LIMITS AS $p \rightarrow \infty$ OF p -LAPLACIAN EIGENVALUE PROBLEMS PERTURBED WITH A CONCAVE OR CONVEX TERM

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ABSTRACT. We investigate the asymptotic behaviour as $p \rightarrow \infty$ of sequences of positive weak solutions of the equation

$$\begin{cases} -\Delta_p u = \lambda_p u^{p-1} + u^{q(p)-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda_p > 0$ and either $1 < q(p) < p$ or $p < q(p)$, with $\lim_{p \rightarrow \infty} q(p)/p = Q \neq 1$. Uniform limits are characterized as positive viscosity solutions of the problem

$$\begin{cases} \min \{ |\nabla u(x)| - \max \{ \Lambda u(x), u^Q(x) \}, -\Delta_\infty u(x) \} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

for appropriate values of $\Lambda > 0$. Due to the decoupling of the nonlinearity under the limit process, the limit problem exhibits an intermediate behavior between an eigenvalue problem and a problem with a power-like right-hand side. Existence and non-existence results for both the original and the limit problem are also obtained.

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1. INTRODUCTION

We analyze the limit as $p \rightarrow \infty$ of the sequence of positive solutions of the following perturbed eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} + u^{q(p)-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian, $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, $\lambda > 0$ and the perturbation is either concave or convex depending on the exponent q . Namely, $q(p)$ is assumed to satisfy the condition, with

$$\lim_{p \rightarrow \infty} \frac{q(p)}{p} = Q \quad \text{for } Q > 0, Q \neq 1. \quad (2)$$

In particular, we shall always assume in the sequel that $p > n$ and $1 < q(p) < p$ or $p < q(p)$ without loss of generality. We shall say that problem (1) is concave whenever $Q < 1$ and convex if $Q > 1$.

The asymptotics of problems having a power-type right-hand side, namely,

$$-\Delta_p u = \lambda u^{\alpha(p)-1} \quad \text{in } \Omega$$

have already been studied in the eigenvalues case, where $\alpha(p) = p$ (see [12, 18, 19]), the subdiffusive case $\alpha(p) < p$ (see [8]), and the superdiffusive case $\alpha(p) > p$ (see [7]).

Hence, it seems natural to consider concave and convex perturbations of eigenvalue problems as $p \rightarrow \infty$ and the associated bifurcation problems.

In the case of the classical Laplacian ($p = 2$) problem (1) has been extensively studied from the point of view of bifurcation theory. Depending on the size of the exponent q , there are two main phenomena: bifurcation from an eigenvalue when $q > 2$ and bifurcation from infinity when $q < 2$. Rabinowitz's global Theorem (see [24]) plays a major role in this theory (see also [3, 9, 10, 25]).

The results in the case $p = 2$ rely strongly on the linearity of the operator and use degree theory. In the case of the p -Laplacian, there are analogous results depending on the size of q relative to p , see [13, 14] and the references therein.

In the present paper, we are interested in the asymptotics as $p \rightarrow \infty$ of problem (1). As we shall see, sequences of positive weak solutions to (1) converge to solutions of the fully nonlinear problem

$$\begin{cases} \min\{|\nabla u_\Lambda(x)| - \max\{\Lambda u_\Lambda(x), u_\Lambda^Q(x)\}, -\Delta_\infty u_\Lambda(x)\} = 0 & \text{in } \Omega, \\ u_\Lambda > 0 & \text{in } \Omega, \\ u_\Lambda = 0 & \text{on } \partial\Omega. \end{cases} \quad (3)$$

Our main concern will be to obtain the corresponding bifurcation results for the limit problem (3). We shall see that the bifurcation diagrams of the limit problem are, somehow, limits of the bifurcation diagrams at level p . For this reason, we shall need to study in a quantitative way the case $p < \infty$. We believe that some of the results we obtain at this level could be new.

Moreover, the decoupling of the nonlinearity under the limit process brings new phenomena not present at the problem at level p . Mainly, the limit problem exhibits an intermediate behavior between an eigenvalue problem and a problem with a power-like right-hand side. The most striking feature is the existence of multiple solutions associated to the first eigenvalue of the infinity Laplacian defined as

$$\Lambda_1(\Omega) := \left(\max_{x \in \Omega} \operatorname{dist}(x, \partial\Omega) \right)^{-1}.$$

whenever they are above or below a certain threshold (depending on $Q < 1$ or $Q > 1$). This is in turn a different behavior from eigenvalue problems themselves.

We summarize the main results of the paper in the following theorem (see Figures 3 and 4).

Theorem 1.1. *Consider a bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary and let $\Lambda > 0$, $Q > 0$, $Q \neq 1$.*

Then, whenever $Q \in (0, 1)$:

- i) *For every $\Lambda \in [0, \Lambda_1(\Omega))$, problem (3) has a unique viscosity solution u_Λ that coincides with w_1 , the unique viscosity solution of*

$$\begin{cases} \min \{ |\nabla w_1(x)| - w_1^Q(x), -\Delta_\infty w_1(x) \} = 0 & \text{in } \Omega, \\ w_1 > 0 & \text{in } \Omega, \\ w_1 = 0 & \text{on } \partial\Omega. \end{cases} \quad (4)$$

Moreover, u_Λ is attained as a limit of solutions of (1) and

$$\|u_\Lambda\|_\infty = \Lambda_1(\Omega)^{-\frac{1}{1-Q}}.$$

- ii) *For $\Lambda = \Lambda_1(\Omega)$, problem (3) has a viscosity solution with $\|u\|_\infty = M$ for every $M \geq \Lambda_1(\Omega)^{-\frac{1}{1-Q}}$ and no solution for $M < \Lambda_1(\Omega)^{-\frac{1}{1-Q}}$. Furthermore, u_Λ is attained as a limit of solutions of (1).*

- iii) *If $\Lambda > \Lambda_1(\Omega)$ problem (3) has no viscosity solutions.*

Moreover, whenever $Q > 1$:

- i) *For every $\Lambda \in [0, \Lambda_1(\Omega))$, problem (3) has a viscosity solution u_Λ such that*

$$\|u_\Lambda\|_\infty = \Lambda_1(\Omega)^{\frac{1}{Q-1}}$$

which is attained as a limit of least energy solutions of (1) (see Section 3.3 for the definition).

- ii) *For $\Lambda = \Lambda_1(\Omega)$, problem (3) has a viscosity solution u_Λ with $\|u_\Lambda\|_\infty = M$ for every $M \leq \Lambda_1(\Omega)^{\frac{1}{Q-1}}$, and no solution for $M > \Lambda_1(\Omega)^{\frac{1}{Q-1}}$. u_Λ is also a first eigenfunction of the infinity Laplacian, namely, it is also a viscosity solution of*

$$\begin{cases} \min \{ |\nabla \psi(x)| - \Lambda_1(\Omega) \psi(x), -\Delta_\infty \psi(x) \} = 0 & \text{in } \Omega, \\ \psi > 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

- iii) *If $\Lambda > \Lambda_1(\Omega)$ problem (3) has no viscosity solutions.*

The paper is organized as follows. In Section 2 we provide some necessary preliminary results. In Section 3 we study problem (1) and obtain quantitative a-priori bounds and existence/non-existence results, tracking down the dependence on p of all the parameters involved. These will be necessary in subsequent sections for the analysis of the limit problem (3).

The limit problem is introduced in Section 4 and then in Section 5 a detailed analysis of the limit problem is performed and bifurcation results are obtained.

Finally, in Section 6 we provide a family of explicit solutions in special domains, illustrative of the phenomena analyzed in the foregoing.

2. PRELIMINARIES

In the following lemma we state that weak solutions of (1) are also viscosity solutions. We omit the proof, that follows analogously to [19, Lemma 1.8] (see also [5]).

Lemma 2.1. *If u is a continuous weak solution of (1), then it is a viscosity solution of the same problem, rewritten as*

$$\begin{cases} F_p(\nabla u, D^2 u) = \lambda u^{p-1} + u^{q(p)-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (5)$$

where

$$F_p(\xi, X) = -|\xi|^{p-2} \cdot \text{trace} \left(\left(Id + (p-2) \frac{\xi \otimes \xi}{|\xi|^2} \right) X \right).$$

The divergence form is more useful from the variational point of view, while (5) is preferred in the viscosity framework. In the sequel we shall always consider the more suitable form of our problem between (1) and (5) without any further reference.

