On the maximum independent set problem in subclasses of subcubic graphs*

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Abstract. It is known that the maximum independent set problem is NP-complete for subcubic graphs, i.e. graphs of vertex degree at most 3. Moreover, the problem is NP-complete for H-free subcubic graphs whenever H contains a connected component which is not a tree with at most 3 leaves. We show that if every connected component of H is a tree with at most 3 leaves and at most 7 vertices, then the problem can be solved for H-free subcubic graphs in polynomial time.

Keywords: Independent set; Polynomial-time algorithm; Subcubic graph

1 Introduction

In a graph, an *independent set* is a subset of vertices no two of which are adjacent. The maximum independent set problem consists in finding in a graph an independent set of maximum cardinality. This problem is generally NP-complete [3]. Moreover, it remains NP-complete even under substantial restriction, for instance, for planar graphs or subcubic graphs (i.e. graphs of vertex degree at most 3). In the present paper, we focus on subcubic graphs in the attempt to identify further restrictions which may lead to polynomial-time algorithms to solve the problem. One such restriction is known to be a bound on the chordality, i.e. on the length of a largest chordless cycle. Graphs of bounded degree and bounded chordality have bounded tree-width [2], and hence the problem can be solved in polynomial time for such graphs. In terms of forbidden induced subgraphs bounded chordality means excluding large chordless cycles, i.e. cycles $C_k, C_{k+1}, \ldots$ for a constant k. More generally, it was recently shown in [6] that excluding large apples (all definitions can be found in the end of the introduction) together with bounded degree leads to a polynomial-time algorithm to solve

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the problem. In both cases (i.e. for graphs without large cycles and for graphs without large apples) the restrictions are obtained by excluding infinitely many graphs. In the present paper, we study subclasses of subcubic graphs obtained by excluding *finitely many* graphs. A necessary condition for polynomial-time solvability of the problem in such classes was given in [1] and can be stated as follows: the maximum independent set problem can be solved in polynomial time in the class of graphs defined by a *finite* set Z of forbidden induced subgraphs *only if* Z contains a graph every connected component of which is a tree with at most three leaves. In other words, for polynomial-time solvability of the problem we must exclude a graph every connected component of which has the form $S_{i,j,k}$ represented in Figure 1. Whether this condition is sufficient for polynomial-time solvability of the problem is a big open question.

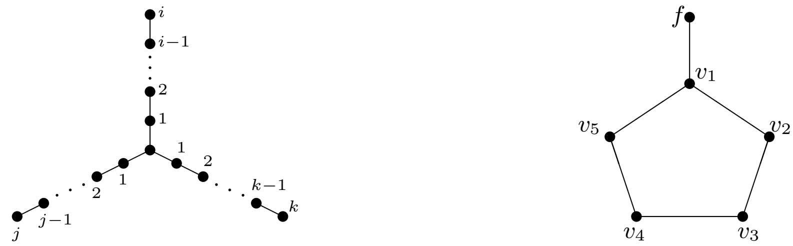

Fig. 1. Graphs $S_{i,j,k}$ (left) and A_5 (right)

Without the restriction on vertex degree, polynomial-time solvability of the problem in classes of $S_{i,j,k}$-free graphs was shown only for very small values of i, j, k. In particular, the problem can be solved for $S_{1,1,1}$-free (claw-free) graphs [11], $S_{1,1,2}$-free (fork-free) graphs [5], and $S_{0,1,1}+S_{0,1,1}$-free ($2P_3$-free) graphs [7]. The complexity of the problem in $S_{0,2,2}$-free (P_5-free) graphs remains an open problem in spite of the multiple partial results on this topic (see e.g. [4, 8–10]).

With the restriction on vertex degree, we can do much better. In particular, we can solve the problem for $S_{1,j,k}$-free graphs of bounded degree for any j and k, because by excluding $S_{1,j,k}$ we exclude large apples. However, nothing is known about classes of $S_{i,j,k}$-free graphs of bounded degree where all three indices i, j, k are at least 2. To make a progress in this direction, we consider best possible restrictions of this type, i.e. we study $S_{2,2,2}$-free graphs of vertex degree at most 3, and show that the problem is solvable in polynomial time in this class. More generally, we show that the problem is polynomial-time solvable in the class of H-free subcubic graphs, where H is a graph every connected component of which is isomorphic to $S_{2,2,2}$ or to $S_{1,j,k}$.

The organization of the paper is as follows. In the rest of this section, we introduce basic definitions and notations. In Section 2 we prove a number of preliminary results. Finally, in Section 3 we present a solution.

