STRONG MAXIMUM PRINCIPLE FOR GENERAL QUASILINEAR PROBLEMS IN NON DIVERGENCE-FORM

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I - Introduction

In this paper we deal with the study of the following question : does a positive non null solution u of the problem

$$(1) \quad \begin{cases} - \sum_{ij} a_{ij}(x, u, \nabla u) \frac{\partial^2 u}{\partial x_i \partial x_j} \geq \mathcal{F}(x, u, \nabla u) \text{ in } \mathbb{R}^N \\ u(x) \geq 0 \quad \forall x, \end{cases}$$

vanish at some point $x_0 \in \mathbb{R}^N$? The answer depends on the behaviour of $\mathcal{F}(x, \eta, p)$ with respect to the behaviour of $\lambda_1(x, \eta, p)$ and $\lambda_N(x, \eta, p)$, respectively the lowest and highest eigenvalue of the matrix $a = (a_{ij}(x, \eta, p))_{ij}$. The answer depends also crucially on the behaviour of $\frac{\lambda_N(x, \eta, p)}{\lambda_1(x, \eta, p)}$. This allows us to consider some general interesting operators. Let us recall that in [PU1] and [PU2] the authors consider some particular operators in divergence form

depending on x and u : $-\sum_i \frac{\partial}{\partial x_i}(a_{ij}(x,u).A(|\nabla u|)\frac{\partial u}{\partial x_j})$. Contrary to the previous works the function $\eta \rightarrow \mathcal{F}(x, \eta, p)$ is not necessarily monotone and the operator coefficients a_{ij} depend on x and u . All the previous papers [VA], [PU], [FE], [MO], [DO] are devoted to quasilinear problems in divergence-form like, for instance,

$$(2) \quad -\operatorname{div}(\varphi(|\nabla u|)\nabla u) + h(u) = 0 \quad \text{in } \mathbb{R}^N,$$

where the function $\eta \rightarrow h(\eta)$ is monotone in some neighbourhood of 0 and is such that $h(0) = 0$. The first result of this kind seems to come from [VA] and [RE]. Roughly speaking, in [VA], [PU], [MO], [DO] the result is linked to the behaviour of some integral like, for instance,

$$(I) \quad \int_0^\delta \frac{ds}{\Phi^{-1}(|H(s)|)} \quad \text{for some real } \delta > 0$$

where

$$H(t) = \int_0^t h(s)ds, \quad \Phi(t) = t^2\varphi(t) - \int_0^t s\varphi(s)ds.$$

Let us recall briefly the main result in the case (2).

If $\int_0^\delta \frac{ds}{\Phi^{-1}(|H(s)|)} = +\infty$ the strong maximum principle holds for (2). And, if $\int_0^\delta \frac{ds}{\Phi^{-1}(|H(s)|)} < +\infty$, any non negative solution of (2), which approaches 0 at infinity, must vanish for all sufficiently large $|x|$. The integral like (I) was first introduced by [VA]. For equation involving a gradient term like, for instance

$$(3) \quad -\operatorname{div}(|\nabla u|^m \cdot \nabla u) + G(|\nabla u|) + h(u) = 0,$$

the previous integral condition does not seem to be appropriate, as it is showed in [FE], where systems are also studied - (for systems cf also [BU]). For (3), instead of the integral condition, the authors use the multiplicity or the uniqueness of solution of an ordinary differential equation associated to (3). In [MO] the previous problem is studied in Orlicz-Sobolev space, using the theory of monotone operators. A precise study in the radial case is given in [DO]. As remarked in [DO], for radial solution u with compact support in $\mathbb{R}^N$, it is not necessary to assume that u is positive : in this work the authors consider nodal radial solutions for some quasilinear elliptic

equations involving p -Laplacian. This remark is still valid in non radial cases. For an exhaustive review and some improvements about this area we can see [PU2]. Finally let us point out that our framework is completely different from the ones of the previous papers, and in the regular case - ($u \in C^2$) - we rediscover the results for operators of divergence form.

Our paper is structured as follows. In a first part we prove theorem 1 giving the strong maximum principle result in the case where u is of class C^2 . As the notion of viscosity solution [CR] is well adapted to the existence theory of operators like $-\sum_{ij} a_{ij}(x, u, \nabla u) \frac{\partial^2 u}{\partial x_i \partial x_j}$, we show that our result is also valid if u is a viscosity solution of class C^1 . The second part is devoted to the study of compact support principle. In a first time we prove theorem 2 for u of class C^2 . And next we show that this result remains valid if u is a viscosity solution of class C^1 . Also we give, for some class of right hand side $\mathcal{F}(\cdot, \cdot, \cdot)$, a necessary and sufficient condition ensuring the strong maximum principle. This is the same kind of condition as the one used in divergence form operators. And finally in our framework, for simplicity, we choose $\Omega = \mathbb{R}^N$. But actually our results are valid for $\Omega \subseteq \mathbb{R}^N \setminus \overline{B(x_0, R)}$ where $B(x_0, R)$ is some ball. In this case we use the same idea as in the previous works, for instance [PU2].

II - Setting of the problem, Notations and Hypothesis.

We consider inequality (1) in the following sense : we assume f and $a = (a_{ij})_{ij}$ as regular as necessary and in a first time, we suppose that u is C^2 . In a second time we assume that u is a viscosity solution of class C^1 . Our goal is to give some sufficient conditions ensuring the strong maximum principle or the so called compact support principle [PU1] for u ; that is to say if u is a non negative solution of (1), then $u(x) > 0$ for any $x \in \mathbb{R}^N$. Also we will study the case when the solution u has a compact support in $\mathbb{R}^N$.

A - The strong maximum principle. Let us consider a function $f : \mathbb{R}_x^N \times \mathbb{R}_\eta \times \mathbb{R}_s \times \mathbb{R}_p^N \rightarrow \mathbb{R}$ sufficiently regular, satisfying $f(x, \eta, o, p) = 0 \quad \forall (x, \eta, p)$ and such that

$$(\mathcal{H}1) \quad \begin{cases} \text{for any } (x, \eta, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \text{ the map} \\ s \rightarrow f(x, \eta, s, p) \text{ is non increasing.} \end{cases}$$

The function giving the right hand side of (1) is defined from f as follows :

$$(\mathcal{H}2) \quad \mathcal{F}(x, s, p) = f(x, s, s, p) \quad \forall (x, s, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N.$$

It is clear that for any (x, p) the map $s \rightarrow \mathcal{F}(x, s, p)$ is not necessarily monotone. About f we assume the following hypothesis.

$$\mathbf{H1} : \frac{-f^-(x, \eta, s, p)}{\lambda_1(x, \eta, p)} \geq g(s) \quad \forall s \in [0, b]$$

for any (x, η, p) belonging to $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ and where $\lambda_1(x, \eta, p)$ is the first eigenvalue of the symmetric matrix $a = (a_{ij})_{ij}$; f^- denotes the negative part of f . The matrix a is in $L^\infty(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N)$ and is assumed to be symmetric non negative.

H'1 : The function $g(\cdot)$ is continuous, decreasing on $[0, b]$ and such that $g(0) = 0$.

$$\mathbf{H2} : \frac{Tr a(x, \eta, p)}{\lambda_1(x, \eta, p)} - 1 \leq \beta(|p|),$$

for any $(x, \eta, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$, the positive continuous function β having the following behaviour :

$$\mathbf{H3} : \begin{cases} \beta(t) \leq At + B & \forall t \geq t_0, \\ A > 0, \quad \text{and } B > 0 & \text{for simplicity.} \end{cases}$$

$$\mathbf{H4} : \begin{cases} \beta(t) \leq c_\beta \text{Log } \frac{1}{t} & \forall t \in]0, t_0], \\ \beta & \text{is non increasing in }]0, t_0] \end{cases}$$

Remarks :

i) we have $\beta(|p|) \geq N - 1, \quad \forall p \in \mathbb{R}^N$

ii) we can replace H2 by the simpler hypothesis

$$\mathbf{H'2} : \frac{N\lambda_N(x, \eta, p)}{\lambda_1(x, \eta, p)} - 1 \leq \beta(|p|).$$

The hypothesis H'2 shows that $\beta(\cdot)$ controls the behaviour of $\frac{\lambda_N(x, \eta, p)}{\lambda_1(x, \eta, p)}$ as $|p|$ goes to zero and to infinity.

iii) we shall see later -(cf. remark 3.1) - that (H1) can be replaced by the following :

$$\mathbf{H''1} : \quad -\frac{f^-(x, \eta, s, p)}{\lambda_1(x, \eta, p)} \geq \frac{g(s)}{\gamma(|p|)} \forall s \in [0, b], \forall x, \forall \eta, \forall p \in \mathbb{R}^N, \text{ with } \gamma(|p|) \geq \gamma_0 > 0 \quad \forall p \in \mathbb{R}^N.$$

H5 : there exists $\delta > 0$ such that

$$\lim_{\lambda \rightarrow 0^+} \frac{g(\lambda)}{\lambda^{1+\delta}} = 0$$

For technical reasons let us introduce the following function

$$\tilde{g}(t) = \begin{cases} g(t) & \forall t \in [0, b], \\ g(b) & \forall t \geq b, \\ g(-t) & \forall t \in [-b, 0], \\ g(-b) & \forall t, t \leq -b. \end{cases}$$

For any λ , $0 < \lambda < b$ let us introduce the two following constants

$$k_0(\lambda, a, T) = k_0(\lambda) = \min \left[\frac{\lambda}{T-a}, -g(\lambda) \cdot (T-a) \right] = -g(\lambda) \cdot (T-a),$$

by H5 for λ small enough.

$$k_1(\lambda, a, T) = k_1(\lambda) = \frac{e^{Q(a, T, \beta, \lambda)}}{(T-a)} \cdot [\lambda + (T-a)^2(-g(\lambda))e^{Q(a, T, \beta, \lambda)}]$$

where $0 < a < T$, $Q(a, T, \beta, \lambda) = \frac{\beta}{a}(k_0(\lambda)) \cdot (T-a) + \frac{A}{a} + \frac{B}{a}(T-a)$.