2.1. Eigenvalue problem for the p -Laplacian. Let us consider the problem

$$\begin{cases} -\Delta_p u &= \lambda |u|^{p-2} u & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases} \quad (6)$$

where $\lambda > 0$ and $1 < p < \infty$. This equation is known as *eigenvalue problem* for the p -Laplacian and has received extensive investigation. One can define the first eigenvalue $\lambda_1(p)$ as

$$\lambda_1(p) := \inf_{v \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla v|^p dx}{\int_{\Omega} |v|^p dx}.$$

Applying the direct method of the Calculus of Variations, it is immediate to see that the infimum is attained, and that a minimizer is a weak solution of (6). It can be proved that every first eigenfunction does not change sign in Ω , and that the first eigenvalue is simple: if u and v are two eigenfunctions associated to $\lambda_1(p)$, then $u = cv$ for some $c \in \mathbb{R}$ (see [4] and the references therein). Moreover, a strong maximum principle holds (see [27]). Therefore, one can always consider strictly positive first eigenfunctions. We also notice that eigenfunctions belong to $L^\infty(\Omega)$ (see [21]) and therefore they are in $C_{loc}^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$ depending on p and n (see [11, 26]).

We shall be interested in the behaviour of the first eigenvalue and of the first eigenfunction as $p \rightarrow \infty$. The following result holds (see [19]).

Proposition 2.2. *Let $\lambda_1(p)$ be the first eigenvalue of the p -Laplacian. Then,*

$$\lim_{p \rightarrow \infty} \lambda_1(p)^{\frac{1}{p}} = \left(\max_{x \in \Omega} \text{dist}(x, \partial\Omega) \right)^{-1} =: \Lambda_1(\Omega),$$

where $\text{dist}(x, \partial\Omega)$ is the distance of x to the boundary of Ω . Moreover, the first eigenfunctions converge uniformly to a viscosity solution of

$$\begin{cases} \min\{|\nabla u| - \Lambda_1(\Omega)u, -\Delta_\infty u\} &= 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega. \end{cases} \quad (7)$$

We recall that $\Delta_\infty u := \langle D^2 u \cdot \nabla u, \nabla u \rangle$ is the *infinity Laplacian*. $\Lambda_1(\Omega)$ is the first eigenvalue of $-\Delta_\infty$, and a positive solution of (7) is a first eigenfunction. We refer to [19] for more details on this topic.

2.2. Morrey's inequality. The following result has been proved in [7, Proposition 3.1].

Proposition 2.3. *Assume $n < p < \infty$ and $u \in W_0^{1,p}(\Omega)$. Then,*

$$\|u\|_{L^\infty(\Omega)} \leq C_p \cdot \left(\int_{\Omega} |\nabla u|^p dx \right)^{\frac{1}{p}}$$

holds with constant

$$C_p = p |B_1(0)|^{-\frac{1}{p}} n^{-\frac{n(p+1)}{p^2}} (p-1)^{\frac{n(p-1)}{p^2}} (p-n)^{\frac{n}{p^2}-1} \lambda_1(p)^{\frac{n-p}{p^2}}. \quad (8)$$

The importance of Proposition 2.3 is that C_p is asymptotically sharp, see [7] for a discussion on this issue.

Remark 2.4. In the sequel we shall strongly exploit the fact that

$$\lim_{p \rightarrow \infty} C_p = \Lambda_1(\Omega)^{-1}.$$

The following well-known result will be necessary in the sequel.

Lemma 2.5. *Assume $n < p < \infty$ and $u \in W_0^{1,p}(\Omega)$. Then $u \in \mathcal{C}^\gamma(\Omega)$, for $\gamma = 1 - \frac{n}{p}$, and*

$$\frac{|u(x) - u(y)|}{|x - y|^\gamma} \leq \tilde{C}_p \cdot \left(\int_\Omega |\nabla u|^p dx \right)^{\frac{1}{p}},$$

where

$$\tilde{C}_p = \frac{2C}{|\partial B_1(0)|^{\frac{1}{p}}} \left(\frac{p-1}{p-n} \right)^{1-\frac{1}{p}}$$

and C is a constant that depends only on n .

3. THE p -LAPLACIAN EQUATION

We consider the equation

$$\begin{cases} -\Delta_p u &= \lambda u^{p-1} + u^{q-1} & \text{in } \Omega \\ u &> 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \quad (9)$$

where $\lambda > 0$, $n < p < \infty$, and either $1 < q < p$, or $p < q < \infty$. In this section we shall derive some estimates for the positive solutions of (9), which will be necessary for the investigation of their behaviour as $p \rightarrow \infty$.

We start showing that the problem does not admit any positive solution for $\lambda \geq \lambda_1(p)$; we point out that this result holds for every $p, q > 1$ with $p \neq q$. This result is sharp, as we shall prove existence of solutions to (9) provided $\lambda < \lambda_1(p)$ in both cases $q < p$ and $q > p$.

3.1. Non-existence for $\lambda \geq \lambda_1(p)$. We shall show here that equation (9) does not admit any positive solution for $\lambda \geq \lambda_1(p)$ for every $p, q > 1$ with $p \neq q$. We would like to stress that the result holds also in the case $1 < p \leq n$ and gives an answer to a conjecture stated in [13, Section 6].

Proposition 3.1. *Let $\lambda \geq \lambda_1(p)$, and either $1 < q < p$ or $1 < p < q$. Then there does not exist any positive solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (9).*

Proof. Suppose that u is a positive solution of (9) for $\lambda \geq \lambda_1(p)$. Then

$$-\Delta_p u = \lambda u^{p-1} + u^{q-1}. \quad (10)$$

Since $u \in L^\infty(\Omega)$, by regularity results (see [11]) $u \in \mathcal{C}_{loc}^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. Denote by ψ a first eigenfunction of the p -Laplacian, so that

$$-\Delta_p \psi = \lambda_1(p) \psi^{p-1}. \quad (11)$$

Notice that also $\psi \in \mathcal{C}_{loc}^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$, since $\psi \in L^\infty(\Omega)$. Consider the functions $w_1 := \frac{(\psi^p - u^p)^+}{u^{p-1}}$ and $w_2 := \frac{(\psi^p - u^p)^+}{\psi^{p-1}}$; let $\{\psi_k\}$ and $\{u_k\}$ be two sequences in $C_c^\infty(\Omega)$ approximating in $W_0^{1,p}(\Omega)$ the functions ψ and u respectively, and use $w_{1,k} := \frac{(\psi_k^p - u^p)^+}{u^{p-1}}$

and $w_{2,k} := \frac{(\psi^p - u_k^p)^+}{\psi^{p-1}}$ as test functions for equations (10) and (11) respectively. Observe that $w_{1,k}, w_{2,k} \in W_0^{1,p}(\Omega)$. We obtain

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla w_{1,k} dx - \int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \nabla w_{2,k} dx \\ &= \int_{\{\psi_k > u\}} \left[p |\nabla u|^{p-2} \nabla u \nabla \psi_k \frac{\psi_k^{p-1}}{u^{p-1}} - (p-1) |\nabla u|^p \frac{\psi_k^p}{u^p} - |\nabla u|^p \right] dx \\ &+ \int_{\{\psi > u_k\}} \left[p |\nabla \psi|^{p-2} \nabla \psi \nabla u_k \frac{u_k^{p-1}}{\psi^{p-1}} - (p-1) |\nabla \psi|^p \frac{u_k^p}{\psi^p} - |\nabla \psi|^p \right] dx \\ &=: \int_{\{\psi_k > u\}} N_{1,k}(x) dx + \int_{\{\psi > u_k\}} N_{2,k}(x) dx. \end{aligned}$$

From Picone's identity (see [2, Theorem 1.1]), $N_{1,k}, N_{2,k} \leq 0$. This implies

$$\lambda \int_{\Omega} (\psi_k^p - u^p)^+ dx - \lambda_1(p) \int_{\Omega} (\psi^p - u_k^p)^+ dx + \int_{\Omega} u^{q-p} (\psi_k^p - u^p)^+ dx \leq 0$$

and sending $k \rightarrow \infty$

$$0 \leq (\lambda - \lambda_1(p)) \int_{\Omega} (\psi^p - u^p)^+ dx + \int_{\Omega} u^{q-p} (\psi^p - u^p)^+ dx \leq 0.$$

This can be the case only if $(\psi^p - u^p)^+ = 0$, which means $u \geq \psi$. But ψ was an arbitrary first eigenfunction of the p -Laplacian, and this leads to a contradiction. Hence, equation (9) does not admit any positive solutions for $\lambda \geq \lambda_1(p)$. $\square$

Remark 3.2. We observe that in the case $p = 2$ the above proof becomes much simpler. If u is a positive weak solution of (9), then

$$\int_{\Omega} \nabla u \nabla v dx = \lambda \int_{\Omega} uv dx + \int_{\Omega} |u|^{q-2} uv dx$$

for every $v \in W_0^{1,2}(\Omega)$. Taking $v = \psi$, where ψ is a first eigenfunction of the Laplacian, one obtains

$$0 \geq (\lambda_1(2) - \lambda) \int_{\Omega} u \psi dx = \int_{\Omega} |u|^{q-2} u \psi dx > 0,$$

a contradiction.

