

# SINGULAR EVOLUTION ON MANIFOLDS, THEIR SMOOTHING PROPERTIES, AND SOBOLEV INEQUALITIES

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ABSTRACT. The evolution equation  $\dot{u} = \Delta_p u$ , posed on a Riemannian manifold, is studied in the singular range  $p \in (1, 2)$ . It is shown that if the manifold supports a suitable Sobolev inequality, the smoothing effect  $\|u(t)\|_\infty \leq C\|u(0)\|_q^\gamma/t^\alpha$  holds true for suitable  $\alpha, \gamma$  and that the converse holds if  $p$  is sufficiently close to 2, or in the degenerate range  $p > 2$ . In such ranges, the Sobolev inequality and the smoothing effect are then equivalent.

## 1. INTRODUCTION

Let  $(M, g)$  be a smooth, connected, complete  $d$ -dimensional ( $d \geq 3$ ) Riemannian manifold of infinite volume. Let  $\nabla$  be the Riemannian gradient,  $\nabla \cdot$  the Riemannian divergence,  $|\cdot|$  the Riemannian length,  $dx$  the Riemannian measure. Lebesgue spaces w.r.t. such measure are indicated by  $L^q(M)$  and the Sobolev spaces  $W^{1,q}(M)$  are then defined as the spaces of those  $L^q(M)$  functions with first order derivatives belonging to the same space. The  $p$ -Laplacian  $\Delta_p$  is the operator formally given by  $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ . For a considerable part of the paper  $p$  will be taken in the *singular* range  $p \in (1, 2)$ , but in one of our main results we shall also discuss the *degenerate* range  $p > 2$ . To define  $\Delta_p$  properly, we notice that it is defined as the subgradient of the convex, lower semicontinuous functional defined as

$$\mathcal{E}_p(u) = \frac{1}{p} \int_M |\nabla u|^p dx$$

on  $W^{1,p}(M)$ , and as  $+\infty$  elsewhere on  $L^2(M)$  (see [10] for an excellent discussion of Sobolev spaces on manifolds). Such operator is hence (see [8]) the generator of a well defined semigroup on  $L^2(M)$ , which we denote by  $u(t)$ , and we notice that  $u(t)$  is a solution to the equation  $\dot{u} = \Delta_p u$  in the semigroup sense.

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In addition the evolution considered is a *nonlinear Markov semigroup* in the sense of [8]. In particular:

- the evolution is well-defined in each  $L^q(M)$  for any  $q \in [1, +\infty]$ ;
- it preserves order in the sense that two initial data are ordered, say  $u(0) \leq v(0)$ , the corresponding solutions are ordered as well, namely  $u(t) \leq v(t)$  for all  $t$ ;
- it is nonexpansive in each  $L^q$  space so that  $\|u(t) - v(t)\|_q \leq \|u(0) - v(0)\|_q$  for all initial data and all  $q \in [1, +\infty]$ . The special choice  $v(0) \equiv 0$  then gives the contraction property  $\|u(t)\|_q \leq \|u(0)\|_q$  for all  $q \in [1, +\infty]$ .

Notice that existence and uniqueness of solutions and the above mentioned properties *do not depend* upon further assumptions on the manifold, but only on the form of the generating functional  $\mathcal{E}_p$ .

Our main result is the following

**Theorem 1.** *Suppose that  $M$  supports the Euclidean-type Sobolev inequality, so that*

$$(1) \quad \|f\|_{2d/(d-2)} \leq C_s \|\nabla f\|_2$$

for any  $f$  in the Sobolev space  $H^1(M)$ . Consider any solution  $u(t)$  to the evolution equation  $\dot{u} = \Delta_p u$ , where  $p$  lies in the singular range  $p \in (1, 2)$ , and take any  $q_0$  satisfying  $q_0 \geq 1$ ,  $q_0 > q_c$  with  $q_c := \frac{d(2-p)}{p}$ . Then for any initial data  $u(0) \in L^{q_0}(M)$  and all  $t > 0$  the smoothing effect

$$(2) \quad \|u(t)\|_\infty \leq C \frac{\|u(0)\|_{q_0}^{pq_0/[pq_0+d(p-2)]}}{t^{d/[pq_0+d(p-2)]}}$$

holds true.

