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RESEARCH REPORT
N° 8381
October 17, 2013
Project-Team Mathrisk

ISSN 0249-6399 ISRN INRIA/RR--8381--FR+ENG

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Research Report n° 8381 — October 17, 2013 — ?? pages

Abstract: We study double barrier reflected BSDEs (DBBSDEs) with jumps and RCLL barriers, and their links with generalized Dynkin games. We provide existence and uniqueness results and prove that for any Lipschitz driver, the solution of the DBBSDE coincides with the value function of a game problem, which can be seen as a generalization of the classical Dynkin problem to the case of g-conditional expectations. Using this characterization, we prove some new results on DBBSDEs with jumps, such as comparison theorems and a priori estimates. We then study DBBSDEs with jumps and RCLL obstacles in the Markovian case and their links with parabolic partial integro-differential variational inequalities (PIDVI) with two obstacles.

Key-words: Double barrier reflected BSDEs, Backward stochastic differential equations with jumps, g-expectation, Dynkin games, comparison theorem, partial integro-differential variational inequalities, viscosity solution

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Equations différentielles stochastiques rétrogrades réfléchies à deux barrières et jeux de Dynkin généralisés

Résumé : Nous étudions des équations différentielles stochastiques rétrogrades réfléchies (EDSR) à deux barrières avec sauts dans le cas où les barrières sont modélisées par des processus stochastiques càdlàg. Nous prouvons l'existence et l'unicité de la solution et nous démontrons que dans le cas où le driver est lipschitzien, la solution coincide avec la fonction valeur d'un jeu différentiel stochastique, qui peut s'écrire comme un jeu de Dynkin généralisé avec g- espérance. Grace à cette caractérisation, nous démontrons des théorèmes de comparaison et des estimations a priori. Dans le cas Markovien, nous étudions le lien avec des inéquations variationnelles intégro-différentielles à 2 obstacles.

Mots-clés : Equations différentielles stochastiques rétrogrades réfléchies à deux barrières; Equations Différentielles stochastiques rétrogrades avec sauts; espérance non-linéaire; jeux de Dynkin; théorème de comparaison; équation intégro-différentielle; solution de viscosité

1 Introduction

Backward stochastic differential equations (BSDEs) were introduced, in the case of a Brownian filtration, by Bismut (1976), then generalized by Pardoux and Peng. They represent a useful tool in mathematical finance and stochastic control. Reflected BSDEs (RBSDEs) are studied by N. EL Karoui et al. ([?, ?]) in the Brownian framework, and extended to the case of jumps by Essaky, Hamadène and Ouknine, Crépey and Matoussi ([?, ?, ?, ?]). In [?], Quenez and Sulem have focused on the case when the obstacle is RCLL only. In particular, they have shown that the solution of an RBSDE with jumps corresponds to the value function of a related optimal stopping problem, generalizing some results of [?]. In [?], we have studied the links between RBSDEs with jumps and RCLL obstacle and parabolic partial integro-differential variational inequalities (PIDVI) in the Markovian case.

In the Brownian framework, double barrier reflected BSDEs (DBBSDEs) have been introduced by Cvitanic and Karatzas in [?] for a regular obstacle, and then studied by several authors [?, ?, ?, ?, ?]. The solutions of such equations are constrained to stay between two processes called barriers or obstacles. The extension to the case of jumps can be found in [?, ?, ?].

In this paper, we study DBBSDEs with jumps and RCLL barriers, and their links with some game problems. We first provide existence and uniqueness results which complete the previous works ([?, ?, ?]). It is well-known that when the driver does not depend on the solution, the solution of the DBBSDE can be characterized as the value function of a Dynkin game problem (see e.g. [?, ?, ?]). We generalize this result to the case of a driver depending on the solution. More generally, we show that for any Lipschitz driver, the solution of the DBBSDE coincides with the value function of a game problem, which can be seen as a generalization of the classical Dynkin problem to the case of g-conditional expectations. Using this characterization, we prove some new results on DBBSDEs, such as comparison theorems and a priori estimates. We then study DBBSDEs with jumps and RCLL obstacles in the Markovian case and their links with parabolic partial integro-differential variational inequalities (PIDVI) with two obstacles.

The paper is organized as follows: In Section ?? we introduce notation and definitions and provide some preliminary results. In Section ??, we study DBBSDEs when the driver does not depend on the solution and their links with Dynkin game problems which completes the previous works on DBBSDEs. In Section ??, we turn to the general case. We state an existence and uniqueness result for DBBSDEs with jumps, RCLL obstacle and general Lipschitz driver, and we prove that the solution of the DBBSDE can be characterized as the value function of a generalized Dynkin game problem with g-conditional expectations. In Section ??, we provide comparison theorems and a priori estimates for DBBSDEs with jumps and RCLL obstacles. In the Markovian case, relations between a DBBSDE with jumps and a PIDVI are studied in Section ??. We show that the solution of a DBBSDE corresponds to a solution of the PIDVI in the viscosity sense. Under additional assumptions, we establish an uniqueness result in the class of continuous functions, with polynomial growth.

2 Definitions and preliminary results

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let W be a one-dimensional Brownian motion and $N(dt, du)$ be a Poisson random measure with compensator $\nu(du)dt$ such that ν is a σ-finite measure on

$\mathbf{R}^*$, equipped with its Borel field $\mathcal{B}(\mathbf{R}^*)$. Let $\tilde{N}(dt, du)$ be its compensated process. Let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be the natural filtration associated with W and N.

Notation. Let $\mathcal{P}$ be the predictable σ-algebra on $[0, T] \times \Omega$.

For each $T > 0$, we use the following notation:

- $L^2(\mathcal{F}_T)$ is the set of random variables ξ which are $\mathcal{F}_T$-measurable and square integrable.
- $\mathbb{H}^2$ is the set of real-valued predictable processes ϕ such that

$$\|\phi\|^2_{\mathbb{H}^2} := E\left[\int_0^T \phi_t^2 dt\right] < \infty.$$

- L^2_ν is the set of Borelian functions $\ell : \mathbf{R}^* \to \mathbf{R}$ such that $\int_{\mathbf{R}^*} |\ell(u)|^2 \nu(du) < +\infty$.
  The set L^2_ν is a Hilbert space equipped with the scalar product

$$\langle \delta, \ell \rangle_\nu := \int_{\mathbf{R}^*} \delta(u)\ell(u)\nu(du) \quad \text{for all } \delta, \ell \in L^2_\nu \times L^2_\nu,$$

  and the norm $\|\ell\|^2_\nu := \int_{\mathbf{R}^*} |\ell(u)|^2 \nu(du)$.
- $\mathbb{H}^2_\nu$ is the set of processes l which are *predictable*, that is, measurable

$$l : ([0, T] \times \Omega \times \mathbf{R}^*,\ \mathcal{P} \otimes \mathcal{B}(\mathbf{R}^*)) \to (\mathbf{R}\ , \mathcal{B}(\mathbf{R})); \quad (\omega, t, u) \mapsto l_t(\omega, u)$$

  such that

$$\|l\|^2_{\mathbb{H}^2_\nu} := E\left[\int_0^T \|l_t\|^2_\nu \, dt\right] < \infty.$$

- $\mathcal{S}^2$ is the set of real-valued RCLL adapted processes ϕ such that

$$\|\phi\|^2_{\mathcal{S}^2} := E(\sup_{0 \leq t \leq T} |\phi_t|^2) < \infty.$$

- $\mathcal{A}^2$ is the set of real-valued non decreasing RCLL predictable processes A with $A_0 = 0$ and $E(A_T^2) < \infty$.
- $\mathcal{T}_0$ denotes the set of stopping times τ such that $\tau \in [0, T]$ a.s.
- For S in $\mathcal{T}_0$, $\mathcal{T}_S$ is the set of stopping times τ such that $S \leq \tau \leq T$ a.s.

Definition 2.1 (Driver, Lipschitz driver) *A function g is said to be a* driver *if*

- $g : [0, T] \times \Omega \times \mathbf{R}^2 \times L^2_\nu \to \mathbf{R}$
  $(\omega, t, y, z, \kappa(\cdot)) \mapsto g(\omega, t, y, z, k(\cdot))$ *is* $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}(L^2_\nu)-$ *measurable,*
- $g(., 0, 0, 0) \in \mathbb{H}^2$.

A driver g is called a Lipschitz driver *if moreover there exists a constant $C \geq 0$ such that $dP \otimes dt$-a.s. , for each (y_1, z_1, k_1), (y_2, z_2, k_2),*

$$|g(\omega, t, y_1, z_1, k_1) - g(\omega, t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + \|k_1 - k_2\|_\nu).$$

Theorem 2.2 (Existence and uniqueness result for BSDEs with jumps, Tang & Li [?]) *Let $T > 0$. For each Lipschitz driver g, and each terminal condition $\xi \in L^2(\mathcal{F}_T)$, there exists a unique solution $(X, \pi, l) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu$ satisfying*

$$-dX_t = g(t, X_{t^-}, \pi_t, l_t(\cdot))dt - \pi_t dW_t - \int_{\mathbf{R}^*} l_t(u)\tilde{N}(dt, du); \qquad X_T = \xi. \tag{2.1}$$

The solution is denoted by $(X(\xi, T), \pi(\xi, T), l(\xi, T))$.

This result can be extended when the terminal time T is replaced by a stopping time $S \in \mathcal{T}_0$. Let $(X(\xi, S), \pi(\xi, S), l(\xi, S))$ (denoted here by (X, π, l)) be the solution of the BSDE associated with driver g, terminal time S and terminal condition $\xi \in L^2(\mathcal{F}_S)$. The solution can be extended on the whole interval $[0, T]$ by setting $X_t = \xi, \pi_t = 0, l_t = 0$ for $t \geq S$. So, $((X_t, \pi_t, l_t); t \leq T)$ is the unique solution of the BSDE with driver $g(t, y, z, k)\mathbf{1}_{\{t \leq S\}}$ and terminal conditions $(T,\, \xi)$.

We refer to [**?**] and to [**?**] for more results on BSDEs with jumps.

Definition 2.3 (Double barrier reflected BSDEs with jumps) *Let $T > 0$ be a fixed terminal time and g be a Lipschitz driver. Let ξ and ζ be two adapted RCLL processes with $\zeta_T = \xi_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \leq \zeta_t$, $\forall t \in [0, T]$ a.s.*

A process $(Y, Z, k(.), \alpha)$ in $\mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{S}^2$ is said to be a solution of the double barrier reflected BSDE (DBBSDE) associated with driver g and barriers ξ, ζ if

$$\alpha = A - A' \text{ with } A, A' \in \mathcal{A}^2$$

$$-\, dY_t = g(t, Y_t, Z_t, k_t(\cdot))dt + dA_t - dA_t^{'} - Z_t dW_t - \int_{\mathbf{R}^*} k_t(u)\tilde{N}(dt, du); \quad Y_T = \xi_T, \tag{2.2}$$

$$\xi_t \leq Y_t \leq \zeta_t,\ 0 \leq t \leq T \text{ a.s.},$$

$$\begin{cases} \int_0^T (Y_t - \xi_t) dA_t^c = 0 \text{ a.s. and } \int_0^T (\zeta_t - Y_t) dA_t^{'c} = 0 \text{ a.s.} \\ \Delta A_\tau^d = \Delta A_\tau^d \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}} \text{ and } \Delta A_\tau^{'d} = \Delta A_\tau^{'d} \mathbf{1}_{\{Y_{\tau^-} = \zeta_{\tau^-}\}} \text{ a.s. } \forall \tau \in \mathcal{T}_0 \text{ predictable} \end{cases} \tag{2.3}$$

Here A^c (resp $A^{'c}$) denotes the continuous part of A (resp $A^{'}$) and A^d (resp $A^{'d}$) its discontinuous part.

We introduce the following definition.

Definition 2.1 *A progressive process (ϕ_t) (resp. integrable) is said to be* left-upper semicontinuous (l.u.s.c.) along stopping times *(resp. along stopping times in expectation) if for all $\tau \in \mathcal{T}_0$ and for each non decreasing sequence of stopping times (τ_n) such that $\tau^n \uparrow \tau$ a.s. ,*

$$\phi_\tau \geq \limsup_{n \to \infty} \phi_{\tau_n} \quad a.s. \quad (resp.\ E[\phi_\tau] \geq \limsup_{n \to \infty} E[\phi_{\tau_n}]). \tag{2.4}$$

Remark 2.4 In Definition **??***, no condition is required at a totally inaccessible stopping time. Since the filtration is generated by W and N, this means that no condition is required at the jump times of N.*
Moreover, when (ϕ_t) is left-limited, then (ϕ_t) is left-upper semicontinuous (l.u.s.c.) along stopping times *if and only if for all predictable stopping time $\tau \in \mathcal{T}_0$,*

$$\phi_\tau \geq \phi_{\tau_-} \quad a.s.$$

3 DBBSDEs with driver g independent of y, z, k and links with Dynkin games

In this section, we suppose that the driver g does not depend on y, z, k, that is

$$g(\omega, t, y, z, k(\cdot)) = g(\omega, t),$$

where g is in $\mathbb{H}^2$. Let ξ and ζ be two adapted RCLL processes with $\zeta_T = \xi_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \le \zeta_t$, $\forall t \in [0,T]$ a.s.

We show that the DBBSDE admits a solution $(Y, Z, k(\cdot), \alpha)$, which is related to the following Dynkin game problem.
For any $S \in \mathcal{T}_0$ and any stopping times $\tau, \sigma \in \mathcal{T}_S$, consider the gain (or payoff):

$$I_S(\tau, \sigma) = \int_S^{\sigma\wedge\tau} g(u)du + \xi_\tau \mathbf{1}_{\{\tau \le \sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma < \tau\}} \tag{3.5}$$

For any $S \in \mathcal{T}_0$, the upper and lower value functions at time S are defined respectively by

$$\overline{V}(S) := ess \inf_{\sigma \in \mathcal{T}_S} ess \sup_{\tau \in \mathcal{T}_S} \mathbb{E}[I_S(\tau, \sigma)|\mathcal{F}_S] \tag{3.6}$$

$$\underline{V}(S) := ess \sup_{\tau \in \mathcal{T}_S} ess \inf_{\sigma \in \mathcal{T}_S} \mathbb{E}[I_S(\tau, \sigma)|\mathcal{F}_S] \tag{3.7}$$

We clearly have the inequality $\underline{V}(S) \le \overline{V}(S)$ a.s.
By definition, we say that there exists a value function at time S for the Dynkin game problem if $\overline{V}(S) = \underline{V}(S)$ a.s.