3.2. The case $q < p$. According to [13, Section 6], equation (9) admits one positive weak solution for $\lambda < \lambda_1(p)$. Moreover, this solution is unique, as the following comparison principle, which generalizes the classical result for the Laplacian (see [6]), holds:

Proposition 3.3. ([1, Lemma 4.1]) *Let $u, v \in W_0^{1,p}(\Omega) \cap C^1(\Omega)$ be two strictly positive functions satisfying*

$$\begin{aligned} -\Delta_p u &\leq \lambda u^{p-1} + u^{q-1}, \\ -\Delta_p v &\geq \lambda v^{p-1} + v^{q-1}. \end{aligned}$$

Then $u \leq v$ in Ω .

In the sequel we study the branch of solutions to problem 9. These results are summarized in Figure 1.

Proposition 3.4. *Let $p > n$, $q < p$ and $u \in W_0^{1,p}(\Omega)$ be a weak solution of (9) for $\lambda < \lambda_1(p)$. Then,*

$$\|u\|_{\infty} \leq (C_p^p \cdot |\Omega| \cdot \lambda_1(p))^{\frac{1}{p-q}} (\lambda_1(p) - \lambda)^{-\frac{1}{p-q}},$$

and

$$\left(\int_{\Omega} |\nabla u|^p dx \right)^{\frac{1}{p}} \leq (C_p^q \cdot |\Omega| \cdot \lambda_1(p))^{\frac{1}{p-q}} (\lambda_1(p) - \lambda)^{-\frac{1}{p-q}},$$

where C_p is defined in (8).

Proof. By definition of weak solution,

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v \, dx = \lambda \int_{\Omega} |u|^{p-2} uv \, dx + \int_{\Omega} |u|^{q-2} uv \, dx$$

for every $v \in W_0^{1,p}(\Omega)$. Taking $v = u$, one obtains

$$\int_{\Omega} |\nabla u|^p \, dx = \lambda \int_{\Omega} |u|^p \, dx + \int_{\Omega} |u|^q \, dx \leq \frac{\lambda}{\lambda_1(p)} \int_{\Omega} |\nabla u|^p \, dx + \int_{\Omega} |u|^q \, dx$$

which implies

$$\int_{\Omega} |\nabla u|^p \, dx \leq \left(\frac{\lambda_1(p)}{\lambda_1(p) - \lambda} \right) \int_{\Omega} |u|^q \, dx \leq \left(\frac{\lambda_1(p)}{\lambda_1(p) - \lambda} \right) \|u\|_{\infty}^q \cdot |\Omega|.$$

By Morrey's inequality (Proposition 2.3) we obtain the result. $\square$

Remark 3.5. Since the solutions of (9) are bounded in $L^{\infty}(\Omega)$, by classical regularity results (see [11, 26]) they actually belong to the class $\mathcal{C}_{loc}^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$ depending on p .

Proposition 3.6. *Let u be a solution of (9) with $q < p$. Then*

$$u \geq w_{1,p} > 0,$$

where $w_{1,p}$ is the solution of (9) for $\lambda = 0$.

Proof. Since

$$-\Delta_p u = \lambda u^{p-1} + u^{q-1} \geq u^{q-1},$$

then u is a supersolution of (9) for $\lambda = 0$. The claim follows by Proposition 3.3. $\square$

In the following result we state the behavior of the branch of solutions to (9).

Proposition 3.7. *Let $q < p$. Let the function $h : [0, \lambda_1(p)) \rightarrow \mathbb{R}$ be defined as*

$$h(\lambda) := \|u_{\lambda}\|_{\infty},$$

where u_{λ} solves (9). Then h is continuous, monotone increasing, and such that

$$\lim_{\lambda \rightarrow \lambda_1(p)} h(\lambda) = +\infty$$

Proof. Let $\lambda \leq \mu$, and u_{λ}, u_{μ} the corresponding solutions of (9). Then

$$-\Delta_p u_{\mu} = \mu u_{\mu}^{p-1} + u_{\mu}^{q-1} \geq \lambda u_{\mu}^{p-1} + u_{\mu}^{q-1}.$$

By the comparison principle stated in Proposition 3.3 we obtain $u_{\lambda} \leq u_{\mu}$ and hence $\|u_{\lambda}\|_{\infty} \leq \|u_{\mu}\|_{\infty}$.

To prove the continuity of $h(\lambda)$ we argue as follows. Consider a sequence $\lambda \rightarrow \hat{\lambda}$, $\lambda, \hat{\lambda} \in (0, \lambda_1(p))$; select a $\mu \in (\hat{\lambda}, \lambda_1(p))$. Let $u_{\lambda}, u_{\hat{\lambda}}, u_{\mu}$ the corresponding solutions to (9). Integrating the equation against the solution, we obtain

$$\begin{aligned} \int_{\Omega} |\nabla u_{\lambda}|^p \, dx &= \lambda \int_{\Omega} u_{\lambda}^p \, dx + \int_{\Omega} u_{\lambda}^q \, dx \\ &\leq \mu \int_{\Omega} u_{\mu}^p \, dx + \int_{\Omega} u_{\mu}^q \, dx = \int_{\Omega} |\nabla u_{\mu}|^p \, dx, \end{aligned}$$

since we have proved that $\lambda \leq \mu$ implies $u_{\lambda} \leq u_{\mu}$. Then,

$$\left(\int_{\Omega} |\nabla u_{\lambda}|^p \, dx \right)^{\frac{1}{p}} \leq \left(\int_{\Omega} |\nabla u_{\mu}|^p \, dx \right)^{\frac{1}{p}} =: C$$

and therefore the u_λ are uniformly bounded in $W_0^{1,p}(\Omega)$. Proposition 2.3, Lemma 2.5, and the Ascoli-Arzelà Theorem yield weak convergence in $W_0^{1,p}(\Omega)$ and uniform convergence of u_λ to some v as $\lambda \rightarrow \hat{\lambda}$ up to a subsequence. Moreover, v is a weak solution of

$$\begin{cases} -\Delta_p v &= \hat{\lambda} v^{p-1} + v^{q-1} & \text{in } \Omega, \\ v &> 0 & \text{in } \Omega, \\ v &= 0 & \text{on } \partial\Omega, \end{cases}$$

and by uniqueness (see Proposition 3.3) we get that $v \equiv u_{\hat{\lambda}}$ and that the whole sequence $u_\lambda \rightarrow u_{\hat{\lambda}}$ uniformly as $\lambda \rightarrow \hat{\lambda}$. Hence, by uniform convergence,

$$|h(\lambda) - h(\hat{\lambda})| = \left| \|u_\lambda\|_\infty - \|u_{\hat{\lambda}}\|_\infty \right| \leq \|u_\lambda - u_{\hat{\lambda}}\|_\infty \rightarrow 0 \quad \text{as } \lambda \rightarrow \hat{\lambda}.$$

Suppose now that there exists a sequence $\lambda_k \rightarrow \lambda_1(p)$ such that the solutions $u_k := u_{\lambda_k}$ satisfy

$$\|u_k\|_\infty \leq M$$

for some $M > 0$ independent of k . Then

$$\int_\Omega |\nabla u_k|^p dx = \lambda_k \int_\Omega |u_k|^p dx + \int_\Omega |u_k|^q dx \leq \lambda_1(p) |\Omega| (M^p + M^q).$$

