Individual Excess Demands*

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Abstract

We characterize the excess demand function in the one-consumer model. We describe the relation between the direct and indirect utility functions, and we use it to derive a simple proof of the Debreu-Mantel-Sonnenschein theorem.

1 Introduction

In consumer theory, an individual demand function $x(p, y)$ is defined as the solution to a simple optimization problem: it maximizes some utility function under a linear budget constraint. The properties that stem from this characterization have been known for more than one century. Under standard smoothness assumptions, individual demand functions are fully characterized by three properties: (i) homogeneity, (ii) Walras Law, and (iii) the Slutsky conditions. In addition, one can define the indirect utility and expenditure functions. Again, it has been known for a long time that a one-to-one relationship exists between demand, direct utility, indirect utility and expenditure functions. Knowing any of these function is sufficient to uniquely recover preferences, hence the other three.

The results summarized above belong by now to the most standard presentation of consumer theory. Equally standard is the fact that, for many applications (among which general equilibrium theory), it can be helpful to consider excess demand functions instead of Marshallian demands. For any Marshallian demand $x(p, y)$, the excess demand $z(p)$

*We thank P. Reny for useful comments. Financial support from the NSF (grant SBR 9729559) is gratefully aknowledged. Errors are ours.

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is defined by $z(p) = x(p, p'\omega) - \omega$ where ω denotes the agent's initial endowment. A very natural question is whether the standard results on Marshallian demands have a counterpart for excess demands. Specifically, can one find necessary and sufficient condition for some arbitrary function $z(p)$ to be an individual excess demand? And does the one-to-one correspondence between the direct utility, indirect utility and demand functions extend to the excess demand framework?

Quite surprisingly, these very natural questions have not received an answer so far. While it is easy to derive necessary conditions on individual excess demand functions, their sufficiency has not been established, and the relationship between the direct utility, indirect utility and demand functions has not been investigated. Standard consumer theory thus exhibits a gap.

The goal of this paper is precisely to fill this gap. We first derive a set of necessary and sufficient conditions that fully characterize excess demand functions. Whenever a smooth function $z(p)$ satisfies these conditions, then it is possible to find an initial endowment and a direct utility function for which $z(p)$ is the excess demand. These conditions, somewhat unsurprisingly, are (i) homogeneity, (ii) Walras Law, and (iii) a variant of the Slutsky conditions; however, the proof of sufficiency (the so-called 'integrability' problem) cannot be transposed from the standard framework, and one has to find a different argument.

It turns out from our results that the one-to-one relationship between the direct utility, indirect utility and demand functions is lost in the excess demand context. For any excess demand function $z(p)$ and any initial endowment, there exist a *continuum* of different direct utility functions from which $z(p)$ can be derived. In a similar way, while it is always possible, starting from some direct utility function (and some initial endowment), to uniquely define the corresponding, indirect utility function, the converse is not true. To any indirect utility function can be associated a *continuum* of different excess demand functions $z(p)$. Each of these satisfy the necessary and sufficient conditions for integrability, hence can be associated with a continuum of direct utility functions.

Finally, our results have direct consequences for several problems in consumer theory and aggregation. In the paper, we consider the well-known Debreu-Mantel-Sonnenschein theorem on aggregate excess demand. The question, here, is whether it is possible, for any arbitrary, smooth function $Z(p)$ that satisfies homogeneity and the Walras Law, to find n individual excess demand functions $z_1(p), ..., z_n(p)$ such that $Z(p)$ decomposes into the sum of the $z_i(p)$. Surprisingly, the existing results do not rely on a formal characterization of individual excess demand functions. Our characterization of individual excess demand yields a *one-line* proof of the theorem. Not only is this proof short, it also allows a precise characterization of the degrees of freedom one has in constructing the individual utilities: in fact, the *indirect* utilities V^i can be chosen arbitrarily; the choice of the corresponding z_i then is unique.

Our result gives a better understanding of the fundamental difference between the two versions (market demand versus excess demand) of Sonnenschein's problem. It also suggests that the characterization of *individual* excess demand is indeed a difficult problem; as a matter of fact, it constitutes the core difficulty in the Debreu-Mantel-Sonnenschein literature.

2 Individual excess demand

2.1 The necessary conditions

Consider a standard consumer model. There are n goods, and the consumer is characterized by his/her utility function $U : \mathbb{R}^n_{++} \longrightarrow \mathbb{R}$, where $\mathbb{R}^n_{++}$ is the open positive orthant of R^n. The excess demand of the consumer is then defined as the solution $z(p)$ of the optimization problem[1]:

$$\begin{cases} \max_z U(z) \\ z \in \mathbb{R}^n_{++},\ p'z = \sum p_i z^i \leq 0 \end{cases} \qquad (\mathfrak{P})$$

We shall say that a C^k function U, with $k \geq 2$, is *standard* if:

- the gradient $D_z U(z)$ is non-zero everywhere, and the restriction of the second derivative $D^2_{zz} U(z)$ of $[D_z U(z)]^\perp$ is negative definite for every z; it follows that U is strictly quasi-concave.
- for every $p \in \mathbb{R}^n_{++}$, problem $(\mathfrak{P})$ has a solution $z\,(p) \in \mathbb{R}^n$ with $p'z\,(p) = 0$