Definition 3.1 (S-saddle point) *Let $S \in \mathcal{T}_0$. A pair $(\tau^*, \sigma^*) \in \mathcal{T}_S^2$ is called an S-saddle point if for each $(\tau, \sigma) \in \mathcal{T}_S^2$, we have*

$$\mathbb{E}[I_S(\tau, \sigma^*)|\mathcal{F}_S] \le \mathbb{E}[I_S(\tau^*, \sigma^*)|\mathcal{F}_S] \le \mathbb{E}[I_S(\tau^*, \sigma)|\mathcal{F}_S] \text{ a.s.}$$

We introduce the following RCLL adapted processes which depend on the process g:

$$\tilde{\xi}_t^g := \xi_t - \mathbb{E}[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t], \quad \tilde{\zeta}_t^g := \zeta_t - \mathbb{E}[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t], \quad 0 \le t \le T. \tag{3.8}$$

They satisfy the important property

$$\tilde{\xi}_T^g = \tilde{\zeta}_T^g = 0 \text{ a.s.}$$

Moreover, this change of variables allows to get rid of the term $\int g(t)dt$, and thus to simplify the notation.

For each RCLL adapted process $\phi = (\phi_t)_{0\le t\le T}$ valued in $\mathbf{R} \cup \{+\infty\}$ with $\phi^- \in \mathcal{S}^2$, we denote by $\mathcal{R}(\phi)$ the Snell envelope of ϕ, defined as the minimal RCLL supermartingale greater or equal to ϕ a.s. By the optimal stopping's results, $\mathcal{R}(\phi)$ is equal to the value function of the optimal stopping problem associated with the reward ϕ.

We state the following result.

Lemma 3.2 *There exists a unique pair of non-negative RCLL supermartingales* (J^g, J'^g) *valued in* $[0,+\infty]$ *satisfying the system*

$$\begin{cases} J^g = \mathcal{R}(J'^g + \tilde{\xi}^g) \\ J'^g = \mathcal{R}(J^g - \tilde{\zeta}^g), \end{cases} \tag{3.9}$$

and satisfying the following minimality property: if H *and* H' *are non-negative RCLL supermartingales valued in* $[0,+\infty]$ *such that* $H \geq H' + \tilde{\xi}^g$ *and* $H' \geq H - \tilde{\zeta}^g$ *, then we have* $J^g \leq H$ *and* $J'^g \leq H'$.

Remark 3.3 If H *and* H' *are non-negative RCLL supermartingales valued in* $[0,+\infty]$ *satisfying* $H = \mathcal{R}(H' + \tilde{\xi}^g)$ *and* $H' = \mathcal{R}(H - \tilde{\zeta}^g)$*, then* $H \geq H' + \tilde{\xi}^g$ *and* $H' \geq H - \tilde{\zeta}^g$ *a.s. Hence,* (J^g, J'^g) *is the* minimal solution of the system (**??**).

The proof is given in the Appendix. We point out that the property $\tilde{\xi}^g_T = \tilde{\zeta}^g_T = 0$ a.s. is used in the proof.

Proposition 3.4 *Let* ξ *and* ζ *be two adapted RCLL processes with* $\zeta_T = \xi_T$ *a.s.,* $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \leq \zeta_t$, $\forall t \in [0,T]$ *a.s. Suppose that* $J^g, J'^g \in \mathcal{S}^2$. *Let* $\overline{Y}$ *be the RCLL adapted process defined by*

$$\overline{Y}_t := J^g_t - J'^g_t + E[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t];\ 0 \leq t \leq T. \tag{3.10}$$

There exists Z, k, α *such that* $(\overline{Y}, Z, k, \alpha)$ *is a solution of DBBSDE* (**??**)*. Moreover,* $\alpha = A - A'$*, where* A *and* A' *are the non-decreasing predictable process associated to the Doob-Meyer decomposition of* J^g *and* J'^g.

Proof. By assumption, J^g and J^g are square integrable supermartingales. The process $\overline{Y}$ is thus well defined. Since (J^g, J'^g) satisfies the system (**??**) and since $\tilde{\xi}^g_T = \tilde{\zeta}^g_T = 0$ a.s. , we have $J^g_T = J'^g_T$ a.s. Hence, $\overline{Y}_T = \xi_T$ a.s. By the Doob-Meyer decomposition of supermartingales, there exist two square integrable martingales M and M', two square integrable nondecreasing predictable RCLL processes A and A' with $A_0 = A'_0 = 0$ such that:

$$dJ^g_t = dM_t - dA_t \tag{3.11}$$

$$dJ'^g_t = dM'_t - dA'_t. \tag{3.12}$$

By the optimal stopping theory (see e.g. Proposition B.1 in [**?**]), the process A^c increases only when the value function J^g is equal to the corresponding reward $J'^g + \tilde{\xi}^g$. Now, $\{J^g_t = J'^g_t + \tilde{\xi}^g\} = \{\overline{Y_t} = \xi_t\}$. Hence, $\int_0^T (\overline{Y_t} - \xi_t) dA^c_t = 0$ a.s. Similarly the process A'^c satisfies $\int_0^T (\overline{Y_t} - \zeta_t) dA'^c_t = 0$ a.s. Moreover, for each predictable stopping time $\tau \in \mathcal{T}_0$ we have $\Delta A^d_\tau = \mathbf{1}_{J^g_{\tau-} = J'^g_{\tau-} + \tilde{\xi}^g_{\tau-}} \Delta A^d_\tau = \mathbf{1}_{\overline{Y}_{\tau-} = \xi_{\tau-}} \Delta A^d_\tau$. Similarly, $\Delta A'^d_\tau = \mathbf{1}_{\overline{Y}_{\tau-} = \zeta_{\tau-}} \Delta A'^d_\tau$ a.s. Define

$$\overline{M}_t := M_t - M'_t + E[\xi_T + \int_0^T g(s)ds|\mathcal{F}_t].$$

By (**??**), (**??**), (**??**), we derive $d\overline{Y}_t = d\overline{M}_t - dA_t + dA'_t - g(t)dt$. Now, by the martingale representation theorem, there exist $Z \in \mathbb{H}^2, k \in \mathbb{H}^2_\nu$ such that:

$$d\overline{M}_t = Z_t dW_t + \int_{\mathbb{R}^*} k_t(u) \tilde{N}(du, dt). \tag{3.13}$$

In other words, $(\overline{Y}, Z, k, \alpha)$ is a solution of DBBSDE (??) with $\alpha = A - A'$. □

From this proposition, we derive the following uniqueness and existence result, as well as the characterization of the solution as the value function of the above Dynkin game problem. Also, we show that, under an additional assumption on the solution, there exists a saddle point.

Theorem 3.5 ***(Existence and uniqueness, characterization)*** *Let ξ and ζ be two adapted RCLL processes with $\zeta_T = \xi_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \leq \zeta_t$, $\forall t \in [0,T]$ a.s. Suppose that $J^g_t, J'^g_t \in \mathcal{S}^2$.*

Then DBBSDE (??) associated with driver process $g(t)$ admits a unique solution $(Y, Z, k(.), \alpha) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{S}^2$, and for each $S \in \mathcal{T}_0$, Y_S is the common value function of the Dynkin game, that is

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad a.s. \tag{3.14}$$

Moreover, if the process α is continuous, then, for each $S \in \mathcal{T}_0$, the pair of stopping times (τ^*_s, σ^*_s) defined by

$$\sigma^*_S := \inf\{t \geq S,\ Y_t = \zeta_t\}; \quad \tau^*_S := \inf\{t \geq S,\ Y_t = \xi_t\}. \tag{3.15}$$

is an S-saddle point for the Dynkin game problem associated with the gain I_S.

Proof. We have already proved the existence. Let $(Y, Z, k(\cdot), \alpha)$ be a solution of the DBBSDE associated with driver process $g(t)$ and obstacles (ξ, ζ). Let us prove that it is unique. We first show the uniqueness of Y.

• Consider first the simpler case when α is continuous. Let A and A' be non-decreasing processes in $\mathcal{A}^2$, which can be taken continuous, such that $\alpha = A - A'$. For each $S \in \mathcal{T}_0$, consider

$$\sigma^*_S := \inf\{t \geq S,\ Y_t = \zeta_t\}; \quad \tau^*_S := \inf\{t \geq S,\ Y_t = \xi_t\}.$$

Note that $\sigma^*_S \in \mathcal{T}_S$ and $\tau^*_S \in \mathcal{T}_S$. Let us show that (τ^*_S, σ^*_S) is an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.

Since Y and ξ are right-continuous processes, we have $Y_{\sigma^*_S} = \xi_{\sigma^*_S}$ and $Y_{\tau^*_S} = \xi_{\tau^*_S}$ a.s. By definition of τ^*_S, for almost every ω, for each $t \in [S(\omega), \tau^*_S(\omega)[$, we have $Y_t(\omega) > \xi_t(\omega)$. Hence, since Y is solution of the DBBSDE, for almost every ω, the nondecreasing function $t \mapsto A_t(\omega)$ is constant on $[S(\omega), \tau^*_S(\omega)[$. The continuity of A implies that $t \mapsto A_t(\omega)$ is constant on $[S(\omega), \tau^*_S(\omega)]$. Similarly, the process A' is constant on $[S, \sigma^*_S]$ a.s.

Consequently, since $(Y, Z, k(.), \alpha)$ is the solution of the DBBSDE associated with driver $g(t)$, the process $(Y_t + \int_0^t g(s)ds, S \leq t \leq \tau^*_S \wedge \sigma^*_S)$ is a martingale. Hence, we have:

$$Y_S = E[Y_{\tau^*_S \wedge \sigma^*_S} + \int_S^{\tau^*_S \wedge \sigma^*_S} g(s)ds \mid \mathcal{F}_S] = E[\xi_{\tau^*_S} \mathbf{1}_{\tau^*_S \leq \sigma^*_S} + \zeta_{\sigma^*_S} \mathbf{1}_{\sigma^*_S < \tau^*_S} + \int_S^{\tau^*_S \wedge \sigma^*_S} g(s)ds \mid \mathcal{F}_S] \quad \text{a.s.}$$

Finally,

$$Y_S = E[I_S(\tau^*_S, \sigma^*_S) \mid \mathcal{F}_S] \quad \text{a.s.}$$

Let $\tau \in \mathcal{T}_S$. We want to show that:

$$Y_S \geq E[I_S(\tau, \sigma^*_S) \mid \mathcal{F}_S] \text{ a.s.} \tag{3.16}$$

Since A' is constant on $[S, \sigma_S^*]$, the process $(Y_t + \int_0^t g(s)ds,\ S \leq t \leq \sigma_S^* \wedge \tau)$ is a supermartingale, hence:

$$Y_S \geq E[Y_{\tau \wedge \sigma_S^*} + \int_S^{\tau \wedge \sigma_S^*} g(s)ds \mid \mathcal{F}_S] \text{ a.s.}$$

Since $Y \geq \xi$ and $Y_{\sigma_S^*} = \zeta_{\sigma_S^*}$ a.s. , we have:

$$Y_{\tau \wedge \sigma_S^*} = Y_\tau \mathbf{1}_{\tau \leq \sigma_S^*} + Y_{\sigma_S^*} \mathbf{1}_{\sigma_S^* < \tau} \geq \xi_\tau \mathbf{1}_{\tau \leq \sigma_S^*} + \zeta_{\sigma_S^*} \mathbf{1}_{\sigma_S^* < \tau} \text{ a.s.}$$

Hence, inequality (**??**) holds. Similarly, one can show that for each $\sigma \in \mathcal{T}_S$, we have

$$Y_S \leq E[I_S(\tau_S^*, \sigma) \mid \mathcal{F}_S] \quad \text{a.s.}$$

Hence, (τ_S^*, σ_S^*) is an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s. The uniqueness of Y follows.

• Consider now the general case. For each $S \in \mathcal{T}_0$ and for each $\varepsilon > 0$, let

$$\tau_S^\varepsilon := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\} \quad \sigma_S^\varepsilon := \inf\{t \geq S,\ Y_t \geq \zeta_t - \varepsilon\}. \tag{3.17}$$

Note that σ_S^ε and $\tau_S^\varepsilon \in \mathcal{T}_S$. Fix $\varepsilon > 0$. For a.e. ω, if $t \in [S(\omega), \tau_S^\varepsilon(\omega)[$, then $Y_t(\omega) > \xi_t(\omega) + \varepsilon$ and hence $Y_t(\omega) > \xi_t(\omega)$. It follows that for a.e. ω, the function $t \mapsto A_t^c(\omega)$ is constant on $[S(\omega), \tau_S^\varepsilon(\omega)]$ and $t \mapsto A_t^d(\omega)$ is constant on $[S(\omega), \tau_S^\varepsilon(\omega)[$. Also, $Y_{(\tau_S^\varepsilon)^-} \geq \xi_{(\tau_S^\varepsilon)^-} + \varepsilon$ a.s. Since $\varepsilon > 0$, it follows that $Y_{(\tau_S^\varepsilon)^-} > \xi_{(\tau_S^\varepsilon)^-}$ a.s. , which implies that $\Delta A^d_{\tau_S^\varepsilon} = 0$ a.s. Hence, the process A is constant on $[S, \tau_S^\varepsilon]$. Furthermore, by the right-continuity of (ξ_t) and (Y_t), we clearly have

$$Y_{\tau_S^\varepsilon} \leq \xi_{\tau_S^\varepsilon} + \varepsilon \quad \text{a.s.}$$

Similarly, one can show that the process A' is constant on $[S, \sigma_S^\varepsilon]$ and that

$$Y_{\sigma_S^\varepsilon} \geq \zeta_{\sigma_S^\varepsilon} - \varepsilon \quad \text{a.s.}$$

Let $\tau \in \mathcal{T}_S$. Since A' is constant on $[S, \sigma_S^\varepsilon]$, the process $(Y_t + \int_0^t g(s)ds,\ S \leq t \leq \tau \wedge \sigma_S^\varepsilon)$ is a supermartingale. Hence

$$Y_S \geq E[Y_{\tau \wedge \sigma_S^\varepsilon} + \int_S^{\tau \wedge \sigma_S^\varepsilon} g(s)ds \mid \mathcal{F}_S] \quad \text{a.s.}$$

We also have that:

$$Y_{\tau \wedge \sigma_S^\varepsilon} = Y_\tau \mathbf{1}_{\tau \leq \sigma_S^\varepsilon} + Y_{\sigma_S^\varepsilon} \mathbf{1}_{\sigma_S^\varepsilon < \tau} \geq \xi_\tau \mathbf{1}_{\tau \leq \sigma_S^*} + (\zeta_{\sigma_S^\varepsilon} - \varepsilon) \mathbf{1}_{\sigma_S^\varepsilon < \tau} \text{ a.s.}$$

We derive

$$Y_S \geq E[I_S(\tau, \sigma_S^\varepsilon) \mid \mathcal{F}_S] - \varepsilon \text{ a.s.} \tag{3.18}$$

Similarly, one can show that for each $\sigma \in \mathcal{T}_S$

$$Y_S \leq E[I_S(\tau_S^\varepsilon, \sigma) \mid \mathcal{F}_S] + \varepsilon \text{ a.s.} \tag{3.19}$$

By (**??**) and (**??**), for each $\varepsilon > 0$

$$ess \sup_{\tau \in \mathcal{T}_s} E[I_S(\tau, \sigma_S^\varepsilon) \mid \mathcal{F}_S] - \varepsilon \ \leq \ Y_S \ \leq \ ess \inf_{\sigma \in \mathcal{T}_S} E[I_S(\tau_S^\varepsilon, \sigma) \mid \mathcal{F}_S] + \varepsilon \ \text{ a.s.} \tag{3.20}$$

$$\overline{V}(S) - \varepsilon \ \leq \ Y_S \ \leq \ \underline{V}(S) + \varepsilon \quad \text{ a.s.}$$

Since $\underline{V}(S) \leq \overline{V}(S)$ a.s. we get

$$\underline{V}(S) = Y_S = \overline{V}(S) \quad \text{ a.s.}$$

This equality holds of each stopping time $S \in \mathcal{T}_0$, which implies the uniqueness of Y.