Hence there exists a function $v \in W_0^{1,p}(\Omega)$ such that, up to a subsequence (not relabeled), $u_k \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$ and uniformly in Ω , so that v is a weak solution of

$$\begin{cases} -\Delta_p v &= \lambda_1(p) v^{p-1} + v^{q-1} & \text{in } \Omega \\ v &= 0 & \text{on } \partial\Omega. \end{cases}$$

By Proposition 3.6 we have

$$v(x) = \lim_{k \rightarrow \infty} u_k(x) \geq w_1(x),$$

where w_1 is the positive solution of

$$\begin{cases} -\Delta_p w_1 &= w_1^{q-1} & \text{in } \Omega \\ w_1 &= 0 & \text{on } \partial\Omega \end{cases}$$

This implies that $v > 0$, a contradiction to Proposition 3.1. Hence,

$$\lim_{\lambda \rightarrow \lambda_1(p)} h(\lambda) = +\infty \quad \square$$

3.3. The case $q > p$. Differently from the case $q < p$, in general one cannot expect uniqueness of positive solutions of (9) (see [20]). However, for $\lambda < \lambda_1(p)$ it is possible to define the notion of *least energy solutions*, which minimize the functional $\varphi : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined as

$$\varphi(v) := \frac{1}{p} \int_\Omega |\nabla v|^p dx - \frac{\lambda}{p} \int_\Omega |v|^p dx - \frac{1}{q} \int_\Omega |v|^q dx$$

among all possible solutions. Observe that $\inf_{W_0^{1,p}(\Omega)} \varphi = -\infty$, so that it is impossible to minimize φ on the whole Sobolev space. To overcome this difficulty, let us define the *Nehari manifold* $\mathcal{N}$ as

$$\mathcal{N} := \{u \in W_0^{1,p}(\Omega) \setminus \{0\} \mid d\varphi(u)(u) = 0\},$$

where $d\varphi(u)(v)$ is the first variation of φ in direction v , namely

$$d\varphi(u)(v) := \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx - \lambda \int_\Omega |u|^{p-2} uv dx - \int_\Omega |u|^{q-2} uv dx.$$

It is clear that every critical point of φ is a weak solution of (9) and belongs to $\mathcal{N}$. The following facts hold true.

Proposition 3.8. *Let $\lambda < \lambda_1(p)$ and $q > p$. Then, for every $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ there exists a unique number $t^*(u) > 0$ such that $t^*(u)u \in \mathcal{N}$. Moreover,*

$$t^*(u) = \left(\frac{\int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} |u|^p dx}{\int_{\Omega} |u|^q dx} \right)^{\frac{1}{q-p}}$$

and

$$\varphi(t^*(u)u) = \max_{t>0} \varphi(tu).$$

Proof. The proof is similar to [15, Proposition 2.1]. Observe that $t^*(u)$ is well defined, since

$$\int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} |u|^p dx \geq (\lambda_1(p) - \lambda) \int_{\Omega} |u|^p dx \geq 0. \quad \square$$

Proposition 3.9. *Let $\lambda < \lambda_1(p)$ and $q > p$. Then the Nehari manifold $\mathcal{N}$ is closed in $W_0^{1,p}(\Omega)$.*

Proof. Since φ is of class $\mathcal{C}^1$, $\mathcal{N} \cup \{0\}$ is closed in $W_0^{1,p}(\Omega)$. Then it is enough to prove that 0 is not an accumulation point for $\mathcal{N}$. Then,

$$\|\nabla u\|_p \geq C\|u\|_q$$

for every $u \in W_0^{1,p}(\Omega)$, where $C = C(p, q) > 0$. Let $v \in \mathcal{N}$; recalling that $t^*(v) = 1$, one has

$$\begin{aligned} \|\nabla v\|_p \geq C\|v\|_q &= C \left(\frac{\|\nabla v\|_p^p - \lambda \|v\|_p^p}{\|v\|_q^q} \right)^{\frac{1}{q-p}} \|v\|_q = C \left(\frac{\|\nabla v\|_p^p - \lambda \|v\|_p^p}{\|v\|_q^p} \right)^{\frac{1}{q-p}} \\ &\geq C \left(1 - \frac{\lambda}{\lambda_1(p)} \right)^{\frac{1}{q-p}} \left(\frac{\|\nabla v\|_p^p}{\|v\|_q^p} \right)^{\frac{1}{q-p}} \geq C^{\frac{q}{q-p}} \left(1 - \frac{\lambda}{\lambda_1(p)} \right)^{\frac{1}{q-p}}. \quad \square \end{aligned}$$

Proposition 3.10. *Let $\lambda < \lambda_1(p)$ and $q > p$. Then the infimum*

$$\inf_{v \in \mathcal{N}} \varphi(v)$$

is attained. Moreover, every minimizer is a strictly positive (or strictly negative) critical point of φ .

Proof. The proof proceeds similarly to [15, Proposition 2.4] and [15, Proposition 2.6]. If a minimizer u was sign-changing, then $u = u^+ + u^-$, with $u^+, u^- \in \mathcal{N}$. But then

$$\varphi(u^+) < \varphi(u^+) + \varphi(u^-) = \varphi(u),$$

a contradiction. Hence, $u \geq 0$ (resp. $u \leq 0$), and $u \not\equiv 0$. The strict positivity (resp. negativity) follows from the Maximum Principle proved in [27]. $\square$

We are now going to prove some estimates for least energy solutions of (9). The results below are summarized in Figure 2.

Proposition 3.11. *Let u be a least energy solution of (9) for $\lambda < \lambda_1(p)$ and $q > p$. Then,*

$$\|\nabla u\|_p \leq |\Omega|^{\frac{1}{p}} \cdot \lambda_1(p)^{\frac{1}{p}} \cdot (\lambda_1(p) - \lambda)^{\frac{1}{q-p}}.$$

Proof. Since $u \in \mathcal{N}$, we have

$$\begin{aligned} \varphi(u) &= \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{p} \int_{\Omega} |u|^p dx - \frac{1}{q} \int_{\Omega} |u|^q dx \\ &= \left(\frac{1}{p} - \frac{1}{q} \right) \left(\int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} |u|^p dx \right) \geq \left(\frac{1}{p} - \frac{1}{q} \right) \left(1 - \frac{\lambda}{\lambda_1(p)} \right) \int_{\Omega} |\nabla u|^p dx. \end{aligned}$$

Let us denote by ψ a first eigenfunction of the p -Laplacian. Then

$$\begin{aligned} \left(\frac{1}{p} - \frac{1}{q}\right) \left(1 - \frac{\lambda}{\lambda_1(p)}\right) \int_{\Omega} |\nabla u|^p dx &\leq \varphi(u) = \inf_{v \in \mathcal{N}} \varphi(v) \leq \varphi(t^*(\psi)\psi) \\ &\leq (t^*(\psi))^p \left(\frac{1}{p} - \frac{1}{q}\right) \left(1 - \frac{\lambda}{\lambda_1(p)}\right) \int_{\Omega} |\nabla \psi|^p dx, \end{aligned}$$

which implies

$$\begin{aligned} \int_{\Omega} |\nabla u|^p dx &\leq (t^*(\psi))^p \int_{\Omega} |\nabla \psi|^p dx \\ &= \int_{\Omega} |\nabla \psi|^p dx \left(\frac{\int_{\Omega} |\nabla \psi|^p dx - \lambda \int_{\Omega} |\psi|^p dx}{\int_{\Omega} |\psi|^q dx} \right)^{\frac{p}{q-p}} \\ &\leq (\lambda_1(p) - \lambda)^{\frac{p}{q-p}} \int_{\Omega} |\nabla \psi|^p dx \left(\frac{\int_{\Omega} |\psi|^p dx}{\int_{\Omega} |\psi|^q dx} \right)^{\frac{p}{q-p}} \\ &\leq |\Omega| (\lambda_1(p) - \lambda)^{\frac{p}{q-p}} \frac{\int_{\Omega} |\nabla \psi|^p dx}{\int_{\Omega} |\psi|^p dx} \\ &= |\Omega| (\lambda_1(p) - \lambda)^{\frac{p}{q-p}} \cdot \lambda_1(p) \end{aligned}$$

and thus

$$\|\nabla u\|_p \leq |\Omega|^{\frac{1}{p}} \cdot \lambda_1(p)^{\frac{1}{p}} \cdot (\lambda_1(p) - \lambda)^{\frac{1}{q-p}}. \quad \square$$

Corollary 3.12. *Let u be a least energy solution of (9) for $\lambda < \lambda_1(p)$ and $q > p$. Then,*

$$\|u\|_{\infty} \leq C_p \cdot |\Omega|^{\frac{1}{p}} \cdot \lambda_1(p)^{\frac{1}{p}} \cdot (\lambda_1(p) - \lambda)^{\frac{1}{q-p}}.$$

The following proposition is valid for every solution of (9), and not only for least energy solutions.