We will see later - (cf Remark 2) - the meaning of $k_0(\lambda)$ and $k_1(\lambda)$. Let us give some examples that we can handle in our framework.

Example 1 : In the following particular case

$$\begin{aligned} a_{ij}(x, s, p) &= \delta_{ij} \\ f(x, \eta, s, p) &= g(s) \end{aligned}$$

the hypothesis (H5) is very close to the integral condition

$$\int_0^1 \frac{ds}{\sqrt{-G(s)}} = +\infty$$

since $\sqrt{-G(s)} = \left[-\int_0^s g(t)dt \right]^{1/2} \approx cs^{1+\theta} - (\theta > 0)$ – in a neighbourhood of 0. This allows us to rediscover the previous results in the divergence form case.

Example 2 :

$$\begin{aligned} Au = & (1 + |\nabla u|^\theta + |\nabla u|^{-\theta}) \frac{\partial^2 u}{\partial x^2} + 2(1 + |\nabla u|^\theta + |\nabla u|^{-\theta}) \frac{\partial^2 u}{\partial y^2} \\ & + (1 + |\nabla u|^\theta + |\nabla u|^{-\theta}) \max[2, \text{Log} \frac{1}{|\nabla u|}] \frac{\partial^2 u}{\partial z^2} \end{aligned}$$

$$f(x, \eta, s, p) = f(x, s, p) \quad \forall (x, \eta, s, p)$$

$$f(x, s, p) = g(s).$$

$$\theta > 0, g(0) = 0, g(s) \leq 0 \quad \forall s \in [0, 1]$$

$$\frac{\lambda_1(p) + \lambda_2(p) + \lambda_3(p)}{\lambda_1(p)} - 1 = 2 + \max[2, \text{Log} \frac{1}{|p|}] = \beta(|p|)$$

$$\frac{f(x, s, p)}{\lambda_1(p)} = \frac{g(s)}{1 + |p|^\theta + |p|^{-\theta}} \geq g(s) \quad \forall s \in [0, 1] \text{ since } g(s) \leq 0 \quad \forall s \in [0, 1].$$

□

Let us point out that if we have for instance

$$\begin{cases} -Au = g(u) & \text{in } \mathbb{R}^3, \\ u(x) \geq 0 & \forall x \in \mathbb{R}^3, \end{cases} \quad \lim_{|x| \rightarrow +\infty} u(x) = 0$$

and if there exists x_0 such that $u(x_0) = 0$, then $\nabla u(x_0) = 0$. In this case

$$\frac{\lambda_1(|\nabla u|) + \lambda_2(|\nabla u|) + \lambda_3(|\nabla u|)}{\lambda_1(|\nabla u|)} - 1 = \beta(|\nabla u|)$$

is not bounded in a neighbourhood of x_0 . This point procures difficulties to state our main result.

Example 3.

Let us consider a regular increasing function

$$\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$$

verifying the following assumption :

$$\forall \lambda > 0 \quad \text{we have} \quad \sup_{t \geq 0} \frac{\varphi(\lambda t)}{\varphi(t)} < +\infty.$$

This hypothesis is satisfied if we have for instance :

$$\lim_{t \rightarrow 0^+} \frac{\varphi(t)}{t^{\theta_1}} = \delta_1 \geq 0, \quad \theta_1 > 0$$

and

$$\lim_{t \rightarrow +\infty} \frac{\varphi(t)}{t^{\theta_2}} = \delta_2 \quad \text{with} \quad \delta_2 > 0 \text{ if } \theta_2 = 0 \text{ and } \delta_2 \geq 0 \text{ if } \theta_2 > 0.$$

Let us consider a function $a : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^+$ such that $0 < a_0 \leq a(x, s) \leq 1 \quad \forall (x, s)$, where a_0 is a positive real number, and a function $h : \mathbb{R} \rightarrow \mathbb{R}$, decreasing on $[0, 1]$ and satisfying $h(0) = 0$. Let $b(\cdot)$ a bounded function defined on $\mathbb{R}^3$ to $\mathbb{R}^+ : |b(x)| \leq \|b\|_\infty \quad \forall x$. The operator that we consider is of the form :

$$\begin{aligned} & -a(x, u) \cdot \varphi(|\nabla u|) \frac{\partial^2 u}{\partial x^2} - \varphi(2|\nabla u|) \frac{\partial^2 u}{\partial y^2} \\ & - \varphi(3|\nabla u|) \cdot \left[\max(1, c_0 \text{Log} \frac{1}{|\nabla u|}) + |\nabla u| \right] \frac{\partial^2 u}{\partial z^2} \\ & = b(x) \cdot h(u) \cdot \varphi(4|\nabla u|) \quad \text{in } \mathbb{R}^3, \text{ with } h(s) \leq 0 \quad \forall s \in [0, 1], \quad h(0) = 0. \end{aligned}$$

We have

$$\frac{b(x)h(s)\varphi(4|p|)}{a(x, s)\varphi(|p|)} \geq \frac{\|b\|_\infty}{a_0} \Phi_{4,\infty} \cdot h(s) = g(s), \quad \forall s \in [0, 1] \text{ with } g(s) \leq 0, \quad \forall s \in [0, 1]$$

$$\begin{aligned} & 3 \cdot \frac{\varphi(3|p|) \cdot \left[\max(1, c_0 \text{Log} \frac{1}{|p|}) + |p| \right]}{a(x, s)\varphi(|p|)} - 1 \leq \\ & 3 \frac{\Phi_{3,\infty}}{a_0} \cdot \left[\max(1, c_0 \text{Log} \frac{1}{|p|}) + |p| \right] - 1 = \beta(|p|) \end{aligned}$$

where, $\lambda > 0$ and $\Phi_{\lambda,\infty} = \sup_{t \geq 0} \frac{\varphi(\lambda t)}{\varphi(t)}$ which is finite from the assumptions made on φ .

Example 4.

We can treat some operators of divergence form in the case $u \in c^2$.
 $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g(s) \leq 0 \quad \forall s \in [0, 1]$
 $g(0) = 0$, g decreasing in $[0, 1]$.

$\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $t \rightarrow \frac{\varphi(t)}{t^2}$ is bounded in $[0, +\infty[$.

Let u be a function belonging to $c^2(\mathbb{R}^2)$ which is a solution of

$$\begin{cases} -\operatorname{div}(|\nabla u|^2 \cdot \nabla u) = \varphi(|\nabla u|) \cdot g(u) & \text{in } \mathbb{R}^2 \\ u(x) \geq 0 \quad \forall x \in \mathbb{R}^2, \quad \lim_{|x| \rightarrow +\infty} u(x) = 0. \end{cases}$$

This equation can be written

$$-\sum_{ij=1}^2 a_{ij}(|\nabla u|) \frac{\partial^2 u}{\partial x_i \partial x_j} = \varphi(|\nabla u|) \cdot g(u) = f(u, \nabla u)$$

where

$$\begin{aligned} a_{11}(|p|) &= |p|^2 + 2p_1^2, & a_{22}(|p|) &= |p|^2 + 2p_2^2 \\ a_{12}(p) &= a_{21}(p) = 2p_1 p_2 \\ \lambda_1(|p|) &= |p|^2, & \lambda_2 &= 3|p|^2, & 2 \frac{\lambda_2(|p|)}{\lambda_1(|p|)} - 1 &= \beta(|p|) = 5 \end{aligned}$$

with $p = (p_1, p_2)$ and $\frac{f(s, p)}{\lambda_1(|p|)} \geq c_0 g(s)$ with $c_0 = \sup_{t \geq 0} \frac{\varphi(t)}{t^2}$.

It is easy to see that we are in the setting of our work.