An elementary consequence of (2) and of the contraction property  $\|u(t)\|_{q_0} \leq \|u(0)\|_{q_0}$  is the following.

**Corollary 1.** *Let  $r > q_0$  with  $q_0 \geq 1$ ,  $q_0 > q_c$ . Then, with the notations and under the assumptions of the above Theorem, one has:*

$$(3) \quad \|u(t)\|_r \leq \frac{\|u(0)\|_{q_0}^{\gamma(r, q_0)}}{t^{\alpha(r, q_0)}}$$

where

$$\begin{aligned} \gamma(r, q_0) &:= \frac{pq_0}{pq_0 + d(p-2)} \left(1 - \frac{q_0}{r}\right) + \frac{q_0}{r}; \\ \alpha(r, q_0) &= \frac{d}{pq_0 + d(p-2)} \left(1 - \frac{q_0}{r}\right). \end{aligned}$$

The above results have been proved in the Euclidean setting with scaling methods (see e.g. the monographs [9], [11] and reference quoted). In the present setting they seem to be new, and they need new methods of proof. Our contribution lies then in the fact that such bounds are related to the validity of a Sobolev inequality only. Such point of view has been adopted in [7], [2] for similar evolution equations in the *degenerate* range  $p > 2$ , but the use of the Sobolev inequality given there seems to be not adaptable to the singular range: the present note therefore closes the gap and allows to cover the whole range of parameters in the class of equations considered. Similar problems were considered for evolutions of porous media type in [3], [5].

We just remind the reader that the validity of the Sobolev inequality we are requiring is related to several geometric and analytic properties. For example it is *equivalent* to the Faber–Krahn inequality [10]. Moreover, we recall that  $M$  is said to be *nonparabolic* if it admits a (minimal) *positive* Green function  $G$  for  $\Delta$ . Conditions for parabolicity and nonparabolicity can be found in [10], pg. 230, and references quoted. It may be interesting to notice that the validity of (1) is equivalent to the following facts:  $M$  is nonparabolic and  $\text{vol}(y : G(x, y) > t) \leq Ct^{-d/(d-2)}$ , where “vol” is the Riemannian volume. This result is due to Carron, see [10]. It is also known ([10], pg. 232) that the Sobolev inequality (1) is valid on any smooth, complete, simply connected manifold of nonpositive sectional curvature.

Our second main result is the following converse.

**Theorem 2.** *Assume that  $p > p_s := 2d/(d+2)$ . Suppose that for any initial data  $u(0) \in L^{q_0}(M)$  and all  $t > 0$  the smoothing effect (3) relative to  $q_0 \in [1, 2)$ ,  $q_0 > q_c$ , and to  $r = 2$  holds true for any solution  $u(t)$  of the equation  $\dot{u} = \Delta_p u$ . Then the Sobolev inequality (1) holds. Therefore, in such range, the smoothing effect (2) is equivalent to the Sobolev inequality (1).*

Notice that the above Theorem covers also the *degenerate* range  $p > 2$ , since the smoothing effect (2) holds in this range as well, as can be proved by means of the techniques of [4] which, although stated in the Euclidean case, work in the present setting as well. This equivalence is new in the degenerate case  $p > 2$  as well.

The Carron’s result mentioned before then proves the following claim.

**Corollary 2.** *If  $p > p_s$  and the smoothing effect (2) holds in the range stated in Theorem 2, then  $M$  is nonparabolic and the bound  $\text{vol}(y : G(x, y) > t) \leq Ct^{-d/(d-2)}$  holds true for all positive  $t$  and all  $x \in M$ .*

## 2. PROOF OF THEOREM 1

We shall divide the proof into several steps.