By the uniqueness of the decomposition of the semimartingale Y and the martingale representation theorem, we derive the uniqueness of (Z, k, α). The proof of thus complete. □

Remark 3.6 The uniqueness of the solution (Y, Z, k, α) of the DBBSDE together with Proposition **??** *yields the equality $Y_t = \overline{Y}_t$, where $\overline{Y}$ is defined by* (**??**).
Moreover, we stress that some arguments of the above proof will be extended to the case of a general driver in the next section.

We now provide a sufficient condition on ξ and ζ for the existence of saddle points. By the last assertion of Theorem **??**, it is sufficient to give a condition which ensures the continuity of the process α.

Theorem 3.7 *Suppose that the assumptions of Th.* **??** *are satisfied and that ξ and $-\zeta$ are l.u.s.c. along stopping times. Let $(Y, Z, k(.), \alpha)$ be the solution of DBBSDE* (**??**).
The process α is then continuous. Also, for each $S \in \mathcal{T}_0$, the pair of stopping times (τ_S^*, σ_S^*) defined by (**??**) *is an S-saddle point.*

Remark 3.8 The assumption made on ξ and ζ is wilder than the one made in the literature where it is also supposed $\xi_t < \zeta_t, t < T$ a.s.

We first prove the following lemma:

Lemma 3.9 *If ξ and $-\zeta$ are l.u.s.c. along stopping times, then the RCLL processes J^g and $J^{'g}$ are l.u.s.c. along stopping times.*

Proof. In order to simplify the notation, we denote J^g by J and $J^{'g}$ by J'. Let us show that J is l.u.s.c. along stopping times, that is for each predictable stopping time τ, we have $\Delta J_\tau \geq 0$. Since the filtration is generated by W and N, the martingales only admit inaccessible jumps. Hence, $\Delta J_\tau = -\Delta A_\tau$. Now, $\Delta A_\tau^d = \mathbf{1}_{\{J_{\tau^-} = J'_{\tau^-} + \tilde{\xi}_{\tau^-}\}} \Delta A_\tau^d$. Hence

$$J_\tau - J_{\tau^-} = \mathbf{1}_{\{J_{\tau^-} = J'_{\tau^-} + \tilde{\xi}_{\tau^-}\}} (J_\tau - J_{\tau^-}) = \mathbf{1}_{\{J_{\tau^-} = J'_{\tau^-} + \tilde{\xi}_{\tau^-}\}} (J_\tau - J'_{\tau^-} - \tilde{\xi}_{\tau^-}).$$

We thus have

$$\Delta J_\tau = \mathbf{1}_{\{\mathbf{\Delta J_\tau \neq 0}\}} (\mathbf{J}_\tau - \mathbf{J}'_{\tau^-} - \tilde{\xi}_{\tau^-}) \geq \mathbf{1}_{\{\mathbf{\Delta J_\tau \neq 0}\}} (\mathbf{J}_\tau - \mathbf{J}'_{\tau^-} - \tilde{\xi}_\tau), \tag{3.21}$$

because, by assumption, $\xi_{\tau^-} \leq \xi_\tau$ a.s. and hence $\tilde{\xi}_{\tau^-} \leq \tilde{\xi}_\tau$ a.s. Suppose we have shown that $\{\Delta J_\tau \neq 0\} \cap \{\Delta J'_\tau \neq 0\} = \emptyset$ a.s. Then, $J'_{\tau^-} = J'_\tau$ a.s. on $\{\Delta J_\tau \neq 0\}$. By inequality (**??**), it follows that $\Delta J_\tau \geq \mathbf{1}_{\{\mathbf{\Delta J_\tau \neq 0}\}} (\mathbf{J}_\tau - J'_\tau - \tilde{\xi}_\tau) \geq \mathbf{0}$ a.s. because $J \geq J' + \xi$.
It remains to show that $\{\Delta J_\tau \neq 0\} \cap \{\Delta J'_\tau \neq 0\} = \emptyset$ a.s. which is equivalent to $\Delta A_\tau \wedge \Delta A'_\tau = 0$

a.s. Note that $J_t = E[A_T - A_t \mid \mathcal{F}_t]$ and $J'_t = E[A'_T - A'_t \mid \mathcal{F}_t]$.
We introduce the following processes:

$$\begin{cases} \tilde{A}_t := A_t - (\Delta A_\tau \wedge \Delta A'_\tau)\mathbf{1}_{\{\mathbf{t}\geq\tau\}} \\ \tilde{A}'_t := A'_t - (\Delta A_\tau \wedge \Delta A'_\tau)\mathbf{1}_{\{\mathbf{t}\geq\tau\}} \end{cases} \tag{3.22}$$

We have $\Delta\tilde{A}_\tau \geq 0$ and $\Delta\tilde{A}'_\tau \geq 0$. Hence $\tilde{A}$ and $\tilde{A}'$ are non decreasing predictable processes with $\tilde{A}_0 = \tilde{A}'_0 = 0$. We set: $\tilde{J}_t := E[\tilde{A}_T - \tilde{A}_t \mid \mathcal{F}_t]$ and $\tilde{J}'_t := E[\tilde{A}'_T - \tilde{A}'_t \mid \mathcal{F}_t]$. We thus have:

$$\begin{cases} \tilde{J}_t = J_t - E[\Delta A_\tau \wedge \Delta A'_\tau \mid \mathcal{F}_t]\mathbf{1}_{\{\mathbf{t}<\tau\}} \\ \tilde{J}'_t = J'_t - E[\Delta A_\tau \wedge \Delta A'_\tau \mid \mathcal{F}_t]\mathbf{1}_{\{\mathbf{t}<\tau\}} \end{cases} \tag{3.23}$$

and the equality $\tilde{J}-\tilde{J}' = J-J'$. It follows that $\tilde{J}$ and $\tilde{J}'$ are non-negative RCLL supermartingales such that $\tilde{\xi} \leq \tilde{J} - \tilde{J}' \leq \tilde{\zeta}$. By the minimality property of $\tilde{J}$ and $\tilde{J}'$ (see Lemma **??**), it follows that $J \leq \tilde{J}$ and $J' \leq \tilde{J}'$. Now we clearly have $J \geq \tilde{J}$ and $J' \geq \tilde{J}'$. Hence, $J = \tilde{J}$ and $J' = \tilde{J}'$ which yields $\Delta A_\tau \wedge \Delta A'_\tau = 0$ a.s. The proof of the lemma is thus complete. □

*Remark 3.10 The first part of the proof is based on the same arguments as those of [**?**]. The arguments of the second part are different and allow to weaken the assumptions.*

Proof of Th. **??**: Let (Y, Z, k, α) be the solution of the DBBSDE (**??**). By Prop. **??**, we have $Y_t = J^g_t - J'^g_t + E[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t]$. Consequently, in order to show that α is continuous, it is sufficient to show the continuity of A, A' (the non decreasing processes associated with the Doob-Meyer decomposition of J and J') because $\alpha = A - A'$. By assumption and by the above lemma, the RCLL processes $\xi, -\zeta, J'^g$ and J^g are l.u.s.c along stopping times in expectation, because they belong to $\mathcal{S}^2$. It follows that $J'^g + \xi$ and $J^g - \zeta$ are l.u.s.c along stopping times in expectation. Now, $J^g = \mathcal{R}(J'^g + \tilde{\xi}^g)$ and $J'^g = \mathcal{R}(J^g - \tilde{\zeta}^g)$. Consequently, by a result of optimal stopping theory (see e.g. Proposition B.10 in [**?**]), the non-decreasing processes A and A' associated with the supermartingales J^g and J'^g are continuous. Moreover, by the second assertion of Th. **??**, for each $S \in \mathcal{T}_0$, the pair (τ^*_S, σ^*_S) is thus an S-saddle point, which ends the proof of the theorem. □

Since $J^g \geq J'^g + \tilde{\xi}^g$ and $J'^g \geq J^g - \tilde{\zeta}^g$, the condition $J^g \in \mathcal{S}^2$ is equivalent to the condition $J'^g \in \mathcal{S}^2$.

We now recall the definition of Mokobodski's condition.

Definition 3.11 (Mokobodski's condition) *Let $\zeta, \xi \in \mathcal{S}^2$. The Mokobodski's condition is defined as follows: there exist two nonnegative RCLL supermartingales H and $H' \in \mathcal{S}^2$ such that:*

$$\xi_t \mathbf{1}_{t<T} \leq H_t - H'_t \leq \zeta_t \mathbf{1}_{t<T} \quad 0 \leq t \leq T \quad \text{a.s.}. \tag{3.24}$$

Proposition 3.12 *Let $g \in I\!H^2$. The following assertions are equivalent:*

(i) $J^g \in \mathcal{S}^2$

(ii) $J^0 \in \mathcal{S}^2$

(iii) Mokobodski's condition holds.

Remark 3.13 In this case (that is when Mokobodski's condition is satisfied), $J^0, J^{'0}$ are the minimal nonnegative RCLL supermartingales satisfying (**??**)*. This follows from Lemma* **??** *applied to* $g = 0$.

Proof. Using Lemma **??**, one can show that $J^g \in \mathcal{S}^2$ if and only if there exist two non-negative supermartingales $H^g, H^{'g} \in \mathcal{S}^2$ such that

$$\tilde{\xi}_t^g \le H_t^g - H_t^{'g} \le \tilde{\zeta}_t^g \quad 0 \le t \le T \quad \text{a.s.} \tag{3.25}$$

Since this equivalence holds for all $g \in I\!H^2$, in particular when $g = 0$, we get (ii) $\Leftrightarrow$ (iii). It remains to show (i) $\Leftrightarrow$ (ii) For this, it is sufficient to show that (**??**) is equivalent to (**??**). Suppose that (**??**) is satisfied. By setting

$$\begin{cases} H_t^g := H_t - \mathbb{E}[\xi_T^+(s)ds|\mathcal{F}_t] - \mathbb{E}[\int_t^T g^+(s)ds|\mathcal{F}_t], 0 \le t \le T \\ H_t^{'g} := H_t' - \mathbb{E}[\xi_T^-(s)ds|\mathcal{F}_t] - \mathbb{E}[\int_t^T g^-(s)ds|\mathcal{F}_t], 0 \le t \le T, \end{cases}$$

(**??**) holds. The converse is clear. The proof is thus complete. □

4 DBBSDEs with Lipschitz driver and links with generalized Dynkin games

In this section, we are given a Lipschitz driver g.

4.1 Existence and uniqueness for DBBSDEs

We state an existence and uniqueness result for DBBSDEs which completes those given in the literature.

Theorem 4.1 *Suppose ξ and ζ are RCLL adapted process in $\mathcal{S}^2$ such that $\xi_t \le \zeta_t$, $0 \le t \le T$ a.s. Suppose that $J^0 \in \mathcal{S}^2$ (or equivalently suppose that Mokobodski's condition is satisfied). Then, DBBSDE (**??**) admits a unique solution $(Y, Z, k(.), \alpha) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{S}^2$.*
If ξ and ζ are l.u.s.c. along stopping times, then the processes A and A' are continuous.

Proof. The proof is based on classical arguments and is given in the appendix.

We prove below that under additional assumptions on the barriers ξ and ζ, the non decreasing processes A and A' are unique.

Proposition 4.2 *Let ξ and ζ be two adapted RCLL processes with $\zeta_T = \xi_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \le \zeta_t$, $\forall t \in [0, T]$ a.s. Let $(Y, Z, k(.), \alpha)$ be the solution of DBBSDE (**??**).*

(i) Suppose that for each predictable stopping time $\tau \in \mathcal{T}_0$, $\xi_{\tau^-} < \zeta_{\tau^-}$ a.s. Then, for each predictable stopping time $\tau \in \mathcal{T}_0$, A^d and $A^{'d}$ are unique. Moreover, we have

$$\Delta A_\tau^d = -\Delta Y_\tau \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}} = (\Delta Y_\tau)^- \ a.s.$$

and

$$\Delta A_\tau^{'d} = \Delta Y_\tau \mathbf{1}_{\{Y_{\tau^-} = \zeta_{\tau^-}\}} = (\Delta Y_\tau)^+ \ a.s.$$

(ii) Suppose that $\xi_t < \zeta_t$, $\forall t \in [0,T[$ *a.s.(or* $\xi_{t^-} < \zeta_{t^-}$, $\forall t \in]0,T]$ *a.s.), then* A^c *and* $A^{'c}$ *are unique.*

Remark 4.3 If ξ *and* ζ *are not predictable, the condition* $\xi_{\tau^-} < \zeta_{\tau^-}$ *a.s. for each predictable stopping time* τ *is weaker than the condition* $\xi_{t^-} < \zeta_{t^-}$, $\forall t \in]0,T]$ *a.s. Also, when this last condition is satisfied, the above proposition yields the uniqueness of* $A, A' \in \mathcal{A}^2$ *such that* $\alpha = Y - Y'$. *It follows that there exists an unique solution* (J, J') *of the system* (**??**).