Example 5. A degenerate problem in divergence form.

$$\begin{aligned} ||p||^\delta &= (p_1^2 + p_2^2)^{\frac{\delta}{2}}, \quad p = (p_1, p_2), \quad \delta > 0 \\ -\operatorname{div}(|\nabla u|^\delta \cdot \nabla u) &= f(x, u, \nabla u) = g(u) \cdot h(x, \nabla u). \end{aligned}$$

We suppose $u \in c^2(\mathbb{R}^2)$. After some elementary computations we see that u satisfies

$$\begin{aligned} &-a_{11}(\nabla u) \frac{\partial^2 u}{\partial x^2} - 2a_{12}(\nabla u) \frac{\partial^2 u}{\partial x \partial y} - a_{22}(\nabla u) \frac{\partial^2 u}{\partial y^2} \\ &= f(x, u, \nabla u), \quad \text{with } a_{11}(p) = \delta p_1^2 \cdot ||p||^{\delta-2} + ||p||^\delta, \end{aligned}$$

$$a_{22}(p) = \delta p_2^2 ||p||^{\delta-2} + ||p||^\delta, \quad a_{12} = a_{21} = 2\delta p_1 p_2 ||p||^{\delta-2},$$

and we have $\lambda_2(p) = (1 + \delta)||p||^\delta$, $\lambda_1(p) = ||p||^\delta$

$$\beta(|p|) = 2 \frac{\lambda_2(p)}{\lambda_1(p)} - 1 = 2\delta + 1$$

$$\frac{f(x, s, p)}{\lambda_1(p)} = \frac{f(s, s, p)}{||p||^\delta} = g(s) \cdot h(x, p)$$

with $h(x, p) \geq 0$, $g(s) \leq 0$ on $[0, a]$ and $\max_{(x,p)} \frac{h(x, p)}{||p||^\delta} \leq k$, some constant. For instance we can take the particular function $h(x, p) = \text{Log}(1 + ||p||^\delta \cdot a(x))$ with $a(x) \geq a_0 > 0$.

Example 6. Let γ be a continuous function $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\gamma(0) = 0$, $\lim_{t \rightarrow +\infty} \gamma(t) = 0$. Let β be a function $\mathbb{R}^+ \rightarrow \mathbb{R}^+$, continuous such that $\beta(t) \geq 2$. Let us set $a_{11}(p) = \gamma(|p|) \cdot \beta(|p|) - \alpha$, $a_{22}(p) = \gamma(|p|) + \alpha$ with $\alpha = \alpha(x, s, p) = \frac{\gamma(|p|)}{1 + \gamma(|p|) + a(x, s, p)}$ and $a(x, s, p) \geq 0 \quad \forall (x, s, p)$, $a_{12} = a_{21} = \{\alpha \cdot [\gamma(\beta - 1) - \alpha]\}^{1/2}$. After some computations we obtain that $\lambda_2(x, s, p) = \gamma(|p|) \cdot \beta(|p|)$, $\lambda_1(x, s, p) = \gamma(|p|)$. Let us suppose that we have $f(x, s, p) = g(s) \cdot h(x, p)$, with $\sup_{(x,p)} \frac{h(x, p)}{\gamma(|p|)} = k > 0 \quad g(s) \leq 0 \quad \forall s \in [0, 1]$, $h(x, p) \geq 0 \quad \forall (x, p)$.

Our assumptions appear to be natural. indeed in the case of divergence form equation - (see examples 1, 4, 5) - we easely check that our hypothesis become equivalent to the standard ones ([VA], [PU1] for instance).

To prove the strong maximum principle for (1), the main idea, used in all the previous works, is to construct local sub or super solution of (1). Roughly speaking this means that we have to compare, in some sense, the operator $A = \sum_{ij} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ with a radial one, for instance. This idea, already used in [TA1, TA2, TA3, TA4], reduces the search of a subsolution of (1) to the existence of a solution for an ordinary differential equation. This method is systematic.

III - Preliminary results : an ordinary differential equation associated to (1). To establish a “positive local sub solution” of (1), we will see that it is enough to prove the existence of a solution for the following ordinary differential equation :

$$(4) \quad \begin{cases} -\omega'' - \frac{\beta}{a}(|\omega'|) \cdot \omega' = \tilde{g}(\omega), & \text{in }]a, T[, \\ \omega(a) = \lambda, \quad \omega(T) = 0, \quad \omega'(t) < 0, \quad \omega''(t) \geq 0, \\ \omega \in C^2([a, T]), \end{cases}$$

where the parameters λ, a, T are choosen in a way somewhat arbitrary. This choice is very crucial to prove the existence result for (4). Let us point out that many of the previous papers use an ordinary differential equation which is somewhat similar to (4). And one of the main differences between our framework and the frameworks of [PU], [FE], [MO] is the presence of the function $\beta(\cdot)$ and its behaviour given by hypothesis H4. We can compare, for instance, with [PU1] Lemma 2, p. 774, [FE] equation (1.10) p. 27, [MO] lemma 8 p. 441.

Remark 2 : Let us consider (4). $k_0(\lambda)$ is an -(a priori) - estimate from below to the derivative $-\omega'(t)$ and $k_1(\lambda)$ is its - (a priori) - estimate from above. We shall see this result in remark 3. We need these estimates to establish the existence of a solution of (4).

For any $\lambda, 0 < \lambda < \lambda_0 \leq b$, let us consider the functional set

$$W_\lambda = \{v \in C^1([a, T]) / 0 \leq v(t) \leq \lambda, \ v \text{ convex decreasing} \\ \text{such that } v(a) = \lambda, \ k_0(\lambda) \leq -v'(t)\}$$

Proposition 1. *Let a, T be two positive real numbers such that*

$$(4.1) \quad 0 < a < T, \quad T \leq a \left[1 + \frac{1}{c_\beta} \left(1 - \frac{1}{1 + \delta} \right) \right].$$

Assume hypothesis H'1 and H6. Then we have the following estimate : For any $\lambda, 0 < \lambda \leq \min(b, \lambda_0) \leq 1$ and for any $v \in W_\lambda$

$$(5) \quad I(\lambda) \leq K(a, T, \beta) \cdot \left[-g(\lambda) \right]^{1-(T-a)\frac{c_\beta}{a}}$$

with

$$I(\lambda) = \int_a^T e^{-\int_a^s \frac{\beta}{a}(-v'(\theta))d\theta} ds \cdot \int_a^T e^{\int_a^T \frac{\beta}{a}(-v'(\theta))d\theta} (-g(\lambda)) ds$$

and where $K(a, T, \beta)$ is a constant independent of λ .

Proof. We set $\tilde{\beta}(t) = \frac{\beta(t)}{a}$. For any $v \in W_\lambda$, let us denote by $[\delta_1 \leq -v' \leq \delta_2]$ the subset $\{\theta \in [a, T] \mid \delta_1 \leq -v'(\theta) \leq \delta_2\}$ of $[a, T]$. It is clear that

$$(6) \quad \int_a^T e^{-\int_a^s \tilde{\beta}(-v'(\theta))d\theta} ds \leq (T - a)$$

We have

$$\int_a^T \tilde{\beta}(-v'(\theta))d\theta = \int_{[k_0(\lambda) \leq -v' < t_0]} \tilde{\beta}(-v')d\theta + \int_{[-v' \geq t_0]} \tilde{\beta}(-v')d\theta$$

because $-v'(\theta) \geq k_0(\lambda) \quad \forall \theta \in [a, T]$. And since $\beta(\cdot)$ is decreasing in $]0, t_0]$ and from (H3) it follows

$$\int_a^T \tilde{\beta}(-v')d\theta \leq \tilde{\beta}(k_0(\lambda)) \cdot (T - a) + \int_a^T \left[\frac{A}{a}(-v') + \frac{B}{a} \right] d\theta$$

i.e.

$$(7) \quad \int_a^T \tilde{\beta}(-v')d\theta \leq \tilde{\beta}(k_0(\lambda)) \cdot (T - a) + \frac{A}{a} + \frac{B}{a}(T - a)$$

since $\lambda_0 < 1$. From (6) and (7) we obtain

$$\begin{aligned} I(\lambda) &\leq \int_a^T e^{-\int_a^s \tilde{\beta}(-v')d\theta} ds \cdot \int_a^T e^{\int_a^T \tilde{\beta}(-v')d\theta} \cdot (-g(\lambda)) ds \\ &\leq -g(\lambda) \cdot (T - a)^2 \cdot e^{(T-a)\tilde{\beta}(k_0(\lambda))} \cdot e^{\frac{A}{a} + \frac{B}{a}(T-a)}. \end{aligned}$$

From (H4) we obtain

$$I(\lambda) \leq -g(\lambda) \cdot (T - a)^2 e^{\frac{A}{a} + \frac{B}{a}(T-a)} \cdot \frac{1}{[k_0(\lambda)]^{(T-a)\frac{c\beta}{a}}}$$

which can be written

$$I(\lambda) \leq K(a, T, \beta) \left[-g(\lambda) \right]^{1-(T-a)\frac{c\beta}{a}}$$

with $K(a, T, \beta) = \frac{(T-a)^2 e^{\frac{A}{a} + \frac{B}{a}(T-a)}}{(T-a)^{(T-a)\frac{c\beta}{a}}}$.

□

Proposition 2. *Let λ, a, T be three real numbers verifying (4.1). Then for any $v \in W_\lambda$ there exists a unique solution $\omega = \omega_v$ of*

$$\begin{cases} -\omega'' - \frac{\beta}{a}(|v'|)\omega' = \tilde{g}(v) & \text{in }]a, T[, \\ \omega(a) = \lambda, \quad \omega(T) = 0, \quad -\omega'(t) > k_0(\lambda), \quad \omega''(t) \geq 0. \end{cases}$$

In addition $\omega \in C^2([a, T])$.

Proof

i) uniqueness. It suffices to establish that $z = 0$ is the unique solution of

$$(8) \quad \begin{cases} -z'' - \frac{\beta}{a}(|v'|)z' = 0, \\ z(a) = z(T) = 0. \end{cases}$$

From (8) we have $z'(t)e^{\int_a^t \frac{\beta}{a}(-v')d\theta} = c_0$, some constant. Thus

$$0 = z(T) - z(a) = \int_a^T z'(s)ds$$

which entails $c_0 = 0$. And then $z = 0$.

ii) existence. We proceed in two steps and use successively the shooting method by introducing the parameter $\omega'_\alpha(a) = -\alpha$, $\alpha > 0$ - (first step) - and a continuity argument of the map $\alpha \rightarrow \omega_\alpha(T)$ - (second step) - .