All graphs in this paper are simple, i.e. undirected, without loops and multiple edges. The vertex set and the edge set of a graph G are denoted by $V(G)$ and

$E(G)$, respectively. For a vertex $v \in V(G)$, we denote by $N(v)$ the neighborhood of v, i.e., the set of vertices adjacent to v, and by $N[v]$ the closed neighbourhood of v, i.e. $N[v] = N(v) \cup \{v\}$. For $v, w \in V(G)$, we set $N[v, w] = N[v] \cup N[w]$. The *degree* of v is the number of its neighbors, i.e., $d(v) = |N(v)|$. The subgraph of G induced by a set $U \subseteq V(G)$ is obtained from G by deleting the vertices outside of U and is denoted $G[U]$. If no induced subgraph of G is isomorphic to a graph H, then we say that G is H-free. Otherwise we say that G contains H. If G contains H, we denote by $[H]$ the subgraph of G induced by the vertices of H and all their neighbours. As usual, by C_p we denote a chordless cycle of length p. Also, an *apple* A_p, $p \geq 4$, is a graph consisting of a cycle C_p and a vertex f which has exactly one neighbour on the cycle. We call vertex f the *stem* of the apple. See Figure 1 for the apple A_5. The size of a maximum independent set in G is called the *independence number* of G and is denoted $\alpha(G)$.

2 Preliminary results

We start by quoting the following result from [6].

Theorem 1. *For any positive integers d and p, the maximum independent set problem is polynomial-time solvable in the class of $(A_p, A_{p+1}, \ldots)$-free graphs with maximum vertex degree at most d.*

We solve the maximum independent set problem for $S_{2,2,2}$-free subcubic graphs by reducing it to subcubic graphs without large apples.

Throughout the paper we let G be an $S_{2,2,2}$-free subcubic graph and $K \geq 1$ a large fixed integer. If G contains no apple A_p with $p \geq K$, then the problem can be solved for G by Theorem 1. Therefore, from now on we assume that G contains an induced apple A_p with $p \geq K$ formed by a chordless cycle $C = C_p$ of length p and a stem f. We denote the vertices of C by $v_1, \ldots, v_p$ (listed along the cycle) and assume without loss of generality that the only neighbour of f on C is v_1 (see Figure 1 for an illustration).

If v_1 is the only neighbour of f in G, then the deletion of v_1 together with f reduces the independence number of G by exactly 1. This can be easily seen and also is a special case of a more general reduction described in Section 2.1. The deletion of f and v_1 destroys the apple A_p. The idea of our algorithm is to destroy all large apples by means of other simple reductions that change the independence number by a constant. Before we describe the reductions in Section 2.1, let us first characterize the local structure of G in the case when the stem f has a neighbor different from v_1.

Lemma 1. *If f has a neighbor g different from v_1, then g has at least one neighbor on C and the neighborhood of g on C is of one of the 8 types represented in Figure 2.*

Proof. First observe that g must have a neighbor among $\{v_{p-1}, v_p, v_2, v_3\}$, since otherwise we obtain an induced $S_{2,2,2}$. If g has only 1 neighbor on C, then clearly we obtain configuration (1) or (2).

Now assume that g has two neighbors on C. Suppose first that g is adjacent neither to v_2 nor to v_p. Then g must be adjacent to at least one of v_{p-1}, v_3. Without loss of generality, we may assume that g is adjacent to v_{p-1} and denote the third neighbor of g by v_j. If $2 < j < p-3$, then we clearly obtain an induced $S_{2,2,2}$ centered at g. Otherwise, we obtain configuration (3) or (4).

Now assume g is adjacent to one of v_2, v_p, say to v_p, and again denote the third neighbor of g by v_j. If $j \in \{p-2, p-1\}$, then we obtain configuration (5) or (6). If $j \in \{2, 3\}$, then we obtain configuration (7) or (8). If $3 < j < p-2$, then G contains an $S_{2,2,2}$ induced by $\{v_{j-2}, v_{j-1}, v_j, v_{j+1}, v_{j+2}, g, f\}$. □

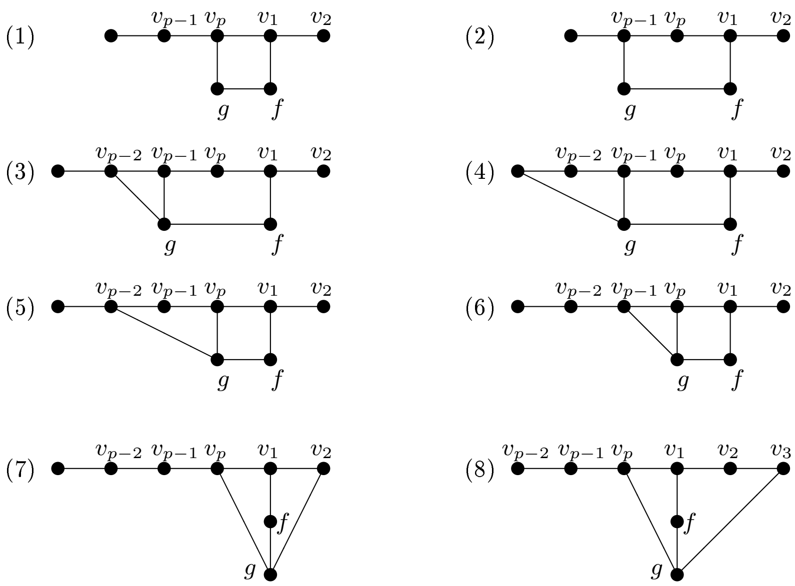

Fig. 2. $A_p + g$

2.1 Graph reductions

***H*-subgraph reduction** Let H be an induced subgraph of G.

Lemma 2. *If* $\alpha(H) = \alpha([H])$, *then* $\alpha(G - [H]) = \alpha(G) - \alpha(H)$.

