v}$ decides to play

Impulse discriminating Viability Kernel

For an evolutionary game $(\mathcal{S}, \Phi)$
And two subsets K and $C \subset K$.

The largest subset D of K containing C , conditionally viable in K outside C under the impulse evolutionary game $(x, \tilde{v}) \mapsto \mathcal{S}_{\tilde{v}}(x)$ is the impulse discriminating kernel of K with target C

$$\text{ImpDisc}(K, C)$$

Continuous example: tax-pollution

$$\begin{aligned} \dot{x} &= ux - p & u &\in [0.5, 1.2] \\ \dot{p} &= v & v &\in [-0.5, 0.5] \end{aligned} \quad (x, p) \in [0.2, 2] \times \mathbb{R}^+$$

Continuous example: tax-pollution

$$\begin{aligned} \dot{x} &= ux - p & u &\in [0.5, 1.2] & (x,p) &\in [0.2, 2] \times \mathbb{R}^+ \\ \dot{p} &= v & v &\in [-0.5, 0.5] \end{aligned}$$

[Bayen Tomlin, CDC 2001]

Properties of the Conditional Viability Kernel

Theorem *The conditional viability kernel $\text{CondImpViab}_{\mathcal{G}}(K, C)$ of a subset K with target $C \subset K$ is the smallest subset $D \in \mathcal{D}(K, C)$ satisfying*

$$\text{CondImpViab}_{\mathcal{G}}(K, D) \subset D$$

and actually is a fixed point

$$\text{CondImpViab}_{\mathcal{G}}(K, C) = \bigcap_{\tilde{v}} \text{ImpViab}_{\tilde{v}} \left(K, \text{CondImpViab}_{\mathcal{G}}(K, C) \right)$$

Unfortunately, the conditional impulse viability kernel $\text{CondImpViab}_{\mathcal{G}}(K, C)$ of K outside the target $C \subset K$ is not itself necessarily conditionally viable outside the target C , i.e., it is not a fixed point of the map $K \mapsto \text{CondImpViab}_{\mathcal{G}}(K, C)$.

Properties of the Discriminating Kernel

$\text{ImpDisc}(K, C)$ is the largest fixed point of the map
 $K \in \mathcal{D}(K, C) \mapsto \text{CondImpViab}_{\mathcal{G}}(K, C)$

$$\text{ImpDisc}(\text{ImpDisc}(K, C), C) = \text{ImpDisc}(K, C)$$

Proposition *Let us consider an evolutionary game $(x, \tilde{v}) \hookrightarrow \mathcal{S}_{\tilde{v}}(x)$*
Then $C \mapsto \text{ImpDisc}(K, C)$ is increasing
$$\text{ImpDisc}(K, \text{ImpDisc}(K, C)) = \text{ImpDisc}(K, C)$$

Conclusions

- **Viability theory introduces new concepts:**
Set valued dynamics, the viability kernel, the capture basin
They can be use to define reachable sets of evolutionary systems
- **The main characterization theorems**
Enables to prove the existence, maximality and uniqueness of the viability kernel, and to compute them.
- **The representation of these sets is algebraic**
- **Open research directions: algorithms, convergence**

Tomorrow's talk...

Fr-P07-5

5:40pm

For more information...

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