If U is standard, it follows from the implicit function theorem that the excess demand function $z(p)$ is well-defined and C^{k-1} on $\mathbb{R}^n_{++}$. It is

[1] In more standard notations, $z\,(p)$ is defined as $x\,(p, p'\omega) - \omega$, where the Marshallian demand x solves

$$\max_x \tilde{U}(x)$$
$$p'x = p'\omega$$

Then U is defined from $\tilde{U}$ by

$$U\,(z) = \tilde{U}\,(z + \omega)$$

zero-homogeneous and satisfies the Walras Law:

$$z(\lambda p) = z(p) \quad \forall \lambda > 0, \ \forall p$$
$$p' z(p) = 0 \quad \forall p$$

and there is a Lagrange multiplier $\lambda(p)$, which is C^{k-1} and homogeneous of degree (-1) on $\mathbb{R}^n_{++}$, such that:

$$D_z U(z(p)) = \lambda(p)\, p \tag{1}$$

Define the indirect utility $V(p)$, for $p \in \mathbb{R}^n_{++}$, by

$$V(p) = \left\{ \max_z U(z) \mid z \in \mathbb{R}^n_{++}, \ p'z \leq 0 \right\} \tag{$\mathfrak{E}$}$$

If U is standard, then V is C^{k-1} in and satisfies $V(p) = U(z(p))$. In addition, $V(p)$ is zero-homogenous and quasi-convex. Note that V cannot be strictly quasi-convex because of homogenety (the level sets $\{V \leq a\}$ are cones). However, it can be proved that, if V is C^2, then $D^2_{pp}V(p)$ has rank $(n-1)$ and the restriction of $D^2_{pp}V(p)$ to $[\text{Span}\{p, D_pV(p)\}]^{\perp}$ is positive definite. Note that $p'D_pV(p) = 0$, by the Euler identity, so p and $D_pV(p)$ are orthogonal vectors; if none of them vanishes, their span is truly two-dimensional.

The envelope theorem applied to ($\mathfrak{E}$) gives:

$$D_pV(p) = -\lambda(p) z(p) \tag{2}$$

Differentiating with respect to p, one gets:

$$D^2_{pp}V = -\lambda D_p z - z.(D_p \lambda)' \tag{3}$$

It follows that the restriction of $D_p z$ to $[z(p)]^{\perp}$ is symmetric, and that its restriction to

$$[\text{Span}\{p, z(p)\}]^{\perp} = [\text{Span}\{p, D_pV(p)\}]^{\perp}$$

is negative definite for every p. Note that this condition is the exact equivalent, in the excess demand case, of Slutsky symmetry and negativeness for Marshallian demands. Also, if $\bar{p}$ is such that $z(\bar{p}) = 0$, then, by equation (3), $D_p z(\bar{p})$ is proportional to $D^2_{pp}V(\bar{p})$.

Note that D_pV is one-to-one. We have even better:

Lemma 1 *If $D_pV(p_1)$ and $D_pV(p_2)$ are collinear, then so are p_1 and p_2.*

Proof. If $D_pV(p_1)$ and $D_pV(p_2)$ are collinear, then so are $z(p_1)$ and $z(p_2)$ by relation (2), so that $p_1' z(p_2) = p_1' z(p_1) = 0$. Set $z(p_1) = \lambda_1 \bar{z}$ and $z(p_2) = \lambda_2 \bar{z}$. Since $z(p_1)$ solves problem ($\mathfrak{P}$) for $p = p_1$, we have $U(z(p_1)) \geq U(z(p_2))$. The converse inequality holds for the same reason, so $U(z(p_1)) = U(z(p_2))$. Since U is standard, the solution of problem ($\mathfrak{P}$) is unique, so $z(p_1)$ must be equal to $z(p_2)$, and p_1, which is collinear to $D_pU(z(p_1)) = D_pU(z(p_2))$ must be collinear to p_2. ■ ■

We summarize these results:

Proposition 2 (Necessary conditions) *If the C^k function $U(z)$ is standard, the indirect utility function $V : \mathbb{R}^n_{++} \longrightarrow \mathbb{R}$ and the excess demand function $z : \mathbb{R}^n_{++} \longrightarrow \mathbb{R}^n$ satisfy the following:*

(a) *V is C^{k-1}, quasi-convex, positively homogenous of degree zero. For every p, we have $D_pV(p) \neq 0$ and $D^2_{pp}V(p)$ has rank $(n-1)$; moreover, the restriction of $D^2_{pp}V(p)$ to $[\text{Span}\{p, D_pV(p)\}]^\perp$ is positive definite. If $D_pV(p_1)$ and $D_pV(p_2)$ are collinear, so are p_1 and p_2.*

(b) *z is C^{k-1} and positively homogenous of degree zero. The restriction of D_pz to $[z(p)]^\perp$ is symmetric, and its restriction to $[\text{Span}\{p, z(p)\}]^\perp$ is negative definite for every p.*

Note that the positivity condition on $D^2_{pp}V(p)$ and the negativity condition on D_pz are void if $n = 2$.