Proposition 3.13. *Let u be a solution of (9) for $\lambda < \lambda_1(p)$ and $q > p$. Then,*

$$\|u\|_{\infty} \geq [C_p^p \cdot |\Omega| \cdot \lambda_1(p)]^{-\frac{1}{q-p}} (\lambda_1(p) - \lambda)^{\frac{1}{q-p}},$$

where C_p is defined in (8).

Proof. The proof is the same as in Proposition 3.4, taking into account the fact that here $q > p$. $\square$

4. THE LIMIT PROBLEM.

In the present section, we characterize uniform limits of solutions of (1) as solutions of a PDE. For the moment we shall assume the uniform convergence, that will be justified in Section 5.

Proposition 4.1. *Assume that $\{\lambda_p\}_p$ verifies $\lim_{p \rightarrow \infty} \lambda_p^{1/p} = \Lambda$ and let $q(p)$ and $Q > 0$, $Q \neq 1$ as in (2). Suppose that $u_{\lambda_p, p} \in W_0^{1,p}(\Omega)$ is a weak solution of*

$$\begin{cases} -\Delta_p u_{\lambda_p, p} = \lambda_p u_{\lambda_p, p}^{p-1} + u_{\lambda_p, p}^{q(p)-1} & \text{in } \Omega \\ u_{\lambda_p, p} > 0 & \text{in } \Omega \\ u_{\lambda_p, p} = 0 & \text{on } \partial\Omega, \end{cases} \quad (12)$$

and that $u_{\lambda_p, p} \rightarrow u_{\Lambda} > 0$ uniformly as $p \rightarrow \infty$. Then, u_{Λ} is a viscosity solution of the following problem

$$\begin{cases} \min \{ |\nabla u_{\Lambda}| - \max \{ \Lambda u_{\Lambda}, u_{\Lambda}^Q \}, -\Delta_{\infty} u_{\Lambda} \} = 0 & \text{in } \Omega, \\ u_{\Lambda} > 0 & \text{in } \Omega \\ u_{\Lambda} = 0 & \text{on } \partial\Omega. \end{cases} \quad (13)$$

FIGURE 1. Regions of non-existence given by Propositions 3.4, 3.6, and 3.1, and branch of solutions of problem (1) for $q < p$ given by Proposition 3.7.

FIGURE 2. Regions of non-existence given by Corollary 3.12, Propositions 3.13 and 3.1, and sketch of the set of solutions of problem (1) for $q > p$.

Proof. Let $x_0 \in \Omega$ and $\phi \in \mathcal{C}^2(\Omega)$ such that $u_\Lambda - \phi$ has a strict local minimum at x_0 . By hypothesis, u_Λ is the uniform limit of $u_{\lambda_p, p}$, so there exists a sequence of points $x_p \rightarrow x_0$ such that $(u_{\lambda_p, p} - \phi)(x_p)$ is a local minimum for each p . As $u_{\lambda_p, p}$ is a continuous weak solution of (12), it is also a viscosity solution and so a supersolution. Then, we get

$$\begin{aligned} -(p-2)|\nabla \phi(x_p)|^{p-4} \left\{ \frac{|\nabla \phi(x_p)|^2}{p-2} \Delta \phi(x_p) + \langle D^2 \phi(x_p) \nabla \phi(x_p), \nabla \phi(x_p) \rangle \right\} \\ = -\Delta_p \phi(x_p) \geq \lambda_p u_{\lambda_p, p}^{p-1}(x_p) + u_{\lambda_p, p}^{q(p)-1}(x_p). \end{aligned}$$

Rearranging terms, we obtain

$$-(p-2) \left[\frac{|\nabla \phi(x_p)|}{\left(\lambda_p u_{\lambda_p, p}^{p-1}(x_p) + u_{\lambda_p, p}^{q(p)-1}(x_p) \right)^{\frac{1}{p-4}}} \right]^{p-4} \left\{ \frac{|\nabla \phi(x_p)|^2}{p-2} \Delta \phi(x_p) + \langle D^2 \phi(x_p) \nabla \phi(x_p), \nabla \phi(x_p) \rangle \right\} \geq 1. \quad (14)$$

Notice that

$$\begin{aligned} \max \left\{ \lambda_p^{\frac{1}{p-4}} u_{\lambda_p, p}^{\frac{p-1}{p-4}}(x_p), u_{\lambda_p, p}^{\frac{q(p)-1}{p-4}}(x_p) \right\} &\leq \left(\lambda_p u_{\lambda_p, p}^{p-1}(x_p) + u_{\lambda_p, p}^{q(p)-1}(x_p) \right)^{\frac{1}{p-4}} \\ &\leq 2^{\frac{1}{p-4}} \max \left\{ \lambda_p^{\frac{1}{p-4}} u_{\lambda_p, p}^{\frac{p-1}{p-4}}(x_p), u_{\lambda_p, p}^{\frac{q(p)-1}{p-4}}(x_p) \right\}, \end{aligned}$$

and hence,

$$\lim_{p \rightarrow \infty} \left(\lambda_p u_{\lambda_p, p}^{p-1}(x_p) + u_{\lambda_p, p}^{q(p)-1}(x_p) \right)^{\frac{1}{p-4}} = \max \{ \Lambda u_\Lambda(x_0), u_\Lambda^Q(x_0) \}.$$

We point out that the latter quantity is positive since $u_\Lambda > 0$ by hypothesis. Then, if we suppose that

$$|\nabla \phi(x_0)| < \max \{ \Lambda u_\Lambda(x_0), u_\Lambda^Q(x_0) \}$$

we obtain a contradiction letting $p \rightarrow \infty$ in (14). Thus, it must be

$$|\nabla \phi(x_0)| - \max \{ \Lambda u_\Lambda(x_0), u_\Lambda^Q(x_0) \} \geq 0. \quad (15)$$

We also have that

$$-\Delta_\infty \phi(x_0) = -\langle D^2 \phi(x_0) \nabla \phi(x_0), \nabla \phi(x_0) \rangle \geq 0, \quad (16)$$

because we would get a contradiction with (14) otherwise.

We can put together (15) and (16) writing

$$\min \{ |\nabla \phi(x_0)| - \max \{ \Lambda u_\Lambda(x_0), u_\Lambda^Q(x_0) \}, -\Delta_\infty \phi(x_0) \} \geq 0$$

and conclude that u_Λ is a viscosity supersolution of equation (13).

It remains to be shown that u_Λ is a viscosity subsolution of the limit equation (13), i.e. we have to show that, for each $x_0 \in \Omega$ and $\phi \in \mathcal{C}^2(\Omega)$ such that $u_\Lambda - \phi$ attains a strict local maximum at x_0 (note that x_0 and ϕ are not the same than before) we have

$$\min \left\{ |\nabla \phi(x_0)| - \max \{ \Lambda u_\Lambda(x_0), u_\Lambda^Q(x_0) \}, -\Delta_\infty \phi(x_0) \right\} \leq 0.$$

We can suppose that

$$|\nabla \phi(x_0)| > \max \{ \Lambda u_\Lambda(x_0), u_\Lambda^Q(x_0) \},$$

because we are done otherwise. As we did before, the uniform convergence of $u_{\lambda_p, p}$ to u_Λ provides a sequence of points $x_p \rightarrow x_0$ which are local maxima of $u_{\lambda_p, p} - \phi$. Recalling the definition of viscosity subsolution we have

$$-(p-2) \left[\frac{|\nabla \phi(x_p)|}{\left(\lambda_p u_{\lambda_p, p}^{p-1}(x_p) + u_{\lambda_p, p}^{q(p)-1}(x_p) \right)^{\frac{1}{p-4}}} \right]^{p-4} \left\{ \frac{|\nabla \phi(x_p)|^2}{p-2} \Delta \phi(x_p) + \langle D^2 \phi(x_p) \nabla \phi(x_p), \nabla \phi(x_p) \rangle \right\} \leq 1,$$

for each fixed p . Letting $p \rightarrow \infty$ we obtain $-\Delta_\infty \phi(x_0) \leq 0$ because in other case we get a contradiction. $\square$

5. ANALYSIS OF THE LIMIT PROBLEM

5.1. A comparison principle for the limit problem when $Q < 1$. We have the following comparison principle for the limit equation

$$\min\{|\nabla u| - \max\{\Lambda u, u^Q\}, -\Delta_\infty u\} = 0$$

for $Q < 1$ and $\Lambda < \Lambda_1(\Omega)$.

