Rough solutions for the periodic Korteweg–de Vries equation

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Abstract

We show how to apply ideas from the theory of rough paths to the analysis of low-regularity solutions to non-linear dispersive equations. Our basic example will be the one dimensional Korteweg–de Vries (KdV) equation on a periodic domain and with initial condition in $\mathcal{F}L^{\alpha,p}$ spaces. We discuss convergence of Galerkin approximations, a modified Euler scheme and the presence of a random force of white-noise type in time.

Keywords: dispersive equations; rough paths; power series solutions.

In this note we start by considering the Cauchy problem for the classical Korteweg–de Vries (KdV) equation:

$$\partial_t u(t,\xi) + \partial_\xi^3 u(t,\xi) + \frac{1}{2}\partial_\xi u(t,\xi)^2 = 0, \quad u(0,\xi) = u_0(\xi), \qquad (t,\xi) \in \mathbb{R}\times\mathbb{T} \tag{1}$$

where the initial condition u_0 belongs to some Sobolev space $H^\alpha(\mathbb{T})$ of the torus $\mathbb{T} = [-\pi,\pi]$. In a remarkable series of papers by Bourgain [2, 17], Kenig–Ponce–Vega [27, 26, 25] and later Colliander–Keel–Staffilani–Takaoka–Tao [5] this equation has been proved to possess global solutions starting from initial conditions in H^α for any $\alpha \geq -1/2$. The existence of solutions in negative Sobolev spaces is possible due to the regularizing effect of the dispersive linear term. This regularization is more effective in the whole line and there global solutions exists for any $\alpha \geq -3/4$ [26, 5, 22]. Other references on the analysis of the KdV equation are [16, 24]. More recently Kappeler–Topalov, taking advantage of the complete integrability of this model, extended the global well-posedness in the periodic setting to any $\alpha \geq -1$ using the inverse scattering method [23].

Inspired by the theory of *rough paths* we will look for an alternative approach to the construction of solutions of eq. (1) and in general for dispersive equations with polynomial non-linearities. Our method turns out to have some similarities with Christ's power series approach [3, 32] and to allow to consider KdV with initial condition in $\mathcal{F}L^{\alpha,p}$ for $1 \leq p \leq +\infty$.

Rough paths (introduced by Lyons in [31, 30]) allow the study of differential equations driven by irregular functions. They have been applied to the path-wise study of stochastic differential equations driven by Brownian motion, by fractional Brownian motion of any index $H > 1/4$ [8] and other stochastic processes [7, 13]. Part of the theory has been reformulated in terms of the sub-Riemmannian geometry of certain Carnot groups [11]. In [18] we showed how to reinterpret the work of Lyons in the terms of a cochain complex of finite increments and a related integration theory. The key step is the introduction of a map Λ (called sewing map) which encodes a basic fact of rough path theory. We exploited this point of view to treat stochastic partial differential equations of evolution type [29, 20] and to study the initial value problem for a partial differential equation modeling the approximate evolution of random vortex filaments in 3d fluids [1].

We would like to show that the concepts of the theory can be used fruitfully for problems not related to stochastic processes. The periodic Korteweg–de Vries equation is used as a case study for our ideas. The point of view here developed should be applicable also to KdV on the full real line or other dispersive semi-linear equations like the modified KdV or the non-linear Schrödinger equation (indeed the results [3, 32] obtained via the power-series method can be understood in terms of rough paths following the lines of the present investigation).

No previous knowledge of rough paths theory is required nor any result on the periodic KdV is used in the following. We took care to make the paper, as much as possible, self-contained. Since the arguments are similar to those used in finite dimensions in [18], the reader can refer to this last paper to gain a wider perspective on the technique and on the stochastic applications of rough paths.

Let us describe the main results of this paper in terms of distributional solutions of KdV: let P_N be the Fourier projector on modes $k \in \mathbb{Z}$ such that $|k| \leq N$ and $\mathcal{N}(\varphi)(t,\xi) = \partial_\xi(\varphi(t,\xi)^2)/2$ for smooth functions φ. Then

Theorem 1 *For any $1 \leq p \leq \infty$, $\alpha > \alpha_*(p) = \max(-1/p, -1/2)$ and $u_0 \in \mathcal{F}L^{\alpha,p}$ there exists a $T_* > 0$ and a continuous function $u(t) \in C([0,T^*], \mathcal{F}L^{\alpha,p})$ with $u(0) = u_0$ for which the distribution $\mathcal{N}(P_N u)$ converge as $N \to \infty$ to a limit which we denote by $\mathcal{N}(u)$ and moreover the distributional equation*

$$\partial_t u + \partial_\xi^3 u + \mathcal{N}(u) = 0$$

is satisfied in $[0,T^*] \times \mathbb{T}$.

This solution is the limit of smooth solutions and of some modified Galerkin approximations. There exists a natural space of continuous functions on $\mathcal{F}L^{\alpha,p}$ for which the nonlinear term can be defined as a distribution and where uniqueness of solutions holds. We show also how to implement the $L^2(\mathbb{T})$ conservation law in the rough path approach and obtain in this way global solutions in $L^2(\mathbb{T})$. Following the lines of the numerical study of SDEs driven by rough paths [9, 14, 12] we analyze an Euler-like time-discretization of the PDE which converges to the above solution. Finally we also prove an existence and uniqueness result under random perturbation of white-noise type in time of the form

$$\partial_t u + \partial_\xi^3 u + \frac{1}{2}\partial_\xi u^2 = \Phi \partial_t \partial_\xi B$$

where Φ is a bounded linear operator from $\mathcal{F}L^{0,\infty}$ to $\mathcal{F}L^{\alpha,p}$ and $\partial_t \partial_\xi B$ a white noise on $\mathbb{R} \times \mathbb{T}$.

Plan. In Sect. 1 we start by recasting the KdV eq. (1) in its mild form and to perform some manipulation to motivate the finite-increment equation which we will study in Sect. 2 where we prove existence and uniqueness of local solutions and discuss the distributional meaning of these solutions. Then we prove a-priori estimates necessary show that global rough solutions exists for initial conditions in $L^2(\mathbb{T})$ (Sect. 2.1). In Sect. 2.2 we prove that the rough solutions are limits of suitable Galerkin approximations. Moreover in Sect. 2.3 we introduce a time discretization scheme and prove its convergence. The equation that we study is just one of a stack of finite-increment equations that can be generated starting from KdV. In Sect. 2.4 we derive the next member of this hierarchy. Finally Sect. 3 addresses the problem of the presence of an additive stochastic forcing. We collect some longer and technical proofs in the Appendix.

Notations. We denote with $\hat{f} : \mathbb{Z} \to \mathbb{C}$ the Fourier coefficients of a real function $f : \mathbb{T} \to \mathbb{R}$: $f(\xi) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{ik\xi}$ and define the space

$$\mathcal{F}L^{\alpha,p} = \{f \in \mathcal{S}'(\mathbb{T}) : \hat{f}(0) = 0, |f|_{\mathcal{F}L^{\alpha,p}} = |\langle\cdot\rangle^\alpha \hat{f}|_{\ell^p} < \infty\}$$

where $\mathcal{S}'(\mathbb{T})$ is the space of real Schwartz distributions on the torus $\mathbb{T}$ and $\langle x \rangle = (1+|x|^2)^{1/2}$. We restrict the space to the mean zero functions since this is the natural setting to discuss periodic KdV. Then $H^\alpha(\mathbb{T})\backslash\mathbb{R} = \mathcal{F}L^{\alpha,2}$. Sometimes we note $\mathcal{F}$ the Fourier transform operator so that $\mathcal{F}f = \hat{f}$. Given two Banach spaces V, W, denote with $\mathcal{L}(V,W)$ the Banach space of bounded linear operators from V to W endowed with the operator norm and with $\mathcal{L}^n V = \mathcal{L}(V^n, V)$. If $X \in \mathcal{L}^n V$ we write $X(a) = X(a,\cdots,a)$ when the operator is applied to n copies of the same argument $a \in V$. The symbol C in the r.h.s. of estimates denotes a positive constant which can be different from line to line and usually we write $A \lesssim B$ for $A \leq CB$.

1 Formulation of the problem

Duhamel's perturbation formula applied to eq. (1) gives

$$u(t) = U(t)u_0 + \frac{1}{2}\int_0^t U(t-s)\mathcal{N}(u(s))\,ds \tag{2}$$

where $U(t)$ is the Airy group of isometries of any $\mathcal{F}L^{\alpha,p}$ given by the solution of the linear part of eq. (1). The group $U(t)$ acts on φ as $\mathcal{F}(U(t)\varphi)(k) = e^{ik^3 t}\widehat{\varphi}(k)$. Conservation of mass for KdV guarantees that if $\widehat{u}_0(0) = 0$ then we will have $\widehat{u}(t,0) = 0$ for any $t \geq 0$. Using as unknown the twisted variable $v(t) = U(-t)u(t)$ we have

$$v(t) = v_0 + \int_0^t \dot{X}_s(v(s))\,ds, \quad t \in [0,T] \tag{3}$$

where $v_0 = u_0$, $\dot{X}_s(v(s)) = \dot{X}_s(v(s),v(s))$ and where the bilinear unbounded operator $\dot{X}_s \in \mathcal{L}^2\mathcal{F}L^{s,p}$ is defined as $\dot{X}_s(\varphi_1,\varphi_2) = U(-s)\partial_\xi[(U(s)\varphi_1)(U(s)\varphi_2)]/2$ so that its Fourier transform reads

$$\mathcal{F}\dot{X}_s(\varphi_1,\varphi_2)(k) = \frac{ik}{2}\sum_{k_1}{}' e^{-i3kk_1k_2 s}\widehat{\varphi}_1(k_1)\widehat{\varphi}_2(k_2)$$

with the convention that $k_2 = k - k_1$ and where we use the convention that the primed summation over $k_1 \in \mathbb{Z}$ does not include the terms when $k_1 = 0$ or $k_1 = k$. In obtaining this equation we have used the algebraic identity $k^3 - k_1^3 - k_2^3 = 3kk_1k_2$ valid for any triple k, k_1, k_2 such that $k = k_1 + k_2$. In the rest of the paper we will concentrate on the study of solution to eq. (3) with the fixed point method.

1.1 Power series solutions and generalized integration

Proceeding formally we can expand any solution to (1) into a series involving only the operators $\dot{X}$ and the initial condition:

$$\begin{aligned} v(t) = v_0 &+ \int_{0\leq\sigma\leq t}\dot{X}_\sigma(v_0,v_0)d\sigma + 2\int_{0\leq\sigma_1\leq\sigma\leq t}\dot{X}_\sigma(v_0,\dot{X}_{\sigma_1}(v_0,v_0))d\sigma d\sigma_1 \\ &+4\int_{0\leq\sigma_2\leq\sigma_1\leq\sigma\leq t}\dot{X}_\sigma(v_0,\dot{X}_{\sigma_1}(v_0,\dot{X}_{\sigma_2}(v_0,v_0)))d\sigma d\sigma_1 d\sigma_2 \\ &+2\int_{0\leq\sigma_{1,2}\leq\sigma\leq t}\dot{X}_\sigma(\dot{X}_{\sigma_2}(v_0,v_0),\dot{X}_{\sigma_1}(v_0,v_0))d\sigma d\sigma_1 d\sigma_2 + \cdots \end{aligned} \tag{4}$$

This perturbative expansion is naturally indexed by (binary) trees representing the various ways of applying $\dot{X}$ to itself and of performing time integrations. A possible approach to define a solution to the differential equation is then to prove that each term of the series is well defined and that the sum of the series converges. There is quite a bit of literature on this method for Navier-Stokes like equations [15, 28, 33, 34, 21] and recently Christ [3] advocated this approach in the context of non-linear dispersive equations and used it to solve a modified periodic non-linear Schrödinger equation in $\mathcal{F}L^{\alpha,p}$ spaces. See also [32] for another recent paper studying power-series solution for the periodic modified KdV equation using Christ's approach.

The advantage of the power series expansion is that the relevant objects are the operators obtained from $\dot{X}$ by successive application and time integrations and applied to the inital condition. For example the first term in the expansion involves the the symmetric bilinear operator X_{ts} defined for any $s \le t$ as

$$\mathcal{F}X_{ts}(\varphi_1,\varphi_2)(k) = \int_s^t \mathcal{F}\dot{X}_\sigma(\varphi_1,\varphi_2)(k)d\sigma = \sum_{k_1}{}' \frac{e^{-i3kk_1k_2s} - e^{-i3kk_1k_2t}}{6k_1k_2}\widehat{\varphi}_1(k_1)\widehat{\varphi}_2(k_2).$$

While $\dot{X}_\sigma$ is usually unbounded in any $\mathcal{F}L^{\alpha,p}$, the operator X_{ts} will be shown to be bounded in $\mathcal{F}L^{\alpha,p}$ for appropriate values of α, p and a similar behavior will hold for higher order operators. The regularizing effect of time integrations is due to the dispersive nature of the linear part combined with absence of resonances in the non-linear term.

On the other end the main disadvantage of the power-series method is that one has to control arbitrary terms of the series and the proof of summability require usually a big analytical and combinatorial effort. Another interesting difference is that in the rough path approach there is available a natural space where uniqueness of solutions holds while in the power-series approach no uniqueness result is available.