2.2 Local integrability

We now consider the converse. Take some C^2 functions $V(p)$ and $z(p)$ satisfying the necessary conditions. Is it possible to find a standard utility function U such that $z(p)$ solves ($\mathfrak{P}$) for all p , and $V(p)$ is the corresponding, indirect utility ? This is the standard 'integration' problem in consumer theory, expressed in the case of excess demand. It divides into two subproblems: the local one, where V and z are considered in a suitably small neighbourhood of a given point $\bar{p}$, and the global one, where V and z are considered on all of $\mathbb{R}^n$. In this section, we address the local problem, and we show that the answer is positive, provided $D_pV(\bar{p}) \neq 0$ and a nondegeneracy condition is met. The situation near a point $\bar{p}$ where $D_pV(\bar{p}) = 0$ is mathematically quite interesting, but will not be investigated in this paper.

Theorem 3 *Let $V(p)$ be a C^k function, $k \geq 3$, defined on some neighbourhood of $\bar{p}$ with $D_pV(\bar{p}) \neq 0$. Assume that, on that neigbourhood, $V(p)$ is quasi-convex, positively homogenous of degree zero, that $D^2_{pp}V(p)$ has rank $(n-1)$ and the restriction of $D^2_{pp}V(p)$ to $[\text{Span}\{p, D_pV(p)\}]^\perp$*

is positive definite. Then there is an open convex cone Γ containing $\bar{p}$ such that V is an indirect utility function on Γ. In fact, take any C^{k-1} function $\lambda(p) > 0$, homogeneous of degree (-1) on Γ, and define:

$$z(p) = -\frac{1}{\lambda(p)} D_p V(p) \tag{4}$$

Then there is a quasi-concave function $U(z)$, defined and C^{k-1} on a convex neighbourhood $\mathcal{N}$ of $z(\bar{p})$, with the following properties:

(1) *For every $z \in \mathcal{N}$, the restriction of $D^2_{zz}U(z)$ to $[D_zU(z)]^{\perp}$ is negative definite*

(2) *For every $p \in \Gamma$, we have:*

$$D_zU(z(p)) = \lambda(p)p, \ p'z(p) = 0 \tag{5}$$

$$V(p) = U(z(p)) = \left\{\max_z U(z) \mid p'z \leq 0, \ z \in \mathcal{N}\right\} \tag{6}$$

The proof will be given in the appendix.

We have similar results for excess demand:

Theorem 4 *Let $z(p)$ be a C^k map, $k \geq 2$, defined on some neighbourhood of $\bar{p}$ into $\mathbb{R}^n$ such that $z(\bar{p}) \neq 0$ and $D_p z(\bar{p})$ has rank $(n-1)$. Assume that, on that neigbourhood, z is homogeneous of degree zero, and that the restriction of $D_p z$ to $[p]^{\perp}$ is symmetric, and that its restriction to $[\mathrm{Span}\{p, z(p)\}]^{\perp}$ is negative definite. Then there is an open convex cone Γ containing $\bar{p}$ such that z is an excess demand function on Γ. In fact, there is a quasi-concave function $U(z)$, defined and C^k on a convex neighbourhood $\mathcal{N}$ of $z(\bar{p})$, with the following properties:*

(1) *For every $z \in \mathcal{N}$, the restriction of $D^2_{zz}U(z)$ to $[D_zU(z)]^{\perp}$ is negative definite*

(2) *For every $p \in \Gamma$, we have:*

$$D_zU(z(p)) = \lambda(p)p, \ p'z(p) = 0 \tag{7}$$

$$V(p) = U(z(p)) = \left\{\max_z U(z) \mid p'z \leq 0, \ z \in \mathcal{N}\right\} \tag{8}$$

The proof will be given in the appendix.

Several remarks are in order here. First, unlike the standard case of market demand, one has existence but not uniqueness. This follows from the fact that, while U is a function of n variables, V and z are functions of $n-1$ variables only (one dimension being lost because of homogeneity): information is lost in going from U to V and z, and cannot be recovered, in sharp constrast with the Marshallian case. Specifically:

- for each excess demand function $z(p)$, there exist a continuum of direct utility functions U for which z is the excess demand. The idea is that U is characterized only on the subset $\mathcal{M} \subset \mathbb{R}^n$ generated by $z(p)$ when p varies in $\Omega \subset S^{n-1}$, which is a submanifold of codimension 1 in $\mathbb{R}^n$. The function U and its derivatives are known only on $\mathcal{M}$, and must be extended outside this set;obviously, this extension can be made in many different ways.

- perhaps more surprisingly, to any homogenous, strictly quasi-convex indirect utility $V(p)$ can be associated a continuum of excess demand functions: just pick up some arbitrary, positive scalar function $\lambda(p)$, then $z(p) = -D_pV(p)/\lambda(p)$ is an excess demand.

It should also be noted that the proof cannot follow the usual path, based on duality. The standard approach, as in Hurwicz and Ozawa [5], relies on the expenditure function, which has no natural equivalent in the case of excess demand. Similarly, proofs based on indirect utilities (i.e., integration of Roy's identity) cannot be used here, precisely because the one-to-one correspondence between direct and indirect utilities does not hold in this context. Hence, although the statement of the result is quite similar to the Slutsky characterization of market demand, its proof relies on a completely different (and somewhat more difficult) approach.