**Step 1: logarithmic Sobolev inequalities.** We shall prove that a suitable family of logarithmic Sobolev inequalities holds true. In fact, let  $p^* := pd/(d-p)$  (recall that  $d \geq 3$  and  $p \in (1, 2)$ ). Define the functional

$$(4) \quad J(r, f) := \int_M \frac{|f|^r}{\|f\|_r^r} \log \left( \frac{|f|}{\|f\|_r} \right) dx$$

in the range  $r \in (0, p^*)$ , for all functions for which it is finite. Then we compute, defining  $\alpha := p^* - r > 0$ :

$$\begin{aligned} rJ(r, f) &= \frac{r}{\alpha} \int_M \log \left( \frac{|f|^\alpha}{\|f\|_r^\alpha} \right) \frac{|f|}{\|f\|_r} dx \\ &\leq \frac{r}{\alpha} \log \left( \frac{\|f\|_{r+\alpha}^{r+\alpha}}{\|f\|_r^{r+\alpha}} \right) = \frac{r(r+\alpha)}{\alpha} \log \left( \frac{\|f\|_{r+\alpha}}{\|f\|_r} \right) \\ &\leq \frac{p^*r}{p^* - r} \log \left( C_s \frac{\|\nabla f\|_p}{\|f\|_r} \right), \end{aligned}$$

where we have used Jensen inequality w.r.t. the probability measure  $(f^r/\|f\|_r^r) dx$  in the first step and the Sobolev inequality and the definition of  $\alpha$  in the last step, where  $C_s$  denotes the Sobolev constant in (1).

Define  $c_{p,r,d} := p^r / (p^* - r) = \left(\frac{1}{r} - \frac{1}{p^*}\right)^{-1}$ . Use then the numerical inequality  $\log x \leq \varepsilon x - \log \varepsilon$  ( $x, \varepsilon > 0$ ) to deduce that

$$rJ(r, f) \leq \frac{\varepsilon C_s^p c_{r,p,d} \|\nabla f\|_p^p}{p \|d_r^p\|} - \frac{c_{r,p,d}}{p} \log \varepsilon.$$

Noticing that  $rJ(r, f) = J(1, |f|^r)$  we can equivalently conclude the present step by writing the following family of logarithmic Sobolev inequalities:

$$(5) \quad \|\nabla f\|_p^p \geq \frac{p \|f\|_r^p}{c_{r,p,d} C_s^p \varepsilon} \left[ J(1, f^r) + \frac{c_{p,r,d}}{p} \log \varepsilon \right]$$

for any  $\varepsilon > 0, r \in \left(0, \frac{pd}{d-p}\right)$ .

**Step 2: inequalities for time-dependent  $L^q$  norms.** We choose from now on data in  $L^\infty(M)$ . This assumption is technical and will be removed later on.

We can adopt the strategy of [7] first to compute, given a smooth, increasing, real valued function of time, say  $q : [0, t) \rightarrow [1, +\infty)$ , where  $t > 0$  is fixed:

$$\begin{aligned} \frac{d}{ds} \log \|u\|_q &= \frac{\dot{q}}{q} J(q, u) \\ &\quad - \left( \frac{p}{q+p-2} \right)^p \frac{(q-1)}{\|u\|_q^q} \|\nabla(|u|^{(q+p-2)/p})\|_p^p \end{aligned}$$

where we omit to indicate the  $s$ -dependence. This can be proved as in [7], whose proof works in the present setting and in the present range of the parameter  $p$  as well. Having at our disposal the new family of logarithmic Sobolev inequalities just proved we can therefore write:

$$\begin{aligned} \frac{d}{ds} \log \|u\|_q &\leq \frac{\dot{q}}{q^2} J(1, |u|^q) \\ &\quad - \left( \frac{p}{q+p-2} \right)^p \frac{(q-1)}{\|u\|_q^q} \frac{p \|u\|_{(q+p-2)r/p}^{q+p-2}}{c_{r,p,d} C_s^p \varepsilon} \times \\ &\quad \times \left[ J\left(1, |u|^{(q+p-2)r/p}\right) + \frac{c_{p,r,d}}{p} \log \varepsilon \right]. \end{aligned}$$