Step 1 : for $\alpha > k_0(\lambda)$, let us consider $\omega = \omega_\alpha$ the unique solution of

$$(9) \quad \begin{cases} -\omega'' - \frac{\beta}{a}(|v'|)\omega' = \tilde{g}(v) & \text{in }]a, T[, \\ \omega(a) = \lambda, \quad \omega'(a) = -\alpha \end{cases}$$

which has the following representation formula

$$(10) \quad \begin{aligned} \omega_\alpha(t) &= \lambda - \alpha \int_a^t e^{-\int_a^s \tilde{\beta}(\theta)d\theta} ds \\ &+ \int_a^t e^{-\int_a^s \tilde{\beta}(\theta)d\theta} \cdot \left[- \int_a^s e^{\int_a^\sigma \tilde{\beta}(\theta)d\theta} g(v(\sigma))d\sigma \right] ds \end{aligned}$$

where we denote by $\tilde{\beta}(\theta)$ the term $\tilde{\beta}(-v'(\theta)) = \frac{\beta}{a}(-v'(\theta))$. Also we have

$$(11) \quad \omega'_\alpha(t) = -\alpha e^{-\int_a^t \tilde{\beta}(\theta) d\theta} + e^{-\int_a^t \tilde{\beta}(\theta) d\theta} \cdot \left[-\int_a^t e^{\int_a^\sigma \tilde{\beta}(\theta) d\theta} g(v(\sigma)) d\sigma \right].$$

Taking

$$(12) \quad \alpha > -g(\lambda) \cdot \int_a^T e^{\int_a^s \tilde{\beta}(\theta) d\theta} ds,$$

we see that $\omega'_\alpha(t) < 0 \quad \forall t \in]a, T[$. From (10) the map $\alpha \in \mathbb{R}^{+,*} \rightarrow \omega_\alpha(T) \in \mathbb{R}$ is continuous and for α_1 large enough we have $\omega_{\alpha_1}(T) < 0$ and (12). To prove that there exists $\bar{\alpha}$ satisfying (12) and $\omega_{\bar{\alpha}}(T) = 0$, it suffices to establish that there exists some α_2 verifying (12) and $\omega_{\alpha_2}(T) > 0$. This means that such a real number α_2 verifies simultaneously (12) and

$$(13) \quad \alpha \cdot \int_a^T e^{-\int_a^s \tilde{\beta}(\theta) d\theta} ds < \lambda + \int_a^T e^{-\int_a^s \tilde{\beta}(\theta) d\theta} \cdot \left[-\int_a^s e^{\int_a^\sigma \tilde{\beta}(\theta) d\theta} g(v(\sigma)) d\sigma \right] ds.$$

From (5) it follows

$$\frac{I(\lambda)}{\lambda} \leq K(a, T, \beta) \left[\frac{-g(\lambda)}{\lambda^\nu} \right]^{1-(T-a)\frac{c_\beta}{a}}$$

with $\nu = 1/(1 - (T-a)\frac{c_\beta}{\alpha})$ or again

$$(14) \quad \frac{I(\lambda)}{\lambda} \leq K(a, T, \beta) \left[\frac{-g(\lambda)}{\lambda^{1+\delta}} \right]^{1-(T-a)\frac{c_\beta}{a}}$$

because a and T are such that $\nu < 1 + \delta$ since from (4.1) we have

$$\frac{T}{a} - 1 = \frac{1}{a}(T-a) \leq \frac{1}{c_\beta} \left(1 - \frac{1}{1+\delta}\right) < \frac{1}{c_\beta} \quad \text{and} \quad \lambda_0 < 1.$$

Consequently, reducing λ_0 if necessary, we obtain from (14) and H6 :

$$(15) \quad \frac{I(\lambda)}{\lambda} \leq K(a, T, \beta) \left[\frac{-g(\lambda)}{\lambda^{1+\delta}} \right]^{1-(T-a)\frac{c_\beta}{a}} < 1, \quad \forall \lambda, \quad 0 < \lambda < \lambda_0.$$

The relations (14) and (15) entail

$$\frac{I(\lambda)}{\lambda} = \frac{-g(\lambda)}{\lambda} \int_a^T e^{\int_a^s \tilde{\beta}(\theta) d\theta} ds \cdot \int_a^T e^{-\int_a^s \tilde{\beta}(\theta) d\theta} ds$$

$$< 1 + \frac{1}{\lambda} \int_a^T e^{-\int_a^s \tilde{\beta}(\theta) d\theta} \left[- \int_a^s e^{\int_a^\sigma \tilde{\beta}(\theta) d\theta} g(v(\sigma)) d\sigma \right] ds;$$

From this there exists some real number α_2 such that (12) and (13) hold simultaneously ; i.e. $\alpha_2 > -g(\lambda) \int_a^T e^{\int_a^s \tilde{\beta}(\theta) d\theta} ds$ and $\omega_{\alpha_2}(T) > 0$. Consequently there exists $\bar{\alpha} \in]\alpha_2 \alpha_1[$ such that $\omega_{\bar{\alpha}}(T) = 0$. And since $\omega_{\bar{\alpha}}(T) = 0$, we have from (10)

$$\bar{\alpha} \cdot (T - a) \geq \bar{\alpha} \cdot \int_a^T e^{-\int_a^s \tilde{\beta}(\theta) d\theta} ds \geq \lambda$$

i.e.

$$\bar{\alpha} \geq \frac{\lambda}{T - a} \geq k_0(\lambda).$$

Since $v \in c^1([a, T])$, from (11) it follows that $\omega_{\bar{\alpha}} \in c^1([a, T])$. It remains to prove that $\omega_{\bar{\alpha}}$ is convex, decreasing. It is clear from (11) that the function $t \rightarrow \pi(t) = e^{\int_a^t \tilde{\beta}(\theta) d\theta} \cdot \omega'_{\bar{\alpha}}(t) = -\bar{\alpha} + \int_a^t -e^{\int_a^\sigma \tilde{\beta}(\theta) d\theta} g(v(\sigma)) d\sigma$ is increasing. From the previous results there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such :

$$\alpha_1 > \alpha_2 > -g(\lambda) \int_a^T e^{\int_a^s \tilde{\beta}(\theta) d\theta} ds.$$

Thus $\bar{\alpha} > -g(\lambda) \int_a^T e^{\int_a^s \tilde{\beta}(\theta) d\theta} ds$, which implies that $\pi(T) < 0$. This means that $\omega'_{\bar{\alpha}}(t) < 0 \quad \forall t$. The decrease of $\omega_{\bar{\alpha}}$ and the equation

$$\omega''_{\bar{\alpha}}(t) = -\tilde{g}(v(t)) - \frac{\beta}{a} (|v'(t)|) \cdot \omega'_{\bar{\alpha}}(t)$$

prove that we have $\omega''_{\bar{\alpha}}(t) \geq 0 \quad \forall t$ i.e. $\omega_{\bar{\alpha}}$ is decreasing convex. □

Comments about hypothesis H5

In the proof of the proposition 2 we have used H5 to establish that there exists some admissible real $\bar{\alpha}$ such that $\omega_{\bar{\alpha}}(T) = 0$, where $\omega_{\bar{\alpha}}(\cdot)$ is given by (10). Let us consider (11) and (13). Throughout this proof we see easily that the optimal condition to prove the existence of $\bar{\alpha}$ is the following : for any

λ , $0 < \lambda < \lambda_0$ and any $v \in W_\lambda$ we have

$$\begin{aligned}
(15.1) \quad & - \int_a^T e^{\int_a^\sigma \frac{\beta}{a}(-v'(\theta))d\theta} g(v(\sigma))d\sigma \cdot \int_a^T e^{-\int_a^s \frac{\beta}{a}(-v'(0))d\theta} ds \\
& < \lambda + \int_a^T e^{\int_a^s \frac{\beta}{a}(-v'(\theta))d\theta} \left[- \int_a^s e^{\int_a^\sigma \frac{\beta}{a}(-v'(\theta))d\theta} g(v(\sigma))d\sigma \right] ds .
\end{aligned}$$

For instance, from (15.1) we see that we can replace (H5) by the more satisfactory hypothesis :

$$\lim_{\lambda \rightarrow 0^+} \frac{\Lambda_{a,T}(\lambda)}{\lambda} = 0$$

with

$$\Lambda_{a,T}(\lambda) = \sup_{v \in W_\lambda} \int_a^T e^{\int_a^\sigma \frac{\beta}{a}(-v'(\theta))d\theta} - g(v(\sigma))d\sigma.$$

□

Now let us consider the convex subset of $c^1([a, T])$:

$$\widetilde{W}_\lambda = \left\{ v \in W_\lambda \mid v(T) = 0, \quad -v'(t) \leq k_1(\lambda) \right\}$$

and the map

$$\Gamma : \widetilde{W}_\lambda \longrightarrow \widetilde{W}_\lambda, \quad v \longmapsto \omega_v = \Gamma(v) \quad \text{the unique solution of}$$

$$(16) \quad \begin{cases} -\omega'' - \frac{\beta}{a}(|v'|) \cdot \omega' = \tilde{g}(v) & \text{in }]a, T[, \\ \omega(a) = \lambda, \quad \omega(T) = 0, \quad -\omega'(t) > k_0(\lambda), \quad \omega''(t) \geq 0, \end{cases}$$

for any λ , $0 < \lambda < \lambda_0$.