Proof.

(i) Since the filtration is generated by W and N, the martingales admit jumps at inaccessible stopping times only. Hence, for each predictable stopping time $\tau \in \mathcal{T}_0$,

$$-\Delta Y_\tau = \Delta A^d_\tau - \Delta A^{'d}_\tau = \Delta A^d_\tau \mathbf{1}_{\{Y_{\tau^-}=\xi_{\tau^-}\}} - \Delta A^{'d}_\tau \mathbf{1}_{\{Y_{\tau^-}=\zeta_{\tau^-}\}}.$$

We have $\{Y_{\tau^-} = \xi_{\tau^-}\} \cap \{Y_{\tau^-} = \zeta_{\tau^-}\} \subset \{\xi_{\tau^-} = \zeta_{\tau^-}\} = \emptyset$ a.s., by assumption on ξ and ζ. The result follows.

(ii) Suppose that $\alpha = \overline{A} - \overline{A}'$, where $\overline{A}$ and $\overline{A}'$ are in $\mathcal{A}^2$. Let $\overline{A}^c$ and $\overline{A}^{'c}$ be the continuous parts of $\overline{A}$ and $\overline{A}'$. We have $dA_t - dA'_t = d\overline{A}_t - d\overline{A}'_t$ which implies $dA^c_t - dA^{'c}_t = d\overline{A}^c_t - d\overline{A}^{'c}_t$. Hence $dA^c_t - d\overline{A}^c_t = dA^{'c}_t - d\overline{A}^{'c}_t$. We thus have

$$dA^c_t - d\overline{A}^c_t = \mathbf{1}_{Y_t=\xi_t}(dA^c_t - d\overline{A}^c_t) = \mathbf{1}_{Y_t=\zeta_t}(dA^{'c}_t - d\overline{A}^{'c}_t) = dA^{'c}_t - d\overline{A}^{'c}_t.$$

Since $\xi_t < \zeta_t$, $0 \le t < T$ a.s., all the members of the above equality are equal to 0. Hence $A^c = \overline{A}^c$ and $A^{'c} = \overline{A}^{'c}$.

□

We introduce the following assumption.

Assumption 4.1 *A lipschitz driver* g *is said to satisfy Assumption* **??** *if the following holds:* $dP \otimes dt$*-a.s for each* $(y, z, k_1, k_2) \in [0,T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2$,

$$g(t, y, z, k_1) - g(t, y, z, k_2) \ge \langle \theta^{y,z,l_1,l_2}_t, k_1 - k_2 \rangle_\nu,$$

with

$$\theta : [0,T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2 \mapsto L^2_\nu \, ; \, (\omega, t, y, z, k_1, k_2) \mapsto \theta^{y,z,k_1,k_2}_t(\omega, .)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}((L^2_\nu)^2)$*-measurable, bounded, and satisfying* $dP \otimes dt \otimes d\nu(u)$*-a.s. , for each* $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,

$$\theta^{y,z,k_1,k_2}_t(u) \ge -1 \quad and \quad |\theta^{y,z,k_1,k_2}_t(u)| \le \psi(u), \tag{4.26}$$

where $\psi \in L^2_\nu$.

This assumption ensures the comparison theorem for BSDEs with jumps (see [**?**]).

We state below some properties on the g-conditional expectation which will be used to characterize the solution of DBSDDE as the value function of a Dynkin game written in terms of g-conditional expectations (see Section **??**).

4.2 Some properties of the g-conditional expectation $\mathcal{E}$

We recall the following definition (see [?]):

Definition 4.4 (g-conditional expectation) *The g-conditional expectation $\mathcal{E}^g$ is defined for all $\tau \in \mathcal{T}_0$ and all $\eta \in L^2(\mathcal{F}_\tau)$ by*

$$\mathcal{E}^g_{t,\tau}(\eta) := X_t(\eta, \tau); \quad 0 \leq t \leq T,$$

where $(X(\eta,\tau), \pi(\eta,\tau), l(\eta,\tau))$ is the solution of the BSDE associated with driver g, terminal time τ and terminal condition η.

When there is no ambiguity on the driver g, $\mathcal{E}^g$ will simply be denoted by $\mathcal{E}$.

Under Assumption **??**, the g-conditional expectation $\mathcal{E}$ is non decreasing, that is if $\eta_1 \leq \eta_2$ a.s., then $\mathcal{E}_{t,\tau}(\eta_1) \leq \mathcal{E}_{t,\tau}(\eta_2)$ a.s.

Definition 4.2 *An RCLL adapted process X_t in $\mathcal{S}^2$ is said to be an $\mathcal{E}$-martingale (resp. $\mathcal{E}$-submartingale, $\mathcal{E}$-supermartingale) if $\mathcal{E}_{\sigma,\tau}(X_\tau) = X_\sigma$ (resp. $\mathcal{E}_{\sigma,\tau}(X_\tau) \leq X_\sigma$, $\mathcal{E}_{\sigma,\tau}(X_\tau) \geq X_\sigma$) a.s. on $\sigma \leq \tau$, for all $\sigma, \tau \in \mathcal{T}_0$.*

Proposition 4.5 *Suppose that g satisfies Assumption (**??**). Let (A_t) be a non decreasing (resp non increasing) RCLL predictable process in $\mathcal{S}^2$ with $A_0 = 0$. Let $(Y,Z,k) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ satisfying the dynamics:*

$$-dY_s = g(s, Y_s, Z_s, k_s)ds + dA_s - Z_s dW_s - \int_{\mathbf{R}^*} k_s(u)\tilde{N}(ds, du).$$

Then the process (Y_t) is an $\mathcal{E}$-supermartingale (resp $\mathcal{E}$-submartingale).

Proof. Suppose A is non decreasing. Let $(X^\tau, \pi^\tau, l^\tau)$ be the solution of the BSDE associated with driver g, terminal time τ, and terminal condition Y_τ, that is

$$-dX^\tau_s = g(s, X^\tau_s, \pi^\tau_s, k^\tau_s)ds - \pi^\tau_s dW_s - \int_{\mathbf{R}^*} k^\tau_s(u)\tilde{N}(ds, du); \qquad X^\tau_\tau = Y_\tau.$$

Since g satisfies Assumption **??** and since $g(s,y,z,k)ds + dA_s \geq g(s,y,z,k)ds$, the comparison theorem for BSDEs (see Theorem 4.2 in [?]) gives that $Y_\sigma \geq X^\tau_\sigma = \mathcal{E}_{\sigma,\tau}(Y_\tau)$ a.s. on $\{\sigma \leq \tau\}$. The case when A is non-increasing can be shown similarly.

□

4.3 Characterization of the solution via generalized Dynkin games

We introduce the following game problem, which can be seen as a Dynkin game written in terms of g-conditional expectations.

For each $\tau, \sigma \in \mathcal{T}_0$, let $I(\tau, \sigma)$ be the $\mathcal{F}_{\tau\wedge\sigma}$-measurable random variable defined by

$$I(\tau, \sigma) = \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}. \tag{4.27}$$

For each stopping time $S \in \mathcal{T}_0$, the upper and lower value functions at time S are defined respectively by

$$\overline{V}(S) := ess \inf_{\sigma \in \mathcal{T}_S} ess \sup_{\tau \in \mathcal{T}_S} \mathcal{E}_{S,\tau\wedge\sigma}(I(\tau, \sigma)) \tag{4.28}$$

$$\underline{V}(S) := ess \sup_{\tau \in \mathcal{T}_S} ess \inf_{\sigma \in \mathcal{T}_S} \mathcal{E}_{S,\tau\wedge\sigma}(I(\tau,\sigma)). \tag{4.29}$$

Recall that $\mathcal{E}_{\cdot,\tau\wedge\sigma}(I(\tau,\sigma)) = X_{\cdot}^{\tau,\sigma}$, with $(X_{\cdot}^{\tau,\sigma}, \pi_{\cdot}^{\tau,\sigma}, l_{\cdot}^{\tau,\sigma})$ being the solution of the BSDE

$$-dX_s^{\tau,\sigma} = g(s, X_s^{\tau,\sigma}, \pi_s^{\tau,\sigma}, l_s^{\tau,\sigma})ds - \pi_s^{\tau,\sigma} dW_s - \int_{\mathbf{R}^*} l_s^{\tau,\sigma}(u)\tilde{N}(ds,du); \qquad X_{\tau\wedge\sigma}^{\tau,\sigma} = I(\tau,\sigma).$$

We clearly have the inequality $\underline{V}(S) \leq \overline{V}(S)$ a.s.
By definition, we say that there *exists a value function* at time S for the game problem if $\overline{V}(S) = \underline{V}(S)$ a.s. We introduce the definition of an S-saddle point for this game problem.

Definition 4.6 *Let $S \in \mathcal{T}_0$. A pair $(\tau^*,\sigma^*) \in \mathcal{T}_S^2$ is called an* S-saddle point *if for each $(\tau,\sigma) \in \mathcal{T}_S^2$ we have*

$$\mathcal{E}_{S,\tau\wedge\sigma^*}(I(\tau,\sigma^*)) \leq \mathcal{E}_{S,\tau^*\wedge\sigma^*}(I(\tau^*,\sigma^*)) \leq \mathcal{E}_{S,\tau^*\wedge\sigma}(I(\tau^*,\sigma)) a.s.$$

We first consider the simpler case when the barriers are *l.u.s.c.* along stopping times. In this case, for each $S \in \mathcal{T}_0$, there exists an S-saddle point and the common value function is equal to Y_S, where Y is the solution of the DBBSDE.

Theorem 4.7 (Existence of an S-saddle point and characterization) *Suppose that g satisfies Assumption (**??**). Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ such that $\xi_t \leq \zeta_t$, $\forall t \in [0,T]$ a.s. Suppose that Mokobodski's condition is satisfied.*
Moreover, suppose that α is continuous (which is the case if ξ and $-\zeta$ are l.u.s.c. along stopping times).
*Let $(Y, Z, k(\cdot), \alpha)$ be the solution of the DBBSDE (**??**). For each $S \in \mathcal{T}_0$, consider*

$$\sigma_S^* := \inf\{t \geq S,\ Y_t = \zeta_t\}; \quad \tau_S^* := \inf\{t \geq S,\ Y_t = \xi_t\}.$$

Then, for each $S \in \mathcal{T}_0$, the pair of stopping times (τ_S^*, σ_S^*) is an S-saddle point and

$$Y_S = \overline{V}(S) = \underline{V}(S) \ a.s.$$

Moreover, $Y_{\sigma_S^*} = \zeta_{\sigma_S^*}$ and $Y_{\tau_S^*} = \xi_{\tau_S^*}$ a.s. and the process $(Y_t, S \leq t \leq \tau_S^* \wedge \sigma_S^*)$ is an $\mathcal{E}$-martingale. The process $(Y_t, S \leq t \leq \tau_S^*)$ is an $\mathcal{E}$-submartingale and the process $(Y_t, S \leq t \leq \sigma_S^*)$ is an $\mathcal{E}$-supermartingale.

Proof. First, note that by Theorem **??** applied with $g(s) = g(s, Y_s, Z_s, k_s)$, we derive that, in the case when ξ and $-\zeta$ are l.u.s.c. along stopping times, then α is continuous.

Suppose now that α is continuous. There exists A, A' continuous belonging to $\mathcal{A}^2$ such that $\alpha = A - A'$. Note that $\sigma_S^* \in \mathcal{T}_S$ and $\tau_S^* \in \mathcal{T}_S$. Since Y and ξ are right-continuous processes, we have $Y_{\sigma_S^*} = \xi_{\sigma_S^*}$ and $Y_{\tau_S^*} = \xi_{\tau_S^*}$ a.s. By definition of τ_S^*, for almost every ω, we have $Y_t(\omega) > \xi_t(\omega)$ for each $t \in [S(\omega), \tau_S^*(\omega)[$. Hence, since Y is solution of the DBBSDE, the process A is constant on $[S, \tau_S^*]$ a.s. Similarly, the process A' is constant on $[S, \sigma_S^*]$ a.s. Now, by assumption, $(Y, Z, k(.), A - A')$ is the solution of DBBSDE (**??**). Hence, by Proposition **??**, the process $(Y_t, S \leq t \leq \tau_S^* \wedge \sigma_S^*)$ is an $\mathcal{E}$-martingale. Hence

$$Y_S = \mathcal{E}_{S,\tau_S^*\wedge\sigma_S^*}(Y_{\tau_S^*\wedge\sigma_S^*}) = \mathcal{E}_{S,\tau_S^*\wedge\sigma_S^*}(\xi_{\tau_S^*}\mathbf{1}_{\tau_S^*\leq\sigma_S^*} + \zeta_{\sigma_S^*}\mathbf{1}_{\sigma_S^*<\tau_S^*}) = \mathcal{E}_{S,\tau_S^*\wedge\sigma_S^*}(I(\tau_S^*,\sigma_S^*)) \quad \text{a.s.}$$

Let $\tau \in \mathcal{T}_S$. We want to show that for each $\tau \in \mathcal{T}_S$

$$Y_S \geq \mathcal{E}_{S,\tau\wedge\sigma_S^*}(I(\tau,\sigma_S^*)) \quad \text{a.s.} \tag{4.30}$$

Since A' is constant on $[S,\sigma_S^*]$, by Proposition **??**, the process $(Y_t, S \leq t \leq \tau \wedge \sigma_S^*)$ is a $\mathcal{E}$ supermartingale. Hence

$$Y_S \geq \mathcal{E}_{S,\tau\wedge\sigma_S^*}(Y_{\tau\wedge\sigma_S^*}).$$

Since $Y \geq \xi$ and $Y_{\sigma_S^*} = \zeta_{\sigma_S^*}$ a.s. , we have

$$Y_{\tau\wedge\sigma_S^*} = Y_\tau \mathbf{1}_{\tau\leq\sigma_S^*} + Y_{\sigma_S^*}\mathbf{1}_{\sigma_S^*<\tau} \geq \xi_\tau \mathbf{1}_{\tau\leq\sigma_S^*} + \zeta_{\sigma_S^*}\mathbf{1}_{\sigma_S^*<\tau} = I(\tau,\sigma_S^*).$$

By the monotonicity property of $\mathcal{E}$, we derive inequality (**??**).
Similarly, one can show that for each $\sigma \in \mathcal{T}_S$, we have:

$$Y_S \leq \mathcal{E}_{S,\tau_S^*\wedge\sigma}(I(\tau_S^*,\sigma)) \quad \text{a.s.}$$

The pair (τ_S^*, σ_S^*) is thus an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s. The proof is thus complete. □

We now turn to the more difficult case when α is not continuous. We show below that, for each $S \in \mathcal{T}_0$, there exists a value function and the common value function is equal to Y_S, where Y is the solution of the DBBSDE. However, there does not necessarily exist an S-saddle point.