Choose then

$$(6) \quad \varepsilon = \frac{q^2}{\dot{q}} \left( \frac{p}{q+p-2} \right)^p \frac{(q-1)p \|u\|_{(q+p-2)r/p}^{q+p-2}}{c_{r,p,d} C_s^p \|u\|_q^q} := \varepsilon_1 \frac{\|u\|_{(q+p-2)r/p}^{q+p-2}}{\|u\|_q^q}$$

where this is also meant to be the definition of  $\varepsilon_1$ , to be used later, so that

$$\frac{d}{ds} \log \|u\|_q \leq \frac{\dot{q}}{q^2} \left[ J(1, |u|^q) - J(q, |u|^{(q+p-2)r/p}) \right] - \frac{\dot{q} c_{r,p,d}}{q^2 p} \log \varepsilon.$$

We choose now  $r$  so that the content of the square bracket in the last formula vanishes identically. This happens precisely when

$$r = \frac{pq}{q+p-2}$$

but we have to check that such choice be compatible with the requested bound  $r \in (0, p^*)$ . This is where the range of  $p, q$  enters, first because the condition  $r > 0$  holds since  $p > 1$  and  $q \geq 1$ , and then because the condition  $r < p^*$  is easily seen

to be equivalent to  $q > d(2-p)/p := q_c$ , as assumed in the statement. Notice that, with this choice of  $r$ , one has:

$$(7) \quad c_{p,r,d} = \frac{pqd}{pq + d(p-2)},$$

a fact which will be used later on.

One thus have, recalling the definition of  $\varepsilon_1$  given in (6) and the choice of  $r$ , that

$$\begin{aligned} \frac{d}{ds} \log \|u\|_q &\leq -\frac{\dot{q}c_{r,p,d}}{q^2p} \left[ \log \varepsilon_1 + \log \left( \frac{\|u\|^{q+p-2}}{\|u\|_q^q} \right) \right] \\ &= -\frac{\dot{q}c_{r,p,d}}{q^2p} \log \varepsilon_1 - \frac{\dot{q}c_{r,p,d}(p-2)}{q^2p} \log \|u\|_q. \end{aligned}$$

Since  $\varepsilon_1$  does not depend on  $u$  but only on  $q, p, d$  and on the Sobolev constant, this is a closed differential inequality for  $\log \|u\|_q$  in which we still have a degree of freedom, namely the choice of  $q$ . We write down explicitly what we have found:

$$\begin{aligned} \frac{d}{ds} \log \|u\|_q &\leq -\frac{\dot{q}(p-2)d}{q[pq + d(p-2)]} \log \|u\|_q \\ &\quad - \frac{\dot{q}d}{q[pq + d(p-2)]} \log \left( \frac{p^p[pq + d(p-2)]q(q-1)}{C_s^p \dot{q}d(q+p-2)^p} \right). \end{aligned}$$

**Step 3: integration.** We make now use of the freedom of choice of  $q$ , setting  $q(s) := q_0 t / (t-s)$  for all  $s \in [0, t)$ , with  $q_0$  as in the statement. Then  $q(s) \rightarrow +\infty$  as  $s \uparrow t$ . Integration of the resulting differential inequality is of course easy but quite tedious and we only sketch the main points. First we rewrite it as

$$\dot{y}(s) + a(s)y + b(s) \leq 0,$$

where

$$\begin{aligned} y(s) &:= \log \|u(s)\|_{q(s)}, \quad a(s) := \frac{\dot{q}(s)(p-2)d}{q(s)[pq(s) + d(p-2)]} \\ b(s) &:= \frac{\dot{q}(s)d}{q(s)[pq(s) + d(p-2)]} \log \left( \frac{p^p[pq(s) + d(p-2)]q(s)(q(s)-1)}{C_s^p \dot{q}(s)d(q(s)+p-2)^p} \right). \end{aligned}$$