Proposition 5.1. *Let $Q < 1$ and $\Lambda < \Lambda_1(\Omega)$. Let $u, v \in \mathcal{C}(\overline{\Omega})$ satisfy*

$$\min\{|\nabla u| - \max\{\Lambda u, u^Q\}, -\Delta_\infty u\} \leq 0 \quad \text{in } \Omega$$

and

$$\min\{|\nabla v| - \max\{\Lambda v, v^Q\}, -\Delta_\infty v\} \geq 0 \quad \text{in } \Omega.$$

Moreover, suppose that $u, v > 0$ in Ω , and $u = 0$ on $\partial\Omega$. Then, $u \leq v$ in Ω .

In the proof of the Comparison Principle we need some partial results.

Lemma 5.2. *Let $u \in \mathcal{C}(\overline{\Omega})$ satisfy*

$$\min\{|\nabla u| - \max\{\Lambda u, u^Q\}, -\Delta_\infty u\} \leq 0 \quad \text{in } \Omega$$

in the viscosity sense for $Q < 1$ and $\Lambda < \Lambda_1(\Omega)$. Moreover, suppose that $u > 0$ in Ω and $u = 0$ on $\partial\Omega$. Then,

$$\|u\|_\infty \leq \Lambda_1(\Omega)^{-\frac{1}{1-Q}}.$$

Proof. Let $x_0 \in \Omega$, and let $\phi \in \mathcal{C}^2(\Omega)$ be such that $u(x_0) = \phi(x_0)$, and $u - \phi$ has a maximum in x_0 . By definition of viscosity subsolution it holds

$$\min\{|\nabla \phi(x_0)| - \max\{\Lambda \phi(x_0), \phi(x_0)^Q\}, -\Delta_\infty \phi(x_0)\} \leq 0.$$

Hence, either $-\Delta_\infty \phi(x_0) \leq 0$, or

$$|\nabla \phi(x_0)| \leq \max\{\Lambda \phi(x_0), \phi(x_0)^Q\} \leq \max\{\Lambda \|u\|_\infty, \|u\|_\infty^Q\} =: C.$$

This means that

$$\min\{|\nabla u| - C, -\Delta_\infty u\} \leq 0$$

in the viscosity sense. Since the function $w(x) = C \operatorname{dist}(x, \partial\Omega)$ is the unique viscosity solution of the problem

$$\begin{cases} \min\{|\nabla w| - C, -\Delta_\infty w\} &= 0 & \text{in } \Omega, \\ w &= 0 & \text{on } \partial\Omega, \end{cases}$$

by comparison (see [16, 17]) it follows that $u \leq w$, which means

$$u(x) \leq \max\{\Lambda \|u\|_\infty, \|u\|_\infty^Q\} \cdot \operatorname{dist}(x, \partial\Omega),$$

and thus

$$\|u\|_\infty \leq \max\{\Lambda \|u\|_\infty, \|u\|_\infty^Q\} \cdot \Lambda_1(\Omega)^{-1}.$$

Since $\Lambda < \Lambda_1(\Omega)$, it must hold

$$\max\{\Lambda \|u\|_\infty, \|u\|_\infty^Q\} = \|u\|_\infty^Q,$$

and hence

$$\|u\|_\infty \leq \Lambda_1(\Omega)^{-\frac{1}{1-Q}}. \quad \square$$

Remark 5.3. Arguing similarly as in the previous theorem, we observe that if u is a subsolution in the case $Q > 1$ and $\Lambda < \Lambda_1(\Omega)$, it holds

$$\|u\|_\infty \geq \Lambda_1(\Omega)^{\frac{1}{Q-1}}.$$

Corollary 5.4. *Let $u \in \mathcal{C}(\overline{\Omega})$ satisfy*

$$\min\{|\nabla u| - \max\{\Lambda u, u^Q\}, -\Delta_\infty u\} \leq 0$$

in the viscosity sense for $Q < 1$ and $\Lambda < \Lambda_1(\Omega)$. Moreover, suppose that $u > 0$ in Ω and $u = 0$ on $\partial\Omega$. Then u satisfies

$$\min\{|\nabla u| - u^Q, -\Delta_\infty u\} \leq 0.$$

Proof. By Lemma 5.2, we have that $\|u\|_\infty \leq \Lambda_1(\Omega)^{-\frac{1}{1-Q}} \leq \Lambda^{-\frac{1}{1-Q}}$, which implies that

$$\max\{\Lambda u, u^Q\} = u^Q. \quad \square$$

We are now in position to prove the Comparison Principle.

Proof of Proposition 5.1. It is clear that v satisfies

$$\min\{|\nabla v| - v^Q, -\Delta_\infty v\} \geq 0$$

in the viscosity sense, since $v^Q \leq \max\{\Lambda v, v^Q\}$. On the other hand, by Corollary 5.4, u is a viscosity subsolution of the same problem. Hence, the claim follows from the comparison result of [8, Theorem 10]. $\square$

As a consequence of the above results, we can characterize very precisely the solution to problem (3) when $\Lambda < \Lambda_1(\Omega)$.

Proposition 5.5. *Let $Q < 1$. Then, for every $\Lambda \in [0, \Lambda_1(\Omega))$ problem (3) has a unique nontrivial solution u_Λ . Moreover, u_Λ coincides with w_1 , the unique nontrivial solution of problem (4), and satisfies*

$$\|u_\Lambda\|_\infty = \Lambda_1(\Omega)^{-\frac{1}{1-Q}}. \quad (17)$$

Proof. By Corollary 5.4 we have that u_Λ satisfies

$$\min\{|\nabla u_\Lambda| - u_\Lambda^Q, -\Delta_\infty u_\Lambda\} \leq 0$$

in the viscosity sense. Since the converse inequality follows trivially from the fact that u_Λ is a supersolution of (3), we actually have that u_Λ is a positive viscosity solution of problem (4). By the results in [8], problem (4) has a unique solution that satisfies (17). $\square$

Remark 5.6. Notice that, as w_1 is the unique solution of (3) for every $\Lambda \in [0, \Lambda_1(\Omega))$, we can take limits $\Lambda \rightarrow \Lambda_1(\Omega)$ in the viscosity sense and get that w_1 also solves (3) for $\Lambda = \Lambda_1(\Omega)$.

5.2. Non-existence for $\Lambda > \Lambda_1(\Omega)$. We prove that the limit equation

$$\min\{|\nabla u| - \max\{\Lambda u, u^Q\}, -\Delta_\infty u\} = 0$$

does not admit any positive solution for $\Lambda > \Lambda_1(\Omega)$. The result holds true in both cases $Q < 1$ and $Q > 1$.

Proposition 5.7. *The limit equation*

$$\min\{|\nabla u| - \max\{\Lambda u, u^Q\}, -\Delta_\infty u\} = 0$$

does not admit any positive solution for $\Lambda > \Lambda_1(\Omega)$.