Theorem 4.8 (Characterization) *Suppose that g satisfies Assumption (**??**). Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ such that $\xi_t \leq \zeta_t$, $\forall t \in [0,T]$ a.s. Suppose that Mokobodski's condition is satisfied. Let $(Y,Z,k(\cdot),\alpha)$ be the solution of the DBBSDE (**??**).*
Then there exists a value function for the generalized Dynkin game and, for each stopping time $S \in \mathcal{T}_0$, we have

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad a.s. \tag{4.31}$$

Proof. For each $S \in \mathcal{T}_0$ and for each $\varepsilon > 0$, let τ_S^ε and σ_S^ε be the stopping times defined by

$$\tau_S^\varepsilon := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\}. \tag{4.32}$$

$$\sigma_S^\varepsilon := \inf\{t \geq S,\ Y_t \geq \zeta_t - \varepsilon\}. \tag{4.33}$$

We first show two lemmas.

Lemma 4.9 • *We have*

$$Y_{\tau_S^\varepsilon} \leq \xi_{\tau_S^\varepsilon} + \varepsilon \quad a.s. \tag{4.34}$$

$$Y_{\sigma_S^\varepsilon} \geq \zeta_{\sigma_S^\varepsilon} - \varepsilon \quad a.s. \tag{4.35}$$

- *The process $(Y_t, S \leq t \leq \tau_S^\varepsilon)$ is an $\mathcal{E}$-submartingale and the process $(Y_t, S \leq t \leq \sigma_S^\varepsilon)$ is an $\mathcal{E}$-supermartingale.*

Proof. The first point follows from the definitions of τ_S^ε and σ_S^ε and the right-continuity of ξ, ζ and Y. Let us show the second point. Note that $\tau_S^\varepsilon \in \mathcal{T}_S$ and $\sigma_S^\varepsilon \in \mathcal{T}_S$. Fix $\varepsilon > 0$. For a.e. ω, if $t \in [S(\omega), \tau_S^\varepsilon(\omega)[$, then $Y_t(\omega) > \xi_t(\omega) + \varepsilon$ and hence $Y_t(\omega) > \xi_t(\omega)$. It follows that almost surely, A^c is constant on $[S, \tau_S^\varepsilon]$ and A^d is constant on $[S, \tau_S^\varepsilon[$. Also, $Y_{(\tau_S^\varepsilon)^-} \geq \xi_{(\tau_S^\varepsilon)^-} + \varepsilon$ a.s. Since $\varepsilon > 0$, it follows that $Y_{(\tau_S^\varepsilon)^-} > \xi_{(\tau_S^\varepsilon)^-}$ a.s., which implies that $\Delta A^d_{\tau_S^\varepsilon} = 0$ a.s. Hence, almost surely, A is constant on $[S, \tau_S^\varepsilon]$, which implies that $(Y_t, S \leq t \leq \tau_S^\varepsilon)$ is an $\mathcal{E}$-submartingale. Similarly, one can show that A' is constant on $[S, \sigma_S^\varepsilon]$, which implies that $(Y_t, S \leq t \leq \sigma_S^\varepsilon)$ is an $\mathcal{E}$-supermartingale. □

Lemma 4.10 *Set $\beta := 3C^2 + 2C$, where C is the Lipschitz constant of f. For each $\varepsilon > 0$ and each $S \in \mathcal{T}_0$, for each $(\tau, \sigma) \in \mathcal{T}_S^2$ we have*

$$\mathcal{E}_{S,\tau\wedge\sigma_S^\varepsilon}(I(\tau, \sigma_S^\varepsilon)) - e^{\frac{\beta(T-S)}{2}}\varepsilon \;\leq\; Y_S \;\leq\; \mathcal{E}_{S,\tau_S^\varepsilon\wedge\sigma}(I(\tau_S^\varepsilon, \sigma)) + e^{\frac{\beta(T-S)}{2}}\varepsilon \quad a.s. \tag{4.36}$$

Proof. Let $\tau \in \mathcal{T}_S$. By Lemma **??**, the process $(Y_t, S \leq t \leq \sigma_S^\varepsilon)$ is an $\mathcal{E}$-supermartingale. Hence,

$$Y_S \geq \mathcal{E}_{S,\tau\wedge\sigma_S^\varepsilon}(Y_{\tau\wedge\sigma_S^\varepsilon}) \quad \text{a.s.} \tag{4.37}$$

Since $Y \geq \xi$ and $Y_{\sigma_S^\varepsilon} \geq \zeta_{\sigma_S^\varepsilon} - \varepsilon$ a.s. (see Lemma (**??**)), we have:

$$Y_{\tau\wedge\sigma_S^\varepsilon} \geq \xi_\tau \mathbf{1}_{\tau\leq\sigma_S^\varepsilon} + (\zeta_{\sigma_S^\varepsilon} - \varepsilon)\mathbf{1}_{\sigma_S^\varepsilon<\tau} = I(\tau, \sigma_S^\varepsilon) - \varepsilon$$

where the last equality follows from the definition of $I(\tau, \sigma)$.

Hence, using (**??**) and the monotonicity property of $\mathcal{E}$, we get

$$Y_S \geq \mathcal{E}_{S,\tau\wedge\sigma_S^\varepsilon}(I(\tau, \sigma_S^\varepsilon) - \varepsilon) \quad \text{a.s.} \tag{4.38}$$

Now, by the a priori estimates on BSDEs (see Proposition A.4, [**?**]), we have

$$|\mathcal{E}_{S,\tau\wedge\sigma_S^\varepsilon}(I(\tau, \sigma_S^\varepsilon) - \varepsilon) - \mathcal{E}_{S,\tau\wedge\sigma_S^\varepsilon}(I(\tau, \sigma_S^\varepsilon))| \leq e^{\frac{\beta(T-S)}{2}}\varepsilon \quad \text{a.s.}$$

for $\beta = 3C^2 + 2C$. It follows that

$$Y_S \;\geq\; \mathcal{E}_{S,\tau\wedge\sigma_S^\varepsilon}(I(\tau, \sigma_S^\varepsilon)) - e^{\frac{\beta(T-S)}{2}}\varepsilon \quad \text{a.s.}$$

Similarly, one can show that

$$Y_S \;\leq\; \mathcal{E}_{S,\tau_S^\varepsilon\wedge\sigma}(I(\tau_S^\varepsilon, \sigma)) + e^{\frac{\beta(T-S)}{2}}\varepsilon \quad \text{a.s.},$$

which ends the proof of Lemma **??**.

□

End of proof of Theorem ??

By Lemma **??**, for each $\varepsilon > 0$, we have

$$ess \sup_{\tau\in\mathcal{T}_s} \mathcal{E}_{S,\tau\wedge\sigma_S^\varepsilon}(I(\tau, \sigma_S^\varepsilon)) - e^{\frac{\beta(T-S)}{2}}\varepsilon \;\leq\; Y_S \leq ess \inf_{\sigma\in\mathcal{T}_S} \mathcal{E}_{S,\tau\wedge\sigma_S^\varepsilon}(I(\tau, \sigma_S^\varepsilon)) + e^{\frac{\beta(T-S)}{2}}\varepsilon \;\; \text{a.s.},$$

which implies that

$$\overline{V}(S) - e^{\frac{\beta(T-S)}{2}}\varepsilon \;\leq\; Y_S \;\leq\; \underline{V}(S) + e^{\frac{\beta(T-S)}{2}}\varepsilon \quad \text{a.s.}$$

Since $\underline{V}(S) \leq \overline{V}(S)$, a.s. we get $\underline{V}(S) = Y_S = \overline{V}(S)$ a.s. The proof of Theorem **??** is thus complete.

□

Remark 4.11 Inequality (**??**) *shows that* $(\tau_S^\varepsilon, \sigma_S^\varepsilon)$ *defined by* (**??**) *and* (**??**) *is an* ε'*-saddle point at time* S *with* $\varepsilon' = e^{\frac{\beta(T-S)}{2}}\varepsilon$.

5 Comparison theorems for DBBSDEs with jumps and a priori estimates

5.1 Comparison theorems

Theorem 5.1 (Comparison theorem for DBBSDEs.) *Let* ξ^1, ξ^2, ζ^1, ζ^2 *be processes in* $\mathcal{S}^2$ *such that* $\xi_t^i \leq \zeta_t^i$, $0 \leq t \leq T$ *a.s. for* $i = 1, 2$. *Suppose that for* $i = 1, 2$, ξ^i, ζ^i *satisfies Mokobodski's condition. Let* g^1 *and* g^2 *be Lipschitz drivers satisfying Assumption* (**??**).
Suppose that

- $\xi_t^2 \leq \xi_t^1$ *and* $\zeta_t^2 \leq \zeta_t^1$, $0 \leq t \leq T$ *a.s.*
- $g^2(t, y, z, k) \leq g^1(t, y, z, k)$, *for all* $(y, z, k) \in \mathbf{R}^2 \times \mathcal{L}_\nu^2$; $dP \otimes dt - a.s.$

Let $(Y^i, Z^i, k^i, \alpha^i)$ *be the solution of the DBBSDE associated with* (ξ^i, ζ^i, g^i) , $i = 1, 2$. *Then,*

$$Y_t^2 \leq Y_t^1, \ \forall t \in [0, T] \ a.s.$$

Proof. The proof is based on the characterization of solutions of DBBSDEs (Theorem **??**). Let $t \in [0, T]$. For each $\tau, \sigma \in T_t$, let us denote by $\mathcal{E}^i_{\cdot, \tau\wedge\sigma}(I^i(\tau, \sigma))$ the unique solution of the BSDE associated with driver g^i, terminal time $\tau \wedge \sigma$ and terminal condition $I^i(\tau, \sigma) := \xi_\tau^i \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma^i \mathbf{1}_{\sigma < \tau}$ for $i = 1, 2$. Since $g^2 \leq g^1$, and $I^2(\tau, \sigma) \leq I^1(\tau, \sigma)$, by the comparison theorem for BSDEs, the following inequality

$$\mathcal{E}^2_{t, \tau\wedge\sigma}(I^2(\tau, \sigma)) \leq \mathcal{E}^1_{t, \tau\wedge\sigma}(I^1(\tau, \sigma)) \ a.s.$$

holds for each τ, σ in $\mathcal{T}_t$. Hence, by taking the essential supremum over τ in $\mathcal{T}_t$ and the essential infimum over σ in $\mathcal{T}_t$, using the characterization Theorem **??**, we get

$$Y_t^2 = \text{ess} \inf_{\sigma \in \mathcal{T}_t} \text{ess} \sup_{\tau \in \mathcal{T}_t} \mathcal{E}^2_{t, \tau\wedge\sigma}(I^2(\tau, \sigma)) \leq \text{ess} \inf_{\sigma \in \mathcal{T}_t} \text{ess} \sup_{\tau \in \mathcal{T}_t} \mathcal{E}^1_{t, \tau\wedge\sigma}(I^1(\tau, \sigma)) = Y_t^1 \ a.s.$$

□

We now provide a strict comparison theorem. The first assertion addresses the particular case when the barriers are left-upper semicontinuous along stopping times and the second one deals with the general case.

Theorem 5.2 (Strict comparison.) *Suppose that the assumptions of Theorem **??** hold and that the driver g^1 satisfies Assumption **??** with*

$$\theta_t^{y,z,k_1,k_2} > -1 \quad dt \otimes dP - \text{a.s.} \tag{5.39}$$

Let S in $\mathcal{T}_0$ and suppose that $Y_S^1 = Y_S^2$ a.s.