Therefore

$$y(s) \leq e^{-A(s)} \left( y(0) - \int_0^s b(\lambda e^{A(\lambda)}) d\lambda \right) := e^{-A(s)} [y(0) - B(s)].$$

We first compute  $A(s)$ . We have:

$$\begin{aligned} A(s) &= \int_0^s a(\lambda) d\lambda = \int_0^s \frac{\dot{q}(\lambda)(p-2)d}{q(\lambda)[pq(\lambda) + d(p-2)]} d\lambda \\ &= \int_{q_0}^{q(s)} \frac{d(p-2)}{\xi[p\xi + d(p-2)]} d\xi = \log \left( \frac{\xi}{p\xi + d(p-2)} \right) \Big|_{q_0}^{q(s)} \\ &= \log \left( \frac{q(s)([pq_0 + d(p-2)])}{q_0[pq(s) + d(p-2)]} \right), \end{aligned}$$

where we have used the change of variable  $\xi = q(\lambda)$ . Notice that

$$A(s) \xrightarrow{s \uparrow t} \log \left( \frac{pq_0 + d(p-2)}{pq_0} \right) := A(t)$$

and that this depends only on the fact that  $q(s) \rightarrow +\infty$  as  $s \uparrow t$ .

Inserting the explicit expression of  $A(s)$  into the definition of  $B(s)$  gives, after elementary algebraic manipulations:

$$\begin{aligned} B(s) &= \frac{d[pq_0 + d(p-2)]}{q_0} \log \left( \frac{p^p q_0 t}{C_s^p d} \right) \int_0^s \frac{\dot{q}(\lambda)}{[pq(\lambda) + d(p-2)]^2} d\lambda \\ &\quad + \frac{d[pq_0 + d(p-2)]}{q_0} \\ &\quad \times \int_0^s \frac{\dot{q}(\lambda)}{[pq(\lambda) + d(p-2)]^2} \log \left[ \frac{(q(\lambda) - 1)(pq(\lambda) + d(p-2))}{q(\lambda)(q(\lambda) + p - 2)^p} \right] d\lambda. \end{aligned}$$

It will suffice for our purposes to notice that the last integral is convergent and remains bounded, and independent of  $t$ , as  $s \uparrow t$ . In fact it coincides with

$$\int_{q_0}^{q(s)} \frac{1}{[p\xi + d(p-2)]^2} \log \left( \frac{(\xi - 1)(p\xi + d(p-2))}{\xi(\xi + p - 2)^p} \right) d\lambda.$$

The integral has no poles in the interval of integration because of the assumptions on  $q_0$  and its convergence as  $s \uparrow t$ , so that  $q(s) \rightarrow +\infty$ , is obvious. We conclude that

$$B(s) = \frac{d(pq_0 + d(p-2))}{q_0 p} \left( \frac{1}{pq_0 + d(p-2)} - \frac{1}{pq(s) + d(p-2)} \right) \log t + G(s),$$

where  $G(s)$  is independent of  $t$  and remains bounded as  $s \uparrow t$ , say  $G(s) \rightarrow \widehat{G}$  as  $s \uparrow t$ . We then have

$$\begin{aligned} y(s) &\leq \frac{q_0[pq(s) + d(p-2)]}{q(s)[pq_0 + d(p-2)]} [y(0) + G(s)] \\ &\quad - \frac{d(pq_0 + d(p-2))}{q_0 p} \left( \frac{1}{pq_0 + d(p-2)} - \frac{1}{pq(s) + d(p-2)} \right) \log t \Big]. \end{aligned}$$

Letting  $s \uparrow t$  we get, with the notation  $y(t) := \lim_{s \uparrow t} y(s)$ :

$$y(t) = \frac{pq_0}{pq_0 + d(p-2)} \left( y(0) + \widehat{G} - \frac{d}{pq_0 \log t} \right).$$

Recalling that  $y(s) = \log \|u(s)\|_{q(s)}$  we recognize, exponentiation both sides and proceeding as in [7], that this is an equivalent form of our statement, at least for essentially bounded data. Such assumption can then be removed exactly as in [7].