Rough path theory allows to bypass the complete power-series expansion exploiting the smallness of the leading terms for small time intervals and using a generalized notion of integration to pass from the approximate solution for an very small time interval to an exact solution for an $O(1)$ interval of time. Instead of writing the series solution from time 0 to time t we write it between two times $s \le t$ denoting by $\delta v_{ts} = v_t - v_s$ with $v_t = v(t)$ (the index notation being more comfortable in the following) so that

$$\delta v_{ts} = X_{ts}(v_s) + r_{ts} \tag{5}$$

where r_{ts} stands for the rest of the series and will be treated as a negligible remainder term. For this to make sense we need that X_{ts} gives itself a small contribution when $|t-s|$ is small. Our first result give then a quantitative control of the size of X_{ts} as a bounded operator in $\mathcal{F}L^{\alpha,p}$. When $2 \le p \le +\infty$ define the set $\mathcal{D} \subset \mathbb{R} \times \mathbb{R}_+$ of pairs (γ, α) by

$$\mathcal{D} = \{\alpha \ge -\frac{1}{2} - \frac{1}{p} + \gamma, 0 \le \gamma < \frac{1}{4}\} \cup \{\alpha > -1 - \frac{1}{p} + 3\gamma, \frac{1}{4} \le \gamma \le \frac{1}{2}\};$$

while when $1 \le p \le 2$

$$\mathcal{D} = \{\alpha \ge -1 + \gamma, 0 \le \gamma < \frac{1}{2p}\} \cup \{\alpha > -1 - \frac{1}{p} + 3\gamma, \frac{1}{2p} \le \gamma \le \frac{1}{2}\}.$$

Then

Lemma 2 *For any couple $(\gamma, \alpha) \in \mathcal{D}$ the operator X_{ts} is bounded from $(\mathcal{F}L^{\alpha,p})^2$ to $\mathcal{F}L^{\alpha,p}$ and $|X_{ts}|_{\mathcal{L}^2\mathcal{F}L^{\alpha,p}} \lesssim_{\gamma,\alpha} |t-s|^\gamma$.*

The proof of this lemma is postponed to Appendix A. A trivial but important observation is that the family of operators $\{X_{ts}\}_{s \le t}$ satisfy the algebraic equation

$$X_{ts}(\varphi_1,\varphi_2) - X_{tu}(\varphi_1,\varphi_2) - X_{us}(\varphi_1,\varphi_2) = 0, \quad s \le u \le t \tag{6}$$

as can be easily checked using the definition.

1.2 The sewing map

The first term in the r.h.s. of eq. (5) is well understood thanks to Lemma 2, while the r term contains all the difficulty. However, due to the particular structure of eq. (5), the r term must satisfy a simple algebraic equation. Indeed, for any triple $s \leq u \leq t$ we have $\delta v_{ts} - \delta v_{tu} - \delta v_{us} = 0$ and substituting eq. (5) in this relation we get

$$r_{ts} - r_{tu} - r_{us} = -X_{ts}(v_s, v_s) + X_{tu}(v_u, v_u) + X_{us}(v_s, v_s). = X_{tu}(\delta v_{us}, v_s) + X_{tu}(v_u, \delta v_{us}) \qquad (7)$$

where we used eq. (6) to simplify the r.h.s.. The main observation contained in the work [18] is that sometimes this equation determines r uniquely. To explain the conditions under which we can solve eq. (7) we need some more notation. Given a normed vector space $(V, |\cdot|)$ introduce the vector space $\mathcal{C}_n V \subset C([0,T]^n; V)$ such that $a \in \mathcal{C}_n V$ iff $a_{t_1,\dots,t_n} = 0$ when $t_i = t_j$ for some $1 \leq i < j \leq n$. We have already introduced the operator $\delta : \mathcal{C}_1 V \to \mathcal{C}_2 V$ defined as $\delta f_{ts} = f_t - f_s$. Moreover, is useful to introduce another operator $\delta : \mathcal{C}_2 V \to \mathcal{C}_3 V$ defined on continuous functions of *two* parameters on a vector space V as $\delta a_{tus} = a_{ts} - a_{tu} - a_{us}$. The two operators satisfy the relation $\delta\delta f = 0$ for any $f \in \mathcal{C}_1 V$. Moreover if $a \in \mathcal{C}_2 V$ is such that $\delta a = 0$ then there exists $f \in \mathcal{C}_1 V$ such that $a = \delta f$. Denote $\mathcal{Z}\mathcal{C}_3 V = \mathcal{C}_3 V \cap \mathrm{Im}\delta$, where $\mathrm{Im}\delta$ is the image of the δ operator. We measure the size of elements in $\mathcal{C}_n V$ for $n = 2, 3$ by Hölder-like norms defined in the following way: for $f \in \mathcal{C}_2 V$ let

$$\|f\|_\mu = \sup_{s,t\in[0,T]} \frac{|f_{ts}|}{|t-s|^\mu}, \quad \text{and} \quad \mathcal{C}_1^\mu V = \{f \in \mathcal{C}_2 V;\ \|f\|_\mu < \infty\}.$$

In the same way, for $h \in \mathcal{C}_3 V$, set

$$\begin{aligned} \|h\|_{\gamma,\rho} &= \sup_{s,u,t\in[0,T]} \frac{|h_{tus}|}{|u-s|^\gamma |t-u|^\rho} \\ \|h\|_\mu &= \inf\left\{ \sum_i \|h_i\|_{\rho_i,\mu-\rho_i};\ h = \sum_i h_i,\ 0 < \rho_i < \mu \right\}, \end{aligned} \qquad (8)$$

where the last infimum is taken over all finite sequences $\{h_i \in \mathcal{C}_3 V\}$ such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0, z)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3 V$, and we set $\mathcal{C}_3^\mu V = \{h \in \mathcal{C}_3 V;\ \|h\|_\mu < \infty\}$. Eventually, let $\mathcal{C}_n^{1+} V = \cup_{\mu>1} \mathcal{C}_n^\mu V$ for $n = 2, 3$ and remark that the same kind of norms can be considered on the spaces $\mathcal{Z}\mathcal{C}_3 V$, leading to the definition of the spaces $\mathcal{Z}\mathcal{C}_3^\mu V$ and $\mathcal{Z}\mathcal{C}_3^{1+} V$. The following proposition is the basic result which allows the solution of equations in the form (7).

Proposition 3 (The sewing map Λ) *There exists a unique linear map $\Lambda : \mathcal{Z}\mathcal{C}_3^{1+} V \to \mathcal{C}_2^{1+} V$ such that $\delta\Lambda = Id_{\mathcal{Z}\mathcal{C}_3 V}$. Furthermore, for any $\mu > 1$, this map is continuous from $\mathcal{Z}\mathcal{C}_3^\mu V$ to $\mathcal{C}_2^\mu V$ and we have*

$$\|\Lambda h\|_\mu \leq \frac{1}{2^\mu - 2} \|h\|_\mu, \qquad h \in \mathcal{Z}\mathcal{C}_3^{1+} V. \qquad (9)$$

This proposition has been first proved in [18]. A simplified proof is contained in [20]. Using the notations just introduced, eq. (7) take the form

$$\delta r_{tus} = X_{tu}(\delta v_{us}, v_s) + X_{tu}(v_u, \delta v_{us}) = [X(\delta v, v) + X(v, \delta v)]_{tus}, \qquad (10)$$

where, by construction, the r.h.s. belongs to $\mathcal{Z}\mathcal{C}_3 \mathcal{F}L^{\alpha,p}$. If we can prove that it actually belongs to $\mathcal{Z}\mathcal{C}_3^z \mathcal{F}L^{\alpha,p}$ for some $z > 1$, Prop. 3 will give us the possibility to state that the unique solution of eq. (10) in $\mathcal{C}_2^{1+} \mathcal{F}L^{\alpha,p}$ is given by $r = \Lambda[X(\delta v, v) + X(v, \delta v)]$.

1.3 Λ equations

Let us come back to our initial problem. Since we aim to work in distributional spaces the rigorous meaning of eq. (3) is a priori not clear (even in a weak sense). By formal manipulations we have been able to recast the initial problem in a finite-difference equation involving the Λ map which reads:

$$\delta v = X(v, v) + \Lambda[X(\delta v, v) + X(v, \delta v)] \tag{11}$$

where we used an abbreviated notation since all the terms have been already described in detail. Note that in this equation the Λ map has replaced the integral so we will call this kind of equations: Λ-*equations*. Instead of solving the integral equation we would like to solve, by fixed-point methods, the Λ-equation (11). Afterwards we will show rigorously that solution to such Λ-equations gives generalized solutions to KdV.

Unfortunately, for the particular form of our X operator, we are not able to show that this equation has solutions. Recall Lemma 2 where we showed that X belongs to $\mathcal{C}_2^\gamma \mathcal{L}^2 \mathcal{F}L^{\alpha,p}$ for $(\gamma, \alpha) \in \mathcal{D}$ so we must expect that $\delta v \in \mathcal{C}_2^\gamma \mathcal{F}L^{\alpha,p}$ since the Λ term will belong (at worst) to $\mathcal{C}_2^{1+}\mathcal{F}L^{\alpha,p}$. But then we have $X(\delta v, v) + X(v, \delta v) \in \mathcal{C}_3^{2\gamma}\mathcal{F}L^{\alpha,p}$ so that we must require $2\gamma > 1$ in order for this term to be in the domain of Λ. The set $\mathcal{D}$ however contains only values of $\gamma \leq 1/2$ so we are not able to study eq. (11) in $\mathcal{C}_1^\gamma \mathcal{F}L^{\alpha,p}$ for $(\gamma, \alpha) \in \mathcal{D}$.

This difficulty can be overcome by truncating the power-series solution (4) to higher order. By looking at eq. (5) we see that the next step in the expansion will generate the following expression

$$\delta v_{ts} = X_{ts}(v_s) + X^2_{ts}(v_s) + r^{(2)}_{ts} \tag{12}$$

where the trilinear operator $X^2 \in \mathcal{C}_2\mathcal{L}^3\mathcal{F}L^{\alpha,p}$ is defined by

$$X^2_{ts}(\varphi_1, \varphi_2, \varphi_3) = 2\int_s^t d\sigma \int_s^{\sigma_1} d\sigma_1\, \dot{X}_\sigma(\varphi_1, \dot{X}_{\sigma_1}(\varphi_2, \varphi_3)) \tag{13}$$

and satisfies the algebraic relation

$$\delta X^2(\varphi_1, \varphi_2, \varphi_3)_{tus} = 2X_{tu}(\varphi_1, X_{us}(\varphi_2, \varphi_3)) \qquad s \leq u \leq t \tag{14}$$

which show that X^2 is related to the previously defined operator X by a “quadratic” relationship (this motivates the abuse of the superscript 2 in the definition of X^2). For the regularity of X^2 we have the following result whose proof is again postponed to Appendix A.

Lemma 4 *There exists unbounded operators $\hat{X}^2, \check{X}^2 \in \mathcal{L}^3\mathcal{F}L^{\alpha,p}$ such that $X^2_{ts} = \hat{X}^2_{ts} + \check{X}^2_{ts}$, $\delta\check{X}^2 = 0$ and when $\alpha > \alpha_*(p) = \max(-1/p, -1/2)$ we have $|\check{X}_{ts}|_{\mathcal{L}^3\mathcal{F}L^{\alpha,p}} \lesssim_{\alpha,\gamma} |t-s|$ and for any couple $(\gamma, \alpha) \in \mathcal{D}$ we have $|\hat{X}^2_{ts}|_{\mathcal{L}^3\mathcal{F}L^{\alpha,p}} \lesssim_{\alpha,\gamma} |t-s|^{2\gamma}$.*

Note that the full operator X^2 can be controlled only in the smaller region $\mathcal{D}' = \mathcal{D} \cap \{\alpha > \alpha_*(p)\}$ of the (γ, α) plane. This limitation has some connection with the low regularity ill-posedness discussed in [4] and is related to the periodic setting. To infer the Λ-equation related to eq. (12) we simply apply to it δ operator and we get

$$\begin{aligned}\delta\hat{r}^{(2)}_{tus} =& 2X_{tu}(\delta v_{us} - X_{us}(v_s, v_s), v_s) + X_{tu}(\delta v_{us}, \delta v_{us}) \\ &+ X^2_{tu}(\delta v_{us}, v_s, v_s) + X^2_{tu}(v_u, \delta v_{us}, v_s) + X^2_{tu}(v_u, v_u, \delta v_{us}).\end{aligned} \tag{15}$$

2 Rough solutions of KdV

From eq. (15) we can write down a second Λ-equation (beside eq. (11)) associated to the KdV equation:

$$\delta v = (\mathrm{Id} - \Lambda\delta)[X(v) + X^2(v)] = X(v) + X^2(v) + v^\flat \tag{16}$$

with $v^\flat = \Lambda[2X(\delta v - X(v), v) + X(\delta v, \delta v) + X^2(\delta v, v, v) + X^2(v, \delta v, v) + X^2(v, v, \delta v)]$. To give a well-defined meaning to eq. (16) it will be enough that the argument of Λ belongs actually to its domain. Given an allowed pair $(\gamma, \alpha) \in \mathcal{D}'$ which fixes the regularity of X and X^2 sufficient (and natural) requirements for v are

$$\sup_{t\in[0,T]} |v_t|_{\mathcal{F}L^{\alpha,p}} < \infty, \qquad \delta v \in \mathcal{C}_1^\gamma \mathcal{F}L^{\alpha,p}, \qquad \delta v - X(v, v) \in \mathcal{C}_2^{2\gamma} \mathcal{F}L^{\alpha,p}. \tag{17}$$

Under these conditions $X(\delta v - X(v, v), v) \in \mathcal{C}_2^{3\gamma} \mathcal{F}L^{\alpha,p}$ and also $X^2(\delta v, v, v)+X^2(v, \delta v, v)+X^2(v, v, \delta v) \in \mathcal{C}_2^{3\gamma} L^{\alpha,p}$. They are in the domain of Λ if $3\gamma > 1$ i.e. $\gamma > 1/3$. This fixes the limiting time regularity for the Λ eq. (16) and it turns out that for any $1 \le p \le \infty$ and $\alpha > \alpha_*(p)$ there is a pair $(\gamma, \alpha) \in \mathcal{D}'$ with $\gamma > 1/3$. This will fix the regularity of the initial data that we are able to handle.