3 Global integrability

We now address the global problem. Take a C^2 function $V : \mathbb{R}^n_{++} \longrightarrow \mathbb{R}$ satisfying condition (a). We associate with it a function $V^* : \mathbb{R}^n_{++} \longrightarrow \mathbb{R}$ defined as follows:

$$V^*(z) = \inf_p \left\{ V(p) \mid p \in \mathbb{R}^n_{++},\ p'z \geq 0 \right\} \qquad (\mathcal{P}^*)$$

The function $V^* : \mathbb{R}^n_{++} \longrightarrow \mathbb{R} \cup \{-\infty\}$ is quasi-concave, positively homogeneous of degree 0, and its domain, $\mathrm{dom} V^* = \{z \mid V^*(z) > -\infty\}$, is a convex set. The optimality condition in problem $(\mathcal{P}^*)$ is $D_pV(p) = \mu z$,with $\mu < 0$. If $\bar{z} = \mu D_pV(\bar{p})$, for some $\mu < 0$, then $\bar{p}'\bar{z} = 0$ by the Euler identity, and the minimum in problem $(\mathcal{P}^*)$ with $z = \bar{z}$ is attained at $p = \bar{p}$.

Introduce the cone:

$$C = \left\{ \mu D_pV(p) \mid \mu < 0,\ p \in \mathbb{R}^n_{++} \right\} \qquad (9)$$

It follows from the above that $C \subset \mathrm{dom} V^*$; if $z \in C$, then $z = \mu D_pV(p)$ for some $\mu < 0$ and $p \in \mathbb{R}^n_{++}$, and $V^*(z) = V(p)$.

Theorem 5 *Let $V : \mathbb{R}^n_{++} \longrightarrow \mathbb{R}$ be a C^2 function satisfying condition (a). Assume that $V(p)$ is quasi-convex, positively homogenous of degree zero, that $D^2_{pp}V(p)$ has rank $(n-1)$ and the restriction of $D^2_{pp}V(p)$ to $[\mathrm{Span}\{p, D_pV(p)\}]^{\perp}$ is positive definite. Given any compact subset $K \subset \mathbb{R}^n_{++}$ where D_pV does not vanish, then V is an indirect utility function on K. In fact, take any C^1 function $\lambda(p) > 0$, homogeneous of degree (-1) on a neighbourhood of K, and define:*

$$z(p) = -\frac{1}{\lambda(p)} D_pV(p) \tag{10}$$

$$\mathcal{M} = \{z(p) \mid p \in K\} \tag{11}$$

Then there is a quasi-concave function $U(z)$, defined and C^2 on a convex open set Ω containing $\mathcal{M}$, with the following properties:

(1) *U is C^2 on a neighbourhood $\mathcal{N}$ of $\mathcal{M}$, and for every $z \in \mathcal{N}$, the restriction of $D^2_{zz}U(z)$ to $[D_zU(z)]^{\perp}$ is negative definite*

(2) *For every $p \in K$, the point $z(p)$ is the unique solution of problem $(\mathcal{P})$, so that:*

$$D_zU(z(p)) = \lambda(p)p, \; p'z(p) = 0 \tag{12}$$

$$V(p) = U(z(p)) = \left\{\max_z U(z) \mid p'z \leq 0, \; z \in \mathbb{R}^n_{++}\right\} \tag{13}$$

4 The Sonnenschein-Mantel-Debreu theorem

Let $Z(p)$ be a C^{∞} map from some open cone Ω of $\mathbb{R}^n_+$ into $\mathbb{R}^n$, homogeneous of degree zero and satisfying the Walras law $p'Z(p) = 0$. The problem raised by Sonnenschein (see [**?**]) is the following: for any $k \geq n$, is it possible to find k excess demand functions $z_1(p), ..., z_k(p)$, such that:

$$Z(p) = z_1(p) + ... + z_k(p) \tag{14}$$

As it is well known, the answer to this question is positive (see [10], [6],[3]). We provide a short proof. Note that we require that Z be C^2, so that our result is more in the Mas-Colell line (see [7])

Theorem 6 *Assume $Z(p)$ is a C^2 map defined on a compact subset $K \subset ; \mathbb{R}^n_{++}$, homogeneous of degree zero and satisfying the Walras law $p'Z(p) = 0$. For any $k \geq n$, we can find k excess demand functions $z_1(p), ..., z_k(p)$, defined on K such that the decomposition (14) holds on K. The corresponding utility functions $U_1(z), ..., U_k(z)$ are quasi-concave, and each U_i is C^2 and strictly quasi-concave in a neighbourhood of $z_i(K)$.*