Remark 3 : As mentioned previously in remark 2, $k_0(\lambda)$ and $k_1(\lambda)$ are a priori estimates of the derivative of the solution $\omega_v = \Gamma(v)$. It is clear for $k_0(\lambda)$. For $k_1(\lambda)$ we proceed from (10) and as in proposition 1 :

$$\bar{\alpha} \cdot \int_a^T e^{-\int_a^s \tilde{\beta}(\theta)d\theta} ds = \lambda + \int_a^T e^{-\int_a^s \tilde{\beta}(\theta)d\theta} \times \left[- \int_a^s e^{\int_a^\sigma \tilde{\beta}(\theta)d\theta} g(v(\sigma))d\sigma \right] ds,$$

or

$$\bar{\alpha} \cdot \int_a^T e^{-\int_a^T \tilde{\beta}(\theta)d\theta} ds \leq \lambda + \int_a^T \left[\int_a^T e^{\int_a^\sigma \tilde{\beta}(\theta)d\theta} \cdot (-g(\lambda))d\sigma \right] ds,$$

or again

$$\bar{\alpha} \cdot \int_a^T e^{-\int_a^T \tilde{\beta}(\theta) d\theta} ds \leq \lambda + (T-a)^2 \cdot (-g(\lambda)) e^{\int_a^T \tilde{\beta}(\theta) d\theta}.$$

But from (7) we have

$$\begin{aligned} \int_a^T \frac{\beta}{a} (-v'(\theta)) d\theta &= \int_a^T \tilde{\beta}(\theta) d\theta \leq \\ \frac{\beta}{a} (k_0(\lambda)) \cdot (T-a) + \frac{A}{a} + \frac{B}{a} (T-a) &= Q(A, T, \beta, \lambda). \end{aligned}$$

Thus

$$\bar{\alpha} \cdot e^{-Q(a, T, \beta, \lambda)} \leq \lambda + (T-a)^2 \cdot (-g(\lambda)) e^{Q(a, T, \beta, \lambda)}$$

i.e.

$$\bar{\alpha} \leq \frac{e^{Q(a, T, \beta, \lambda)}}{T-a} \left[\lambda + (T-a)^2 \cdot (-g(\lambda)) \cdot e^{Q(a, T, \beta, \lambda)} \right] = k_1(\lambda)$$

□

Proposition 3. *Let λ_0 be a positive real number, small enough and such that $\lambda_0 < b$. Then for any λ , $0 < \lambda \leq \lambda_0$, for any $(a, T) \in (\mathbb{R}^+)^2$ such that $0 < a < T$, $\frac{T}{a} - 1 \leq \frac{1}{c_\beta} (1 - \frac{1}{1+\delta})$, there exists a c^2 -function w belonging to $\widetilde{W}_\lambda$, which is a solution of*

$$(\mathcal{E}) \quad \begin{cases} -\omega'' - \frac{\beta}{a} (|\omega'|) \cdot \omega' = g(\omega) & \text{in }]a, T[\\ \omega(a) = \lambda, \quad \omega(T) = 0, \quad -\omega'(t) > k_0(\lambda) \end{cases}$$

Proof.

We use the fixed point theorem of Schauder in $\widetilde{W}_\lambda$ equipped with the natural topology of $c^1([a, T])$. It is clear that for any $v \in \widetilde{W}_\lambda$, the solution $\Gamma(v) = \omega_v$ of (16) satisfies

$$(17) \quad \sup [|\omega''(t)| \mid t \in [a, T]] \leq \frac{\beta_\infty}{a} k_1(\lambda) + |g(\lambda)|$$

with

$$\beta_\infty = \sup [\beta(t) \mid k_0(\lambda) \leq t \leq k_1(\lambda)]$$

i) Γ is a compact operator. Let v_n be a sequence belonging to $\widetilde{W}_\lambda$. The sequence $\omega_n = \Gamma(v_n)$ verifies :

$$(18) \quad \begin{cases} \|\omega'_n\|_\infty &\leq k_1(\lambda), \\ \|\omega''_n\|_\infty &\leq \frac{\beta_\infty}{a}k_1(\lambda) + |g(\lambda)|, \quad \text{from (17),} \\ \|\omega_n\|_\infty &\leq \lambda. \end{cases}$$

From Arzela-Ascoli Theorem there exists a subsequence - (labelled again) - ω_n which converges to some ω in $c^1([a, T])$. And it is easy to see that $\omega \in \widetilde{W}_\lambda$; i.e. Γ is compact.

ii) Γ is continuous on $\widetilde{W}_\lambda$. Let v_n be a sequence in $\widetilde{W}_\lambda$ converging to $v \in \widetilde{W}_\lambda$. Let ω be the solution corresponding to $v : \omega = \Gamma(v)$. Let us set $\omega_n = \Gamma(v_n)$. Thanks to (18) there exists a subsequence of $\{\omega_n\}_n$ converging to ω : it is easy to pass to the limit in the equation (16). As ω is the unique solution of (16) corresponding to v , it follows that any subsequence of ω_n which converges, tends to ω as n goes to infinity. This means that the whole sequence ω_n tends to ω ; i.e. Γ is continuous on $\widetilde{W}_\lambda$ equipped with the natural topology of $c^1([a, T])$.

The Schauder Theorem can be applied : there exists $\omega \in \widetilde{W}_\lambda$ which is a solution of

$$\begin{cases} -\omega'' - \frac{\beta}{a}(|\omega'|)\omega' = \tilde{g}(\omega) \text{ in } [a, T] , \\ \omega(a) = \lambda, \omega(T) = 0, \quad -\omega'(t) \geq k_0(\lambda) \end{cases}$$

and thus ω is a solution of $(\mathcal{E})$ since $0 \leq \omega(t) \leq \lambda \leq b$ and $\tilde{g}(s) = g(s) \forall s \in [0, b]$. The c^2 -regularity of ω follows from $(\mathcal{E})$. □

IV - Strong Maximum Principle

A - The regular case

Theorem 1. *Let us assume that the matrix $a(x, \eta, p)$ is continuously differentiable with respect to the variable p in $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$. Assume the hypothesis H1 to H5, $\mathcal{H}1$ and suppose that*

$$(19) \quad \begin{cases} -\sum_{ij} a_{ij}(x, u(x)) \nabla u(x) \frac{\partial^2 u}{\partial x_i \partial x_j} \geq \mathcal{F}(x, u, \nabla u) \quad \text{in } \mathbb{R}^N, \\ u(x) \geq 0 \quad \forall x \in \mathbb{R}^N, \end{cases}$$

has a c^2 solution. Then we have either $u(x) = 0 \quad \forall x \in \mathbb{R}^N$ or $u(x) > 0 \quad \forall x \in \mathbb{R}^N$.

Proof. We argue by contradiction. Let us suppose there exists $x_0 \in \mathbb{R}^N$ such that $u(x_0) = 0$ and $u \not\equiv 0$. Then there is some $x_1 \in \mathbb{R}^N$ such that $u(x_1) > 0$ and there exists $T_0 > 0$ such that we have : $u(x) > 0 \quad \forall x \in \overline{B(x_1, T_0)}$, where $B(x_1, T_0)$ is the open ball centred at x_1 , with radius T_0 . We set

$$T_1 = T_0 \left[1 + \frac{1}{c_\beta} \left(1 - \frac{1}{1 + \delta} \right) \right].$$

Consider λ such that $0 < \lambda < \min \{ u(x) / x \in \partial B(x_1, T_0) \}$. We choose also $0 < \lambda < b$. Considering, if necessary, the change of function $\tilde{u}(x) = u(x_1 + x)$ in (19), we can suppose that $x_1 \equiv 0$ i.e. $u(0) > 0$: there is no loss of generality. From Proposition 3, consider ω a solution of

$$(20) \begin{cases} -\omega'' - \frac{\beta}{T_0} (|\omega'|) \cdot \omega' = g(\omega) & \text{in }]T_0, T_1[\\ \omega(T_0) = \lambda, \quad \omega(T_1) = 0, \quad -\omega'(t) < 0, \quad \omega \in c^2([T_0, T_1]), \quad \omega''(t) \geq 0. \end{cases}$$

Setting $\omega(x) = \omega(|x|)$, we obtain from (20)

$$-\omega''(|x|) - \frac{\beta(|\omega'(|x|)|)}{|x|} \cdot \omega'(|x|) \leq g(\omega(|x|))$$

$$\forall x, \quad T_0 < |x| < T_1 \quad \text{i.e.} \quad \forall x \in B(0, T_1) \setminus \overline{B(0, T_0)}.$$