To define solutions of eq. (16) let us introduce a suitable space to enforce the all the conditions in eq. (17). For any $0 < \eta \le \gamma$ and consider the complete metric space $\mathcal{Q}_\eta$ whose elements are triples $(y, y', y^\sharp)$ where $y \in \mathcal{C}_1^\eta \mathcal{F}L^{\alpha,p}$, $y' \in \mathcal{C}_1^\eta \mathcal{F}L^{\alpha,p}$, $y'_0 = y_0$ and $y^\sharp \in \mathcal{C}_2^{2\eta} \mathcal{F}L^{\alpha,p}$. Additional requirement is that they satisfy the equation $\delta y = X(y', y') + y^\sharp$. The distance $d_{\mathcal{Q},\eta}$ on $\mathcal{Q}_\eta$ is defined by

$$d_{\mathcal{Q},\eta}(y, z) = |y_0 - z_0| + \|y - z\|_\eta + \|y' - z'\|_\eta + \|y^\sharp - z^\sharp\|_\eta.$$

for any two elements $y, z \in \mathcal{Q}_\eta$. With abuse of notations we will denote a triple $(y, y', y^\sharp) \in \mathcal{Q}_\eta$ using simply its first component. Moreover we denote with y_0 also the constant path in $\mathcal{Q}_\eta$ with value $(y_0, 0, 0)$. The main result of the paper is the following theorem.

Theorem 5 *For any $\alpha > \alpha_*(p)$ take $\gamma > 1/3$ such that $(\gamma, \alpha) \in \mathcal{D}'$. Then, for any $v_0 \in \mathcal{F}L^{\alpha,p}$ and for a sufficiently small interval of time $[0, T^*]$ where T^* depends only on the norm of v_0, there exists a unique $v \in \mathcal{C}^\gamma \mathcal{F}L^{\alpha,p}$ such that $v(0) = v_0$ and*

$$v_t = v_s + X_{ts}(v_s) + X^2_{ts}(v_s) + o(|t - s|)$$

for all $0 \le s \le t \le T^*$. If we write $v = \Theta(X, X^2, v_0)$ then $\Theta : \mathcal{C}_2^\gamma \mathcal{L}^2 \mathcal{F}L^{\alpha,p} \times \mathcal{C}_2^{2\gamma} \mathcal{L}^3 \mathcal{F}L^{\alpha,p} \times \mathcal{F}L^{\alpha,p} \to \mathcal{Q}_\gamma$ is a locally Lipschitz map wrt all its arguments.

Since $(\gamma, \alpha) \in \mathcal{D}$ the operators X and X^2 are regular enough so that the proof of the theorem can follow essentially the pattern of a similar result in [18]. A proof is given in Appendix B. Since $v \in \mathcal{Q}_\gamma$ we have that $v^\flat \in \mathcal{C}_2^{3\gamma} \mathcal{F}L^{\alpha,p}$ with $3\gamma > 1$, then v satisfy the property

$$v_t = v_0 + \lim_{|\Pi|\to 0} \sum_i X_{t_{i+1}t_i}(v_{t_i}) + X^2_{t_{i+1}t_i}(v_{t_i}) \tag{18}$$

where $\Pi = \{0 = t_0 \le t_1 \le \cdots \le t_n = t\}$ is a partition of $[0, t]$ and $|\Pi| = \max |t_{i+1} - t_i|$ its size. The limit is in $\mathcal{F}L^{\alpha,p}$. The proof is simple: by eq. (16)

$$\sum_{i=0}^{n-1} X_{t_{i+1}t_i}(v_{t_i}) + X^2_{t_{i+1}t_i}(v_{t_i}) = \sum_{i=0}^{n-1} (\delta v)_{t_{i+1}t_i} - \sum_{i=0}^{n-1} v^\flat_{t_{i+1}t_i} = v_t - v_0 - \sum_{i=0}^{n-1} v^\flat_{t_{i+1}t_i}$$

and the second term goes to zero as $|\Pi| \to 0$. Below we will also prove that an Euler scheme related to eq. (18) converges as the size of the partition goes to zero. A nice property of these solutions is the following (already noted by Christ for power series solutions of NLS [3]).

Corollary 6 *Let v be the unique solution of the Λ-equation given by Thm. 5 and let $u(t) = U(t)v(t)$. Let P_N be the Fourier projector on modes k such that $|k| \le N$ and let $\mathcal{N}(\varphi)(t, \xi) = \partial_\xi(\varphi(t, \xi)^2)/2$ for smooth functions φ. Then in the sense of distributions in $\mathcal{C}([0, T^*], \mathcal{S}'(\mathbb{T}))$ we have convergence of $\mathcal{N}(P_N u)$ to a limit which we denote by $\mathcal{N}(u)$ and moreover the distributional equation*

$$\partial_t u + \partial_\xi^3 u + \mathcal{N}(u) = 0$$

is satisfied.

Proof. We start by proving that $\mathcal{N}(P_N u) \to \mathcal{N}(u)$ distributionally. It is enough to prove for any $0 \leq s \leq t \leq T^*$ the convergence of $V_t = \int_0^t U(-r)\mathcal{N}(P_N u(r))dr$ in $\mathcal{F}L^{\alpha,p}$ since any smooth test function can be approximated in time by step functions. Now using eq. (16) we have

$$\delta V_{ts} = \int_s^t \dot{X}_r(P_N v(r), P_N v(r))dr = Y_{ts}(v_s, v_s) + Y^2_{ts}(v_s, v_s, v_s) + V^\flat_{ts} \tag{19}$$

where $V^\flat$ is a remainder term, $Y_{ts} = X_{ts}(P_N \times P_N)$ and

$$Y^2_{ts}(\varphi_1, \varphi_2, \varphi_3) = 2\int_s^t \dot{X}_r(P_N \varphi_1, P_N X_{rs}(\varphi_2, \varphi_3))dr.$$

From the regularity proofs for X and X^2 the following facts are easy to prove: (i) Y, Y^2 enjoy at least the same regularity of X and X^2; (ii) as $N \to \infty$ they are equibounded in $\mathcal{C}^\gamma \mathcal{L}^2 \mathcal{F}L^{\alpha,p}$ and $\mathcal{C}^{2\gamma}\mathcal{L}^3\mathcal{F}L^{\alpha,p}$ respectively; (iii) $Y_{ts} \to X_{ts}$ and $Y^2_{ts} \to X^2_{ts}$ in the strong operator norm for fixed t, s. For $V^\flat_{ts}$ we have then the following equation

$$\delta V^\flat_{tus} = 2Y_{tu}(\delta v_{us} - X_{us}(v_s, v_s), v_s) + Y_{tu}(\delta v_{us}, \delta v_{us})$$

$$+ Y^2_{tu}(\delta v_{us}, v_s, v_s) + Y^2_{tu}(v_u, \delta v_{us}, v_s) + Y^2(v_u, v_u, \delta v_{us})$$

(cfr. eq. (15)). Using the sewing map we have that the functions $V^\flat$ are also equibounded in $\mathcal{C}^{3\gamma}_3 \mathcal{F}L^{\alpha,p}$. For fixed $0 \leq s \leq t \leq T^*$ we have

$$\delta V_{ts} - V^\flat_{ts} = Y_{ts}(v_s) + Y^2_{ts}(v_s) \to X_{ts}(v_s) + X^2_{ts}(v_s) = \delta v_{ts} - v^\flat_{ts}$$

so $\limsup_{N\to\infty} |\delta(V - v)_{ts}|_{\mathcal{F}L^{\alpha,p}} = \limsup_{N\to\infty} |V^\flat_{ts} - v^\flat_{ts}|_{\mathcal{F}L^{\alpha,p}} \lesssim |t-s|^{3\gamma}$ and since $3\gamma > 1$ this implies that $\limsup_{N\to\infty} \sup_{0\leq t\leq T^*} |V_t - (v_t - v_0)|_{\mathcal{F}L^{\alpha,p}} = 0$ proving that $V \to v - v_0$ in $\mathcal{C}^0_1 \mathcal{F}L^{\alpha,p}$. This gives the distributional convergence of $\mathcal{N}(P_N u)$. If we call $\mathcal{N}(u)$ the limit we have $V_t = \int_0^t U(-r)\mathcal{N}(u(r))dr$ and $u(t) = U(t)u(0) + \int_0^t U(t-r)\mathcal{N}(u(r))dr$ which is the mild form of the differential equation. ■

Some remarks are in order. The function v is γ-Hölder continous in $\mathcal{F}L^{\alpha,p}$, $|u(t) - u(s)|_{\mathcal{F}L^{\alpha,p}} \leq |(U(t) - U(s))u(s)|_{\mathcal{F}L^{\alpha,p}} + |v(t) - v(s)|_{\mathcal{F}L^{\alpha,p}}$ so that by dominated convergence the function u is only continuous $\mathcal{F}L^{\alpha,p}$ without any further regularity. It is not difficult to prove that for (γ, α) in the interior of $\mathcal{D}$ the solution v is actually in $\mathcal{C}^\gamma \mathcal{F}L^{\alpha+\varepsilon,p}$ for some small $\varepsilon > 0$. In this case it is clear that $u \in \mathcal{C}^{\varepsilon/3}\mathcal{F}L^{\alpha,p}$ (cfr. the discussion of convergence of Galerkin approximation below and eq. (42)).

Fix p, α and γ such that $\gamma > 1/3$ and $(\gamma, \alpha) \in \mathcal{D}'$. Then an interesting property of the space $\mathcal{Q}_\gamma$ is that for any continuous function z in $\mathcal{F}L^{\alpha,p}$ such that $y(t) = U(-t)z(t)$ is in $\mathcal{Q}_\gamma$ the distribution $\mathcal{N}(P_N z)$ converge to a limit $\mathcal{N}(z)$. This follows from the proof of the previous corollary. Indeed for general elements $y \in \mathcal{Q}_\gamma$ the analog of eq. (19) reads

$$\delta V_{ts} = \int_s^t U(-r)\mathcal{N}(P_N z(r))dr = Y_{ts}(y_s, y_s) + Y^2_{ts}(y_s, y'_s, y'_s) + V^\flat_{ts}$$

and by the convergence of the couple Y, Y^2 to X, X^2 we have $\delta V \to \delta V^\infty$ in $\mathcal{C}^\gamma_1 \mathcal{F}L^{\alpha,p}$ where δV^∞ is given by the Λ-equation $\delta V^\infty = (1 - \Lambda\delta)[X(y, y) + X^2(y, y', y')]$. We can then define the distribution $\mathcal{N}(z)$ by letting $\int_0^t U(-r)\mathcal{N}(z(r))dr = V^\infty_t$ and have $\mathcal{N}(P_N z) \to \mathcal{N}(z)$ in weak sense.

2.1 L^2 conservation law and global solutions

It is well known that the KdV equation formally conserves the $L^2(\mathbb{T})$ norm of the solution. This conservation law can be used to show existence of global solution when the initial condition is in $L^2(\mathbb{T})$. Denote with $\langle \cdot, \cdot \rangle$ the L^2 scalar product.

Lemma 7 *For $\varphi \in L^2$ we have $\langle \varphi, X_{ts}(\varphi,\varphi)\rangle = 0$ and $2\langle \varphi, X^2_{ts}(\varphi,\varphi,\varphi)\rangle + \langle X_{ts}(\varphi,\varphi), X_{ts}(\varphi,\varphi)\rangle = 0$.*

Proof. For smooth test functions we have $\langle \varphi_1, \dot{X}_s(\varphi_2,\varphi_2)\rangle = \int_{\mathbb{T}} (U(s)\varphi_1)(\xi)(U(s)\varphi_2)(\xi)\partial_\xi (U(s)\varphi_2)(\xi) d\xi$ and an integration by parts gives

$$\langle \varphi_1, \dot{X}_s(\varphi_2,\varphi_2)\rangle = -\int_{\mathbb{T}} \partial_\xi[(U(s)\varphi_1)(\xi)(U(s)\varphi_2)(\xi)](U(s)\varphi_2)(\xi)d\xi = -2\langle \varphi_2, \dot{X}_s(\varphi_1,\varphi_2)\rangle$$

this gives directly that $\langle \varphi, X_{ts}(\varphi,\varphi)\rangle = 0$. Moreover

$$\begin{aligned}\langle \varphi, X^2_{ts}(\varphi,\varphi,\varphi)\rangle &= 2\int_s^t d\sigma \int_s^\sigma d\sigma_1 \langle \varphi, \dot{X}_\sigma(\varphi, \dot{X}_{\sigma_1}(\varphi,\varphi))\rangle \\ &= -\int_s^t d\sigma \int_s^\sigma d\sigma_1 \langle \dot{X}_{\sigma_1}(\varphi,\varphi), \dot{X}_\sigma(\varphi,\varphi)\rangle \\ &= -\frac{1}{2}\int_s^t d\sigma \int_s^t d\sigma_1 \langle \dot{X}_{\sigma_1}(\varphi,\varphi), \dot{X}_\sigma(\varphi,\varphi)\rangle = -\frac{1}{2}\langle X_{ts}(\varphi,\varphi), X_{ts}(\varphi,\varphi)\rangle\end{aligned} \tag{20}$$

and conclude by density. ■

Theorem 8 *If v is a solution of eq. (16) in $[0,T_*]$ with initial condition $v_0 \in L^2$ then $|v_t|^2_{L^2(\mathbb{T})} = |v_0|^2_{L^2(\mathbb{T})}$ for any $t \in [0,T_*]$.*