Proof. Since any independent set of G contains at most $\alpha([H])$ vertices in $[H]$, we know that $\alpha(G - [H]) \geq \alpha(G) - \alpha([H])$. Now let S be an independent set in $G - [H]$ and A an independent set of size $\alpha(H)$ in H. Then $S \cup A$ is an independent set in G and hence $\alpha(G) \geq \alpha(G - [H]) + \alpha(H)$. Combining the two inequalities together with $\alpha(H) = \alpha([H])$, we conclude that $\alpha(G - [H]) = \alpha(G) - \alpha(H)$. □

The deletion of $[H]$ in the case when $\alpha(H) = \alpha([H])$ will be called the *H-subgraph reduction*. For instance, if a vertex v has degree 1, then the deletion of v together with its only neighbour is the H-subgraph reduction with $H = \{v\}$.

***Φ*-reduction** Let us denote by Φ the graph represented on the left of Figure 3. The transformation replacing Φ by Φ' as shown in Figure 3 will be called *Φ-reduction.*

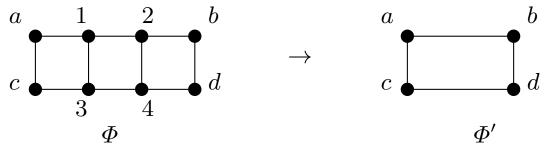

Fig. 3. Φ-reduction

Lemma 3. *By applying the Φ-reduction to an $S_{2,2,2}$-free subcubic graph G, we obtain an $S_{2,2,2}$-free subcubic graph G' such that $\alpha(G') = \alpha(G) - 2$.*

Proof. Let S be an independent set in G. Clearly it contains at most two vertices in $\{a, b, c, d\}$ and at most two vertices in $\{1, 2, 3, 4\}$. Denote $X = S \cap \{1, 2, 3, 4\}$. If the intersection $S \cap \{a, b, c, d\}$ contains at most one vertex or one of the pairs $\{a, d\}$, $\{b, c\}$, then $S - X$ is an independent set in G' of size at least $\alpha(G) - 2$. If $S \cap \{a, b, c, d\} = \{a, b\}$, then X contains at most one vertex and hence $S-(X \cup \{b\})$ is an independent set in G' of size at least $\alpha(G)-2$. Therefore, $\alpha(G') \geq \alpha(G) - 2$.

Now let S' be an independent set in G'. Then the intersection $S' \cap \{a, b, c, d\}$ contains at most two vertices. If $S' \cap \{a, b, c, d\} = \{a, d\}$, then $S' \cup \{2, 3\}$ is an independent set of size $\alpha(G') + 2$ in G. Similarly, if $S' \cap \{a, b, c, d\}$ contains at most one vertex, then G contains an independent set of size at least $\alpha(G') + 2$. Therefore, $\alpha(G) \geq \alpha(G') + 2$. Combining the two inequalities, we conclude that $\alpha(G') = \alpha(G) - 2$.

Now let us show that G' is an $S_{2,2,2}$-free subcubic graph. The fact that G' is subcubic is obvious. Assume to the contrary that it contains an induced subgraph H isomorphic to $S_{2,2,2}$. If H contains none of the edges ab and cd, then clearly H is also an induced $S_{2,2,2}$ in G, which is impossible. If S contains both edges ab and cd, then it contains $C_4 = (a, b, c, d)$, which is impossible either. Therefore, H has exactly one of the two edges, say ab. If vertex b has degree 1 in H, then by replacing b by vertex 1 we obtain an induced $S_{2,2,2}$ in G. By symmetry, a also is not a vertex of degree 1 in H. Therefore, we may assume, without loss of generality, that a has degree 3 and b has degree 2 in H. Let us denote by x the only neighbour of b in H. Then $(H - \{b, x\}) \cup \{1, 2\}$ is an induced $S_{2,2,2}$ in G. This contradiction completes the proof. □

***AB*-reduction** The AB-reduction deals with two graphs A and B represented in Figure 4. We assume that the vertices v_i belong to the cycle $C = C_p$, and the vertices p_j are outside of C.