Proof. The proof follows the ideas of [19, Theorem 3.1]. Suppose by contradiction that there exists a $\Lambda > \Lambda_1(\Omega)$ such that the equation has a positive solution u_Λ . By definition of $\Lambda_1(\Omega)$, there exists a ball $B \subset\subset \Omega$ of radius Λ^{-1} . Let $\delta : B \rightarrow \mathbb{R}$ be defined as

$$\delta(x) := \text{dist}(x, \partial B).$$

For every $c > 0$ the function $v := c\delta$ satisfies the equation

$$\begin{cases} \min\{|\nabla v| - \Lambda v, -\Delta_\infty v\} = 0 & \text{in } B \\ v = 0 & \text{on } \partial B. \end{cases}$$

On the other hand,

$$\min\{|\nabla u_\Lambda| - \Lambda u_\Lambda, -\Delta_\infty u_\Lambda\} \geq 0 \quad \text{in } B$$

and $u_\Lambda > 0$ on ∂B . By the comparison principle (see [19])

$$v = c\delta \leq u_\Lambda$$

in B , which yields a contradiction if we take $c \rightarrow \infty$. Thus, $\Lambda \leq \Lambda_1(\Omega)$. $\square$

5.3. Existence for $\Lambda < \Lambda_1(\Omega)$. For a fixed $\Lambda < \Lambda_1(\Omega)$ we obtain solutions as limits.

Theorem 5.8. *Let $Q < 1$ and fix $\Lambda \in (0, \Lambda_1(\Omega))$. Let $\{\lambda_p\}_p$ be a sequence such that $\lim_{p \rightarrow \infty} \lambda_p^{1/p} = \Lambda$ and consider $\{u_{\lambda_p, p}\}_p$, the corresponding sequence of positive weak solutions of (12). Then,*

$$u_{\lambda_p, p} \rightarrow u_\Lambda \quad \text{uniformly as } p \rightarrow \infty,$$

with $u_\Lambda(x)$ the unique positive viscosity solution of problem (13). Moreover, $u_\Lambda \equiv w_1$, the unique positive viscosity solution of (4).

Proof. Since $\lim_{p \rightarrow \infty} \lambda_p^{1/p} = \Lambda$, we can assume $\lambda_p < \lambda_1(p)$ without loss of generality. Then, Proposition 3.4, Lemma 2.5, and the Ascoli-Arzelà Theorem give the existence of u_Λ and the uniform convergence up to a subsequence. Once we know that a subsequence converges uniformly and that the limit does not degenerate (see Proposition 3.6 and the results in [8]), Proposition 4.1 gives the characterization of the limit as a solution of (13). Finally, we have that the whole sequence converges by uniqueness of the limit, see Proposition 5.5. $\square$

Theorem 5.9. *Let $Q > 1$ and fix $\Lambda \in (0, \Lambda_1(\Omega))$. Let $\{\lambda_p\}_p$ be a sequence such that $\lim_{p \rightarrow \infty} \lambda_p^{1/p} = \Lambda$ and consider $\{u_{\lambda_p, p}\}_p$, the corresponding sequence of least energy solutions of (12). Then, up to a subsequence,*

$$u_{\lambda_p, p} \rightarrow u_\Lambda \quad \text{uniformly as } p \rightarrow \infty,$$

with $u_\Lambda(x)$ a positive viscosity solution of problem (13).

Proof. Since $\lim_{p \rightarrow \infty} \lambda_p^{1/p} = \Lambda$, we can assume $\lambda_p < \lambda_1(p)$ without loss of generality. Then, Proposition 3.11, Corollary 3.12, Lemma 2.5, and the Ascoli-Arzelà Theorem give the existence of u_Λ and the uniform convergence up to a subsequence. As before, once we know that a subsequence converges uniformly and that the limit does not degenerate, Proposition 4.1 will give the characterization of the limit as a solution of (13).

For the proof of the positivity of the limit, notice that u_Λ is ∞ -superharmonic and non-negative (it is a uniform limit of p -superharmonic positive functions). Then, Proposition 3.13 and the Harnack inequality for ∞ -superharmonic functions (see [22, 23]) imply $u_\Lambda > 0$ inside Ω . $\square$

5.4. The case $\Lambda = \Lambda_1(\Omega)$. In the case $\Lambda = \Lambda_1(\Omega)$ we have to distinguish between the two cases $Q < 1$ and $Q > 1$.

Proposition 5.10. *Let $Q < 1$ and $\Lambda = \Lambda_1(\Omega)$. Then every solution of*

$$\min\{|\nabla u| - \max\{\Lambda u, u^Q\}, -\Delta_\infty u\} = 0$$

satisfies

$$\|u\|_\infty \geq \Lambda_1(\Omega)^{-\frac{1}{1-Q}}.$$

Proof. Let u be a solution of the equation. Let ψ be a positive eigenfunction with $\|\psi\|_\infty = \Lambda_1(\Omega)^{-\frac{1}{1-Q}}$. This yields $\psi^Q \geq \Lambda_1(\Omega)\psi$ and then

$$\min\{|\nabla \psi| - \Lambda_1(\Omega)\psi, -\Delta_\infty \psi\} = 0$$

implies

$$\min\{|\nabla \psi| - \psi^Q, -\Delta_\infty \psi\} \leq 0.$$

On the other hand u satisfies

$$\min\{|\nabla u| - u^Q, -\Delta_\infty u\} \geq 0.$$

By the comparison principle in [8, Theorem 10] it follows that $\psi \leq u$ and hence

$$\|u\|_\infty \geq \|\psi\|_\infty = \Lambda_1(\Omega)^{-\frac{1}{1-Q}}. \quad \square$$

Proposition 5.11. *Let $Q < 1$ and $\Lambda = \Lambda_1(\Omega)$. Then for every $M \geq \Lambda_1(\Omega)^{-\frac{1}{1-Q}}$ there exists a solution of*

$$\min\{|\nabla u| - \max\{\Lambda u, u^Q\}, -\Delta_\infty u\} = 0$$

which satisfies

$$\|u\|_\infty = M.$$

Proof. The claim follows from Remark 5.6 in the case $M = \Lambda_1(\Omega)^{-\frac{1}{1-Q}}$. Suppose now that $M > \Lambda_1(\Omega)^{-\frac{1}{1-Q}}$ and consider a sequence $\{M_k\}_k$ such that $M_k \rightarrow M$ as $k \rightarrow \infty$. Fix a sequence $\{p_k\}$ such that $p_k \rightarrow \infty$ as $k \rightarrow \infty$. Let w_{1,p_k} be the solution of (9) for $\lambda = 0$. Since by the results in [8] we know that $\|w_{1,p_k}\|_\infty \rightarrow \Lambda_1(\Omega)^{-\frac{1}{1-Q}}$ as $k \rightarrow \infty$, we can assume that $M_k \geq \|w_{1,p_k}\|_\infty$. Under this condition, by Proposition 3.7 and the Intermediate Value Theorem, for every k there exists a $\lambda_k \in [0, \lambda_1(p_k))$ for which there exists a solution v_k of

$$\begin{cases} -\Delta_p v_k &= \lambda_k v_k^{p-1} + v_k^{q(p_k)-1} & \text{in } \Omega, \\ v_k &= 0 & \text{on } \partial\Omega, \end{cases}$$

with $\|v_k\|_\infty = M_k$. Passing to a subsequence (not relabeled), we can suppose that $\lim_{k \rightarrow \infty} \lambda_k^{\frac{1}{p_k}} = \Lambda$. From Proposition 2.2, $\Lambda \leq \Lambda_1(\Omega)$. The functions v_k converge uniformly as $k \rightarrow \infty$ to $v > 0$ (the positivity follows from Proposition 3.6 and the results in [8]), a viscosity solution of

$$\min\{|\nabla v| - \max\{\Lambda v, v^Q\}, -\Delta_\infty v\} = 0$$

such that $\|v\|_\infty = M$. If $\Lambda < \Lambda_1(\Omega)$, this would be a contradiction to Lemma 5.2, since $M > \Lambda_1(\Omega)^{-\frac{1}{1-Q}}$. Hence, $\Lambda = \Lambda_1(\Omega)$, and the claim is proved. $\square$

Proposition 5.12. *Let $Q > 1$ and $\Lambda = \Lambda_1(\Omega)$. Then for every $M \leq \Lambda_1(\Omega)^{\frac{1}{Q-1}}$ there exists a solution of*

$$\min\{|\nabla u| - \max\{\Lambda u, u^Q\}, -\Delta_\infty u\} = 0$$

which satisfies

$$\|u\|_\infty = M.$$

Proof. Let ψ be a positive solution of

$$\min\{|\nabla \psi| - \Lambda \psi, -\Delta_\infty \psi\} = 0$$

with $\|\psi\|_\infty = M$. Since $M \leq \Lambda_1(\Omega)^{\frac{1}{Q-1}}$, we have that $\Lambda \psi \geq \psi^Q$, and hence ψ solves

$$\min\{|\nabla \psi| - \max\{\Lambda \psi, \psi^Q\}, -\Delta_\infty \psi\} = 0. \quad \square$$