Proof. We will prove that $\delta\langle v, v\rangle = 0$. Let us compute explicitly this finite increment:

$$[\delta\langle v,v\rangle]_{ts} = \langle v_t, v_t\rangle - \langle v_s, v_s\rangle = 2\langle \delta v_{ts}, v_s\rangle + \langle \delta v_{ts}, \delta v_{ts}\rangle$$

Substituting in this expression the Λ-equation (16) we get

$$\begin{aligned}[\delta\langle v,v\rangle]_{ts} &= 2\langle X_{ts}(v_s,v_s) + X^2_{ts}(v_s,v_s,v_s) + v^\flat_{ts}, v_s\rangle \\ &\quad + \langle X_{ts}(v_s,v_s), X_{ts}(v_s,v_s)\rangle + 2\langle X_{ts}(v_s,v_s), v^\sharp_{ts}\rangle + \langle v^\sharp_{ts}, v^\sharp_{ts}\rangle\end{aligned}$$

where we set $v^\sharp = X^2(v,v,v) + v^\flat$. Lemma 7 implies that $\langle v_s, X_{ts}(v_s,v_s)\rangle = 0$ and allows to cancel the X^2 term with the quadratic X term. After the cancellations the increment of the L^2 norm squared is $[\delta\langle v,v\rangle]_{ts} = 2\langle v^\flat_{ts}, v_s\rangle + 2\langle X_{ts}(v_s,v_s), v^\sharp_{ts}\rangle + \langle v^\sharp_{ts}, v^\sharp_{ts}\rangle$. Each term on the r.h.s. of this expression belongs at least to $\mathcal{C}_2^{3\eta}\mathbb{R}$ and since $3\eta > 1$ this implies that the function $t \mapsto |v_t|^2_{L^2(\mathbb{T})}$ is an Hölder function of index greater than 1 hence it must be constant. ■

Corollary 9 *If $v_0 \in L^2$ there exist a unique global solutions to the Λ-equation (16).*

Proof. By Thm. 5 there exists a unique local solution up to a time T_* which depends only on $|v_0|_{L^2}$. Since $|v_{T_*}|_{L^2} = |v_0|_{L^2}$ we can start from T_* and extend uniquely this solution to the interval $[0, 2T_*]$ and then on any interval. ■

It would be interesting to try to adapt the I-method of Colliander–Steel–Staffilani–Takaoka–Tao [5] to extend the global well-posedness at least in the case $p = 2$ for any $\alpha > \alpha_*(2) = -1/2$. The handling of correction terms to the conservation law seems however to require some efforts and we prefer to leave this study to a further publication.

2.2 Galerkin approximations

Recall that P_N is the projection on the Fourier modes $|k| \leq N$. In [6] it is proven that the solutions of the approximate KdV equation

$$\partial_t u^{(N)} + \partial_\xi^3 u^{(N)} + \frac{1}{2} P_N \partial_\xi (u^{(N)})^2 = 0, \qquad u^{(N)}(0) = P_N u_0 \tag{21}$$

do not converge even weakly to the flow of the full KdV equation. In the same paper the authors propose a modified finite dimensional scheme and prove its convergence in $H^\alpha(\mathbb{T})$ for any $\alpha \geq -1/2$. Here we would like to propose a different scheme inspired by the rough path analysis. By partial series expansion for the twisted variable $v^{(N)}(t) = U(-t)u^{(N)}(t)$ is not difficult to show that the unique solution of equation (21) satisfy the Λ-equation

$$\delta v^{(N)} = (\mathrm{Id} - \Lambda\delta)[X^{(N)}(v^{(N)}) + X^{(N),2}(v^{(N)})] \tag{22}$$

where $X^{(N)} = P_N X(P_N \times P_N)$ and where the trilinear operator $X^{(N),2}$ is defined as

$$X^{(N),2}_{ts}(\varphi_1, \varphi_2, \varphi_3) = 2\int_s^t d\sigma \int_s^\sigma d\sigma_1 P_N \dot{X}_\sigma(P_N\varphi_1, P_N \dot{X}_{\sigma_1}(P_N\varphi_2, P_N\varphi_3)) \tag{23}$$

so that $\delta X^{(N),2}(\varphi_1, \varphi_2, \varphi_3) = 2X^{(N)}(\varphi_1, X^{(N)}(\varphi_2, \varphi_3))$. These are just multi-linear operators in a finite-dimensional space and to have convergence of the Galerkin approximation it would be enough that both converge in norm to their infinite-dimensional analogs X, X^2. A decomposition for $X^{(N),2}$ analogous to that of X^2 described in Lemma 4 holds $X^{(N),2}_{ts} = \hat{X}^{(N),2}_{ts} + \check{X}^{(N),2}_{ts}$ and we will prove in the appendix that

Lemma 10 *For any pair (γ, α) in the interior of $\mathcal{D}$ we have that as $N \to \infty$, $X^{(N)} \to X$ in $\mathcal{C}_2^\gamma \mathcal{L}^2 \mathcal{F}L^{\alpha,p}$ and $\hat{X}^{(N),2} \to X^2$ in $\mathcal{C}_2^{2\gamma} \mathcal{L}^3 \mathcal{F}L^{\alpha,p}$.*

Unfortunately it is not difficult to see that $\check{X}^{(N),2}$ cannot converge in norm, indeed we have

$$\mathcal{F}\check{X}^{(N),2}_{ts}(\varphi_1, \varphi_2, \varphi_3)(k) = (t-s)\sum_{k_1} \frac{I_{0<|k|,|k_1|,|k_2|\leq N}}{3ik_1} \widehat{\varphi}_1(k_1)[\widehat{\varphi}_2(-k_1)\widehat{\varphi}_3(k) + \widehat{\varphi}_3(-k_1)\widehat{\varphi}_2(k)]$$

and there is no way to make this converge in norm to $\check{X}^2_{ts} = \check{X}^{(\infty),2}_{ts}$ due to the explicit dependence of the cutoff on $|k|, |k_2|$ which cannot be compensated by the regularity of the test functions or by the $1/k_1$ factor. A way to remove this difficulty is to modify the finite dimensional ODE in order to remove this operator in the Λ-equation. This is possible since $\check{X}^{(N),2}_{ts}$ is proportional to $t-s$ so that it admits an obvious differential counterpart. Let use define the trilinear operator $\Gamma^{(N)}$ as

$$\mathcal{F}\Gamma^{(N)}(\varphi_1, \varphi_2, \varphi_3)(k) = \sum_{k_1} \frac{(I_{0<|k|,|k_1|,|k_2|} - I_{0<|k|,|k_1|,|k_2|\leq N})}{3ik_1} \widehat{\varphi}_1(k_1)[\widehat{\varphi}_2(-k_1)\widehat{\varphi}_3(k) + \widehat{\varphi}_3(-k_1)\widehat{\varphi}_2(k)]$$

and note that

$$\check{X}^2_{ts}(\varphi_1, \varphi_2, \varphi_3) - \check{X}^{(N),2}_{ts}(\varphi_1, \varphi_2, \varphi_3) = \int_s^t U(-r)\Gamma(U(r)\varphi_1, U(r)\varphi_2, U(r)\varphi_3)dr.$$

Then the modified Galerkin scheme

$$\partial_t u^{(N)} + \partial_\xi^3 u^{(N)} + \frac{1}{2} P_N \partial_\xi (u^{(N)})^2 - \Gamma^{(N)}(u^{(N)}) = 0, \qquad u^{(N)}(0) = P_N u_0 \tag{24}$$

is still finite dimensional since $\Gamma^{(N)} P_N^{\times 3} = P_N \Gamma^{(N)} P_N^{\times 3}$ and is equivalent to the Λ-equation

$$\begin{aligned} \delta v^{(N)} &= (\mathrm{Id} - \Lambda\delta)[X^{(N)}(v^{(N)}) + X^{(N),2}(v^{(N)}) - \check{X}^{(N),2}(v^{(N)}) + \check{X}^2(v^{(N)})] \\ &= (\mathrm{Id} - \Lambda\delta)[X^{(N)}(v^{(N)}) + \tilde{X}^{(N),2}(v^{(N)})] \end{aligned} \tag{25}$$

where $\tilde{X}^{(N),2} = \hat{X}^{(N),2} + \check{X}^2$ now do converge in norm to X^2 and satisfy the correct algebraic relations. As a consequence of the Lipschitz continuity of the solution of the Λ-equation (25) w.r.t X, X^2 and v_0 implies the following convergence result.

Corollary 11 *Let $1 \le p \le +\infty$ and $\alpha > \alpha_*(p)$, then for any $u_0 \in \mathcal{F}L^{\alpha,p}$ as $N \to \infty$ the Galerkin approximations $v_t^{(N)} = U(-t)u_t^{(N)}$ obtained by the ODE (24) converges in $\mathcal{C}_1^\gamma \mathcal{F}L^{\alpha,p}$ to the solution v of the Λ-equation (16) up to a strictly positive time T^* which depends only on the norm of v_0, X, X^2.*

It is whortwhile to note that this result imply that $\sup_{t\in[0,T^*]} |u^{(N)}(t) - u(t)|_{\mathcal{F}L^{\alpha,p}} \to 0$ as $N \to \infty$ while for the finite dimensional scheme devised in [6] the convergence holds only in the sense that $\sup_{t\in[0,T^*]} |P_{\sqrt{N}}(u^{(N)}(t) - u(t))|_{H^\alpha} \to 0$ i.e. only for a very low frequency part of the solution. It is interesting to remark that the modified ODE (24) remains an Hamiltonian flow on $P_N(H^{-1/2}(\mathbb{T})\backslash\mathbb{R})$ endowed with the symplectic structure given by $\Omega(u,v) = \sum_{0<|k|\le N} u(-k)v(k)/(ik)$. Its Hamiltonian is given by

$$\begin{aligned} H(u) = \frac{1}{2} \sum_{0<|k|\le N} k^2 u(-k)u(k) + \frac{1}{6} \sum_{\substack{k_1+k_2+k_3=0 \\ 0<|k_i|\le N}} u(k_1)u(k_2)u(k_3) \\ - \frac{1}{12} \sum_{0<|k|,|k_1|\le N} \frac{u(-k)u(k)}{ik} \frac{u(-k_1)u(k_1)}{ik_1} I_{|k-k_1|>N}. \end{aligned}$$

2.3 A discrete time scheme

The solution described by the Λ-equation (16) can be approximated by a discrete Euler-like scheme defined as follows. For any $n > 0$ let $y_0^n = v_0$ and

$$y_i^n = X_{i/n,j/n}(y_{i-1}^n) + X^2_{i/n,j/n}(y_{i-1}^n)$$

for $i \ge 1$. The combination of this scheme with the Galerkin approximation discussed before provide an implementable numerical approximation scheme for the solutions of KdV with low regularity initial conditions. Indeed the next theorem can be combined with Corollary 11 to obtain effective rates of convergence.

Theorem 12 *Let $\Delta_i^n = y_i^n - y_{i/n}$ and let $T > 0$ be the existence time of the solution described in Thm. 5, then*

$$\sup_{0\le i<j\le nT} \frac{|\Delta_i^n - \Delta_j^n|_{\mathcal{F}L^{\alpha,p}}}{|i-j|^\gamma} = O(n^{1-3\gamma}). \tag{26}$$

Proof. Let T be the existence time for the solution v described in theorem 5 and let $N = \lfloor nT \rfloor$. We begin by proving some uniform bounds on the sequence $\{y_i^n, i = 0, \dots, N\}$. Let

$$q_{ij}^n = X_{i/n,j/n}(y_i^n) + X^2_{i/n,j/n}(y_i^n)$$

so that $y_k^n = \sum_{i=0}^{k-1} q_{i,i+1}$. Given $0 \le i < j \le N$ let $\tau_0^1 = i, \tau_1^1 = j$ and define recursively τ_l^k, $k > 0$, $l = 0, \dots, 2^k$ such that $\tau_{2l}^{k+1} = \tau_l^k$ for $l = 0, \dots, 2^k$ and $\tau_{2l+1}^{k+1} < \lfloor (\tau_l^k + \tau_{l+1}^k)/2 \rfloor$ for $l = 0, \dots, 2^k - 1$.