Lemma 4. *If G contains an induced subgraph isomorphic to A, then*

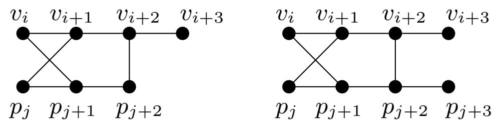

Fig. 4. Induced subgraphs A (left) and B (right)

- *either A can be extended to an induced subgraph of G isomorphic to B in which case p_{j+2} can be deleted without changing $\alpha(G)$*
- *or the deletion of $N[v_i] \cup N[p_j]$ reduces the independence number by 2.*

Proof. Assume first that A can be extended to an induced B (by adding vertex p_{j+3}). Consider an independent set S containing vertex p_{j+2}. Then S contains neither p_{j+1} nor p_{j+3} nor v_{i+2}. If neither p_j nor v_i belongs to S, then p_{j+2} can be replaced by p_{j+1} in S. Now assume, without loss of generality, that v_i belongs to S. Then $v_{i+1} \notin S$ and therefore we may assume that $v_{i+3} \in S$, since otherwise p_{j+2} can be replaced by v_{i+2} in S. If p_{j+3} has one more neighbour x in S (different from p_{j+2}), then vertices $v_i, v_{i+2}, v_{i+3}, p_{j+1}, p_{j+2}, p_{j+3}$ and x induce an $S_{2,2,2}$ in G (because the 3 endpoints are in S and the internal vertices have degree 3 in A). Therefore, we conclude that p_{j+2} is the only neighbour of p_{j+3} in S, in which case p_{j+2} can be replaced by p_{j+3} in S. Thus, for any independent S in G containing vertex p_{j+2}, there is an independent set of size $|S|$ which does not contain p_{j+2}. Therefore, the deletion of p_{j+2} does not change the independence number of G.

Now let us assume that A cannot be extended to B. Clearly, every independent set S in $G - N[v_i, p_j]$ can be extended to an independent set of size $|S| + 2$ in G by adding to S vertices v_i and p_j. Therefore, $\alpha(G) \geq \alpha(G - N[v_i, p_j]) + 2$.

Conversely, consider an independent set S in G. If it contains at most 2 vertices in $N[v_i, p_j]$, then by deleting these vertices from S we obtain an independent set of size at least $|S| - 2$ in $G - N[c_i, p_j]$.

Suppose now that S contains more than 2 vertices in $N[v_i, p_j]$. Let us show that in this case it must contain exactly three vertices in $N[v_i, p_j]$, two of which are v_{i+1} and p_{j+1}. Indeed, $N[v_i, p_j]$ contains at most 6 vertices: v_{i-1}, v_i, v_{i+1}, p_j, p_{j+1} and possibly some vertex x. Moreover, if x exists, then it is adjacent to v_{i-1}, since otherwise an $S_{2,2,2}$ arises induced either by vertices $x, p_j, p_{j+1}, p_{j+2}, v_{i+2}$, v_{i-1}, v_i (if p_{j+2} is not adjacent to v_{i-1}) or by vertices $p_j, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}$, v_{i-1}, p_{j+2} (if p_{j+2} is adjacent to v_{i-1}). Therefore, S cannot contain more than three vertices in $N[v_i, p_j]$, and if it contains tree vertices, then two of them are v_{i+1} and p_{j+1}. As a result, S contains neither v_{i+2} nor p_{j+2}. If each of v_{i+2} and p_{j+2} has one more neighbour in S (different from v_{i+1} and p_{j+1}), then A can be extended to B, which contradicts our assumption. Therefore, we may assume without loss of generality that p_{j+1} is the only neighbour of p_{j+2} in S. In this case, the deletion from $N[v_i, p_j]$ of the three vertices of S and adding to it vertex p_{j+2} results in an independent set of size $|S| - 2$ in $G - N[v_i, p_j]$.

Therefore, $\alpha(G - N[v_i, p_j]) \geq \alpha(G) - 2$. Combining with the inverse inequality, we conclude that $\alpha(G - N[v_i, p_j]) = \alpha(G) - 2$. □

Other reductions Two other reductions that will be helpful in the proof are the following.

- The A^*-*reduction* applies to an induced A^* (Figure 5) and consists in deleting vertex p_{j+2}.
- The *House-reduction* applies to an induced *House* (Figure 5) and consists in deleting the vertices of the triangle $v_{i+2}, v_{i+3}, p_{j+2}$.

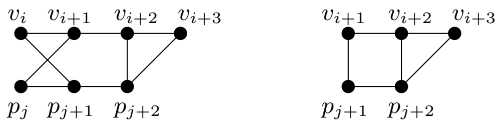

Fig. 5. Induced subgraphs A^* (left) and *House* (right)

Lemma 5. *The A^*-reduction does not change the independence number, and the House-reduction reduces the independence number by exactly 1.*

Proof. Assume G contains an induced A^* and let S be an independent set containing p_{j+2}. If S does not contain v_{i+1}, then p_{j+2} can be replaced by v_{i+2}, and if S contains v_{i+1}, then p_{j+2} can be replaced by p_{j+1}. Therefore, G has an independent set of size $|S|$ which does not contain p_{j+2} and hence the deletion of p_{j+2} does not change the independence number.