Proof. Normalize prices by imposing that $p'p = 1$, so that prices belong to $\mathbb{R}^n_{++} \cup S^{n-1}$. Since $p'Z(p) = 0$, we can think of $Z(p)$ as a tangent vector to S^{n-1} at p; the space of all such vectors (the tangent space T_pS^{n-1} to S^{n-1} at p) has dimension $(n-1)$. Pick k functions $V_1(p), ..., V_k(p)$, of class C^3, satisfying condition (a), and such that, at every $p \in K$, the convex hull of $DV_1(p), ..., DV_k(p)$ contains 0 in its interior; this is possible provided $k \geq n$. Using a smooth partition of unity, we can then find C^2 functions $\mu_1 < 0, ..., \mu_k < 0$ on K such that

$$Z(p) = \sum_{i=1}^{k} \mu_i(p) D_pV_i(p)$$

Define $z_i(p) = -D_pV_i(p)/\lambda_i(p)$ where $\lambda_i(p) = 1/\mu_i(p)$. By Theorem 4, z_i is an individual excess demand, and the corresponding utility function $U_i(z)$ satisfy conditions (1) and (2). ■

4.1 Comments

A few comments are in order at this stage. Note, first, that in general only $(n-1)$ functions are required to span the positive orthant of the $(n-1)$-dimensional sphere S^{n-1}. It is only when $Z(\overline{p}) = 0$ that n functions are required, a well-known fact. Secondly, an interesting by-product of this proof is that it describes the degrees of freedom available in the choice of individual utility functions. Basically, *indirect* utilities can be picked up almost arbitrarily (the only constraint being that $Z(p)$ belongs to their *negative* span). However, once they have been chosen, then only one set of $\lambda_i(p)$ is acceptable; in other words, among the continuum of individual excess demands that can be associated with V_i, only one can be used. Then any of the direct utilities corresponding to the z_i can be adopted.

Finally, this argument sheds some light on the differences between the two problems raised by Sonnenschein, i.e. aggregate excess demand versus aggregate market demand. In the market demand case, a given function $X(p)$ must be decomposed as the sum of n individual Marshallian demands:

$$X(p) = x^1(p) + ... + x^n(p)$$

where $x_i(p)$ is the solution of

$$V^i(p) = \max U^i(x^i)$$
$$p.x^i = 1, \quad x^i \geq 0$$

In particular, the envelope theorem gives $D_pV^i(p) = -\alpha_i.x^i(p)$, where

α_i is the Lagrange multiplier. It follows that :

$$X(p) = -\frac{1}{\alpha_1(p)} D_pV^1(p) - ... - \frac{1}{\alpha_n(p)} D_pV^n(p)$$
$$= -\mu_1(p) D_pV^1(p) - ... - \mu_n(p) D_pV^n(p)$$

Again, $X(p)$ must be a linear combination of n gradients. The difference, however, is that while the coefficients μ_i could be freely chosen in the excess demand case, in the market demand context they have to satisfy the budget constraint, which implies that:

$$\mu_i(p) = \frac{1}{p.D_pV^i(p)} \quad \forall i$$

These additional restrictions considerably increase the difficulty of the problem, which now amounts to a system of nonlinear partial differential equations:

$$\frac{D_pV^1(p)}{p'.D_pV^1(p)} + ... + \frac{D_pV^n(p)}{p'.D_pV^n(p)} = -X(p) \tag{15a}$$

This system has been solved in [1], in the case when the rigth-hand side $X(p)$ is real analytic. The case when $X(p)$ is smooth, C^∞ say, but not real analytic, remains open. It is also an open question whether solutions to (15a) can be found globally: the mathematical technique used in [1] constructs solutions in some neighbourhood of any given point $\bar{p}$.

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A Proof of Theorem 3

A.1 Step 1: Constructing U_A.

Define $z(p)$ from (4). It is C^{k-1}, and by the Euler identity, we have:

$$p'z = -\frac{1}{\lambda}p'D_pV = 0$$

so that z satisfies the Walras law. In addition, $z(p)$ is zero-homogeneous of degree zero, so that

$$(D_pz)'p = 0 \tag{16}$$

and

$$p'D_pz + z' = 0 \tag{17}$$

Without loss of generality, assume that $\|\bar{p}\| = 1$. It follows from equations (16) and (17) that $p \in \text{Ker } D_pz(p)$ and $z(p) \notin \text{Range } D_pz(p)$, and it follows from relation (3) that $D_pz(p)$ has rank $(n-1)$.Consider the map ϕ defined on $S^{n-1} \times \mathbb{R}$ by $\phi(p,t) = t\ z(p)$. Then $D_{p,t}\phi(p)$ has rank n,and it follows from the inverse function theorem that ϕ is a C^{k-1} diffeomorphism from some neighbourhood $\mathcal{N} \times [1-\varepsilon,\ 1+\varepsilon]$ of $(\overline{p}, 1)$ in $S^{n-1} \times \mathbb{R}$ onto a neigbourhood of $z(\overline{p})$ in R^n. In particular, the image

$$\mathcal{M} = \{z(p) \mid p \in \mathcal{N}\} = \{\phi(p,1) \mid p \in \mathcal{N}\}$$

is diffeomorphic to $\mathcal{N}$.