By H2 and H1 we obtain

$$\begin{aligned} -\omega''(|x|) &= \left[\frac{\text{Tr}(a(x, u(x), \nabla \omega(x)))}{\frac{1}{|x|^2} \sum_{ij} a_{ij}(x, u(x), \nabla \omega(x)) x_i x_j} - 1 \right] \frac{\omega'(|x|)}{|x|} \\ &\leq g(\omega(|x|)) \leq \frac{-f^-(x, u(x), \omega(x), \nabla \omega(x))}{\lambda_1(x, u(x), \nabla \omega(x))} \\ &\leq -\frac{f^-(x, u(x), \omega(x), \nabla \omega(x))}{\frac{1}{|x|^2} \sum_{ij} a_{ij}(x, u(x), \nabla \omega(x)) x_i x_j} \\ &\leq \frac{f(x, u(x), \omega(x), \nabla \omega(x))}{\frac{1}{|x|^2} \sum_{ij} a_{ij}(x, u(x), \nabla \omega(x)) x_i x_j} \quad \forall x \in B(0, T_1) \setminus \overline{B(0, T_0)} \end{aligned}$$

i.e.

$$\begin{aligned}
& -\frac{\omega''(|x|)}{|x|^2} \cdot \sum_{ij} a_{ij}(x, u(x), \nabla\omega(x)) x_i x_j \\
& - \left[\text{Tr}(a, (x, u(x), \nabla\omega(x)) - \sum_{ij} \frac{a_{ij}(x, u(x), \nabla\omega(x))}{|x|^2} \right] \frac{\omega'(|x|)}{|x|} \\
& \leq f(x, u(x), \omega(x), \nabla\omega(x))
\end{aligned}$$

and since $\omega \in c^2$, we obtain

$$(20.1) \quad \left\{ \begin{array}{l} -\sum_{ij} a_{ij}(x, u(x), \nabla\omega(x)) \frac{\partial^2 \omega}{\partial x_i \partial x_j} \leq f(x, u(x), \omega(x), \nabla\omega(x)) \\ \omega / \partial B(0, T_0) = \lambda, \quad \omega / \partial B(0, T_1) = 0, \quad \frac{\partial \omega}{\partial \nu} < 0 \quad \text{on } \partial B(0, T_1), \end{array} \right.$$

where ν stands for the external normal unit vector to $\partial B(0, T_1)$. Since $k_0(\lambda) \leq |\nabla\omega(x)| \leq k_1(\lambda)$ the matrix $a(x, u(x), \nabla\omega(x))$ is uniformly elliptic. By the comparison principle given in [GT] p. 207-208, we obtain

$$\omega(x) \leq u(x) \quad \forall x \in B(0, T_1) \setminus \overline{B(0, T_0)}$$

We claim that $u(x) > 0 \quad \forall x \in \overline{B(0, T_1)}$. For this purpose suppose that there exists $x_2 \in \partial B(0, T_1)$ such that $u(x_2) = 0$. We have $\nabla u(x_2) = 0$ since $u(x) \geq 0, \forall x$. For $t > 0$ small enough we have

$$\frac{\omega(x_2 - t\nu) - \omega(x_2)}{t} \leq \frac{u(x_2 - t\nu) - u(x_2)}{t}$$

where ν stands for the exterior normal unit vector to the boundary $\partial B(0, T_1)$ at x_2 . Passing to the limit as t goes to 0, we obtain

$$0 < -\omega'(T_1) = -\frac{\partial \omega}{\partial \nu}(x_2) \leq -\frac{\partial u(x_2)}{\partial \nu} = 0 \quad \text{since } \nabla u(x_2) = 0$$

It is a contradiction. Thus $u(x) > 0 \quad \forall x \in \overline{B(0, T_1)}$. Let us consider T_2 such that

$$T_2 = T_1 \left[1 + \frac{1}{c_\beta} \left(1 - \frac{1}{1 + \delta} \right) \right].$$

As previously we can establish that $u(x) > 0 \quad \forall x \in \overline{B(0, T_2)}$. So there is a sequence of positive real numbers T_n such that

$$\begin{aligned} T_n &= T_{n-1} \left[1 + \frac{1}{c_\beta} \left(1 - \frac{1}{1 + \delta} \right) \right] \\ &= T_0 \left[1 + \frac{1}{c_\beta} \left(1 - \frac{1}{1 + \delta} \right) \right]^n \end{aligned}$$

with $u(x) > 0 \quad \forall x \in \overline{B(0, T_n)}$. For n large enough we have $x_0 \in B(0, T_n)$. This entail that $u(x_0) > 0$. Thus it is a contradiction. $\square$

Remark 3.1. For convenience we have simplified the hypothesis H1. Instead of H1 we can suppose the more general assumption

$$\mathbf{H''1} : - \frac{f^-(x, \eta, s, p)}{\lambda_1(x, \eta, p)} \geq \frac{g(s)}{\gamma(|p|)} \quad \forall s \in [0, b] ,$$

$\forall (x, \eta, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$, $\gamma(|p|) \geq \gamma_0 > 0 \quad \forall p \in \mathbb{R}^N$. In fact to see this, from the proposition 3 there exists a c^2 function ω such that

$$\begin{cases} -\omega'' - \delta_0 \cdot \frac{\beta(|\omega'|)\gamma(|\omega'|)}{a} \omega' = \delta_0 \cdot g(\omega) \text{ in } [a, T] \\ \omega(a) = \lambda, \quad \omega(T) = 0, \quad \omega'(t) < 0, \quad \omega''(t) \geq 0 \end{cases}$$

where $\delta_0 > 0$ is a some real number such that $\delta_0 \cdot \gamma(|p|) \geq 1 \quad \forall p$. To prove that ω satisfy (20.1) it is sufficient to see that we have

$$-\delta_0 \cdot \gamma(|\omega'|) \omega'' - \delta_0 \cdot \frac{\beta(|\omega'|)\gamma(|\omega'|) \omega'}{a} \leq \delta_0 \cdot g(\omega)$$

since $-\omega''(t) \leq 0$ and $\delta_0 \cdot \gamma(|\omega'|) \geq 1$, i.e.

$$-\omega''(|x|) - \frac{\beta(\omega'(|x|))}{a} \cdot \omega'(|x|) \leq \frac{g(\omega)}{\gamma(|\omega'|)} \leq - \frac{f^-(x, u(x), \omega(x), \nabla \omega(x))}{\lambda_1(x, u(x), \nabla \omega(x))}.$$

The sequel remain to be unchanged. $\square$

B - The case of viscosity solution. Let us recall the following

Definition. Assume u of class c^1 . We say that u is a viscosity super solution of (1) if, for any c^2 test function φ such that $u - \varphi$ has a local minimizer x_0 , we have

$$(*) \quad -\Sigma a_{ij}(x_0, u(x_0), \nabla u(x_0)) \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x_0) \geq \mathcal{F}(x_0, u(x_0), \nabla u(x_0))$$

Corollary 1. Assume that the inequality (H1) is strict and u is a c^1 viscosity supersolution of (1). Then under the assumptions (H1) to (H5) we have either $u(x) = 0 \quad \forall x \in \mathbb{R}^N$ or $u(x) > 0 \quad \forall x \in \mathbb{R}^N$.

Proof. We proceed as previously. We argue by contradiction. Suppose that $u(x_0) = 0$ and $u \not\equiv 0$. Then there exists a ball $B(x_1, T_0)$ such that $u(x) > 0 \quad \forall x \in \overline{B(x_1, T_0)}$. We set

$$(**) \quad T_1 = T_0 \left[1 + \frac{1}{c_\beta} \left(1 - \frac{1}{1 + \delta} \right) \right].$$

Consider $\lambda > 0$ such that $0 < \lambda < \min \left[b, \min(u(x)/x \in \partial B(x_1, T_0)) \right]$. Let us consider $\varphi = \omega$ and let x_2 be such that

$$u(x_2) - \omega(x_2) = \min \left[u(x) - \omega(x) / x \in \overline{B(x_1, T_1)} \setminus B(x_1, T_0) \right].$$

i) if $x_2 \in \partial B(x_0, T_0)$ we have $u(x) > 0 \quad \forall x \in B(x_1, T_1)$ since we have

$$u(x) \geq \omega(x) + u(x_2) - \omega(x_2) > \omega(x) > 0 \quad \forall x \in \overline{B(x_1, T_1)} \setminus B(x_1, T_0)$$

ii) if $x_2 \in \partial B(x_1, T_1)$, we proceed as previously and we prove that there is a contradiction.

iii) if $x_2 \in B(x_1, T_1) \setminus \overline{B(x_1, T_0)}$, then $u - \omega$ has a strict minimizer x_2 and from the very definition of viscosity supersolution we obtain (*) at x_2 which is a contradiction since ω is a strict subsolution of (1).

Finally we have proved that $u(x) > 0 \quad \forall x \in \overline{B(x_1, T_1)}$. Now from T_1 we define T_2 as in (**); and so on. The last part of the proof remain to be unchanged.

V - Solutions with compact support.

A - The regular case.