Assume G contains an induced *House* and let S be a maximum independent set in G. Then obviously at most one vertex of the triangle $v_{i+2}, v_{i+3}, p_{j+2}$ belongs to S. On the other hand, S must contain at least one vertex of this triangle. Indeed, if none of the three vertices belong to S, then each of them must have a neighbour in S (else S is not maximum), but then both v_{i+1} and p_{j+1} belong to S, which is impossible. Therefore, every maximum independent set contains exactly one vertex of the triangle, and hence the deletion of the triangle reduces the independence number by exactly 1. □

3 Solving the problem

In the subgraph of G induced by the vertices having at least one neighbor on $C = C_p$, every vertex has degree at most 2 and hence every connected component in this subgraph is either a path or a cycle. Let F be the component of this subgraph containing the stem f. In what follows we analyze all possible cases for F and show that in each case the apple A_p can be destroyed by means of graph reductions described above or by other simple reductions.

Lemma 6. *If F is a cycle, then A_p can be destroyed by graph reductions that change the independence number by a constant.*

Proof. If F is a triangle, then, according to Lemma 1, the neighbors of F in C are three consecutive vertices of C. In this case, F together with two consecutive vertices of C form a House and hence the deletion of F reduces the independence number of G by exactly one.

Assume F is a cycle of length 4 induced by vertices f_1, f_2, f_3, f_4. With the help of Lemma 1 it is not difficult to see that the neighbors of F in C must be consecutive vertices, say $v_i, \ldots, v_{i+3}$, and the only possible configuration, up to symmetry, is this: v_i is a neighbor of f_1, v_{i+1} is a neighbor of f_2, v_{i+2} is a neighbor of f_4, v_{i+3} is a neighbor of f_3. In this case, the deletion of vertex v_{i+1} does not change the independence number of G. To show this, consider an independent set S containing vertex v_{i+1}. Then S does not contain $2, v_i, v_{i+2}$. If $f_4 \in S$, then $f_1, f_3 \notin S$, in which case v_{i+1} can be replaced by f_2 in S. So, assume $f_4 \notin S$. If $f_3 \notin S$, then we can assume that $v_{i+3} \in S$ (else v_{i+1} can be replaced by f_3 in S), in which case v_{i+1}, v_{i+3} can be replaced by v_{i+2}, f_3. So, assume $f_3 \in S$, and hence $v_{i+3} \notin S$. But now v_{i+1} can be replaced by v_{i+2} in S. This proves that for every independent set S containing v_{i+1}, there is an independent set of the same size that does not contain v_{i+1}. Therefore, the deletion of v_{i+1} does not change the independence number of G.

Assume F is a cycle of length 5 induced by vertices f_1, f_2, f_3, f_4, f_5. With the help of Lemma 1 it is not difficult to verify that the neighbors of F in C must be consecutive vertices, say $v_i, \ldots, v_{i+4}$, and the only possible configuration, up to symmetry, is this this: f_1 is adjacent to v_i, f_2 is adjacent to v_{i+1}, f_3 is adjacent to v_{i+3}, f_4 is adjacent to v_{i+4}, f_5 is adjacent to v_{i+2}. But then the vertices $f_2, f_3, f_4, f_5, v_{i+2}, v_{i+4}, v_{i+5}$ induce an $S_{2,2,2}$.

If F is a cycle of length more than 5, then an induced $S_{2,2,2}$ can be easily found. □

Lemma 7. *If F is a path with at least 5 vertices, then A_p can be destroyed by graph reductions that change the independence number by a constant.*

Proof. Assume F has at least 5 vertices $f_1, \ldots, f_5$. Denote the neighbour of f_3 on C by v_i. Assume v_{i-1} has a neighbour in $\{f_1, f_5\}$, say f_1 (up to symmetry). By Lemma 1, f_2 is adjacent either to v_{i-2} or v_{i+1}.

Let first f_2 be adjacent to v_{i+1}. Then either f_1 is not adjacent to v_{i-2}, in which case the vertices $v_{i-2}, \ldots, v_{i+1}, f_1, f_2, f_3$ induce an A, or f_1 is adjacent to v_{i-2}, in which case f_4 is adjacent to v_{i+2} (by Lemma 1) and hence the vertices $v_i, \ldots, v_{i+3}, f_2, f_3, f_4$ induce an A. In either case, we can apply Lemma 4.

Suppose now that f_2 is adjacent to v_{i-2}. Then f_1 is not adjacent to v_{i+1}, since otherwise f_4 is adjacent to v_{i+2} (by Lemma 1), in which case the vertices $v_{i+1}, \ldots, v_{i+4}, f_1, f_3, f_4$ induce an $S_{2,2,2}$. As a result, vertices $v_{i-2}, \ldots, v_{i+1}$, f_1, f_2, f_3 induce an A and we can apply Lemma 4.

The above discussion shows that v_{i-1} has no neighbour in $\{f_1, f_5\}$. By symmetry, v_{i+1} has no neighbour in $\{f_1, f_5\}$. Then each of v_{i-1} and v_{i+1} has a neighbour in $\{f_2, f_4\}$, since otherwise $f_1, \ldots, f_5, v_i$ together with v_{i-1} or with

v_{i+1} induce an $S_{2,2,2}$. Up to symmetry, we may assume that v_{i-1} is adjacent to f_2, while v_{i+1} is adjacent to f_4.