We want a function $U(z)$ such that $U(z(p)) = V(p)$ and $D_zU(z(p)) = \lambda(p)p$. In other words, we are prescribing the values of U and its first derivatives on $\mathcal{M}$. There is a compatibility condition to be satisfied, namely that we get the same value for $D_pU(z(p))$. This yields:

$$D_pV = \lambda(D_pz)'p$$

Substituting relations (17) and (4), we find an identity, so the compatibility condition holds. Now look at the problem in the (p,t)-coordinates, that is, consider the function $\tilde{U} = U \circ \phi$: we know $\tilde{U}(p,1) = V(p)$ and the partial derivative $\frac{\partial \tilde{U}}{\partial t}(p,1)$, and we want to define $\tilde{U}(p,t)$ in a neighbourhood of $t = 1$. This can be done in many ways, for instance by setting:

$$\tilde{U}_A(p,t) = V(p) + (t-1)\,A(p,t) \tag{18}$$

where $A : \mathbb{R}^n \to \mathbb{R}$ is any C^{k-1} function such that $A(p,1) = \frac{\partial \tilde{U}}{\partial t}(p,1)$. In particular, if we evaluate the situation at $\bar{p}$, we find that the matrix of second derivatives takes the form:

$$D^2\tilde{U}_A(\bar{p},1) = \begin{matrix} D^2_{pp}\tilde{U}_A & \begin{matrix} a_{1,n} \\ \cdots \\ a_{n-1,n} \end{matrix} \\ a_{n,1} \dots a_{n,n-1} & a_{nn} \end{matrix} \tag{19}$$

where the n terms $a_{n,i} = a_{i,n}, 1 \leq i \leq n$, can be chosen freely. We now show that if A, and hence the $a_{i,n}$, are appropriately chosen, the extension $U_A = \tilde{U}_A \circ \phi^{-1}$ is quasi-concave.

A.2 Step 2: U_A is strictly quasi-concave

A.2.1 Substep 2a: investigating $D^2_{zz}U_A(z)$ on the tangent space $T_z\mathcal{M}$

We have $U = U_A$ on $\mathcal{M}$. Differentiating the relation $D_zU_A(z(p)) = \lambda(p)p$, we get:

$$D^2_{zz}U_A(z)\,D_pz = pD_p\lambda + \lambda I$$

and hence, for any $\xi \in T_pS^{n-1}$:

$$(D_pz\;\xi)'\,D^2_{zz}U_A(z)\,(D_pz\;\xi) = (p'D_pz\;\xi)\,(D_p\lambda\;\xi) + \lambda\,(\xi'D_pz\;\xi)$$

Now, by relation (17), $p'D_pz\;\xi = -z'\xi$. So the first term on the right vanishes if $D_pz\;\xi \in [p]^\perp$, or, equivalently, if $\xi \in [z(p)]^\perp$. We are then left with the second term, which we compute:

$$\begin{aligned}\xi'D_pz\;\xi &= \xi'\left(-\frac{1}{\lambda}D^2_{pp}V + \frac{1}{\lambda^2}D_pV\,(D_p\lambda)'\right)\xi \\ &= -\frac{1}{\lambda}\xi'D^2_{pp}V\xi + \frac{1}{\lambda^2}\,(\xi', D_pV)\,(\xi, D_p\lambda) \\ &= -\frac{1}{\lambda}\xi'D^2_{pp}V\xi - \frac{1}{\lambda}\,(\xi', z)\,(\xi, D_p\lambda)\end{aligned}$$

and the last term on the right vanishes again if $z'\xi = 0$. So we are left with the following:

$$\xi \in T_pS^{n-1},\; z'\xi = 0 \Longrightarrow (D_pz\;\xi)'\,D^2_{zz}U_A(z)\,(D_pz\;\xi) = -\frac{1}{\lambda}\xi'D^2_{pp}V\xi$$

and the quadratic form on the right-hand side is negative definite on $[\mathrm{Span}\,\{p, z(p)\}]^\perp$, by condition (a). Since $\xi \in T_pS^{n-1}$, we have $(\xi, p) = 0$, and so:

$$0 \neq \xi \in T_pS^{n-1},\; z'\xi = 0 \Longrightarrow (D_pz\;\xi)'\,D^2_{zz}U_A(z)\,(D_pz\;\xi) < 0$$

Set $\eta = D_p z\ \xi$. Note that $D_z U_A(z)\,\eta = D_z U_A D_p z\ \xi = D_p V \xi = -\lambda z' \xi$, so that $z'\xi = 0$ if and only if $D_z U_A(z)\,\eta = 0$. Rewrite the preceding result in terms of η :

$$0 \neq \eta \in T_p\mathcal{M},\ D_z U_A(z)\,\eta = 0 \Longrightarrow \eta' D^2_{zz} U_A(z)\,\eta < 0$$

which is the desired result.