1. *Suppose that α^1, α^2 are continuous. For $i = 1, 2$, let* $\tau_i^* = \tau_{i,S}^* := \inf\{s \geq S;\, Y_s^i = \xi_s^i\}$ *and* $\sigma_i^* = \sigma_{i,S}^* := \inf\{s \geq S;\, Y_s^i = \zeta_s^i\}$. *Then*

$$Y_t^1 = Y_t^2, \;\; S \leq t \leq \tau_1^* \wedge \tau_2^* \wedge \sigma_1^* \wedge \sigma_2^* \;\; a.s.$$

   and

$$g^2(t, Y_t^2, Z_t^2, k_t^2) \;=\; g^1(t, Y_t^2, Z_t^2, k_t^2) \qquad S \leq t \leq \tau_1^* \wedge \tau_2^* \wedge \sigma_1^* \wedge \sigma_2^*,\; dP \otimes dt - a.s. \tag{5.40}$$

2. *Consider the case when α^1, α^2 are not necessarily continuous. For $\varepsilon > 0$, define*

$$\tau_i^\varepsilon := \inf\{t \geq S,\; Y_t^i \leq \xi_t^i + \varepsilon\} \quad \text{and} \quad \tilde{\tau}_i := \lim_{\varepsilon \downarrow 0} \uparrow \tau_i^\varepsilon \;\; i = 1, 2.$$

$$\sigma_i^\varepsilon := \inf\{t \geq S,\; Y_t^i \leq \zeta_t^i - \varepsilon\} \quad \text{and} \quad \tilde{\sigma}_i := \lim_{\varepsilon \downarrow 0} \uparrow \sigma_i^\varepsilon \;\; i = 1, 2.$$

   Then, for each $\varepsilon > 0$,

$$Y_t^1 = Y_t^2, \;\; S < \tilde{\tau}_1 \wedge \tilde{\tau}_2 \wedge \tilde{\sigma}_1 \wedge \tilde{\sigma}_2. \quad a.s. \tag{5.41}$$

   Moreover,

$$g^2(t, Y_t^2, Z_t^2, k_t^2) \;=\; g^1(t, Y_t^2, Z_t^2, k_t^2) \qquad S \leq t \leq \tilde{\tau}_1 \wedge \tilde{\tau}_2 \wedge \tilde{\sigma}_1 \wedge \tilde{\sigma}_2,\; dP \otimes dt - a.s.$$

Proof. We adopt the same notation as in the proof of the comparison theorem.
Suppose first that α^1 and α^2 are continuous. By Theorem **??**, for $i = 1, 2$, (τ_i^*, σ_i^*) is a saddle point for the game problem associated with $g = g^i$, $\xi = \xi^i$ and $\zeta = \zeta^i$. By Theorem **??**, $(Y_t^i, S \leq t \leq \tau_i^* \wedge \sigma_i^*)$ is an $\mathcal{E}^g$ martingale. Hence we have

$$Y_t^i = \mathcal{E}^g_{t, \tau_i^* \wedge \sigma_i^*}(I(\tau_i^*, \sigma_i^*)), \;\; S \leq t \leq \tau_i^* \wedge \sigma_i^* \;\; \text{a.s.}$$

Set $\theta^* = \tau_1^* \wedge \tau_2^* \wedge \sigma_1^* \wedge \sigma_2^*$.
We thus have

$$Y_t^i = \mathcal{E}^i_{t, \theta^*}(Y^i_{\theta^*}), \;\; S \leq t \leq \theta^* \;\; \text{a.s. for } i = 1, 2.$$

By hypothesis, $Y_S^1 = Y_S^2$. Now, Assumption (**??**) allows us to apply the strict comparison theorem for non reflected BSDEs with jumps (see [**?**], Th 4.4) for terminal time θ^*. Hence, we get $Y_t^1 = Y_t^2, \;\; S \leq t \leq \theta^*$ a.s., and equality (**??**), which provides the desired result.

Consider now the general case.
Let $\varepsilon > 0$. By Lemma **??**, $(Y_t^i, S \leq t \leq \tau_i^\varepsilon \wedge \sigma_i^\varepsilon)$ is an $\mathcal{E}^g$ martingale. Hence we have

$$Y_t^i = \mathcal{E}^g_{t, \tau_i^\varepsilon \wedge \sigma_i^\varepsilon}(I(\tau_i^\varepsilon, \sigma_i^\varepsilon)), \;\; S \leq t \leq \tau_i^\varepsilon \wedge \sigma_i^\varepsilon \;\; \text{a.s.}$$

By the same arguments as above with τ_1^*,τ_2^* and σ_1^*,σ_2^* replaced by τ_1^ε,τ_2^ε and σ_1^ε,σ_2^ε respectively, we derive $Y_t^1 = Y_t^2, \;\; S \leq t \leq \tau_1^\varepsilon \wedge \tau_2^\varepsilon \wedge \sigma_1^\varepsilon \wedge \sigma_2^\varepsilon$ a.s., and equality (**??**) holds on $[S, \tau_1^\varepsilon \wedge \tau_2^\varepsilon \wedge \sigma_1^\varepsilon \wedge \sigma_2^\varepsilon]$, $dt \otimes dP$-a.s. By letting ε tend to 0, we obtain the desired result.

5.2 A priori estimates

Using the characterization of the solution of the DBBSDE (see Theorem **??**), we prove the following estimates.

Proposition 5.1 *Let $\xi^1, \xi^2, \zeta^1, \zeta^2 \in \mathcal{S}^2$ such that $\xi^i_t \leq \zeta^i_t$, $0 \leq t \leq T$ a.s. Suppose that for $i = 1, 2$, ξ^i, ζ^i satisfy Mokobodski's condition. Let g^1, g^2 be Lipschitz drivers satisfying Assumption **??** with Lipschitz constant $C > 0$. For $i = 1, 2$, let Y^i be the solution of the DBBSDE associated with driver g^i, terminal time T and barriers ξ^i, ζ^i. For $s \in [0, T]$, let $\overline{Y} := Y^1 - Y^2$, $\overline{\xi} := \xi^1 - \xi^2$, $\overline{\zeta} = \zeta^1 - \zeta^2$ and $\overline{g}_s := \sup_{y,z,k} |g^1(s, y, z, k) - g^2(s, y, z, k)|$. Let $\eta, \beta > 0$ be such that $\beta \geq \frac{3}{\eta} + 2C$ and $\eta \leq \frac{1}{C^2}$. Then for each t, we have:*

$$\overline{Y}_t^2 \leq e^{\beta(T-t)} \mathbb{E}[\sup_{s \geq t} \overline{\xi_s}^2 + \sup_{s \geq t} \overline{\zeta_s}^2 |\mathcal{F}_t] + \eta \mathbb{E}[\int_t^T e^{\beta(s-t)} \overline{g}_s^2 ds |\mathcal{F}_t] \ a.s. \tag{5.42}$$

Remark 5.2 Note that η and β are universal constants, i.e. they do not depend on T, ξ^1, ξ^2, g^1, g^2.

Proof. For $i = 1, 2$ and for each $\tau, \sigma \in \tau_0$, let $(X^{i,\tau,\sigma}, \pi^{i,\tau,\sigma}, l^{i,\tau,\sigma})$ be the solution of the BSDE associated with driver g^i, terminal time $\tau \wedge \sigma$ and terminal condition $I^i(\tau, \sigma)$, where $I^i(\tau, \sigma) = \xi^i_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta^i_\sigma \mathbf{1}_{\sigma < \tau}$. Set $\overline{X}^{\tau,\sigma} := X^{1,\tau,\sigma} - X^{2,\tau,\sigma}$ and $\overline{I}^{\tau,\sigma} := I^1(\tau, \sigma) - I^2(\tau, \sigma) = \overline{\xi}_\tau \mathbf{1}_{\tau \leq \sigma} + \overline{\zeta}_\sigma \mathbf{1}_{\sigma < \tau}$. By a priori estimate on BSDEs (see Proposition $A.4$ in [**?**]), we have a.s.:

$$(\overline{X}_t^{\tau,\sigma})^2 \leq e^{\beta(T-t)} \mathbb{E}[\overline{I}(\tau, \sigma)^2 \mid \mathcal{F}_t] + \eta \mathbb{E}[\int_t^T e^{\beta(s-t)} [(g^1 - g^2)(s, X_s^{2,\tau,\sigma}, \pi_s^{2,\tau,\sigma}, l_s^{2,\tau,\sigma})]^2 ds \mid \mathcal{F}_t] \tag{5.43}$$

from which we derive that

$$(\overline{X}_t^{\tau,\sigma})^2 \leq e^{\beta(T-t)} \mathbb{E}[\sup_{s \geq t} \overline{\xi}_s^2 + \sup_{s \geq t} \overline{\zeta}_s^2 |\mathcal{F}_t] + \eta \mathbb{E}[\int_t^T e^{\beta(s-t)} \overline{g}_s^2 ds |\mathcal{F}_t] \quad a.s. \tag{5.44}$$

Now, by using the inequality (**??**), we obtain that for each $\varepsilon > 0$ and for all stopping times τ, σ,

$$Y_t^1 - Y_t^2 \leq X_t^{1,\tau^{\epsilon,1},\sigma} - X_t^{2,\tau,\sigma^{\epsilon,2}} + 2e^{\frac{\beta(T-t)}{2}} \epsilon. \tag{5.45}$$

By applying this inequality to $\tau = \tau^{\epsilon,1}, \sigma = \sigma^{\epsilon,2}$ we get

$$Y_t^1 - Y_t^2 \leq X_t^{1,\tau^{\epsilon,1},\sigma^{\epsilon,2}} - X_t^{2,\tau^{\epsilon,1},\sigma^{\epsilon,2}} + 2e^{\frac{\beta(T-t)}{2}} \epsilon \tag{5.46}$$

which implies that:

$$Y_t^1 - Y_t^2 \leq |X_t^{1,\tau^{\epsilon,1},\sigma^{\epsilon,2}} - X_t^{2,\tau^{\epsilon,1},\sigma^{\epsilon,2}}| + 2e^{\frac{\beta(T-t)}{2}} \epsilon \tag{5.47}$$

By (**??**) and (**??**), we have:

$$Y_t^1 - Y_t^2 \leq \sqrt{e^{\beta(T-t)} \mathbb{E}[\sup_{s \geq t} \overline{\xi_s}^2 + \sup_{s \geq t} \overline{\zeta_s}^2 |\mathcal{F}_t] + \eta \mathbb{E}[\int_t^T e^{\beta(s-t)} \overline{g}_s^2 ds |\mathcal{F}_t]} + 2e^{\frac{\beta(T-t)}{2}} \epsilon$$

By symmetry, the last inequality is also verified by $Y_t^2 - Y_t^1$. The result follows.

Proposition 5.3 *Suppose that the assumptions of Proposition **??** hold. For each t, we have:*

$$\overline{Y}_t^2 \leq e^{\beta(T-t)}\mathbb{E}[\sup_{s\geq t}{\xi_s}^2 + \sup_{s\geq t}{\zeta_s}^2|\mathcal{F}_t] + \eta\mathbb{E}[\int_t^T e^{\beta(s-t)}g(s,0,0,0)^2 ds|\mathcal{F}_t] \ a.s. \tag{5.48}$$

Proof. Let $X_t^{\tau,\sigma}$ be the solution of the BSDE associated with driver g, terminal time $\tau\wedge\sigma$ and terminal condition $I(\tau,\sigma)$. By applying inequality (**??**) with $g^1 = g$, $\xi_1 = \xi$, $\zeta_1 = \zeta$, $g^2 = 0$, $\xi^2 = 0$ and $\zeta^2 = 0$, we get:

$$(X_t^{\tau,\sigma})^2 \leq e^{\beta(T-t)}\mathbb{E}[I(\tau,\sigma)^2|\mathcal{F}_t] + \eta\mathbb{E}[\int_t^T e^{\beta(s-t)}(g(s,0,0,0))^2|\mathcal{F}_t]. \tag{5.49}$$

By using the same procedure as in the proof of Proposition **??**, the result follows.

6 Relation with partial integro-differential variational inequalities (PIDVI)

We now restrict ourselves to the Markovian case. Let $b:\mathbb{R}\to\mathbb{R}$, $\sigma:\mathbb{R}\to\mathbb{R}$ be continuous mappings, globally Lipschitz and $\beta:\mathbb{R}\times\mathbb{R}^*\to\mathbb{R}$ a measurable function such that for some non negative real C, and for all $e\in\mathbb{R}$

$$|\beta(x,e)| \leq C(1\wedge|e|), \quad x\in\mathbb{R}$$
$$|\beta(x,e)-\beta(x',e)| \leq C|x-x'|(1\wedge|e|), \quad x,x'\in\mathbb{R}.$$

For each $(t,x)\in[0,T]\times\mathbb{R}$, let $\{X_s^{t,x}, t\leq s\leq T\}$ be the unique $\mathbb{R}$-valued solution of the SDE with jumps:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r + \int_t^s\int_{\mathbb{R}^*}\beta(X_{r^-}^{t,x},e)\tilde{N}(dr,de),$$

and set $X_s^{t,x} = x$ for $s\leq t$. We consider the DBBSDE associated with obstacles $\xi^{t,x}$, $\zeta^{t,x}$ and driver $g(s,X_s^{t,x},.)$ of the following form:

$$\begin{cases}\xi_s^{t,x} := h_1(s,X_s^{t,x}), \ \ s<T\\ \zeta_s^{t,x} := h_2(s,X_s^{t,x}), \ \ s<T\\ \xi_T^{t,x} = \zeta_T^{t,x} := f(X_T^{t,x})\\ g(s,X_s^{t,x}(\omega),y,z,k) := \varphi(s,X_s^{t,x}(\omega),y,z,\int_{\mathbb{R}^*}k(e)\gamma(x,e)\nu(de))\mathbf{1}_{s\geq t}\end{cases}$$

where h, g,φ, and γ are as follows.

- $g\in\mathcal{C}(\mathbb{R})$ and has at most polynomial growth at infinity.
- $h_1,h_2:[0,T]\times\mathbb{R}\to\mathbb{R}$ are jointly continuous in t and x and there exist $p\in\mathbb{N}$ and a real constant, still denoted by C, such that

$$\begin{cases}|h_1(t,x)|\leq C(1+|x|^p), \forall t\in[0,T], x\in\mathbb{R}, \ \ s<T\\ |h_2(t,x)|\leq C(1+|x|^p), \forall t\in[0,T], x\in\mathbb{R}, \ \ s<T\end{cases} \tag{6.50}$$

- $h_1(T,x) \leq f(x)$ and $\quad h_2(T,x) \geq f(x) \;\; \forall x \in \mathbb{R}$
- The processes $\xi^{t,x}$ and $\zeta^{t,x}$ satisfy Mokobodski's condition.
- $\gamma : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^*)$-measurable and
  $|\gamma(x,e) - \gamma(x',e)| < C|x-x'|(1 \wedge |e|), x, x' \in \mathbb{R}, e \in \mathbb{R}^*$
  $0 \leq \gamma(x,e) \leq C(1 \wedge |e|),\ e \in \mathbb{R}^*$
- $\varphi : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ is continuous in t, uniformly with respect to x, y, z, k and continuous in x, uniformly with respect to y, z, k.

  (i) $|\varphi(t,x,0,0,0)| \leq C(1+|x|^p), x \in \mathbb{R}$

  (ii) $|\varphi(t,x,y,z,q) - \varphi(t,x',y',z',q')| \leq C(|y-y'| + |z-z'| + |q-q'|)$, $0 \leq t \leq T$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}$, $q, q' \in \mathbb{R}$

  (iii) $q \to \varphi(t,x,y,z,q)$ is non decreasing, for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^3$.