If f_1 is adjacent to v_{i-2} or f_5 is adjacent to v_{i+2}, then an induced Φ arises, in which case we can apply the Φ-reduction. Therefore, we can assume that f_1 is adjacent to v_{i-3}, while f_5 is adjacent to v_{i+3}.

We may assume that vertex v_{i-2} has no neighbour x different from v_{i-3}, v_{i-1}, since otherwise x must be adjacent to f_1 (else vertices $x, v_{i-2}, v_{i-1}, v_i, v_{i+1}, f_1, f_2$ induce an $S_{2,2,2}$), in which case $v_{i-3}, \ldots, v_i, x, f_1, f_2$ induce an A and we can apply the AB-reduction. Similarly, we may assume that vertex f_1 has no neighbour x different from v_{i-3}, f_2. But then $d(f_1) = d(v_{i-2}) = 2$ and we can apply the H-subgraph reduction with $H = \{v_{i-2}, f_1\}$. □

Lemma 8. *If F is a path with 4 vertices, then A_p can be destroyed by graph reductions that change the independence number by a constant.*

Proof. Let F be a path (f_1, f_2, f_3, f_4). Without loss of generality we assume that f_2 is adjacent to v_i and f_3 to v_j with $j > i$. By Lemma 1, $j = i+1$ or $j = i+2$.

Case $j = i+1$. Assume f_1 is adjacent to v_{i+2}. Then vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}, f_1, f_2, f_3$ induce either the graph A (if f_1 is not adjacent to v_{i+3}) or the graph A^* (if f_1 is adjacent to v_{i+3}), in which case we can apply either Lemma 4 or Lemma 5. Therefore, we may assume that f_1 is not adjacent to v_{i+2}, and by symmetry, f_4 is not adjacent to v_{i-1}. Then by Lemma 1, f_1 must have a neighbour in $\{v_{i-2}, v_{i-1}\}$ and f_4 must have a neighbour in $\{v_{i+2}, v_{i+3}\}$.

Assume that f_4 is adjacent to v_{i+3}. If v_{i+2} has a neighbour x outside of the cycle C, then x is not adjacent to f_4 (else F has more than 4 vertices) and hence $v_{i-1}, v_i, v_{i+1}, v_{i+2}, x, f_3, f_4$ induce an $S_{2,2,2}$. Therefore, the degree of v_{i+2} in G is 2. Similarly, the degree of f_4 in G is two. But now we can apply the H-subgraph reduction with $H = \{v_{i+2}, f_4\}$. This allows us to assume that f_4 is not adjacent to v_{i+3}, and by symmetry, f_1 is not adjacent to v_{i-2}. But then f_1 is adjacent to v_{i-1} and f_4 is adjacent to v_{i+2}, in which case we can apply the Φ-reduction to the subgraph of G induced by $v_{i-1}, v_i, v_{i+1}, v_{i+2}, f_1, f_2, f_3, f_4$.

Case $j = i+2$. If f_1 or f_4 is adjacent to v_{i+1}, then an induced graph A arises, in which case we can apply Lemma 4. Then f_1 must be adjacent to v_{i-1}, since otherwise it adjacent to v_{i-2} (by Lemma 1), in which case vertices $v_{i-2}, f_1, f_2, f_3, f_4, v_i, v_{i+1}$ induce an $S_{2,2,2}$. By symmetry, f_4 is adjacent to v_{i+3}.

If f_1 is adjacent to v_{i-2}, then we can apply the *House*-reduction to the subgraph of G induced by $v_{i-2}, v_{i-1}, v_i, f_1, f_2$, and if f_1 is adjacent to v_{i-3}, then vertices $v_{i-3}, f_1, f_2, f_3, f_4, v_i, v_{i+1}$ induce an $S_{2,2,2}$. Therefore, we may assume by Lemma 1 that f_1 has degree 2 in G. By symmetry, f_4 has has degree 2. Also, to avoid an induced $S_{2,2,2}$, we conclude that v_{i+1} has degree 2. But now we apply the H-subgraph reduction with $H = \{f_1, v_i, v_{i+2}, f_4\}$, which reduces the independence number of G by 4. □

Lemma 9. *If F is a path with 3 vertices, then A_p can be destroyed by graph reductions that change the independence number by a constant.*

Proof. Assume F is a path (f_1, f_2, f_3). Without loss of generality let f_2 be adjacent to v_1. Since G is $S_{2,2,2}$-free, each of f_1 and f_3 must have at least one neighbor in $\{v_{p-1}, v_p, v_2, v_3\}$. Denote $L = \{v_{p-1}, v_p\}$ and $R = \{v_2, v_3\}$.