A.3 Substep 2b: investigating $D^2_{zz}U_A(z)$ on the whole space

From $\tilde{U}_A = U_A \circ \varphi$ we get, at $z = \varphi(p,t)$:

$$(D_z\varphi)'\, D^2_{zz} U_A(z)\, D_z\varphi = D^2 \tilde{U}_A(p,t) - D_z U_A(z)\, D^2_{zz}\varphi$$

Note that the last term, which involves only first derivatives of U_A, does not depend on the choice of A, provided it is evaluated at a point $z \in \mathcal{M}$; this is because of the relations $U_A(z(p)) = V(p)$ and $D_z U(z(p)) = \lambda(p)p$. Evaluating everything at $\bar{z} = \phi(\bar{p},1)$, the above equation takes the form:

$$(D_z\varphi)'\, D^2_{zz} U_A(\bar{z})\, D_z\varphi = D^2 \tilde{U}_A(\bar{p},1) + M \tag{20}$$

where M is a fixed operator and $D^2\tilde{U}_A(\bar{p},1)$ is given by formula (19). We have shown, in the preceding substep, that the restriction of $D^2_{zz}U_A(z)$ to $T_z\mathcal{M} \cap [p]^{\perp}$ is negative definite. This means that the restriction of $D^2\tilde{U}_A(\bar{p},1) + M$ to $E \cap F$ is negative definite, where $E = [D_z\varphi(\bar{p},1)]^{-1}(T_z\mathcal{M})$ and $F = [D_z\varphi(\bar{p},1)]^{-1}\left([p]^{\perp}\right)$. From the definition of φ, it follows that E is simply the hyperplane $t = 0$. Going back to formula (19), we find that:

$$D^2\tilde{U}_A(\bar{p},1) + M = \begin{array}{cc} Q & \begin{array}{c} a_{1,n}+m_{1,n} \\ \cdots \\ a_{n-1,n}+m_{n-1,n} \end{array} \\ a_{1,n}+m_{1,n} \ldots a_{n-1,n}+m_{n-1,n} & a_{n,n}+m_{n,n} \end{array}$$

where the restriction of Q to $F \cap E$ is negative definite. It is then easy to pick the $a_{i,n}$ so that the restriction of $D^2\tilde{U}_A(\bar{p},1) + M$ to F is negative definite. Going back to equation (20), we find that the restriction of $D^2_{zz}U_A(\bar{z})$ to $[\bar{p}]^{\perp}$ is negative definite. Since $[\bar{p}]^{\perp}$ is just $[D_p U_A(\bar{z})]^{\perp}$, this means that U_A is strictly quasi-concave..■

B Proof of Theorem 4

Proof. Introduce the differential one-form

$$\omega = \sum z^i(p)dp_i$$

Since the restriction of $D_p z$ to $[z(p)]^{\perp}$ is symmetric, we must have $\omega \wedge d\omega = 0$. By the Darboux theorem (see [?]), there are C^{k+1} functions $\mu(p)$ and $V(p)$, defined in some neighbourhood of $\bar{p}$, such that $\omega = -\mu dV$. By the convex Darboux theorem (see [4] and the references therein; the proof has to be adapted to the homogeneous case, which is straigthforward), we can take the functions μ and V to be homogeneous, with $\mu > 0$ and $D^2_{pp}V$ positive definite on $[p]^{\perp}$.

We can therefore apply theorem 3, taking $\lambda(p) = 1/\mu(p)$. We find a C^k utility function U satisfying (a) and (b), the corresponding excess demand function being precisely $z(p)$.■

C Proof of Theorem 5

Choose the function λ. By theorem 3, for every point $z \in \mathcal{M}$ there is a neighbourhood $\mathcal{N}_z \subset C$, the cone defined by (9) and a C^2 function $U_z : \mathcal{N}_z \to \mathbb{R}$ satisfying conditions (1) and (2) in that neighbourhood. Since $\mathcal{M}$ is compact, it can be covered by finitely many such neigbourhoods, say $\mathcal{N}_1, ..., \mathcal{N}_m$, associated with the functions $U_1, ..., U_m$. Take a smooth partition of unity associated with this covering, that is, m functions $\varphi_1, ..., \varphi_m$ of class C^2 such that $\varphi_i \geq 0$, $\varphi_i(z) = 0$ if $z \notin U_i$, and $\sum \varphi_i = 1$. Set $U = \sum \varphi_i U_i$; the function U then is defined over $\cup \mathcal{N}_i$, which is a neighbourhood of $\mathcal{M}$, and satisfies conditions (1) and (2) in that neighbourhood.

Looking at formula (9), which defines the cone C, we find that $\mathcal{M} \subset C$. If $z \in \mathcal{M}$, then $z = z(p)$ for some p, and $U(z) = V(p) = V^*(z)$.

$$z \in \mathcal{M} \Longrightarrow U(z) = V^*(z) \tag{21}$$

On the other hand, if $z \in \cup \mathcal{N}_i$, but $z \notin \mathcal{M}$, we must have $U(z) < U(z(p)) = V(p)$ whenever $p'z = 0$, otherwise z would be the minimizer of problem $(\mathcal{P})$. Since $V^*(z) \leq V(p)$ whenever $p'z = 0$, it follows that

$$\begin{aligned} &z \in \mathcal{M} \Longrightarrow U(z) = V^*(z) \\ &z \in \cup \mathcal{N}_i\ ,\ z \notin \mathcal{M},\ p'z = 0 \Longrightarrow U(z) < V^*(z) \end{aligned}$$

For the latter, note that since $z \in \cup \mathcal{N}_i$, the value $U(z)$ is well-defined. Since $z \in C$, there exists some $\lambda > 0$ such that $\lambda z \in \mathcal{M}$, and since $z \notin \mathcal{M}$, we must have $\lambda \neq 1$. We have $\lambda z = z(p)$ for some p, and hence:

$$\begin{aligned} V^*(z) &= V^*(z(p)) \text{ for } V^* \text{ is 0-homogeneous} \\ U(z) &< V(p) \text{ for } p'z = \frac{1}{\lambda} p' z(p) = 0 \end{aligned}$$

Since $V^*(z(p)) = V(p)$, the conclusion follows.