By Theorem **??**, for each $(t,x) \in [0,T] \times \mathbb{R}$, there exists a unique process

$$(Y^{t,x}, Z^{t,x}, K^{t,x}, \alpha^{t,x}) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2$$

of progressively measurable processes, which solves the following DBBSDE

$$\begin{cases} \alpha_s^{t,x} = A_s^{t,x} - A_s^{'\,t,x},\ \text{where } A^{t,x}, A^{'\,t,x} \in \mathcal{A}^2 \\ Y_s^{t,x} = f(X_T^{t,x}) + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, K_r^{t,x}(\cdot)) + A_T^{t,x} - A_s^{t,x} + A_s^{'t,x} - A_T^{'t,x} \\ \qquad - \int_s^T Z_r^{t,x} dW_r - \int_s^T \int_{\mathbb{R}^*} K^{t,x}(r,e) \tilde{N}(dr,de) \\ \xi_s^{t,x} \leq Y_s^{t,x} \leq \zeta_s^{t,x},\ 0 \leq s \leq T \text{ a.s.,} \\ \int_0^T (Y_t - \xi_t) dA_t^c = 0 \text{ a.s. and } \ \Delta A_t^d = \Delta A_t^d \mathbf{1}_{Y_{t^-} = \xi_{t^-}} \text{ a.s.} \\ \int_0^T (\zeta_t - Y_t) dA_t^{'c} = 0 \text{ a.s. and } \ \Delta A_t^{'d} = \Delta A_t^{'d} \mathbf{1}_{Y_{t^-} = \zeta_{t^-}} \text{ a.s.} \end{cases}$$

The non decreasing property of φ and the assumption on γ ensure that Assumption **??** holds. Moreover, by definition, $\xi^{t,x}$ and $-\zeta^{t,x}$ are l.u.s.c. along stopping times. It follows that the process $\alpha^{t,x}$ is continuous and the processes $A^{t,x}$, $A^{'\,t,x}$ can be chosen continuous. We define:

$$u(t,x) := Y_t^{t,x}, \quad t \in [0,T],\ x \in \mathbb{R}. \tag{6.51}$$

which is a deterministic quantity. Note that $Y_s^{t,x} = Y_t^{t,x}$ for $0 \leq s \leq t$.

Lemma 6.1 *The function u is continuous in (t,x) and has at most polynomial growth at infinity.*

Proof. The result follows from the above a priori estimates (see Propositions **??** and **??**) and the same arguments as those used in the proofs of Lemma 3.1 and Lemma 3.2 in [**?**]. □

6.1 Existence of a viscosity solution

We now consider the related obstacle problem for a parabolic PIDE. Roughly speaking, a solution of the obstacle problem is a function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ which satisfies, for each $x \in \mathbb{R}$ the equality $u(T,x) = f(x)$, and which, for each $(t,x) \in [0,T) \times \mathbb{R}$, satisfies:

$$\begin{cases} \text{either } h_1(t,x) = u(t,x) = h_2(t,x) \\ \text{or } h_1(t,x) < u(t,x) < h_2(t,x) \text{ and } \mathcal{H}u = 0 \\ \text{or } h_1(t,x) = u(t,x) < h_2(t,x) \text{ and } \mathcal{H}u \geq 0 \\ \text{or } h_1(t,x) < u(t,x) = h_2(t,x) \text{ and } \mathcal{H}u \leq 0 \end{cases} \tag{6.52}$$

or, equivalently, the three following conditions:

$$\begin{cases} h_1(t,x) \leq u(t,x) \leq h_2(t,x) \\ \text{if } u(t,x) < h_2(t,x) \text{ then } \mathcal{H}u \geq 0 \\ \text{if } h_1(t,x) < u(t,x) \text{ then } \mathcal{H}u \leq 0 \end{cases} \tag{6.53}$$

where

- $L := A + K$
- $A\phi(x) := \frac{1}{2}\sigma^2(x)\frac{\partial^2 \phi}{\partial x^2}(x) + b(x)\frac{\partial \phi}{\partial x}(x)$, $\phi \in C^2(\mathbb{R})$
- $K\phi(x) := \int_{\mathbb{R}^*}\left(\phi(x + \beta(x,e)) - \phi(x) - \frac{\partial \phi}{\partial x}(x)\beta(x,e)\right)\nu(de)$, $\phi \in C^2(\mathbb{R})$
- $B\phi(x) := \int_{\mathbb{R}^*}(\phi(x + \beta(x,e)) - \phi(x))\gamma(x,e)\nu(de)$
- $\mathcal{H}\phi(t,x) := -\frac{\partial \phi}{\partial t}(t,x) - L\phi(t,x) - g(t,x,\phi(t,x),(\sigma\frac{\partial \phi}{\partial x})(t,x), B\phi(t,x))$.

Here, in order to simplify the notation, the map φ is simply denoted by g.

We now prove that the solution of the double barrier reflected BSDE is solution of the above obstacle problem, by using the classical definition of viscosity solutions ([?]).

Definition 6.2 • A continuous function u is said to be a *viscosity subsolution* of (**??**) if $u(T,x) \leq f(x), x \in \mathbb{R}$, and if for any point $(t_0,x_0) \in [0,T) \times \mathbb{R}$, we have $h_1(t_0,x_0) \leq u(t_0,x_0) \leq h_2(t_0,x_0)$ and, for any $\phi \in C^{1,2}([0,T] \times \mathbb{R})$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its minimum at (t_0,x_0), if $u(t_0,x_0) > h_1(t_0,x_0)$, then $(\mathcal{H}\phi)(t_0,x_0) \leq 0$.

• A continuous function u is said to be a *viscosity supersolution* of (**??**) if $u(T,x) \geq f(x), x \in \mathbb{R}$, and if for any point $(t_0,x_0) \in [0,T) \times \mathbb{R}$, we have $h_1(t_0,x_0) \leq u(t_0,x_0) \leq h_2(t_0,x_0)$ and, for any $\phi \in C^{1,2}([0,T] \times \mathbb{R})$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its maximum at (t_0,x_0), if $u(t_0,x_0) < h_2(t_0,x_0)$ then $(\mathcal{H}\phi)(t_0,x_0) \geq 0$.

Theorem 6.3 *The function u, defined by* (**??**)*, is a viscosity solution (i.e. both a viscosity sub- and supersolution) of the obstacle problem* (**??**)*.*

Proof. Following the same arguments as in the proof of Theorem 3.4 in [?], one can show that u is viscosity subsolution of (**??**). By symmetry, we derive that u is also a viscosity supersolution of (**??**). □

6.2 Uniqueness of the viscosity solution

We need the following additional assumptions to prove a uniqueness result for (**??**).

Assumption 6.2 *1. For each $R > 0$, there exists a continuous function $m_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $m_R(0) = 0$ and*

$$|g(t,x,v,p,q)-g(t,y,v,p,q)| \leq m_R(|x-y|(1+|p|)), 0 \leq t \leq T, |x|, |y| \leq R, |v| \leq R, p, q \in \mathbb{R}.$$

2. $|\gamma(x,e) - \gamma(y,e)| \leq C|x-y|(1 \wedge |e|^2), x, y \in \mathbb{R}, e \in \mathbb{R}^*$.

3. There exists $r > 0$ such that for any $x \in \mathbb{R}$, $t \in [0,T]$, $u, v \in \mathbb{R}$, $p \in \mathbb{R}$, $l \in \mathbb{R}$:

$$g(t,x,v,p,l) - g(t,x,u,p,l) \geq r(u-v) \text{ when } u \geq v.$$

Theorem 6.3 (Comparison principle) *Under Assumption **??**, if U is a viscosity subsolution and V is a viscosity supersolution of the obstacle problem (**??**), then $U(t,x) \leq V(t,x)$, for each $(t,x) \in [0,T] \times \mathbb{R}$.*

Proof. The proof is based on similar arguments as that of Theorem 3.6 in **??** except that we have to consider an additional case. We give below a sketch of the proof.
It is sufficient to show that, for a fixed $K > 0$, $M_K := \sup_{x \in [-K,K], t \in [0,T]}(U - V)$, is negative. Let $K > 0$. To simplify notation, M_K is denoted by M.

We approximate M by dedoubling the variables. We consider the following function:

$$\psi^{\epsilon,\eta}(t,s,x,y) := U(t,x) - V(s,y) - \frac{|x-y|^2}{\epsilon^2} - \frac{|t-s|^2}{\epsilon^2} - \eta^2(|x|^2 + |y|^2).$$

where ϵ, η are small parameters devoted to tend to 0, for x, y in $[-K, K]$. Let $M^{\epsilon,\eta}$ be a maximum of $\psi^{\epsilon,\eta}(t,s,x,y)$. This maximum is reached at some point $(t^{\epsilon,\eta}, s^{\epsilon,\eta}, x^{\epsilon,\eta}, y^{\epsilon,\eta})$ in the compact set $([0,T]^2 \times \overline{B_{R_\eta}}^2)$, where B_{R_η} is a ball with a large radius R_η.
For ϵ, η small enough, we have:

$$0 < \frac{M}{2} \leq M^{\epsilon,\eta} \leq U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - V(s^{\epsilon,\eta}, y^{\epsilon,\eta}). \tag{6.54}$$

We define:

$$\Psi_1(t,x) := V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) + \frac{|x - y^{\epsilon,\eta}|^2}{\epsilon^2} + \frac{|t - s^{\epsilon,\eta}|^2}{\epsilon^2} + \eta^2(|x|^2 + |y^{\epsilon,\eta}|^2);$$

$$\Psi_2(s,y) := U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - \frac{|x^{\epsilon,\eta} - y|^2}{\epsilon^2} - \frac{|t^{\epsilon,\eta} - s|^2}{\epsilon^2} - \eta^2(|x^{\epsilon,\eta}|^2 + |y|^2).$$

As $(t,x) \to (U - \Psi_1)(t,x)$ reaches its maximum at $(t^{\epsilon,\eta}, x^{\epsilon,\eta})$ and U is a subsolution, we have the two following cases:

- $t^{\epsilon,\eta} = T$ and then $U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) \leq f(x^{\epsilon,\eta})$,

- $t^{\epsilon,\eta} \neq T$, $h_1(t^{\epsilon,\eta}, x^{\epsilon,\eta}) \leq U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) \leq h_2(t^{\epsilon,\eta}, x^{\epsilon,\eta})$ and, if $U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) > h_1(t^{\epsilon,\eta}, x^{\epsilon,\eta})$, we then have:

$$-\frac{\partial \Psi_1}{\partial t}(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - L\Psi_1(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - g\left(t^{\epsilon,\eta}, x^{\epsilon,\eta}, U(t^{\epsilon,\eta}, x^{\epsilon,\eta}), (\sigma\frac{\partial \Psi_1}{\partial x})(t^{\epsilon,\eta}, x^{\epsilon,\eta}), B\Psi_1(t^{\epsilon,\eta}, x^{\epsilon,\eta})\right) \leq 0. \tag{6.55}$$

As $(s,y) \to (\Psi_2 - V)(s,y)$ reaches its maximum at $(s^{\epsilon,\eta}, y^{\epsilon,\eta})$ and V is a supersolution, we have the two following cases:

- $s^{\epsilon,\eta} = T$ and $V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \geq f(y^{\epsilon,\eta})$,
- $t^{\epsilon,\eta} \neq T$, $h_1(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \leq V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \leq h_2(s^{\epsilon,\eta}, y^{\epsilon,\eta})$ and, if $V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) < h_2(s^{\epsilon,\eta}, y^{\epsilon,\eta})$ then

$$-\frac{\partial \Psi_2}{\partial t}(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - L\Psi_2(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - g(s^{\epsilon,\eta}, y^{\epsilon,\eta}, V(s^{\epsilon,\eta}, y^{\epsilon,\eta}), (\sigma\frac{\partial \Psi_2}{\partial x})(s^{\epsilon,\eta}, y^{\epsilon,\eta})), B\Psi_2(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \geq 0.$$

We have:

$$\lim_{\eta\to 0}\lim_{\epsilon\to 0} M^{\epsilon,\eta} = M\,; \qquad \frac{|x^{\epsilon,\eta} - y^{\epsilon,\eta}|^2}{\epsilon^2} \xrightarrow{\epsilon\to 0} 0\,; \qquad \frac{|t^{\epsilon,\eta} - s^{\epsilon,\eta}|^2}{\epsilon^2} \xrightarrow{\epsilon\to 0} 0.$$

Using that $\psi^{\epsilon,\eta}(t^{\epsilon,\eta}, s^{\epsilon,\eta}, x^{\epsilon,\eta}, y^{\epsilon,\eta}) \geq \psi^{\epsilon,\eta}(0,0,0,0)$, we obtain:

$$U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - \frac{|t^{\epsilon,\eta} - s^{\epsilon,\eta}|^2}{\epsilon^2} - \frac{|x^{\epsilon,\eta} - y^{\epsilon,\eta}|^2}{\epsilon^2} - \eta^2(|x^{\epsilon,\eta}|^2 + |y^{\epsilon,\eta}|^2)$$
$$\geq U(0,0) - V(0,0). \tag{6.56}$$

and, equivalently,

$$\frac{|t^{\epsilon,\eta} - s^{\epsilon,\eta}|^2}{\epsilon^2} + \frac{|x^{\epsilon,\eta} - y^{\epsilon,\eta}|^2}{\epsilon^2} + \eta^2(|x^{\epsilon,\eta}|^2 + |y^{\epsilon,\eta}|^2)$$
$$\leq ||U||_\infty + ||V||_\infty - U(0,0) - V(0,0). \tag{6.57}$$

Consequently, we can find a constant C such that:

$$|x^{\epsilon,\eta} - y^{\epsilon,\eta}| + |t^{\epsilon,\eta} - s^{\epsilon,\eta}| \leq C\epsilon, |x^{\epsilon,\eta}|, |y^{\epsilon,\eta}| \leq \frac{C}{\eta}. \tag{6.58}$$

As $[0,T]$ is bounded and by (**??**), extracting a subsequence if necessary, we may suppose that for each η the sequences $(t^{\epsilon,\eta})_\epsilon$ and $(s^{\epsilon,\eta})_\epsilon$ converge to a common limit t^η, and from (**??**) we may also suppose, extracting again, that for each η, the sequences $(x^{\epsilon,\eta})_\epsilon$ and $(y^{\epsilon,\eta})_\epsilon$ converge to a common limit x^η.

We have to consider four cases.