Case (a): f_1 and f_3 have both a neighbor in R. Due to the symmetry, we may assume without loss of generality that f_1 is adjacent to v_2, while f_3 is adjacent to v_3. Then we may further assume that f_1 is adjacent to v_4, since otherwise vertices $v_1, v_2, v_3, v_4, f_1, f_2, f_3$ induced either an A (if f_3 is not adjacent to v_4) or an A^* (if f_3 is adjacent to v_4), in which case we can apply either Lemma 4 or Lemma 5. But now the deletion of f_3 does not change the independence number of G. Indeed, let S be an independent set containing f_3. If $f_1 \in S$, then f_3 can be replaced by v_3. If $f_1 \notin S$, then we can assume that $v_1 \in S$ (else f_3 can be replaced by f_2), in which case f_3, v_1 can be replaced by f_2, v_2.

The above discussion allows us to assume, without loss of generality, that f_1 has no neighbor in R, while f_3 has no neighbor in L.

Case (b): f_3 is adjacent to v_3. Then we may assume that f_3 is not adjacent to v_2, since otherwise we can apply the *House*-reduction to the subgraph of G induced by v_1, v_2, v_3, f_3, f_2. Let us show that in this case

- *the degree of v_2 is 2.* Assume to the contrary v_2 has a third neighbour x. Then x is not adjacent to v_{p-1}, since otherwise G contains an $S_{2,2,2}$ induced either by $v_{p-1}, x, v_2, v_1, f_2, v_3, v_4$ (if x is not adjacent to v_4) or by $v_{p-2}, v_{p-1}, x, v_2, v_1, v_4, v_5$ (if x is adjacent to v_4). This implies that x is adjacent to v_p, since otherwise $x, v_2, v_1, f_2, f_3, v_p, v_{p-1}$ induced an $S_{2,2,2}$. As a result, f_1 is adjacent to v_{p-1}. Due to the degree restriction, x may have at most one neighbour in $\{v_{p-3}, v_{p-2}, v_4, v_5\}$. By symmetry, we may assume without loss of generality that x has no neighbour in $\{v_4, v_5\}$. Also, f_3 has no neighbour in $\{v_4, v_5\}$, since otherwise this neighbour together with $v_{p-1}, f_1, f_2, f_3, v_1, v_2$ would induce an $S_{2,2,2}$. But now $x, v_2, v_3, v_4, v_5, f_3, f_2$ induce an $S_{2,2,2}$. This contradiction complete the proof of the claim.

If f_3 also has degree two, then we can apply the H-subgraph reduction with $H = \{v_3, f_3\}$. Therefore, may assume that f_3 has one more neighbour, which must be, by Lemma 1, either v_4 or v_5. If f_3 is adjacent to f_5, then $f_1, f_2, f_3, v_5, v_6, v_3, v_2$ induce an $S_{2,2,2}$. Therefore, f_3 is adjacent to v_4. But now v_3 can be deleted without changing the independence number. Indeed, let S be an independent set containing v_3. If S does not contain v_1, then v_3 can be replaced by v_2, and if S contains v_1, then v_1, v_3 can be replaced by v_2, f_3.

Cases (a) and (b) reduce the analysis to the situation when f_1 is adjacent to v_p and non-adjacent to v_{p-1}, while f_3 is adjacent to v_2 and non-adjacent to v_3. If f_3 is adjacent to v_4, then vertices $v_p, v_1, v_2, v_3, v_4, f_1, f_2, f_3$ induced the graph Φ, in which case we can apply Lemma 3. Therefore, we can assume by Lemma 1 that the degree of f_3 is 2, and similarly the degree of f_1 is 2. But now we can apply the H-subgraph reduction with $H = \{f_1, v_1, f_3\}$, which reduces the independence number of G by 3. □

Lemma 10. *If F is a path with 2 vertices, then A_p can be destroyed by graph reductions that change the independence number by a constant.*

Proof. If F is a path with 2 vertices, we deal with the eight cases represented in Figure 2. It is easy to see that in cases (1) and (7), every maximum independent set must contain exactly one of f, g and thus by deleting f, g we reduce the independence number by exactly 1.

In case (5), the deletion of f, g also reduces the independence number by exactly 1. Indeed, let S be a maximum independent set containing neither f nor g. Since S is maximum it must contain v_1, v_{p-2} and hence it does not contain v_p, v_{p-1}. But then $(S \setminus \{v_1\}) \cup \{v_p, f\}$ is an independent set larger than S, contradicting the choice of S. Therefore, every maximum independent set contains exactly one of f and g and hence $\alpha(G - \{f, g\}) = \alpha(G) - 1$.

In case (2), the deletion of the set $X = \{v_{p-1}, v_p, v_1, f, g\}$ reduces the independence number of the graph by exactly 2. Indeed, any independent set of G contains at most two vertices in X, and hence $\alpha(G - X) \geq \alpha(G) - 2$. Assume now that S is a maximum independent set in $G - X$. If $v_2 \notin S$, then $S \cup \{v_1, g\}$ is an independent set in G of size $\alpha(G - X) + 2$. Now assume $v_2 \in S$. By symmetry, $v_{p-2} \in S$. Assume v_p has a neighbour x in S. Then x is adjacent neither to v_{p-2} nor to v_2, as all three vertices belong to S. Also, x cannot be adjacent to both v_{p-3} and v_3, since otherwise an induced $S_{2,2,2}$ can be easily found. But if x is not adjacent, say, to v_3, then $x, v_p, v_1, v_2, v_3, f, g$ induce an $S_{2,2,2}$. This contradiction shows that v_p has no neighbours in S. Therefore, $S \cup \{v_p, f\}$ is an independent set in G of size $\alpha(G - X) + 2$, and hence $\alpha(G) \geq \alpha(G - X) + 2$. Combining the two inequalities, we conclude that $\alpha(G - X) = \alpha(G) - 2$.