Lemma 7 *There is a neighbourhood $\mathcal{N}$ of $\mathcal{M}$, with $\mathcal{N} \subset \cup \mathcal{N}_i$, such that the restriction of U to $\mathcal{N}$ is quasi-concave in the following sense:*

$$z = \alpha z_1 + (1-\alpha) z_2,\ 0 < \alpha < 1,\ z_1 \in \mathcal{N},\ z_2 \in \mathcal{N}, \Longrightarrow\ U(z) \geq \min\{U(z_1), U(z_2)\} \tag{22}$$

Proof. Argue by contradiction. If condition (22) does not hold, there must be three sequences $z_1^k, z^k, z_2^k, k \to \infty$, with the following property:

$$z_1^k \to z_1 \in \mathcal{M},\ z^k \to z \in \mathcal{M},\ z_2^k \to z_2 \in \mathcal{M} \tag{23}$$

$$z^k = (1-\alpha^k) z_1^k + \alpha^k z_2^k,\ 0 < \alpha^k < 1 \tag{24}$$

$$U\left(z^k\right) < \min\left\{U\left(z_1^k\right), U\left(z_2^k\right)\right\} \tag{25}$$

There are three possible cases: ■

1. The three points z_1, z, z_2 are distinct. Then we get $U(z) \leq \min\{U(z_1), U(z_2)\}$ in the limit. Since $U = V^*$ on $\mathcal{M}$, and V^* is quasi-concave, the reverse inequality holds as well, and we get $V^*(z) = \min\{V^*(z_1), V^*(z_2)\}$. Without loss of generality, assume $V^*(z_1) \leq V^*(z_2)$, so that $V^*(z) = V^*(z_1)$. From the quasi-concavity of V^* it follows that $V^*(z_1 + t(z - z_1)) = V^*(z_1)$ for all $t \in [0, 1]$. Differentiating once, we get $D_z U(z_1)(z_2 - z_1) = 0$, so $(z_2 - z_1) \in [D_z U(z_1)]^{\perp}$. Differentiating twice, we get $(z_2 - z_1)' D_{zz}^2 U(z_1)(z_2 - z_1) = 0$. But $D_{zz}^2 U(z_1)$ is positive definite on $[D_z U(z_1)]^{\perp}$, and we have a contradiction.

2. Two of the points z_1, z, z_2 coincide. The middle one, z, must be one of them; without loss of generality, say the other one is z_1. So $z = z_1 \neq z_2$. Condition (25) then yields

   $$U(z_1) \leq U(z_2), \tag{26}$$

   and condition (24) yields $\alpha^k \to 0$. Substituting into condition (25), we get

   $$D_z U(z_1)(z_2 - z_1) \leq 0, \tag{27}$$

   If the strict inequality holds in (27), then there is a point z_3 between z_1 and z_2 where $U(z_3) < U(z_1) \leq U(z_2)$, contradicting the quasi-concavity of U. If $D_z U(z_1)(z_2 - z_1) = 0$, then, since $D_{zz}^2 U(z_1)$ is negative definite on $[D_z U(z_1)]^{\perp}$, again there will be a point z_3 between z_1 and z_2 where $U(z_3) < U(z_1) \leq U(z_2)$, and again we derive a contradiction.

3. The three points coincide: $z_1 = z = z_2$. But then, for k large enough, the points $z_1^k,\ z^k,\ z_2^k$ enter a convex neighbourhood of z where U is strictly quasi-concave, contradicting relation (25)■

We now extend U to a quasi-concave function $\bar{U}$, defined on a convex set containing $\mathcal{N}$, and coinciding with U on $\mathcal{N}$. This is an easy matter: for any number a, consider the set

$$\Omega(a) = [z \in \mathcal{N} \mid U(z) > a] \subset \mathbb{R}^n_{++}$$

and let $\mathrm{co}[\Omega(a)]$ be its convex hull. It is an open and convex subset of $\mathbb{R}^n_{++}$. If $a > b$, then $\mathrm{co}[\Omega(a)] \subset \mathrm{co}[\Omega(b)]$. Set $\Omega = \cup \mathrm{co}[\Omega(a)]$, which is a convex open set, and define a function $\bar{U} : \nleq \to \mathbb{R}$ by:

$$\bar{U}(z) = \sup\{a \mid z \in \mathrm{co}[\Omega(a)]\}$$

It is quasi-concave by construction. The Lemma implies that if $z \in \mathcal{N}$ and $z \in \mathrm{co}[\Omega(a)]$, then $z \in \Omega(a)$, so $\bar{U}(z) = U(z)$. This concludes the proof.■