FIGURE 3. Regions of non-existence given by Propositions 5.5, 5.10 and 5.7 and branch of solutions of problem (3) for $Q < 1$

FIGURE 4. Regions of non-existence given by Remark 5.3 and Proposition 5.7 and branch of solutions of problem (3) for $Q > 1$

6. EXPLICIT SOLUTIONS FOR THE LIMIT PROBLEM IN SPECIAL DOMAINS

We first introduce some notation necessary in the sequel. We define the *ridge set* of Ω ,

$$\begin{aligned} \mathcal{R} &= \{x \in \Omega : \text{dist}(x, \partial\Omega) \text{ is not differentiable at } x\} \\ &= \{x \in \Omega : \exists x_1, x_2 \in \partial\Omega, x_1 \neq x_2, \text{ s.t. } |x - x_1| = |x - x_2| = \text{dist}(x, \partial\Omega)\} \end{aligned}$$

and its subset $\mathcal{M}$, the *set of maximal distance* to the boundary,

$$\mathcal{M} = \{x \in \Omega : \text{dist}(x, \partial\Omega) = \|\text{dist}(\cdot, \partial\Omega)\|_\infty\}.$$

In this section we are going to show that in bounded domains satisfying the geometric condition $\mathcal{M} \equiv \mathcal{R}$ it is possible to find a curve of explicit positive solutions corresponding to the solutions already found in previous sections (see Figures 3 and 4). Some examples of domains satisfying the geometric condition are the ball, the annulus and the stadium (convex hull of two balls of the same radius). A square or an ellipse do not verify the condition.

Theorem 6.1. *Suppose that Ω is a bounded domain such that $\mathcal{M} \equiv \mathcal{R}$ and let $\Lambda > 0$, $Q > 0$, $Q \neq 1$. Consider problem (3) and solutions of the form*

$$u(x) = a \cdot \text{dist}(x, \partial\Omega), \quad a > 0. \quad (18)$$

Then, whenever $Q \in (0, 1)$:

- i) For every $\Lambda \in [0, \Lambda_1(\Omega))$ problem (3) has a unique viscosity solution, given by (18) with $a = \Lambda_1(\Omega)^{-\frac{Q}{1-Q}}$.*
- ii) For $\Lambda = \Lambda_1(\Omega)$, problem (3) has exactly one viscosity solution of the form (18) with $\|u\|_\infty = M$ for every $M \geq \Lambda_1(\Omega)^{-\frac{1}{1-Q}}$ and no solution for $M < \Lambda_1(\Omega)^{-\frac{1}{1-Q}}$.*
- iii) If $\Lambda > \Lambda_1(\Omega)$ problem (3) has no viscosity solutions.*

Moreover, whenever $Q > 1$:

- i) For every $\Lambda \in [0, \Lambda_1(\Omega))$ problem (3) has a unique viscosity solution of the form (18) corresponding to $a = \Lambda_1(\Omega)^{\frac{Q}{Q-1}}$.*
- ii) For $\Lambda = \Lambda_1(\Omega)$, problem (3) has exactly one viscosity solution of the form (18) with $\|u\|_\infty = M$ for every $M \geq \Lambda_1(\Omega)^{\frac{1}{Q-1}}$ and no solution for $M < \Lambda_1(\Omega)^{\frac{1}{Q-1}}$.*
- iii) If $\Lambda > \Lambda_1(\Omega)$ problem (3) has no viscosity solutions.*

Proof. First of all, since (18) is regular outside $\mathcal{R} \equiv \mathcal{M}$ it can be checked by direct computation that

$$-\Delta_\infty u(x) = 0 \quad \forall x \in \Omega \setminus \mathcal{R},$$

in the classical sense. Hence, we need make sure that

$$|\nabla u(x)| - \max \{ \Lambda u(x), u^Q(x) \} \geq 0 \quad \forall x \in \Omega \setminus \mathcal{R}.$$

Indeed, plugging (18) into the latter expression (recall that $x \notin \mathcal{R}$ so the derivatives are classical), we find that

$$|\nabla u(x)| - \max \{ \Lambda u(x), u^Q(x) \} = a - \max \{ \Lambda a \text{dist}(x, \partial\Omega), a^Q \text{dist}(x, \partial\Omega)^Q \}.$$

Since we can choose points $x \notin \mathcal{R} \equiv \mathcal{M}$ arbitrarily close to $\mathcal{M}$, we find the following necessary condition for a ,

$$a - \max \{ a \Lambda \Lambda_1(\Omega)^{-1}, a^Q \Lambda_1(\Omega)^{-Q} \} \geq 0. \quad (19)$$

Now, we turn our attention to the ridge set $\mathcal{R}$. First, observe that cones as in (18) are always supersolutions of (3) in the ridge set, since they cannot be touched from below with $\mathcal{C}^2$ functions at those points.

Hence, we only have to consider the subsolution case. So, let $x_0 \in \mathcal{R}$ and $\phi \in \mathcal{C}^2$ such that $u - \phi$ has a local maximum at x_0 . We aim to prove that

$$\min \{ |\nabla \phi(x_0)| - \max \{ \Lambda u(x_0), u^Q(x_0) \}, -\Delta_\infty \phi(x_0) \} \leq 0. \quad (20)$$

It is well known (see for instance [17, Lemma 6.10]) that u in (18) satisfies

$$\min \{ |\nabla u(x)| - a, -\Delta_\infty u(x) \} = 0.$$

Thus, by definition of viscosity subsolution we have that either $|\nabla \phi(x_0)| \leq a$ or $-\Delta_\infty \phi(x_0) \leq 0$. In the latter case, (20) holds and there is nothing to prove. Thus, we can suppose in the sequel that $-\Delta_\infty \phi(x_0) > 0$ and $|\nabla \phi(x_0)| \leq a$. Then, since $x_0 \in \mathcal{R} \equiv \mathcal{M}$, we have $u(x_0) = a \Lambda_1(\Omega)^{-1}$ and

$$|\nabla \phi(x_0)| - \max \{ \Lambda u(x_0), u^Q(x_0) \} \leq a - \max \{ a \Lambda \Lambda_1(\Omega)^{-1}, a^Q \Lambda_1(\Omega)^{-Q} \}.$$

Recalling (19), we discover that the only possibility is

$$a - \max \{ a \Lambda \Lambda_1(\Omega)^{-1}, a^Q \Lambda_1(\Omega)^{-Q} \} = 0. \quad (21)$$

The rest of the proof is devoted to study the number of positive solutions of this equation.

We rewrite (21) as $\Phi_\Lambda(a) = 1$ where $\Phi_\Lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is given by

$$\Phi_\Lambda(a) = \max\{\Lambda\Lambda_1(\Omega)^{-1}, \Lambda_1(\Omega)^{-Q}a^{Q-1}\}.$$

From here, it is easy to check that:

- (1) $\Phi_\Lambda(a) \neq 1$ for all $a > 0$ if $\Lambda > \Lambda_1(\Omega)$.
- (2) Consider the case $Q < 1$. Then:
 - (a) If $\Lambda = \Lambda_1(\Omega)$, then every $a \geq \Lambda_1(\Omega)^{-\frac{Q}{1-Q}}$ satisfies $\Phi_\Lambda(a) = 1$.
 - (b) If $\Lambda < \Lambda_1(\Omega)$, then $a = \Lambda_1(\Omega)^{-\frac{Q}{1-Q}}$ is the only solution of $\Phi_\Lambda(a) = 1$.
- (3) Consider the case $Q > 1$. Then:
 - (a) If $\Lambda = \Lambda_1(\Omega)$, then every positive $a \leq \Lambda_1(\Omega)^{\frac{Q}{Q-1}}$ satisfies $\Phi_\Lambda(a) = 1$.
 - (b) If $\Lambda < \Lambda_1(\Omega)$, then $a = \Lambda_1(\Omega)^{\frac{Q}{Q-1}}$ is the only solution of $\Phi_\Lambda(a) = 1$.

For the non-existence result in the statement of the theorem, we use Proposition 5.7. $\square$

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