In the case the maximum principle does not hold, a natural question is: under what conditions any non null solution of (1) has a compact support? In the sequel our purpose is to give a satisfactory answer to the above question. The inequality we are going to treat is of the form

$$(22) \quad \begin{cases} -\sum_{ij} a_{ij}(x, u(x), \nabla u) \frac{\partial^2 u}{\partial x_i \partial x_j} \leq \mathcal{F}(x, u, \nabla u) & \text{in } \mathbb{R}^N \\ u(x) \geq 0 & \forall x \in \mathbb{R}^N. \end{cases}$$

As in section II-A, $\mathcal{F}$ is defined from some function $f : \mathbb{R}_x^N \times \mathbb{R}_\eta \times \mathbb{R}_s \times \mathbb{R}_p^N \rightarrow \mathbb{R}$ as follows $\mathcal{F}(x, s, p) = f(x, s, s, p) \forall x, \forall s, \forall p$. About f we assume the following hypotheses :

$\tilde{H1}$: $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$, continuous ; there exist a, b , $-b < 0 < a$ such that $\forall x \in \mathbb{R}^N, \forall \eta, \forall p \in \mathbb{R}^N$ the function $s \rightarrow f(x, \eta, s, p)$ is decreasing and $f(x, \eta, 0, p) = 0 \quad \forall x, \forall p, \forall \eta$.

$\tilde{H2}$: There exist two continuous functions $\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+, g : \mathbb{R} \rightarrow \mathbb{R}$ such that $\gamma(t) > 0 \quad \forall t > 0, g(0) = 0 \quad g(s) \leq 0 \quad \forall s \in [0, a]$,

$$\frac{f(x, \eta, s, p)}{\lambda_N(x, \eta, p)} \leq \frac{g(s)}{\gamma(|p|)} \quad \forall x, \forall \eta, \forall p, \forall s \in [0, a] .$$

$\tilde{H3}$: $\int_0^a \frac{ds}{H^{-1}(-G(s))} < +\infty$ where H and G are defined as following:
 $h(t) = t(\gamma(t) + \epsilon), \forall t > 0$ with $\epsilon > 0$ arbitrary small ;
 $H(t) = \int_0^t h(s)ds, \quad G(s) = \int_0^s g(\theta)d\theta.$

To simplify the computations let us set

$$\begin{aligned} \tilde{a}(x, p) &= a(x, u(x), p) \quad \forall x, \forall p \\ \tilde{f}(x, s, p) &= f(x, u(x), s, p) \quad \forall x, \forall s, \forall p \end{aligned}$$

We can compare the function h of this framework with the ones used in the previous papers : [VA], [PU], [FE], [MO], [DO].

Example :

$$a_{ij} = 0 \quad \forall i \neq j \quad a_{ii} = a_i(|p|), \quad 0 \leq a_1 \leq a_2 \leq \dots \leq a_N ,$$

$\mathcal{F}(x, s, p) = f(x, s, p) = c(|p|) \cdot g(s), \quad c(|p|) > 0, \quad g(s) \leq 0 \quad \forall s \in [0, 1],$
 $\gamma(|p|) = a_N(|p|)/c(|p|)$. For instance we can take $c(|p|) = a_n(|p|)$ i.e. $\gamma(|p|) = 1$.

Remark 4.

1) Consider a regular function $v(x) = \varphi(|x|)$. Then

$$\begin{aligned} \sum_{ij} \tilde{a}_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} &= \varphi''(|x|) \cdot \sum_{ij} \frac{\tilde{a}_{ij} x_i x_j}{|x|^2} \\ &+ \frac{\varphi'(|x|)}{|x|} \left[Tr(\tilde{a}) - \sum_{ij} \frac{\tilde{a}_{ij} x_i x_j}{|x|^2} \right] \end{aligned}$$

2)

$$Tr(\tilde{a}) - \sum_{ij} \frac{\tilde{a}_{ij} x_i x_j}{|x|^2} \geq \sum_{i=1}^{N-1} \lambda_i \geq (N-1)\lambda_1$$

where $0 \leq \lambda_1(\tilde{a}) \leq \dots \leq \lambda_N(\tilde{a})$, stand for the eigenvalues of the matrix $\tilde{a} = (a_{ij})$. In a first step we use the same idea as in the previous papers [VA], [PU].

Proposition 4 : *There exists a c^2 -function $\omega : [0, A] \rightarrow \mathbb{R}$ such that*

$$\begin{cases} -\gamma(|\omega'(r)|) \cdot \omega''(r) \geq g(\omega(r)) & \text{in }]0, A[\\ \omega(0) = a, \quad \omega'(A) = \omega(A) = 0, \quad \omega''(A) = 0 \\ \omega'(r) < 0 \quad \forall r \in]0, A[, \quad \omega''(r) \geq 0 \quad \forall r \in]0, A[, \end{cases}$$

with $A = \int_0^a \frac{ds}{H^{-1}(-G(s))}$.

Remark 4.1. Contrary to [VA] and [PU] we use in proposition 4 an inequality i.e. it is not necessary to solve an equation.

Proof of Proposition 4.

Let $\epsilon > 0$ be a small real number. Consider the problem

$$\begin{cases} -[\gamma(|\omega'(r)|) + \epsilon]\omega''(r) = g(\omega(r)) & \text{in }]0, A[\\ \omega(0) = a, \quad \omega'(A) = \omega(A) = 0 \\ \omega'(r) < 0 \quad \forall r \in]0, A[. \end{cases}$$

By a monotony argument we define ω as follows - (cf [VA], [PU], [MO], [DO])

$$(23) \quad r = \int_{\omega(r)}^a \frac{ds}{H^{-1}(-G(s))}$$

which give by derivation

$$(23.1) \quad 1 = -\omega'(r) / H^{-1}(-G(\omega(r)))$$

i.e.

$$(24) \quad -G(\omega(r)) = H(-\omega'(r))$$

or again

$$-g(\omega(r)) \cdot \omega'(r) = h(-\omega'(r)) \cdot (-\omega''(r)).$$

But from (23.1) $\omega'(r) < 0 \forall r$. Thus

$$-\omega''(r) \frac{h(-\omega'(r))}{-\omega'(r)} = g(\omega(r)) \quad r \in]0, A[$$

i.e.

$$(25) \quad -(\epsilon + \gamma(|\omega'|)) \cdot \omega'' = g(\omega) \quad \text{in }]0, A[$$

and $\omega(0) = a$, $\omega(A) = 0$, $\omega'(A) = 0$ by (22) and (24). And since $g(0) = 0$, $\omega''(A) = 0$ follows from (25); since $0 \leq \omega(r) \leq a$, (25) imply $\omega''(r) \geq 0$, $\forall r$ and then we obtain

$$-\gamma(|\omega'|) \cdot \omega'' \geq g(\omega) \quad \text{in }]0, A[.$$

□

Proposition 5. *The function $v(x) = \omega(|x| - R)$ satisfies*

$$\begin{cases} -\sum_{ij} a_{ij}(x, u(x), \nabla v(x)) \frac{\partial^2 v}{\partial x_i \partial x_j} \geq \tilde{f}(x, v(x), \nabla v(x)), \\ x \in B^c(0, R) = \{x / |x| > R\}, \end{cases}$$

where $R > 0$ is large enough.

Proof.

We denote again by ω the extension by zero of ω to $\mathbb{R}^+$; from proposition 4, this extension ω belongs to $c^2(\mathbb{R}^+)$. Since $u(x) \geq 0$ and $\lim_{|x| \rightarrow +\infty} u(x) = 0$, there exists $R > 0$ such that : $\forall x \in D(R) = \{x / |x| > R\}$ we have $0 \leq u(x) \leq a$. Let us set $v(x) = \omega(|x| - R) = v(|x|) \quad \forall x \in D(R)$. It is clear that we have

$$\begin{cases} -\gamma(|v'|)v'' \geq g(v), & \text{in }]R, R+A[\\ v(R) = a, \quad v'(A+R) = v(A+R) = v''(A+R) = 0 \\ v'(|x|) < 0 \quad R < |x| < R+A. \end{cases}$$

Using the remark 4 and the monotony of v , we obtain

$$\begin{aligned} -v''(|x|) &= \frac{v'(|x|)}{|x|} \left[Tr(\tilde{a}(x, \nabla v)) - \sum_{ij} \frac{\tilde{a}_{ij}(x, \nabla v) x_i x_j}{|x|^2} \right] \\ &\times \frac{1}{\frac{1}{|x|^2} \sum_{ij} \tilde{a}_{ij}(x, \nabla v) x_i x_j} \geq \frac{g(v)}{\gamma(|\nabla v|)}. \end{aligned}$$

Since $0 \leq v(|v|) \leq a$ and from (H2) we obtain

$$(26) \quad \begin{cases} -v''(|x|) - \frac{v'(|x|)}{|x|} \left[Tr(\tilde{a}) - \sum_{ij} \frac{\tilde{a}_{ij} x_i x_j}{|x|^2} \right] \frac{1}{\frac{1}{|x|^2} \sum \tilde{a}_{ij} x_i x_j} \\ \geq \frac{g(v)}{\gamma(|\nabla v|)} \geq \frac{f(x, u(x), v, \nabla v)}{\lambda_N(x, u(x), \nabla v)} \geq \frac{\tilde{f}(x, v, \nabla v)}{\frac{1}{|x|^2} \sum \tilde{a}_{ij} x_i x_j} \end{cases}$$

since $\tilde{f}(x, v(x), \nabla v(x)) \leq 0 \quad \forall |x| \geq R$, because $0 \leq v(x) \leq a$. That is to say

$$\begin{cases} -\sum_{ij} \tilde{a}_{ij}(x, \nabla v) \frac{\partial^2 v}{\partial x_j \partial x_j} \geq \tilde{f}(x, v(x), \nabla v(x)), \\ \forall x, \quad |x| > R. \end{cases}$$

□

Remark 5. If $\gamma(0) = 0$ then the term $\frac{g(v(x))}{\gamma(|\nabla v(x)|)}$ in (26) means $\frac{g(v(x))}{\gamma(|\nabla v(x)|)}$ if $R < |x| < R + A$ and 0 if $|x| \geq R + A$ since $g(0) = 0$. Now we are able to prove the main result of this section.