In case (3), we may delete g without changing the independence number, because in any independent set S containing g, vertex g can be replaced either by v_{p-1} (if S does not contain v_p) or by f (if S contains v_p). In case (6), we apply the $House$-reduction.

In cases (4) and (8), we find another large apple A' whose stem f' belongs to a path F' with at least 3 vertices. In case (4), A' is induced by the cycle $v_1, \ldots, v_{p-3}, g, f$ with stem $f' = v_{p-1}$, and in case (8) the apple is induced by the cycle $v_3, \ldots, v_p, g$ with stem $f' = v_1$. In both cases, the situation can be handled by one of the previous lemmas. □

Theorem 2. *Let H be a graph every connected component of which is isomorphic either to $S_{2,2,2}$ or to $S_{1,j,k}$. The maximum independent set problem can be solved for H-free graphs of maximum vertex degree at most 3 in polynomial time.*

Proof. First, we show how to solve the problem in the case when $H = S_{2,2,2}$. Let $G = (V, E)$ be an $S_{2,2,2}$-free subcubic graph and let K be a large fixed constant. We start by checking if G contains an apple A_p with $p \geq K$. To this end, we detect every induced $S_{1,k,k}$ with $k = K/2$, which can be done in time n^K. If G is $S_{1,k,k}$-free, then it is obviously A_p-free for each $p \geq K$. Assume a copy of $S_{1,k,k}$ has been detected and let x, y be the two vertices of this copy at distance k from the center of $S_{1,k,k}$. We delete from G all vertices of $V(S_{1,k,k}) - \{x, y\}$ and all their neighbours, except x and y, and determine if in the resulting graph there is a path connecting x to y. It is not difficult to see that this procedure can be implemented in polynomial time.

Assume G contains an induced apple A_p with $p \geq K$. If the stem of the apple has degree 1 in G, we delete it together with its only neighbour, which destroys the apple and reduces the independence number of G by exactly one. If the stem has degree more than 1, we apply one of the lemmas of Section 3 to destroy A_p and reduce the independence number of G. It is not difficult to see that all the reductions used in the lemmas can be implemented in polynomial time.

Thus in polynomial time we reduce the problem to a graph G' which does not contain any apple A_p with $p \geq K$, and then we find a maximum independent set in G' with the help of Theorem 1. This also shows that in polynomial time we can compute $\alpha(G)$, since we know the difference between $\alpha(G)$ and $\alpha(G')$. To find a maximum independent set in G, we take an arbitrary vertex $v \in V(G)$. If $\alpha(G - v) = \alpha(G)$, then there is a maximum independent set in G that does not contain v and hence v ignored (deleted). Otherwise, v belongs to every maximum independent set in G and hence it must be included in the solution. Therefore, in polynomial time we can find a maximum independent set in G. This completes the proof of the theorem in the case when $H = S_{2,2,2}$.

By Theorem 1 we also know how to solve the problem in the case when $H = S_{1,j,k}$. Now we assume that H contains $s > 1$ connected components. Denote by S any of the components of H and let H' be the graph obtained from H by deleting S. Consider an H-free graph G. If G does not contain a copy of S, the problem can be solved for G by the first part of the proof. So, assume G contains a copy of S. By deleting from G the vertices of $[S]$ we obtain a graph G' which is H'-free and hence the problem can be solved for G' by induction on s. The number of vertices in $[S]$ is bounded by a constant independent of $|V(G)|$ (since $|V(S)| < |V(H)|$ and every vertex of S has at most three neighbours in G), and hence the problem can be solved for G in polynomial time as well, which can be easily seen by induction on the number of vertices in $[S]$. □

4 Conclusion

Unless $P = NP$, the maximum independent set problem can be solved in polynomial time for H-free subcubic graphs *only if* every connected component of H has the form $S_{i,j,k}$ represented in Figure 1. Whether this condition is sufficient for polynomial-time solvability of the problem is a challenging open question. In this paper, we contributed to this topic by solving the problem in the case when every connected component of H is isomorphic either to $S_{2,2,2}$ or to $S_{1,j,k}$. Our poof also shows that, in order to answer the above question, one can be restricted to H-free subcubic graphs where H is connected. In other words, one can consider $S_{i,j,k}$-free, or more generally, $S_{k,k,k}$-free subcubic graphs. We believe that the answer is positive for all values of k and hope that our solution for $k = 2$ can base a foundation for algorithms for larger values of k.

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