To prove the Corollary, just use the elementary inequality

$$\|u(t)\|_r \leq \|u(t)\|_\infty^{1-(q/r)} \|u(t)\|_q^{q/r}$$

the smoothing effect, and the contraction property.  $\square$

### 3. PROOF OF THEOREM 2

Take an initial datum  $u(0) \in W^{1,p}(M)$ . We consider the equality

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 = -\|\nabla u(t)\|_p^p.$$

Integrating it over time we get

$$\frac{1}{2} (\|u(t)\|_2^2 - \|u(0)\|_2^2) = -\int_0^t \|\nabla u(s)\|_p^p ds.$$

It is known (cf. [6]) that  $\|\nabla u(s)\|_p^p$  decreases in time, so that

$$\frac{1}{2} (\|u(t)\|_2^2 - \|u(0)\|_2^2) \geq -t \|\nabla u(0)\|_p^p.$$

The assumption on the range of  $p$  is easily shown to be equivalent to the condition  $q_c < 2$ , where  $q_c$  is the quantity defined in Theorem 1. We can therefore use inequality (3), which is a consequence of the smoothing effect (2) only, to write

$$\|u(t)\|_2 \leq C \frac{\|u(0)\|_{q_0}^{\gamma(2, q_0)}}{t^{\alpha(2, q_0)}}$$

provided  $q_0 \in (q_c, 2)$ . Combining the latter inequalities we get

$$C \frac{\|u(0)\|_{q_0}^{2\gamma(2, q_0)}}{t^{2\alpha(2, q_0)}} + 2t \|\nabla u(0)\|_p^p \geq \|u(0)\|_2^2.$$

The above inequality is of the form

$$(8) \quad f(t) := A t^{-2\alpha(2, q_0)} + B t \geq D$$

where

$$A = C \|u_0\|_{q_0}^{2\gamma(2, q_0)}, \quad B = 2 \|\nabla(u(0))\|_p^p, \quad D = \|u(0)\|_2^2.$$

The real function  $f$  has a unique minimum for positive  $t$ , namely

$$t = \hat{t} := \left( \frac{2\alpha(2, q)A}{B} \right)^{1/[1+2\alpha(2, q_0)]}$$

and one has

$$f(\hat{t}) = K A^{1/[1+2\alpha(2, q)]} B^{2\alpha(2, q_0)/[1+2\alpha(2, q_0)]},$$

$K$  being a positive numerical constant.

Insert the value  $t = \hat{t}$  into (8), and notice that this gives, once substituting the values of  $A$ ,  $B$  and  $D$ :

$$\|u(0)\|_2^2 \leq K \|u(0)\|_{q_0}^{2\gamma(2, q_0)/[1+2\alpha(2, q_0)]} \|\nabla(u(0))\|_p^{2p\alpha(2, q_0)/[1+2\alpha(2, q_0)]}$$

where again  $K$  is a numerical constant. This Gagliardo–Nirenberg inequality is in turn known to be equivalent to the Sobolev inequality (1), see [1].  $\square$

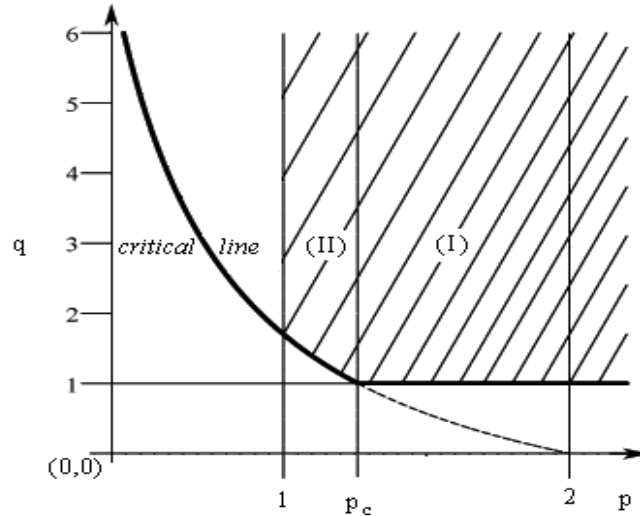
## Summary of the results

We conclude our paper by giving a graphic description of our results:

the plane  $(p, q)$  is split in various parts described below,  $p$  lying on the horizontal axis and  $q$  on the vertical axis.