Then we have $|\tau^k_{l+1} - \tau^k_l| \leq 1 \vee (j-i)/2^k$ and $y^n_j - y^n_i = \sum_{l=0}^{2^K} q^n_{\tau^k_l, \tau^k_{l+1}}$ where K is such that $(j-i)/2^K \leq 1$. Using the triangular array τ^k_l we rewrite the above expression as a telescopic sum:

$$
\begin{aligned}
y^n_j - y^n_i &= q^n_{ij} + \sum_{k=1}^{K} \sum_{l=0}^{2^{K-1}} (q^n_{\tau^k_{2l}, \tau^k_{2l+2}} - q^n_{\tau^k_{2l}, \tau^k_{2l+1}} - q^n_{\tau^k_{2l+1}, \tau^k_{2l+2}}) \\
&= q^n_{ij} + \sum_{k=1}^{K} \sum_{l=0}^{2^{k-1}} (\delta q^n)_{\tau^k_{2l}, \tau^k_{2l+1}, \tau^k_{2l+2}}
\end{aligned}
$$

Up to T, the solution v satisfy the equation $\delta v = X(v) + X^2(v) + v^\flat$ where $\sup_{t\in[0,T]} |v_t|_\alpha + \|v^\flat\|_{3\gamma} \leq C$ so $v_{j/n} - v_{i/n} = \sum_{l=i}^{j-1} p_{l,l+1} + \sum_{l=i}^{j-1} v^\flat_{l/n,(l+1)/n}$ where $p_{ij} = X_{i/n,j/n}(v_{i/n}) + X^2_{i/n,j/n}(v_{i/n})$. Then

$$
\Delta^n_j - \Delta^n_i = q^n_{ij} - p_{ij} + \sum_{k=1}^{K} \sum_{l=0}^{2^{k-1}} (\delta q^n - \delta p)_{\tau^k_{2l}, \tau^k_{2l+1}, \tau^k_{2l+2}} - r_{ij}
$$

where $r_{ij} = \sum_{l=i}^{j-1} v^\flat_{l/n,(l+1)/n}$. This last term is readily estimated by

$$
|r_{ij}| \leq \sum_{l=i}^{j-1} \|v^\flat\|_{3\gamma} n^{-3\gamma} \leq C \left(\frac{j-i}{n}\right) n^{1-3\gamma}
$$

uniformly in n. Let

$$
M^n_\ell = \sup_{0\leq i<j\leq \ell} \left(\frac{j-i}{n}\right)^{-1} |\Delta^n_j - \Delta^n_i - q^n_{ij} + p_{ij} + r_{ij}|
$$

we want to show that $M^n_N \leq A n^{1-3\gamma}$ uniformly in n for some constant A depending only on the data of the problem: this will imply the statement of the theorem since then

$$
|\Delta^n_i| \leq |q^n_{0i} - p_{0i}| + |r_{0i}| + A n^{1-3\gamma} \left(\frac{i}{n}\right) \leq |r_{0i}| + A n^{1-3\gamma} \left(\frac{i}{n}\right) \leq C n^{1-3\gamma}
$$

for any $i \leq N$ and

$$
|\Delta^n_j - \Delta^n_i| \leq |q^n_{ij} - p^{ij}| + |r_{ij}| + A n^{1-3\gamma} \left(\frac{j-i}{n}\right) \leq C n^{1-3\gamma} \left(\frac{j-i}{n}\right)^\gamma
$$

for any $0 \leq i < j \leq N$.

We proceed by induction on ℓ. For $\ell = 1$ the statement is clearly true since $\Delta^n_1 - \Delta^n_0 - q^n_{0,1} + p_{0,1} + r_{0,1} = 0$, moreover for the same reason we have, for all l that $\Delta^n_{l+1} - \Delta^n_l - q^n_{l,l+1} + p_{l,l+1} + r_{l,l+1} = 0$. Assume then that $M^n_{\ell-1} \leq A$ for some $\ell > 0$. The basic observation is that when $|i - j| \leq \ell$ the sums

$$
\Delta^n_j - \Delta^n_i - q^n_{ij} + p_{ij} + r_{ij} = \sum_{k=1}^{K} \sum_{l=0}^{2^{k-1}} (\delta q^n - \delta p)_{\tau^k_{2l}, \tau^k_{2l+1}, \tau^k_{2l+2}}
$$

can be estimated in terms of $M^n_{\ell-1}$ and various norms of v,X,X^2 much like in the proof of Thm. 5. The bound has the form

$$
|(\delta q^n - \delta p)_{\tau^k_{2l}, \tau^k_{2l+1}, \tau^k_{2l+2}}| \leq C \left(\frac{j-i}{2^k n}\right)^{3\gamma} (1 + M^n_{\ell-1})^3 (M^n_{\ell-1} + n^{1-3\gamma})
$$

where $C = C(v_0, X, X^2)$ and where the factor $n^{1-3\gamma}$ is due to the previous estimate on r_{ij}. Then since $3\gamma > 1$ and

$$\sum_{k=1}^{K}\sum_{l=0}^{2^{k-1}}\left(\frac{j-i}{2^k n}\right)^{3\gamma} \leq \left(\frac{j-i}{n}\right)^{3\gamma}\sum_{k=1}^{\infty}2^{k(1-3\gamma)} \leq C\left(\frac{j-i}{n}\right)^{3\gamma} \leq C\left(\frac{\ell}{n}\right)^{3\gamma-1}\left(\frac{j-i}{n}\right)$$

we get

$$M_\ell^n \leq C(1+M_{\ell-1}^n)^4\sum_{k=1}^{K}\sum_{l=0}^{2^{k-1}}\left(\frac{j-i}{2^k n}\right)^{3\gamma} \leq C(1+M_{\ell-1}^n)^3(M_{\ell-1}^n + n^{1-3\gamma})\left(\frac{\ell}{n}\right)^{3\gamma-1}$$

Let $m_\ell^n = n^{3\gamma-1}M_\ell^n$, then, for n large enough $m_\ell^n \leq F(m_{\ell-1}^n)$ where F is the increasing map

$$\mathbb{R}_+ \ni m \mapsto F(m) = C(1+m)^4\left(\frac{\ell}{n}\right)^{3\gamma-1} \in \mathbb{R}_+$$

which, for ℓ/n small enough has a unique attracting fix-point under iteration starting from 0. In particular the iterations stay bounded and if we set $x_{i+1} = F(x_i)$, $x_0 = 0$ we have $A = \sup_i F(x_i) < \infty$ and $m_\ell^n \leq x_\ell \leq A$. By repeating this argument it is easy to prove that the bound holds for all $\ell \leq nT$, i.e. in the whole existence interval found in Thm. 5. ∎

Remark 13 *With a bit more of work it is possible to prove the existence of the solution stated in Thm. 5 using directly the discrete approximation as done by Davie [9] for rough differential equations.*

2.4 Higher order Λ equations

Further expansion of eq. (12) generate a hierarchy of Λ-equations for KdV. The next one is given by

$$\delta v = (\mathrm{Id} - \Lambda\delta)[X(v) + X^2(v) + X^{3a}(v) + X^{3b}(v)] \tag{27}$$

where X^{3a}, X^{3b} are operators increments respectively defined as

$$X_{ts}^{3a}(\varphi_1,\varphi_2,\varphi_3,\varphi_4) = \int_s^t d\sigma_1 \dot{X}_{\sigma_1}(\varphi_1, \int_s^{\sigma_1} d\sigma_2 \dot{X}_{\sigma_2}(\varphi_2, \int_s^{\sigma_2} d\sigma_3 \dot{X}_{\sigma_3}(\varphi_3,\varphi_4))) \tag{28}$$

and

$$X_{ts}^{3b}(\varphi_1,\varphi_2,\varphi_3,\varphi_4) = \int_s^t d\sigma_1 \dot{X}_{\sigma_1}(\int_s^{\sigma_1} d\sigma_2 \dot{X}_{\sigma_2}(\varphi_1,\varphi_2), \int_s^{\sigma_1} d\sigma_3 \dot{X}_{\sigma_3}(\varphi_3,\varphi_4)) \tag{29}$$

which satisfy the following relations with X and X^2:

$$\delta X^{3a}(\varphi_1,\varphi_2,\varphi_3,\varphi_4) = X(\varphi_1, X^2(\varphi_2,\varphi_3,\varphi_4)) + X^2(\varphi_1,\varphi_2,X(\varphi_3,\varphi_4)) \tag{30}$$

and

$$\begin{aligned}\delta X^{3b}(\varphi_1,\varphi_2,\varphi_3,\varphi_4) = X(X(\varphi_1,\varphi_2),X(\varphi_3,\varphi_4)) + X^2(\varphi_3,\varphi_4,X(\varphi_1,\varphi_2)) \\ + X^2(\varphi_1,\varphi_2,X(\varphi_3,\varphi_4))\end{aligned} \tag{31}$$

As we report elsewhere [19] the Hopf algebra of rooted trees is the natural language to describe this hierarchy of equations and the algebraic relations between the various operators. In this special case however these relations can be easily checked by direct computations. Using Lemmas 2 and 4 we can show that the r.h.s. of the eqns. (30) and (31) belongs to the domain of Λ and so the equations can be used to express X^{3a}, X^{3b} in function of X, X^2 and prove that

Corollary 14 *For $(\gamma,\alpha) \in \mathcal{D} \cap \{\alpha \geq -1/p\}$ with $\gamma > 1/3$ we have $X^{3a}, X^{3b} \in \mathcal{C}_2^\gamma \mathcal{L}^4 \mathcal{F}L^{\alpha,p}$.*

3 Additive stochastic forcing

As another application of this approach we would like to discuss the presence of an additive random force in the KdV eq. (1):

$$\partial_t u + \partial_\xi^3 u + \frac{1}{2}\partial_\xi u^2 = \Phi \partial_t \partial_\xi B \tag{32}$$

where $\partial_t\partial_\xi B$ a white noise on $\mathbb{R}\times\mathbb{T}$ and where Φ is a linear operator acting on the space variable which is diagonal in Fourier space: $\Phi e_k = \lambda_k e_k$ where $\{e_k\}_{k\in\mathbb{Z}}$ is the orthonormal basis $e_k(\xi) = e^{ik\xi}/\sqrt{2\pi}$ and such that $\lambda_0 = 0$. In this way the noise does not affect the zero mode. In the rest of this section fix $1 \le p \le +\infty$, $\alpha > \alpha_*(p)$ and $\gamma \in (1/3, 1/2)$ such that $(\gamma, \alpha) \in \mathcal{D}'$ and assume that

$$|\lambda|_{\ell^{\alpha,p}} = \sum_{k\in\mathbb{Z}} |k|^{\alpha p} |\lambda_k|^p < \infty. \tag{33}$$

The transformed integral equation (analogous to eq. (3)) associated to (32) is

$$v_t = v_0 + w_t + \int_0^t \dot{X}_s(v_s, v_s)\, ds \tag{34}$$

where $\widehat{w}_t(k) = \lambda_k \beta_t^k$ and $\{\beta_\cdot^k\}_{k\in\mathbb{Z}_*}$ is a family of complex-valued centered Brownian motions such that $\beta^{-k} = \overline{\beta^k}$ and with covariance $\mathbb{E}[\overline{\beta}_t^k \beta_s^q] = \delta_{k,q}(t\wedge s)$. The relation between the initial noise B and the family $\{\beta_\cdot^k\}_{k\in\mathbb{Z}_*}$ is given by $\beta_t^k = \langle e_k, \int_0^t U(-s)\partial_\xi\partial_s B(s,\cdot) ds\rangle_{H^0}$. Eq. (34) can be expanded in the same way as we have done before and the first interesting Λ-equation which appears is the following:

$$\delta v = (\mathrm{Id} - \Lambda\delta)[X(v) + \delta w + X^2(v) + X^w(v)]. \tag{35}$$

Here the random operator $X^w_{ts} : \mathcal{F}L^{\alpha,p} \to \mathcal{F}L^{\alpha,p}$ is given by

$$X^w_{ts}(\varphi) = \int_s^t d\sigma \dot{X}_\sigma(\varphi, \delta w_{\sigma s}) \tag{36}$$

and satisfy the equation $\delta X^w_{tus} = X_{tu}(\varphi, \delta w_{us})$. For any couple of integers n, m we have

$$\begin{aligned}
\mathbb{E}|\delta w_{ts}|^{2mn}_{\mathcal{F}L^{\alpha,2n}} &= \mathbb{E}\left[\sum_k |k|^{\alpha 2n} |\delta\widehat{w}_{ts}(k)|^{2n}\right]^m = \sum_{k_1,..,k_m} \mathbb{E}\left[\prod_{i=1}^m |k_i|^{\alpha 2n} |\delta\widehat{w}_{ts}(k_i)|^{2n}\right] \\
&\le \sum_{k_1,..,k_m} \prod_{i=1}^m \left[|k_i|^{\alpha 2nm}\, \mathbb{E}\, |\delta\widehat{w}_{ts}(k_i)|^{2nm}\right]^{1/m} = \left\{\sum_k \left[|k|^{\alpha 2nm}\, \mathbb{E}\, |\delta\widehat{w}_{ts}(k)|^{2nm}\right]^{1/m}\right\}^m \\
&\lesssim_{nm} \left\{\sum_k |k|^{\alpha 2n} |\lambda_k|^{2n}\right\}^m |t-s|^{nm} = \|\lambda\|^{2nm}_{\ell^{\alpha,2n}} |t-s|^{nm}
\end{aligned} \tag{37}$$

where we used the Gaussian bound

$$\mathbb{E}\, |\delta\widehat{w}_{ts}(k)|^{2nm} \le C_{nm} \left(\mathbb{E}\, |\delta\widehat{w}_{ts}(k)|^2\right)^{nm} \le C_{nm} |\lambda_k|^{2nm}.$$

By interpolation this gives $\mathbb{E}\, |\delta w_{ts}|^r_{\mathcal{F}L^{\alpha,p}} \lesssim |\lambda|^r_{\ell^{\alpha,p}} |t-s|^{r/2}$ for all $r \ge p \ge 2$ which is finite by assumption (33). By the standard Kolmogorov criterion this implies that a.s. $\delta w \in \mathcal{C}_2^\rho \mathcal{F}L^{\alpha,p}$ for any $\rho < 1/2$ and a-fortiori $\delta w \in \mathcal{C}_2^\gamma \mathcal{F}L^{\alpha,p}$ by choosing $\rho \in [\gamma, 1/2)$. To prove that a sufficiently regular version of the Gaussian stochastic process X^w exists we will use a generalization of the classic Garsia-Rodemich-Rumsey lemma which has been proved in [18].

Lemma 15 *For any* $\theta > 0$ *and* $p \geq 1$*, there exists a constant* C *such that for any* $R \in \mathcal{C}_2 V$ *(*$(V, |\cdot|)$ *some Banach space), we have*

$$\|R\|_\theta \leq C(U_{\theta+2/p,p}(R) + \|\delta R\|_\theta), \tag{38}$$

where $U_{\theta,p}(R) = \left[\iint_{[0,T]^2} \left(\frac{|R_{ts}|}{|t-s|^\theta}\right)^p dtds\right]^{1/p}$.