Theorem 2. Assume $u \in C^2(\mathbb{R}^N)$ and hypothesis $\tilde{H}1 - \tilde{H}3$. We have $0 \leq u(x) \leq v(x) = \omega(|x| - R) \quad \forall x, |x| > R$. And thus $u(x) = 0 \quad \forall x, |x| > R$.

Proof. We argue by contradiction. Let us suppose that there exists $x_0, |x_0| > R$ such that

$$(27) \quad u(x_0) - v(x_0) = \max_{|x| > R} (u(x) - v(x)) > 0 .$$

As u and v are C^2 , we obtain

$$(28) \quad \begin{cases} \nabla u(x_0) = \nabla v(x_0) \\ D^2(u(x_0) - v(x_0)) \leq 0. \end{cases}$$

There exists a neighbourhood $\vartheta_1(x_0)$ such that

$$(29) \quad D^2(u(x) - v(x)) \leq 0 \quad \forall x \in \vartheta_1(x_0).$$

As $a_0 = (\tilde{a}_{ij}(x_0, \nabla u(x_0)))_{ij}$ is a non negative matrix it follows

$$-Tr(a_0 \cdot D^2(u(x) - v(x))) \geq 0 \quad \forall x \in \vartheta_1(x_0) ,$$

i.e.

$$(30) \quad -\sum_{ij} \tilde{a}_{ij}(x_0, \nabla u(x_0)) \frac{\partial^2(u(x) - v(x))}{\partial x_i \partial x_j} \geq 0, \quad \forall x \in \vartheta_1(x_0).$$

But from (21) and proposition 5 it follows

$$\begin{aligned} & -\sum_{ij} \tilde{a}_{ij}(x, \nabla v(x)) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{ij} \tilde{a}_{ij}(x, \nabla u(x)) \frac{\partial^2 u}{\partial x_i \partial x_j} \\ & \geq \tilde{f}(x, v(x), \nabla v(x)) - \tilde{f}(x, u(x), \nabla u(x)) = \delta(x) \end{aligned}$$

where

$$\begin{aligned} \delta(x) &= \tilde{f}(x, v(x), \nabla v(x)) - \tilde{f}(x, v(x), \nabla v(x_0)) \\ &- [\tilde{f}(x, u(x), \nabla u(x)) - \tilde{f}(x, u(x), \nabla u(x_0))] \\ &+ [\tilde{f}(x, v(x), \nabla v(x_0)) - \tilde{f}(x, u(x), \nabla u(x_0))] . \end{aligned}$$

For any $\epsilon > 0 \quad \exists \vartheta_2(x_0)$ such that

$$(31) \quad \sup_{\vartheta_2(x_0)} |\tilde{f}(x, v(x), \nabla v(x)) - \tilde{f}(x, v(x), \nabla v(x_0))| < \epsilon$$

and there exists $\vartheta_3(x_0)$ such that

$$(32) \quad \sup_{\vartheta_3(x_0)} |\tilde{f}(x, u(x), \nabla u(x)) - \tilde{f}(x, u(x), \nabla u(x_0))| < \epsilon.$$

But, since $u(x) - v(x) \leq 0 \quad \forall x, \quad |x| = R$, we have from (27)

$$u(x_0) - v(x_0) = \max_{\overline{D(R)}} (u(x) - v(x)) > 0;$$

and as $s \rightarrow \tilde{f}(x, s, p)$ is decreasing $\forall (x, p)$, there exist $\vartheta_4(x_0)$ and $\alpha > 0$ such that $u(x) - v(x) > 0 \quad \forall x \in \vartheta_4(x_0)$, and

$$(33) \quad \tilde{f}(x, v(x), \nabla v(x_0)) - \tilde{f}(x, u(x), \nabla u(x_0)) \geq \alpha \quad \forall x \in \vartheta_4(x_0),$$

because $\nabla u(x_0) = \nabla v(x_0)$. Let us set

$$\vartheta(x_0) = \bigcap_{i=1}^4 \vartheta_i(x_0).$$

Then thanks to (31) - (33) the following inequality holds :

$$(34) \quad \delta(x) \geq \alpha - 2\epsilon > 0 \quad \forall x \in \vartheta(x_0), \quad \text{for } \epsilon \text{ small enough.}$$

This gives

$$-\sum_{ij} \tilde{a}_{ij}(x, \nabla v(x)) \frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum_{ij} \tilde{a}_{ij}(x, \nabla u(x)) \frac{\partial^2 u}{\partial x_i \partial x_j} > 0 \quad \forall x \in \vartheta(x_0)$$

which entails

$$-\sum_{ij} \tilde{a}_{ij}(x_0, \nabla u(x_0)) \frac{\partial^2 (v(x_0) - u(x_0))}{\partial x_i \partial x_j} > 0, \quad \text{since } \nabla u(x_0) = \nabla v(x_0).$$

As $x \rightarrow -\sum_{ij} \tilde{a}_{ij}(x_0, \nabla u(x_0)) \frac{\partial^2}{\partial x_i \partial x_j} (v(x) - u(x))$ is continuous, $\exists \mathcal{W}(x_0)$ such that

$$(35) \quad -\sum_{ij} \tilde{a}_{ij}(x_0, \nabla u(x_0)) \frac{\partial^2}{\partial x_i \partial x_j} (v(x) - u(x)) > 0, \quad \forall x \in \mathcal{W}(x_0).$$

Finally for any $x \in \vartheta(x_0) \cap \mathcal{W}(x_0)$, (30) is in contradiction with (35). $\square$

B - The viscosity case.

Definition. Assume u of class c^1 . We say that u is a viscosity subsolution of (22) if, for any c^2 test function φ such that $u - \varphi$ has a local maximizer x_0 , we have

$$(36) \quad - \sum_{ij} \tilde{a}_{ij}(x_0, \nabla u(x_0)) \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x_0) \leq \tilde{f}(x_0, u(x_0), \nabla u(x_0))$$

Corollary 2. Assume that inequality $(\tilde{H}2)$ is strict and u is a c^1 viscosity subsolution of (22). Then under the assumptions $(\tilde{H}1)$ to $(\tilde{H}3)$, there exists $R > 0$ such that $u(x) = 0 \quad \forall x, |x| > R$.

Proof. We proceed as in the Theorem 2, arguing by contradiction. Let us consider Proposition 5 and $\varphi = v$. We have proved that

$$(37) \quad \tilde{f}(x, v(x), \nabla v(x)) < - \sum_{ij} \tilde{a}_{ij}(x, \nabla v(x)) \frac{\partial^2 v(x)}{\partial x_i \partial x_j} \quad \forall x, |x| > R$$

since $(\tilde{H}2)$ is strict. The function v and the real number R are defined in proposition 5. Let x_0 be such that

$$u(x_0) - v(x_0) = \max_{|x| > R} (u(x) - v(x)) > 0.$$

Since $\nabla u(x_0) = \nabla v(x_0)$, from (36) we obtain

$$(38) \quad \begin{aligned} - \sum_{ij} \tilde{a}_{ij}(x_0, \nabla v(x_0)) \frac{\partial^2 v}{\partial x_i \partial x_j}(x_0) &\leq \tilde{f}(x_0, u(x_0), \nabla v(x_0)) \\ &\leq \tilde{f}(x_0, v(x_0), \nabla v(x_0)) \end{aligned}$$

since we have $(\tilde{H}1)$. Thus (38) is in contradiction with (37) at x_0 , and thus we have $u(x) = 0 \quad \forall x, |x| > R$. $\square$

VI - Necessary and sufficient conditions to obtain the strong maximum principle.

Let us point out that our assumptions (H"1) in section II and ($\tilde{H}2$) in section V are quite satisfactory. Indeed let us consider, for instance, the equation

$$(E) \quad \begin{cases} -\sum_{ij} a_{ij}(x, \nabla u(x)) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u(x), \nabla u(x)) \\ u(x) \geq 0 \quad \text{in } \mathbb{R}^N. \end{cases}$$

And assume that we have : there exist g and γ such that

$$(\mathcal{H}) \quad \frac{g(s)}{\gamma(|p|)} \leq \frac{-f^-(x, s, p)}{\lambda_1(x, p)}, \quad \frac{f(x, s, p)}{\lambda_N(x, p)} \leq c \cdot \frac{g(s)}{\gamma(|p|)}$$

for any s and p , where c is some positive constant. We have :

$$\frac{g(s)}{\gamma(|p|)} \leq \frac{-f^-(x, s, p)}{\lambda_1(x, p)} \leq \frac{f(x, z, p)}{\lambda_N(x, p)} < c \cdot \frac{g(s)}{\gamma(|p|)}.$$

The other assumptions remain to be unchanged. From theorem 1 and theorem 2, the integral condition $\int_0^{+\infty} \frac{ds}{H^{-1}(-G(s))} = +\infty$, - (cf $\tilde{H}3$) - is necessary and sufficient to obtain the strong maximum principle for (E).

Acknowledgments : I am grateful to Pierre Cardaliaguet for his valuable remarks especially those concerning viscosity solutions.

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