**THE CRITICAL LINE:** the critical line  $q = d(2 - p)/p = q_c$  defines the critical value  $p_c = 2d/(d+1)$  when it intersects the horizontal line  $q = 1$ . The critical line characterizes the two regions (I) and (II), separating the fast and the degenerate diffusion zones from the very fast diffusion zone. We remark that the same curve explains the case of the Euclidean setting for the  $p$ -Laplacian without coefficients, and has first appeared in the monograph [11], where it is obtained by scaling techniques and with the help of explicit source-type or self-similar solutions; in the Riemannian setup, scaling techniques are useless, and, as far as we know, no explicit source type solution is known.

(I) **THE SUPERCRITICAL CASE :**  $p_c < p < 2$ . In this zone the smoothing effect



(2) holds, for any initial data belonging to  $L^{q_0}$ , for any  $q_0 \geq 1$ : this is the result of Theorem 1. Moreover in this zone the smoothing effect (2) is equivalent to the Sobolev inequality (1), and this is part of the result of Theorem 2. This equivalence holds in the degenerate range  $p > 2$  as well, by Theorem 2 and the results of [4] adapted to the present case. The equivalence is well-known to hold in the linear case  $p = 2$  also.

(II) THE SUBCRITICAL CASE (or Very Fast Diffusion):  $1 < p < p_c$ . In this zone the  $L^{q_0} - L^\infty$  smoothing effect (2) holds for any initial data belonging to  $L^{q_0}$ , for any  $q_0 \geq q_c = d(2 - p)/p$ : this is the result of Theorem 1. Theorem 2 moreover shows that a certain converse does hold in the interval  $p \in (p_s, p_c]$ .

#### REFERENCES

- [1] D. Bakry, T. Coulhon, M. Ledoux and L. Saloff-Coste, *Sobolev inequalities in disguise*, Indiana Univ. Math. J., **44** (1995), 1033–1074.
- [2] M. Bonforte, F. Cipriani and G. Grillo, *Ultracontractivity and approach to equilibrium for supercritical parabolic equations on riemannian manifolds*, Adv. Differential Equations, **8** (2003), 843–872.
- [3] M. Bonforte and G. Grillo, *Asymptotics of the porous media equation via Sobolev inequalities*, J. Funct. Anal., **225** (2005), 33–62.
- [4] M. Bonforte and G. Grillo, *Super and ultracontractive bounds for doubly nonlinear evolution equations*, Rev. Mat. Iberoamericana, **22** (2006), 111–129.
- [5] M. Bonforte, G. Grillo and J. L. Vazquez, *Fast Diffusion Flow on Manifolds of Nonpositive Curvature*, preprint 2006.
- [6] H. Brezis, “Operateurs Maximaux Monotones,” North-Holland Publishing Company, Amsterdam, London, 1973.
- [7] F. Cipriani, G. Grillo, *Uniform bounds for solutions to quasilinear parabolic equations*, J. Diff. Equations., **177** (2001), 209–234.
- [8] F. Cipriani, G. Grillo, *Nonlinear Dirichlet forms, nonlinear Markov semigroups, and applications to minimal surfaces*, J. reine angew. Math., **562** (2003), 201–235.
- [9] E. Dibenedetto “Degenerate Parabolic Equations,” Springer Verlag, New York, Berlin, 1993.
- [10] E. Hebey, “Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities,” American Mathematical Society, 2000.



- [11] J. L. Vazquez, "Smoothing and Decay Estimates for Nonlinear Diffusion Equations, Equations of Porous Medium Type," Oxford Lecture Series in Mathematics and Applications 33, Oxford University Press, New York, 2006.

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