The operator X^w behaves not worse than X^2:

Lemma 16 *Under condition (33) we have* $X^w \in \mathcal{C}_2^{2\gamma} \mathcal{L}\mathcal{F}L^{\alpha,p}$ *a.s..*

Proof. After an integration by parts, X^w can be rewritten as

$$X^w_{ts}(\varphi) = X_{ts}(\varphi, \delta w_{ts}) - \int_s^t X_{\sigma s}(\varphi, dw_\sigma) = X_{ts}(\varphi, \delta w_{ts}) + I_{ts}(\varphi) \tag{39}$$

The first term in the r.h.s. belongs to $\mathcal{C}_2^{2\gamma} \mathcal{L}\mathcal{F}L^{\alpha,p}$ path-wise:

$$|X_{ts}(\varphi, \delta w_{ts})|_{\mathcal{F}L^{\alpha,p}} \leq |X_{ts}|_{\mathcal{L}\mathcal{F}L^{\alpha,p}} |\delta w_{ts}|_{\alpha,p} |\varphi|_{\alpha,p} \leq \|X\|_{\mathcal{C}_2^\gamma \mathcal{L}\mathcal{F}L^{\alpha,p}} \|\delta w\|_{\mathcal{C}_1^\gamma \mathcal{F}L^{\alpha,p}} |\varphi|_{\mathcal{F}L^{\alpha,p}} |t-s|^{2\gamma}.$$

Let us estimate the random operator $I_{ts} : \varphi \mapsto I_{ts}(\varphi)$. Its Fourier kernel is

$$\mathcal{F}I_{ts}(\varphi)(k) = \int_s^t \sum_{k_1} \frac{e^{-i3kk_1k_2\sigma} - e^{-i3kk_1k_2 s}}{6k_1k_2} \lambda_{k_2} \widehat{\varphi}(k_1) d\beta^{k_2}_\sigma = \sum_{k_1} \frac{|kk_1k_2|^\gamma}{6k_1k_2} \lambda_{k_2} \widehat{\varphi}(k_1) J(k, k_1, k_2)$$

where

$$J_{ts}(k, k_1, k_2) = \int_s^t \frac{e^{-i3kk_1k_2\sigma} - e^{-i3kk_1k_2 s}}{|kk_1k_2|^\gamma} d\beta^{k_2}_\sigma$$

so

$$|I_{ts}|_{\mathcal{L}\mathcal{F}L^{\alpha,p}} \leq |\mathcal{Y}_2(Q)|_{\mathcal{F}L^p} |\lambda|_{\ell^{\alpha,p}} \sup_{k,k_1} |J_{ts}(k, k_1, k_2)|$$

with

$$Q(k, k_1, k_2) = \frac{|k|^{\alpha+\gamma}}{|k_1k_2|^{1+\alpha-\gamma}} I_{k_1k_2 \neq 0}.$$

The majorizing kernel Q is the same appearing in the estimates for X so that we already know that $\|\mathcal{Y}_2(Q)\|_{\mathcal{F}L^p} < \infty$ for all allowed pairs (γ, α). It remains to show that $\mathbb{E} \sup_{k,k_1} |J_{ts}(k, k_1, k_2)|^n \lesssim |t-s|^{n(1+\gamma)}$ for arbitrarily large n. It is then enough to bound

$$\mathbb{E} \sup_{k,k_1} |J_{ts}(k, k_1, k_2)|^{2n} \leq \sum_{k,k_1} \mathbb{E}\, |J_{ts}(k, k_1, k_2)|^{2n} \lesssim_n \sum_{k,k_1} \left[\int_s^t \frac{|e^{-i3kk_1k_2\sigma} - e^{-i3kk_1k_2 s}|^2}{|kk_1k_2|^{2\gamma}} d\sigma\right]^n$$

$$\lesssim_n |t-s|^{n+2\gamma'} \sum_{k_1,k_2} \frac{1}{|kk_1k_2|^{2n(\gamma-\gamma')}}$$

where the sum is finite for n large enough (depending on $\gamma - \gamma'$). Then choosing p sufficiently large, we have $U_{2\gamma+2/p,p}(I) < \infty$ a.s.. Moreover $\delta I_{tus}(\varphi) = X_{tu}(\varphi, \delta w_{us})$ so that δI can be bounded path-wise in $\mathcal{C}_3^{2\gamma} \mathcal{L}\mathcal{F}L^{\alpha,p}$. Then Lemma 15 implies that $\|I\|_{2\gamma} \leq C(U_{2\gamma+2/p,p}(I) + \|\delta I\|_{2\gamma}) < \infty$ a.s. ending the proof. ■

Then, modifying a bit the proof of Thm. 5, is not difficult to prove the following.

Theorem 17 *For any* $1 \leq p \leq +\infty$ *and* $\alpha > \alpha_*(p)$ *eq. (35) has a unique local path-wise solution in* $\mathcal{C}^\gamma \mathcal{F}L^{\alpha,p}$ *for any initial condition in* $\mathcal{F}L^{\alpha,p}$*.*

When $p = 2$ we obtain solutions for noises with values in $H^\alpha(\mathbb{T})$ for any $\alpha > -1/2$. In this way we essentially cover and extend the results of De Bouard-Debussche-Tsutsumi [10]. Their approach consist in modifying Bourgain's method to handle Besov spaces in order to compensate for the insufficient Sobolev time regularity of Brownian motion.

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A Regularity of some operators

Some elementary results needed in the proofs of this appendix are the subject of the next few lemmas. We have to deal with n-multilinear operators $\mathcal{Y}_n(m) : (\mathcal{F}L^p)^n \to \mathcal{F}L^p$ associated to multipliers $m : \mathbb{R}^n \to \mathbb{C}$ as

$$\mathcal{F}[\mathcal{Y}_n(m)(\psi_1, \dots, \psi_n)](k_0) = \sum_{k_0+k_1+\cdots+k_n=0} m(k_0, k_1, \dots, k_n)\hat{\psi}_1(k_1)\cdots\hat{\psi}_n(k_n).$$

Recall that we have the interpolation inequalities

$$|\mathcal{Y}_n(m(t))|_{\mathcal{L}^n\mathcal{F}L^{\alpha,p(t)}} \le |\mathcal{Y}_n(m_1)|^t_{\mathcal{L}^n\mathcal{F}L^{\alpha,p_1}} |\mathcal{Y}_n(m_2)|^{1-t}_{\mathcal{L}^n\mathcal{F}L^{\alpha,p_2}}$$

for any positive multiplier $m(t) = m_1^t m_2^{1-t}$ and $t \in [0,1]$ such that $\mathcal{Y}_n(m_1) \in \mathcal{L}^n\mathcal{F}L^{\alpha,1}$ and $\mathcal{Y}_n(m_\infty) \in \mathcal{L}^n\mathcal{F}L^{\alpha,\infty}$ and where $1/p(t) = t/p_1 + (1-t)/p_2$. The $\mathcal{Y}$ operators can be bounded in $\mathcal{L}^n\mathcal{F}L^p$ in terms of the multipliers as stipulated by the following

Lemma 18 *For $1/p + 1/q = 1$ we have*

$$|\mathcal{Y}_n(m)|_{\mathcal{L}^n\mathcal{F}L^p} \le \sup_{k_0} \Big[\sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} |m(k_0, k_1, \dots, k_n)|^q \Big]^{1/q}$$

and, for any $p \ge n/(n-1)$,

$$|\mathcal{Y}_n(m)|_{\mathcal{L}^n\mathcal{F}L^p} \le \Big\{ \sum_{k_0} \Big[\sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} |m|^{\hat{q}} \Big]^{p/\hat{q}} \Big\}^{1/p}$$

where $\hat{q} = p/[p - n/(n-1)]$.

Proof. By duality it is enough to bound the linear form

$$F = {}_{\mathcal{F}L^q}\langle \psi_0, \mathcal{Y}_n(m)(\psi_1, \dots, \psi_n)\rangle_{\mathcal{F}L^p} = \sum_{k_0+k_1+\cdots+k_n=0} m(k_0, k_1, \dots, k_n)\hat{\psi}_0(k_0)\cdots\hat{\psi}_n(k_n)$$

for any $\psi_0 \in \mathcal{F}L^q$:

$$|F| \le \sum_{k_0} |\hat{\psi}_0| \Big[\sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} |m|^q \Big]^{1/q} \Big[\sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} |\hat{\psi}_1 \cdots \hat{\psi}_n|^p \Big]^{1/p}$$

$$\le \sup_{k_0} \Big[\sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} |m|^q \Big]^{1/q} \sum_{k_0} |\hat{\psi}_0| \Big[\sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} |\hat{\psi}_1 \cdots \hat{\psi}_n|^p \Big]^{1/p}$$

$$\le \sup_{k_0} \Big[\sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} |m|^q \Big]^{1/q} |\psi_0|_{\mathcal{F}L^q} |\psi_1|_{\mathcal{F}L^p} \cdots |\psi_n|_{\mathcal{F}L^p}$$

For the second inequality we have

$$|F| \leq \sum_{k_0} |\hat{\psi}_0| \sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} |m| \prod_{k=1}^{n} |\hat{\psi}_k|$$

$$\leq \sum_{k_0} |\hat{\psi}_0| [\sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} |m|^{\hat{q}}]^{1/\hat{q}} [\sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} \prod_{a=1}^{n} |\hat{\psi}_a|^{\hat{p}}]^{1/\hat{p}}$$

Now choosing $n\hat{p} = (n-1)p$ we have

$$\sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} \prod_{k=1}^{n} |\hat{\psi}_k|^{\hat{p}} = \sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} \prod_{j=1}^{n} (\prod_{\substack{a=1,..,n \\ a\neq j}} |\hat{\psi}_a|^{\hat{p}/(n-1)})$$

$$\leq \prod_{j=1}^{n} [\sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} (\prod_{\substack{k=1,..,n \\ k\neq j}} |\hat{\psi}_k|^{n\hat{p}/(n-1)})]^{1/n}$$

$$= \prod_{j=1}^{n} [\sum_{k_1,\ldots,\hat{k}_j,\ldots,k_n} (\prod_{\substack{a=1,..,n \\ a\neq j}} |\hat{\psi}_a|^{n\hat{p}/(n-1)})]^{1/n} = \prod_{j=1}^{n} |\psi_j|_{\mathcal{F}L^p}^{\hat{p}}$$

so that

$$|F| \leq \prod_{j=1}^{n} |\psi_j|_{\mathcal{F}L^p} \sum_{k_0} |\hat{\psi}_0| [\sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} |m|^{\hat{q}}]^{1/\hat{q}}$$

$$\leq \prod_{j=1}^{n} |\psi_j|_{\mathcal{F}L^p} |\psi_0|_{\mathcal{F}L^q} \{ \sum_{k_0} [\sum_{\substack{k_0+k_1+\cdots+k_n=0 \\ k_0 \text{ fixed}}} |m|^{\hat{q}}]^{p/\hat{q}} \}^{1/p}$$

with $\hat{q} = p/[p - n/(n-1)]$. ■

Lemma 19 *Fix any $a, b \in \mathbb{R}$ and let $m = \int_s^t \int_s^\sigma e^{ia\sigma} e^{ib\sigma_1} d\sigma d\sigma_1 - I_{a+b=0}(t-s)/(ib)$. If $\gamma \in [0, 1/2]$ we have*

$$|m| \lesssim \frac{|t-s|^{2\gamma}}{|b|^\gamma |a|^{1-\gamma} |a+b|^{1-2\gamma}} + \frac{|t-s|^{2\gamma}}{|a|^{1-\gamma}|b|^{1-\gamma}}.$$

Proof. An explicit integration gives

$$m = \frac{e^{i(a+b)s} - e^{i(a+b)t}}{b(a+b)} + \frac{e^{ibs+iat} - e^{i(a+b)s}}{ab} - I_{a+b=0} \frac{t-s}{ib}.$$

Then if $a + b \neq 0$

$$|m| \leq \frac{1}{|b||a+b|} + \frac{1}{|a||b|} \tag{40}$$

moreover $|m| \leq \int_s^t \left| \int_s^\sigma e^{ib\sigma_1} d\sigma_1 \right| d\sigma \leq |b|^{-1}|t-s|$ and symmetrically $|m| \leq |t-s||a|^{-1}$. These last bounds imply that $|m| \leq |t-s||ab|^{-1/2}$ and, since $2\gamma \in [0,1]$, interpolating between this bound and eq. (40) we get

$$|m| \leq \frac{|t-s|^{2\gamma}}{|ab|^\gamma} \left(\frac{1}{|b||a+b|} + \frac{1}{|a||b|} \right)^{1-2\gamma} \leq \frac{|t-s|^{2\gamma}}{|b|^\gamma |a|^{1-\gamma}|a+b|^{1-2\gamma}} + \frac{|t-s|^{2\gamma}}{|a|^{1-\gamma}|b|^{1-\gamma}} \tag{41}$$

When $a + b = 0$ then $m = (1 - e^{ia(t-s)})/a^2$ which gives $|m| \lesssim \frac{|t-s|^{2\gamma}}{|a|^{2-2\gamma}} \lesssim \frac{|t-s|^{2\gamma}}{|a|^{1-\gamma}|b|^{1-\gamma}}$. ■

A.1 Proof of Lemma 2

Proof. Estimating the norm of X_{ts} is equivalent to estimating in $\mathcal{F}L^p$ the norm of the operator $\mathcal{Y}_2(\Psi^1_{ts}) \in \mathcal{L}^2\mathcal{F}L^p$ where $\Psi^1_{ts} : \mathbb{R}^3 \to \mathbb{C}$ is the multiplier

$$\Psi^1_{ts}(k,k_1,k_2) = \frac{|k|^\alpha}{|k_1|^\alpha |k_2|^\alpha} \frac{e^{-i3kk_1k_2 s} - e^{-i3kk_1k_2 t}}{k_1 k_2} I_{kk_1k_2\neq 0}$$