1st case: there exists a subsequence of (t^η) such that $t^\eta = T$ for all η (of this subsequence)

2nd case: there exists a subsequence of (t^η) such that $t^\eta \neq T$ and for all η belonging to this subsequence, there exist a subsequence of $(x^{\epsilon,\eta})_\epsilon$ and a subsequence of $(t^{\epsilon,\eta})_\epsilon$such that

$$U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - h_1(t^{\epsilon,\eta}, x^{\epsilon,\eta}) = 0.$$

3rd case: there exists a subsequence such that $t^\eta \neq T$, and for all η belonging to this subsequence, there exist a subsequence of $(y^{\epsilon,\eta})_\epsilon$ and a subsequence of $(s^{\epsilon,\eta})_\epsilon$such that

$$V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - h_2(s^{\epsilon,\eta}, y^{\epsilon,\eta}) = 0.$$

Last case: we are left with the case when, for a subsequence of η we have $t^\eta \neq T$, and for all η belonging to this subsequence, there exist a subsequence of $(x^{\epsilon,\eta})_\epsilon$, $(y^{\epsilon,\eta})_\epsilon$, $(t^{\epsilon,\eta})_\epsilon$ and $(s^{\epsilon,\eta})_\epsilon$ such that

$$\begin{cases} U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - h_1(t^{\epsilon,\eta}, x^{\epsilon,\eta}) > 0 \\ h_2(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) > 0. \end{cases}$$

The first, second and fourth case are identical to the three cases considered for reflected BSDEs (see [**?**]). The third one, which didn't appear in the case of reflected BSDEs, can be treated similarly to the second one. □

Corollary 6.1 (Uniqueness) *Under Assumption* **??**, *there exists a unique solution of the obstacle problem (***??***) in the class of continuous functions with polynomial growth.*

7 Appendix

Proof of Lemma ??:
Set $J_\cdot^{(0)} = 0$ and $J_\cdot'^{(0)} = 0$ and define recursively for each $n \in \mathbb{N}$, the supermartingales:

$$\begin{cases} J^{(n+1)} := \mathcal{R}(J'^{(n)} + \tilde{\xi}^g) \\ J'^{(n+1)} := \mathcal{R}(J^{(n)} - \tilde{\zeta}^g) \end{cases} \tag{7.59}$$

which belong to $\mathcal{S}^2$. For sake of simplicity, the exposant g is omitted in the definition of $J^{(n)}$. Since $\tilde{\xi}^g_T = \tilde{\zeta}^g_T = 0$ a.s. , it follows that, for each n, $J^{(n)}_T = J'^{(n)}_T = 0$ a.s. We show the following result, from which Lemma **??** follows.

Proposition 7.1 *(i) The sequences $(J^{(n)}, n \in I\!N)$ and $(J^{'(n)}, n \in I\!N)$ are non decreasing sequences of nonnegative RCLL supermartingales.*

(ii) Let

$$J^g := \lim \uparrow J^{(n)} \quad \text{and} \quad J'^g := \lim \uparrow J'^{(n)}.$$

J^g and J'^g are RCLL supermartingales valued in $[0, +\infty]$ with $J^g_T = J'^g_T = 0$ a.s. and satisfy

$$\begin{cases} J^g = \mathcal{R}(J'^g + \tilde{\xi}^g) \\ J'^g = \mathcal{R}(J^g - \tilde{\zeta}^g). \end{cases} \tag{7.60}$$

(iii) the families J and J' are minimal in the following sense: if H and H' are two nonnegative supermartingale families such that $H \geq H' + \xi$ and $H' \geq H - \zeta$, then we have $J \leq H$ and $J' \leq H'$.

Remark 7.1 Note that here, J^g and $J^{'g}$ can take infinite values, which was not the case in the previous literature, where the Mokobodski's condition was assumed.

We point out that the property $\tilde{\xi}^g_T = \tilde{\zeta}^g_T = 0$ a.s. ensures that for each n, $J^{(n)}_T = J'^{(n)}_T = 0$ a.s. If we had not made the change of variable, then $\tilde{\xi}^g, \tilde{\zeta}^g$ would be replaced by ξ, ζ in the definitions of $J^{(n)}$ and $J^{'(n)}$. In that case, $\xi_T = \zeta_T$ a.s. but would not necessarily be equal to 0, and we would have $J^{(n)}_T = -J'^{(n)}_T = 0$ a.s. if n is even, and ξ_T otherwise. Then, the sequences

$(J_T^{(n)})_{n\in I\!N}$ *and* $(J_T^{\prime(n)})_{n\in I\!N}$ *do not converge a.s. if* $P(\xi_T \neq 0). > 0$ *Also, the non negativity property of the sequences* $(J^{(n)}, n \in I\!N)$ *and* $(J^{'(n)}, n \in I\!N)$ *and their non decreasing property would not necessarily hold.*

For completeness, we give the proof of this proposition.

Proof. (i) We have $J^{(0)} = 0$ and $J^{'(0)} = 0$. Suppose that $J^{'(n)}, J^{(n)}$ are well defined and nonnegative. Then $J^{(n+1)}$, $J^{'(n+1)}$ are well defined since $(J^{'(n)} + \xi)^-$ and $(J^{(n)} - \zeta)^-$ belong to $\mathcal{S}^2$. Also, $J_t^{(n+1)} \geq \mathbb{E}[J_T^{'(n)} + \tilde{\xi}_T^g | \mathcal{F}_t] \geq 0$ a.s. since $\tilde{\xi}_T^g = 0$ a.s. Similarly, because $\tilde{\zeta}_T^g = 0$ a.s., $J_t^{'(n+1)} \geq 0$ a.s. By classical results, $J^{(n)}$ and $J^{'(n)}$ are RCLL supermartingales.

Let us prove that $J^{(n)}$ and $J^{'(n)}$ are non decreasing sequences. We have $J^{(1)} \geq 0 = J^{(0)}$ and $J^{'(1)} \geq 0 = J^{'(0)}$. Suppose that $J^{(n)} \geq J^{(n-1)}$ and $J^{'(n)} \geq J^{'(n-1)}$. We then have:

$$\begin{cases} \mathcal{R}(J^{\prime(n)} + \tilde{\xi}^g) \geq \mathcal{R}(J^{'(n-1)} + \tilde{\xi}^g) \\ \mathcal{R}(J^{(n)} - \tilde{\zeta}^g) \geq \mathcal{R}(J^{(n-1)} - \tilde{\zeta}^g) \end{cases} \tag{7.61}$$

which leads to $J^{(n+1)} \geq J^{(n)}$ and $J^{'(n+1)} \geq J^{(n)}$.

(ii) By some results of Dellacherie-Meyer (see Th. 18, Ch. VI in [?]), J^g and J'^g are indistinguishable from non negative RCLL supermartingales valued in $[0, +\infty]$, as the non decreasing limits of non negative RCLL supermartingales. For each $n \in \mathbb{N}$, we have:

$$J^{(n+1)} = \mathcal{R}(J^{'(n)} + \tilde{\xi}^g) \leq \mathcal{R}(J^{'g} + \tilde{\xi}^g).$$

By letting n tend to $+\infty$, we get that

$$J^g \leq \mathcal{R}(J^{'g} + \tilde{\xi}^g). \tag{7.62}$$

Now, for each $n \in \mathbb{N}$, $J^{(n+1)} \geq J^{'(n)} + \tilde{\xi}^g$. By letting n tend to $+\infty$, we derive that $J^g \geq J^{'g} + \tilde{\xi}^g$. By the supermartingale property of J^g and the characterization of $\mathcal{R}(J^{'g} + \tilde{\xi}^g)$ as the smallest supermartingale greater than $J^{'g} + \tilde{\xi}^g$, it follows that $J^g \geq \mathcal{R}(J^{'g} + \tilde{\xi}^g)$. This with (**??**) yields that $J^g = \mathcal{R}(J^{'g} + \tilde{\xi}^g)$. By similar arguments, one easily derives that $J^{'g} = \mathcal{R}(J^g - \tilde{\zeta}^g)$.

(iii) Note first that by (**??**), $J^g \geq J^{'g} + \xi$ and $J^{'g} \geq J^g - \zeta$.Let J^g and $J^{'g}$ be two nonnegative supermartingale families such that $J^g \geq J^{'g} + \xi$ and $J^{'g} \geq J^g - \zeta$. Let us first show that for each $n \in I\!N$,

$$J^{(n)} \leq H \quad \text{and} \quad J^{'(n)} \leq H'. \tag{7.63}$$

by induction. It clearly holds for $J^{(0)}$ and $J^{'(0)}$. Let us suppose that, for some fixed $n \in I\!N$, inequalities (**??**) hold. Using the inequality $H' + \xi \leq H$, we thus derive that $J^{'(n)} + \xi \leq H' + \xi \leq H$. Since the operator $\mathcal{R}$ is non decreasing, we get $J^{(n+1)} = \mathcal{R}(J^{'(n)} + \xi) \leq \mathcal{R}(H)$. Now, since H is a supermartingale, Since H is a supermartingale, we have $\mathcal{R}(H) = H$, and hence $J^{(n+1)} \leq H$. By similar arguments, we also have $J^{'(n+1)} \leq H'$, which ensures that Property (**??**) holds at rank $n + 1$.

By letting n tend to $+\infty$ in (**??**), we get that $J^g \leq H$ and $J'^g \leq H'$, which ends the proof.

□

Proof of Theorem ??:

For $\beta > 0$, $\phi \in I\!H^2$, and $l \in I\!H_\nu^2$, we introduce the norms $\|\phi\|_\beta^2 := E[\int_0^T e^{\beta s} \phi_s^2 ds]$, and $\|l\|_{\nu,\beta}^2 := E[\int_0^T e^{\beta s} \|l_s\|_\nu^2 \, ds]$.

Let $I\!H^2_{\beta,\nu}$ (below simply denoted by $I\!H^2_\beta$) the space $I\!H^2 \times I\!H^2 \times I\!H^2_\nu$ equipped with the norm $\|Y,Z,k(\cdot)\|^2_\beta := \|Y\|^2_\beta + \|Z\|^2_\beta + \|k\|^2_{\nu,\beta}$.

We define a mapping Φ from $I\!H^2_\beta$ into itself as follows. Given $(U,V,l) \in I\!H^2_\beta$, by Theorem **??** there exists a unique process $(Y,Z,k) = \Phi(U,V,l)$ solution of the DBBSDE associated with driver process $g(s) = g(s,U_s,V_s,l_s)$. Note that $(Y,Z,k) \in I\!H^2_\beta$. Let $\alpha = A - A'$ be the associated finite variation process. Let us show that Φ is a contraction and hence admits a unique fixed point (Y,Z,k) in $I\!H^2_\beta$, which corresponds to the unique solution of DBBSDE (**??**). The associated finite variation process α is then uniquely determined in terms of (Y,Z,k). Let (U^2,V^2,l^2) be another element of $I\!H^2_\beta$ and define $(Y^2,Z^2,k^2) = \Phi(U^2,V^2,l^2)$. Let $\alpha_2 = A^2 - A'^2$ be the associated finite variation process. Set $\overline{U} = U - U^2$, $\overline{V} = V - V^2$, $\overline{l} = l - l^2$ and, $\overline{Y} = Y - Y^2$, $\overline{Z} = Z - Z^2$, $\overline{k} = k - k^2$. By Itô's formula, for any $\beta > 0$, we have

$$\begin{aligned}
&\overline{Y}_0^2 + E\int_0^T e^{\beta s}[\beta \overline{Y}_s^2 + \overline{Z}_s^2 + \|\overline{k}_s^2\|]\,ds \\
&= 2E\int_0^T e^{\beta s}\overline{Y}_s[g(s,U_s,V_s,l_s) - g(s,U_s^2,V_s^2,l_s^2)]\,ds + 2E[\int_0^T e^{\beta s}\overline{Y}_{s^-}\,dA_s - \int_0^T e^{\beta s}\overline{Y}_{s^-}\,dA_s^2] \qquad (7.64)\\
&- 2E[\int_0^T e^{\beta s}\overline{Y}_{s^-}\,dA'_s - \int_0^T e^{\beta s}\overline{Y}_{s^-}\,dA'^2_s].
\end{aligned}$$

Now, we have a.s.

$$\overline{Y}_s dA_s^c = (Y_s - \xi_s)dA_s^c - (Y_s^2 - \xi_s)dA_s^c = -(Y_s^2 - \xi_s)dA_s^c \le 0$$

and by symmetry, $\overline{Y}_s dA_s^{2c} \ge 0$ a.s. Also, we have a.s.

$$\overline{Y}_{s^-}\Delta A_s^d = (Y_{s^-} - \xi_{s^-})\Delta A_s^d - (Y_{s^-}^2 - \xi_{s^-})\Delta A_s^d = -(Y_{s^-}^2 - \xi_{s^-})\Delta A_s^d \le 0$$

and $\overline{Y}_{s^-}\Delta {A^2}_s^d \ge 0$ a.s. Similarly, we have a.s.

$$\overline{Y}_s dA_s^{'c} = (Y_s - \zeta_s)dA_s^{'c} - (Y_s^2 - \zeta_s)dA_s^{'c} = -(Y_s^2 - \zeta_s)dA_s^{'c} \ge 0$$

and by symmetry, $\overline{Y}_s dA'^{2c}_s \le 0$ a.s. Also, we have a.s.

$$\overline{Y}_{s^-}\Delta A_s^{'d} = (Y_{s^-} - \zeta_{s^-})\Delta A_s^{'d} - (Y_{s^-}^2 - \zeta_{s^-})\Delta A_s^{'d} = -(Y_{s^-}^2 - \zeta_{s^-})\Delta A_s^{'d} \ge 0$$

and $\overline{Y}_{s^-}\Delta {A'^2}_s^d \le 0$ a.s.

Consequently, the second and the third term of (**??**) is non positive. By using the Lipschitz property of g and the inequality $2Cyu \le 2C^2y^2 + \frac{1}{2}u^2$, we get that

$$\beta\|\overline{Y}\|^2_\beta + \|\overline{Z}\|^2_\beta + \|\overline{k}\|^2_{\nu,\beta} \le 6C^2\|\overline{Y}\|^2_\beta + \frac{1}{2}(\|\overline{U}\|^2_\beta + \|\overline{V}\|^2_\beta + \|\overline{l}\|^2_{\nu,\beta}).$$

Choosing $\beta = 6C^2 + 1$, we deduce $\|(\overline{Y},\overline{Z},\overline{k})\|^2_\beta \le \frac{1}{2}\|(\overline{U},\overline{V},\overline{l})\|^2_\beta$.
The last assertion of the theorem follows from Theorem **??**.

□

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ISSN 0249-6399