We must prove that, for $(\gamma,\alpha) \in \mathcal{D}$ we have $|\mathcal{Y}_2(\Psi^1_{ts})|_{\mathcal{L}^2\mathcal{F}L^p} \lesssim |t-s|^\gamma$. To begin we bound

$$|\Psi^1_{ts}(k,k_1,k_2)| \lesssim |t-s|^\gamma |k|^{2\gamma-1} \left(\frac{|k|}{|k_1||k_2|}\right)^{1+\alpha-\gamma} I_{kk_1k_2\neq 0}$$

for any $\gamma \in [0,1]$. We first prove a bound for this operator for $p = 1, 2, \infty$ and then conclude by interpolation. For $p = 1$, using Lemma 18, we have to bound

$$|\mathcal{Y}_2(\Psi^1_{ts})|_{\mathcal{L}^2\mathcal{F}L^1} \lesssim |t-s|^\gamma \sup_{k,k_1} |k|^{2\gamma-1} \left(\frac{|k|}{|k_1||k_2|}\right)^{1+\alpha-\gamma} I_{kk_1k_2\neq 0}$$

which is finite if $2\gamma - 1 \leq 0$ and $1+\alpha-\gamma \geq 0$ since from $k_1 + k_2 = k$ we infer that $\max(|k_1|,|k_2|) \geq |k|/3$. For $p = \infty$ using again Lemma 18 we get

$$|\mathcal{Y}_2(\Psi^1_{ts})|_{\mathcal{L}^2\mathcal{F}L^\infty} \lesssim |t-s|^\gamma \sup_{k\neq 0} |k|^{\alpha+\gamma} \sum_{k_1\neq 0,k} \frac{1}{|k_1|^{1+\alpha-\gamma}|k_2|^{1+\alpha-\gamma}}$$

which is finite if $\gamma \leq 1/2$ and $1+\alpha-\gamma > 1$ or by Cauchy-Schwartz if $\alpha+\gamma \leq 0$ and $1/2+\alpha-\gamma > 0$. Interpolating the $p = 1$ and $p = \infty$ bounds we get that for $p = 2$ the norm is finite also for $\gamma = 1/2$ and $\alpha > 0$. But when $p = 2$ we have at our disposal another inequality, namely,

$$|\mathcal{Y}_2(\Psi^1_{ts})|^2_{\mathcal{L}^2\mathcal{F}L^2} \lesssim |t-s|^{2\gamma} \sum_{k\neq 0} |k|^{2(2\gamma-1)} \sup_{k_1\neq 0,k} \left(\frac{|k|}{|k_1||k_2|}\right)^{2(1+\alpha-\gamma)}$$

if $1+\alpha-\gamma \geq 0$ and $\gamma < 1/4$ this quantity is finite since can be bounded by $\lesssim |t-s|^{2\gamma} \sum_k |k|^{2(2\gamma-1)} \lesssim 1$. Putting all together we get that for $p = 2$ the norm is bounded when $\gamma < 1/4$ and $1+\alpha-\gamma \geq 0$ or when $\gamma \in [1/4, 1/2]$ and $\alpha \geq -3/2 + 3\gamma$. Again by interpolation we obtain the bound for all $1 \leq p \leq +\infty$ and $(\gamma,\alpha) \in \mathcal{D}$. ■

A.2 Proof of Lemma 4

Proof. We start the argument as in the proof of Lemma 2. The Fourier transform of X^2 reads

$$\mathcal{F}X^2_{ts}(\varphi_1,\varphi_2,\varphi_3)(k) = -2 \sum_{k_1,k_{21}}{}' \frac{kk_2}{4} \int_s^t \int_s^\sigma e^{-i3kk_1k_2\sigma - i3k_2k_{21}k_{22}\sigma_1}\, d\sigma d\sigma_1\, \widehat{\varphi}_1(k_1)\widehat{\varphi}_2(k_{21})\widehat{\varphi}_3(k_{22})$$

then letting

$$\Phi^2_{ts}(k,k_1,k_{21},k_{22}) = -\frac{kk_2}{2} I_{kk_1k_2k_{21}k_{22}\neq 0} \int_s^t \int_s^\sigma e^{-i3kk_1k_2\sigma - i3k_2k_{21}k_{22}\sigma_1}\, d\sigma d\sigma_1;$$

we have $X^2_{ts} = \mathcal{Y}_3(\Phi^2_{ts})$. Indeed it is enough to restrict the sums over the set of k_i for which $kk_1k_2k_{21}k_{22} \neq 0$. The multiplier Φ^2_{ts} has two different behaviors depending on the quantity $h = kk_1k_2 + k_2k_{21}k_{22}$ being zero or not. We let $\Phi^2_{ts} = \hat{\Phi}^2_{ts} + \breve{\Phi}^2_{ts}$ where

$$\breve{\Phi}^2_{ts} = \frac{kk_2}{i3k_2k_{21}k_{22}} I_{h=0,kk_1k_2k_{21}k_{22}\neq 0}(t-s) = -\frac{t-s}{i3k_1} I_{h=0,kk_1k_2k_{21}k_{22}\neq 0}$$

$$= i\frac{t-s}{3k_1}(I_{k=k_{21},k_1=-k_{22},k_2\neq 0} + I_{k=k_{22},k_1=-k_{21},k_2\neq 0})$$

since the factorization $h = k_2(k_1+k_{21})(k-k_{21})$ implies that in the expression for $\check{\Phi}^2_{ts}$ the only relevant contributions come from the case where $k = k_{21}$ and $k_1 = -k_{22}$ or $k = k_{22}$ and $k_1 = -k_{21}$ and thus defining the operators $\hat{X}^2_{ts} = \mathcal{Y}_3(\hat{\Phi}^2_{ts})$ and $\check{X}^2_{ts} = \mathcal{Y}_3(\check{\Phi}^2_{ts})$ we have $X^2_{ts} = \hat{X}^2_{ts} + \check{X}^2_{ts}$ with $\delta\check{X}^2_{tus} = 0$.

Bound for $\check{X}^2$. When $1 \le p \le 2$ the operator $\check{X}^2_{ts}$ is bounded in $\mathcal{L}^3\mathcal{F}L^{\alpha,p}$ when $\alpha \ge -1/2$. While when $p > 2$ is bounded if $[\sum_{k\neq 0}(|k|^{-1-2\alpha})^{p/(p-2)}]^{(p-2)/p}$ is finite. This happens if $\alpha > -1/p$. In both cases we have $|\check{X}^2_{ts}|_{\mathcal{L}^3\mathcal{F}L^{\alpha,p}} \lesssim |t-s|^2$. Note that, due to some cancellations, the symmetric part of $\check{X}^2$ is more regular and is bounded in any $\mathcal{F}L^{\alpha,p}$ as soon as $\alpha \ge -1/2$. Indeed

$$\mathcal{F}\check{X}^2_{ts}(\varphi,\varphi,\varphi)(k) = 2(t-s)\sum_{k_1}\frac{I_{0<|k|,|k_1|,|k_2|}}{3ik_1}\widehat{\varphi}(k_1)\widehat{\varphi}(-k_1)\widehat{\varphi}(k) = -\frac{2(t-s)}{3ik}\widehat{\varphi}(k)\widehat{\varphi}(-k)\widehat{\varphi}(k). \quad (42)$$

We will not exploit here this better regularity since $\hat{X}^2$ will in any case limit the overall regularity of X^2 to $\alpha > -1/p$ when $3\gamma > 1$.

Bound for $\hat{X}^2$. For any $0 \le \gamma \le 1/2$, Lemma 19 gives

$$|\hat{\Phi}^2_{ts}(k,k_1,k_{21},k_{22})| \lesssim (A_1+A_2)|t-s|^{2\gamma} I_{k_1,k_{21},k_{22}\neq 0,k}$$

where $A_1 = |k||k_2||kk_1k_2|^{\gamma-1}|k_2k_{21}k_{22}|^{\gamma-1}$ and

$$A_2 = |k|^{1-\gamma}|k_1|^{-\gamma}|k_{22}|^{\gamma-1}|k_{21}|^{\gamma-1}|k-k_{21}|^{2\gamma-1}|k-k_{22}|^{2\gamma-1}|k-k_1|^{2\gamma-1}.$$

So in order to bound $\hat{X}^2_{ts}$ in $\mathcal{F}L^{\alpha,p}$ with a quantity of order $|t-s|^{2\gamma}$ it will be enough to bound separately the multipliers

$$\Theta_1 = A_1|k|^{\alpha}|k_1|^{-\alpha}|k_{21}|^{-\alpha}|k_{22}|^{-\alpha}I_{k_1,k_{21},k_{22}\neq 0,k}$$

and

$$\Theta_2 = A_2|k|^{\alpha}|k_1|^{-\alpha}|k_{21}|^{-\alpha}|k_{22}|^{-\alpha}I_{k_1,k_{21},k_{22}\neq 0,k}$$
$$= I_{k_1,k_{21},k_{22}\neq 0,k}\left(\frac{|k|}{|k_1||k_{21}||k_{22}|}\right)^{1+\alpha-\gamma}\left(\frac{|k_1|}{|k-k_1||k-k_{21}||k-k_{22}|}\right)^{1-2\gamma}$$

in $\mathcal{F}L^p$. The expression A_1 is nicely factorized in its dependence on the couple k_{21}, k_{22} and the multiplier Θ_1 can be easily bounded using (twice and iteratively) the same arguments as in Lemma 2. We will concentrate on the multiplier Θ_2 which requires different estimates.

Since $k_1 + k_{21} + k_{22} = k$ and $(k-k_1)-(k-k_{21})-(k-k_{22}) = -2k_1$ at least one of $|k_1|, |k_{21}|, |k_{22}|$ is larger than $|k|/4$ and similarly one of $|k-k_1|, |k-k_{21}|, |k-k_{22}|$ is larger than $|k_1|/2$. Using this fact, when $1+\alpha-\gamma \ge 0$ and $1-2\gamma \ge 0$, we have

$$\Theta_2 \lesssim I_{k_a,k_b,k_c\neq 0,k}\left(\frac{1}{|k_a||k_b|}\right)^{1+\alpha-\gamma}\left(\frac{1}{|k-k_d||k-k_e|}\right)^{1-2\gamma}$$

where (a,b,c) and (d,e,f) are two permutations of $(1,21,22)$ (depending on k_1, k_{21}, k_{22}) so that we must have $\{a,b\}\cap\{d,e\} \neq \emptyset$. It is clear then that $\Theta_2 \lesssim 1$ so that $\mathcal{Y}_3(\Theta_2)$ is bounded in $\mathcal{L}^3\mathcal{F}L^{\alpha,1}$ when $1+\alpha-\gamma \ge 0$ and $1-2\gamma \ge 0$. The bound in $\mathcal{L}^3\mathcal{F}L^{\alpha,\infty}$ will follow by showing that $\sup_k\sum_{k_1,k_{21}}\Theta_2(k,k_1,k_{21},k_{22})$ is finite. If $1+\alpha-\gamma > 1$ and $1-2\gamma \ge 0$ we can bound this quantity by

$$\sup_k\sum_{k_1,k_2\neq 0,k}\left(\frac{1}{|k_1||k_2|}\right)^{1+\alpha-\gamma}\left(\frac{1}{|k-k_1|}\right)^{1-2\gamma} \lesssim 1.$$

When $1+\alpha-\gamma \geq 0$ a bound in any $\mathcal{L}^3\mathcal{F}L^{\alpha,p}$ for $p \in [1,+\infty]$ is obtained from a bound of

$$\sup_k \sum_{k_1,k_{21}} [\Theta_2(k,k_1,k_{21},k_{22})]^q \lesssim \sum_{n_1,n_2} \left(\frac{1}{|n_1|}\right)^{q(1+\alpha-\gamma)} \left(\frac{1}{|n_1||n_2|}\right)^{q(1-2\gamma)}.$$

This last quantity is then finite for any $\gamma < 1/(2p)$. By interpolation of these various estimates we obtain again that $\mathcal{Y}_3(\Theta_2)$ is bounded in $\mathcal{L}^3\mathcal{F}L^{\alpha,p}$ for any $(\gamma,\alpha) \in \mathcal{D}$. Note that the extra factors in the majorizing sums can be used to show that actually in the interior of the region $\mathcal{D}$ the operator X^2 in bounded from $\mathcal{F}L^{\alpha,p}$ to $\mathcal{F}L^{\alpha+\varepsilon,p}$ for some $\varepsilon > 0$. ■

A.3 Proof of Lemma 10

Proof. Let us work out the details for X, first. The Fourier multiplier associated to $X^{(N)}_{ts}$ differs from that of X_{ts} for a factor of the form $I_{|k|\leq N,|k_1|\leq N,|k_2|\leq N}$ so the difference $X_{ts} - X^{(N)}_{ts}$ has a Fourier multiplier containing the factor

$$I_{kk_1k_2\neq 0}(1 - I_{|k|\leq N,|k_1|\leq N,|k_2|\leq N}) \leq I_{kk_1k_2\neq 0}(I_{|k_1|>N} + I_{|k_2|>N} + I_{|k|>N}) \lesssim \frac{|kk_1k_2|^\varepsilon}{N^\varepsilon} I_{kk_1k_2\neq 0}$$

for any $\varepsilon > 0$. To prove the convergence in $\mathcal{C}^\gamma_2\mathcal{F}L^{\alpha,p}$ is then enough to prove that X is bounded in the space $\mathcal{C}^\gamma_2\mathcal{L}((\mathcal{F}L^{\alpha-\varepsilon,p})^2, \mathcal{F}L^{\alpha+\varepsilon,p})$ for some $\varepsilon > 0$. When (γ,α) is in the interior of $\mathcal{D}$, exploiting the gaps in the inequalities involved in the proof of Lemma 2, it is not difficult to show that this is the case for sufficiently small $\varepsilon > 0$. Then

$$|(X_{ts} - X^{(N)}_{ts})(\varphi_1,\varphi_2)|_{\mathcal{F}L^{\alpha,p}} \lesssim N^{-\varepsilon}|\varphi_1|_{\mathcal{F}L^{\alpha,p}}|\varphi_2|_{\mathcal{F}L^{\alpha,p}}.$$

A similar argument works for the convergence of $\hat{X}^{(N),2}$ to $\hat{X}^{(2)}$ since in this case the multiplier associated to the difference $\hat{X}^2 - \hat{X}^{(N),2}$ has a factor of the form

$$I_{kk_1k_2k_{21}k_{22}\neq 0}(1 - I_{|k|,|k_1|,|k_2|,|k_{21}|,|k_{22}|\leq N}) \lesssim \frac{|kk_1k_2k_{21}k_{22}|^\varepsilon}{N^\varepsilon} I_{kk_1k_2k_{21}k_{22}\neq 0} \lesssim \frac{|kk_1k_{21}k_{22}|^{2\varepsilon}}{N^\varepsilon} I_{kk_1k_2k_{21}k_{22}\neq 0}$$

where we used the fact that $k_2 = k_{21} + k_{22}$ to remove it from the r.h.s. Again an inspection of the proof of the regularity of $\hat{X}^2$ confirms that in the interior of $\mathcal{D}$ we have a small gain of regularity which can be converted in an estimate of $\hat{X}^2$ in $\mathcal{C}^{2\gamma}\mathcal{L}^3((\mathcal{F}L^{\alpha-\varepsilon,p})^2; \mathcal{F}L^{\alpha+\varepsilon})$ and obtain the norm convergence of $\hat{X}^{(N),2}$ to $\hat{X}^2$ with a polynomial speed: $|\hat{X}^2_{ts} - \hat{X}^{(N),2}_{ts}|_{\mathcal{L}^3\mathcal{F}L^{\alpha,p}} \lesssim N^{-\varepsilon/2}|t-s|^{2\gamma}$ for some $\varepsilon > 0$. ■

B Proof of Theorem 5

Proof. Fix $\eta \in (0,\gamma)$ and define a map $\Gamma : \mathcal{Q}_\eta \to \mathcal{Q}_\eta$ by $z = \Gamma(y)$ where

$$z_0 = y_0, \qquad \delta z = X(y,y) + X^2(y,y,y) + z^\flat \tag{43}$$

with

$$z^\flat = \Lambda[2X(y^\sharp,y) + X(\delta y,\delta y) + X^2(\delta y,y,y) + X^2(y,\delta y,y) + X^2(y,y,\delta y)].$$

and we set $z' = y$ and $z^\sharp = X^2(y,y,y) + z^\flat$. For the map Γ to be well defined it is enough that $(\gamma,\alpha) \in \mathcal{D}$ and that $\eta > 1/3$. Indeed if $y \in \mathcal{Q}_\eta$ then all the terms in the argument of the Λ-map belongs to $\mathcal{C}^{3\eta}_2\mathcal{F}L^{\alpha,p}$.

We will proceed in three steps. First we will prove that there exists $T_* \in (0,1]$ depending only on $|v_0|$ such that for any $T \leq T_*$, Γ maps a closed ball B_T of $\mathcal{Q}_\eta$ in itself. Then we will prove that if T is sufficiently small, Γ is actually a contraction on this ball proving the existence of a fixed-point. Finally uniqueness will follow from a standard argument.

Boundedness. The key observation to prove (below) that Γ is a contraction for T sufficiently small is that $\Gamma(y)$ belongs actually to $\mathcal{Q}_\gamma$, indeed $z' = y \in \mathcal{C}_1^\gamma \mathcal{F}L^{\alpha,p}$ and $z^\sharp \in \mathcal{C}_2^{2\gamma}\mathcal{F}L^{\alpha,p}$.

Let us assume that $T \leq 1$. First we will prove that

$$d_{\mathcal{Q},\gamma}(\Gamma(y), 0) = d_{\mathcal{Q},\gamma}(z, 0) \leq C(1 + d_{\mathcal{Q},\eta}(y, 0))^3. \tag{44}$$

Since $3\eta > 1$ using Prop. 3 we have

$$\begin{aligned}\|z^\flat\|_{3\eta} &\leq C\|2X(y^\sharp, y) + X(\delta y, \delta y) + X^2(\delta y, y, y) + X^2(y, \delta y, y) + X^2(y, y, \delta y)\|_{3\eta} \\ &\leq C\Big[2\|X(y^\sharp, y)\|_{3\eta} + \|X(\delta y, \delta y)\|_{3\eta} \\ &\quad +\|X^2(\delta y, y, y)\|_{3\eta} + \|X^2(y, \delta y, y)\|_{3\eta} + \|X^2(y, y, \delta y)\|_{3\eta}\Big]\end{aligned} \tag{45}$$

The first term is bounded by $\|X(y^\sharp, y)\|_{3\eta} \leq \|X\|_\eta \|y^\sharp\|_{2\eta}\|y\|_\infty$ where the Hölder norm $\|X\|_\eta$ is considered with respect to the operator norm on $\mathcal{L}((\mathcal{F}L^{\alpha,p})^2, \mathcal{F}L^{\alpha,p})$. Now $\|X\|_\eta \leq \|X\|_\gamma$ since we assume $\eta < \gamma$, while $\|y^\sharp\|_{2\eta} \leq d_{\mathcal{Q},\eta}(y, 0)$ and finally remark that $\|y\|_\infty = |y_0| + \|y\|_\eta \leq d_{\mathcal{Q},\eta}(y, 0)$ since we assumed $T \leq 1$. So $\|X(y^\sharp, y)\|_{3\eta} \leq C d_{\mathcal{Q},\eta}(y, 0)^2$. Each of the other terms in eq. (45) can be bounded similarly. Then

$$\|z^\flat\|_{3\eta} \leq C(1 + d_{\mathcal{Q},\eta}(y, 0))^3 \tag{46}$$

This implies

$$\begin{aligned}\|z^\sharp\|_{2\gamma} &= \|X^2(y, y, y) + z^\flat\|_{2\gamma} \leq \|X^2(y, y, y)\|_{2\gamma} + \|z^\flat\|_{2\gamma} \\ &\leq C\|y\|_\infty^3 + \|z^\flat\|_{2\gamma}\end{aligned}$$

By hypothesis we have $2\gamma < 1 < 3\eta$ so $\|z^\flat\|_{2\gamma} \leq \|z^\flat\|_{3\eta}$ and again we obtain $\|z^\sharp\|_{2\gamma} \leq C(1+d_{\mathcal{Q},\eta}(y, 0))^3$. Finally we have to bound z' and z:

$$\begin{aligned}\|z'\|_\gamma = \|y\|_\gamma &\leq \|X(y', y') + y^\sharp\|_\gamma \leq C\|y'\|_\infty^2 + \|y^\sharp\|_{2\gamma} \\ &\leq C(|y_0|^2 + \|y'\|_\gamma^2) + \|y^\sharp\|_{2\gamma} \leq C(1 + d_{\mathcal{Q},\eta}(y, 0))^2\end{aligned}$$

Similarly we can bound z as $\|z\|_\gamma \leq C(1 + d_{\mathcal{Q},\eta}(y, 0))^3$. Putting these bounds together we prove eq. (44). Let $\varepsilon = \gamma - \eta > 0$ and observe that

$$\begin{aligned}d_{\mathcal{Q},\eta}(\Gamma(y), 0) &= |z_0| + \|z\|_\eta + \|z'\|_\eta + \|z^\sharp\|_{2\eta} \\ &\leq |z_0| + T^\varepsilon\Big[\|z\|_\gamma + \|z'\|_\gamma + \|z^\sharp\|_{2\gamma}\Big] \\ &\leq |y_0| + T^\varepsilon d_{\mathcal{Q},\gamma}(\Gamma(y), 0)\end{aligned}$$

So

$$d_{\mathcal{Q},\eta}(\Gamma(y), 0) \leq |y_0| + CT^\varepsilon(1 + d_{\mathcal{Q},\eta}(y, 0))^3 \tag{47}$$

Let $B_T = \{y \in \mathcal{Q}_\eta : d_{\mathcal{Q},\eta}(y, 0) \leq b_T, y_0 = v_0\}$ be a closed ball of elements in $\mathcal{Q}_\eta$ with initial condition v_0. Let $b_T > 0$ be the solution of the algebraic equation $b_T = |v_0| + CT^\varepsilon(1+b_T)^3$ which exists whenever $T \leq T_*(|v_0|)$ where $T_*(|v_0|)$ is a strictly positive time which depends only on $|v_0|$. If $T \leq T_*(|v_0|)$ and $y \in B_T$ then by eq. (47) we have $d_{\mathcal{Q},\eta}(\Gamma(y), 0) \leq |v_0| + CT^\varepsilon(1 + b_T)^3 = b_T$ which implies that Γ maps B_T onto itself.

Contraction. By arguments similar to those used above we can prove that if $y^{(1)}, y^{(2)}$ are two elements of $\mathcal{Q}_\eta$ then

$$d_{\mathcal{Q},\gamma}(\Gamma(y^{(1)}), \Gamma(y^{(2)})) \leq C d_{\mathcal{Q},\eta}(y^{(1)}, y^{(2)})(1 + d_{\mathcal{Q},\eta}(y^{(1)}, 0) + d_{\mathcal{Q},\eta}(y^{(2)}, 0))^2 \tag{48}$$

Then

$$d_{\mathcal{Q},\eta}(\Gamma(y^{(1)}), \Gamma(y^{(2)})) \leq |y_0^{(1)} - y_0^{(2)}| + CT^\varepsilon d_{\mathcal{Q},\eta}(y^{(1)}, y^{(2)})(1 + d_{\mathcal{Q},\eta}(y^{(1)}, 0) + d_{\mathcal{Q},\eta}(y^{(2)}, 0))^2 \tag{49}$$

Taking $y_1, y_2 \in B_T$ we have

$$d_{\mathcal{Q},\eta}(\Gamma(y^{(1)}), \Gamma(y^{(2)})) \leq C(1 + 2b_T)^2 T^\varepsilon d_{\mathcal{Q},\eta}(y^{(1)}, y^{(2)}) \tag{50}$$

Let $T_{**} < T_*$ and such that $C(1 + 2b_T)^2 T_{**}^\varepsilon < 1$, eq. (50) implies that Γ is a strict contraction on $B_{T_{**}}$ with a unique fixed-point (in $B_{T_{**}}$). A standard argument allows to extend the solution to the larger time interval T_*.

Uniqueness. Now assume to have two solutions $v^{(1)}, v^{(2)} \in \mathcal{Q}_\eta$ of eq. (16), i.e. such that $v^{(i)} = \Gamma(v^{(i)})$, $v_0^{(i)} = v_0$ for $i = 1, 2$. Eq. (48) applied to the distance between $\Gamma(v^{(1)})$ and $\Gamma(v^{(2)})$ implies

$$d_{\mathcal{Q},\gamma}(v^{(1)}, v^{(2)}) \leq C d_{\mathcal{Q},\eta}(v^{(1)}, v^{(2)})(1 + d_{\mathcal{Q},\eta}(v^{(1)}, 0) + d_{\mathcal{Q},\eta}(v^{(2)}, 0))^2$$

If we denote $\mathcal{Q}_{\eta,T}$ the space $\mathcal{Q}_\eta$ considered for functions on $[0, T]$ and with $d_{\mathcal{Q},\eta,T}(\cdot, \cdot)$ the corresponding distance, then we have for $S \leq T$ $d_{\mathcal{Q},\eta,S}(y, 0) \leq d_{\mathcal{Q},\eta,T}(y, 0)$ for any $y \in \mathcal{Q}_{\eta,T}$. Then

$$d_{\mathcal{Q},\eta,S}(v^{(1)}, v^{(2)}) \leq C d_{\mathcal{Q},\eta,S}(v^{(1)}, v^{(2)})(1 + d_{\mathcal{Q},\eta,S}(v^{(1)}, 0) + d_{\mathcal{Q},\eta,S}(v^{(2)}, 0))^2$$

and so

$$d_{\mathcal{Q},\eta,S}(v^{(1)}, v^{(2)}) \leq CS^\varepsilon d_{\mathcal{Q},\eta,S}(v^{(1)}, v^{(2)})(1 + d_{\mathcal{Q},\eta,S}(v^{(1)}, 0) + d_{\mathcal{Q},\eta,S}(v^{(2)}, 0))^2$$

so when S is small enough that $CS^\varepsilon(1 + d_{\mathcal{Q},\eta,S}(v^{(1)}, 0) + d_{\mathcal{Q},\eta,S}(v^{(2)}, 0))^2 \leq 1/2$ we must have that $d_{\mathcal{Q},\eta,S}(v^{(1)}, v^{(2)}) = 0$ and then $v_t^{(1)} = v_t^{(2)}$ for any $t \in [0, S]$. By repeating this argument we conclude that the two solution coincide in the whole interval $[0, T]$.

Lipschitz continuity. Modulo some technicalities due to the local character of the solution, the proof of the Lipschitz continuity of the map Θ follows the line of the finite dimensional situation. See [18] for details. ■

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