LONG PERSUASION GAMES

FRANÇOISE FORGES
FRÉDÉRIC KOESSLER

CESIFO WORKING PAPER NO. 1669
CATEGORY 9: INDUSTRIAL ORGANISATION
FEBRUARY 2006

CESifo *Working Paper No. 1669*

LONG PERSUASION GAMES

Abstract

This paper characterizes geometrically the set of all Nash equilibrium payoffs achievable with unmediated communication in persuasion games, i.e., games with an informed expert and an uninformed decisionmaker in which the expert's information is certifiable. The first equilibrium characterization is provided for unilateral persuasion games, and the second for multistage, bilateral persuasion games. As in Aumann and Hart (2003), we use the concepts of diconvexification and dimartingale. A leading example illustrates both geometric characterizations and shows how the expert, whatever his type, can increase his equilibrium payoff compared to all equilibria of the unilateral persuasion game by delaying information certification.

JEL Code: C72, D82.

Keywords: cheap talk, communication, diconvexification, dimartingale, disclosure of certifiable information, jointly controlled lotteries, long conversation, persuasion, verifiable types.

Françoise Forges
University of Paris-Dauphine
CEREMADE
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16
France
Francoise.Forges@dauphine.fr

Frédéric Koessler
University of Cergy-Pontoise
THEMA
33 Boulevard du Port
95011 Cergy-Pontoise
France
Frederic.Koessler@eco.u-cergy.fr

February 1, 2006
We thank Sergiu Hart and seminar participants at U. Paris-Dauphine (Microeconomic Workshop), U. Paris 1 (CIDWorkshop), HEC Paris (Finance and Economics Seminar), U. Cergy-Pontoise (Microeconomic Workshop), U. Toulouse 1 (Economic Theory Seminar), Hebrew U. Jerusalem (Center for Rationality, Game Theory Seminar), U. Tel Aviv (Game Theory and Mathematical Economic Research Seminar), U. Haifa (Technion, Game Theory Seminar), Paris Game Theory Seminar (IHP), CORE, the 16th Summer Festival on Game Theory at Stony Brook, the Workshop on Stochastic Methods in Game Theory at Erice, U. Saint-Etienne (CREUSET), U. Copenhagen (Theory Seminar), the Workshop on Strategic Communication and Networks at U. Valencia, and UCL for useful comments. Financial support from an ACI grant by the French Ministry of Research is gratefully acknowledged. This research was partly carried out while the second author was a fellow at the Institute for Advanced Studies, Hebrew University of Jerusalem, Israel.

1 Introduction

As is now well known in the literature on cheap talk games (i.e., games with costless, non-binding, and unmediated communication), repeated communication generally allows to reach outcomes that cannot be implemented with unilateral or single-period communication, even if only one player is privately informed (see Aumann and Hart, 2003, Forges, 1984, 1990a, Krishna and Morgan, 2004, and Simon, 2002). In this paper we study this feature in "sender-receiver" communication games with partially verifiable types, also called persuasion games, in which the informed player (the expert, or "sender") has the ability to voluntarily certify partial or full information to the uninformed decisionmaker (the "receiver"). We characterize the set of all Nash equilibrium payoffs achievable with unmediated communication, by allowing players to talk for many periods. At each stage of this communication phase, the sender can certify part of his information.

This possibility of certifying information, in addition to make cheap talk claims, is justified by many concrete interactive decision situations. For example, players may present physical proofs such as documents, observable characteristics of a product, endowments or costs. Alternatively, in economic or legal interactions there may be labels, penalties for perjury, false advertising and warranty violations, or accounting principles that allow agents to submit substantive evidence of their information. Interesting phenomena similar to those obtained in the cheap talk case arise in games with strategic information certification. We show that several bilateral communication stages and delayed information certification allow to convey substantive information and lead to equilibrium outcomes that are not achievable when only one signalling stage is permitted. A leading example is analyzed in Section 2.

Our study is closely related to Aumann and Hart (2003) who characterized Nash equilibrium payoffs of long cheap talk games, i.e., the subset of communication equilibrium payoffs (Forges, 1986, 1990b; Myerson, 1982, 1986) that use only plain conversation. A communication equilibrium is a Nash equilibrium of an extension of the game allowing the players to communicate for several periods, with the help of a mediator, before they make their decisions. Here, we characterize the analog of that subset for certification equilibria (Forges and Koessler, 2005). A certification equilibrium is defined as a communication equilibrium, except that each player can also transmit reports from a type-dependent set, i.e., can send certified information into the communication system.

Our general model, presented in Section 3, is a one-side incomplete information game with an expert (the informed player) and a decision maker (the uninformed player). A common prior probability distribution first selects the expert's type in a finite set. The decision maker chooses his action without observing the expert's type. However, before the action phase, but after the expert learns his type, the players are able to directly communicate with each other. The payoff of each player only depends on the expert's type and on the decision maker's action. Communication is assumed strategic, non-binding (no commitment and no contract are allowed), payoff-irrelevant, and unmediated. In addition, players are not able to observe private payoff-irrelevant signals ("private sunspots") and there is no extraneous noise in communication, which thus takes place "face-to-face". However, randomized strategies are allowed in both the communication and action phases.

Contrary to usual cheap talk games (Crawford and Sobel, 1982; Ben-Porath, 2003; Gerardi, 2004; Krishna and Morgan, 2004), our communication games allow the set of messages available to the expert to be type-dependent, which reflects the ability to certify his information. We will assume that the expert has always the opportunity to remain silent, i.e., to send a meaningless message to the decision maker. Furthermore, to guarantee that our geometric characterization be sufficient for an equilibrium, we will require that players have access to a rich language and that information is fully certifiable. More precisely, we make the following assumption: for any set of types containing his real type, the expert has a sufficiently large set of messages allowing him to certify that his real type belongs to that set.

In the associated one-shot communication game the expert learns his type and sends a message to the decision maker, who then chooses an action. Such games are sometimes called persuasion or disclosure games (see, e.g., Milgrom, 1981; Milgrom and Roberts, 1986; Seidmann and Winter, 1997). To the best of our knowledge, this literature has always focused on one-shot information revelation with very specific assumptions on players' preferences, like single-peakedness, strict concavity and monotonicity. Our first result (Theorem 1) is a full characterization of Nash equilibrium payoffs of one-shot communication games with certifiable information. Roughly, equilibrium payoff vectors are obtained by convexifying the graph of the equilibrium payoff correspondence of the basic game without communication (the silent game), by keeping the payoff of the informed player constant and individually rational. Several geometric illustrations involving full, partial and/or no information revelation are provided.

In a multistage communication game, the talking phase has an arbitrarily large number of periods. In each communication period *both* players simultaneously send a message that depends on the history of play up to that period. The informed player's message may also depend on his private information. As in Hart (1985) and Aumann and Hart (2003), our equilibrium characterization makes use of the mathematical concepts of diconvexification and dimartingale. In Theorem 2 we show that the set of equilibrium payoffs of any multistage communication game can be characterized in terms of starting points of dimartingales converging to the graph of the equilibrium payoff correspondence of the silent game, and staying in an adapted set of individually rational payoffs for the informed player during the whole process. Individual rationality must indeed be formulated in a stage-dependent way in our model. This is the main difference with Aumann and Hart's (2003) characterization. Our representation can also be formulated by using the diconvexification operator. However, by contrast to Aumann and Hart (2003), the graph of the equilibrium payoff correspondence of the multistage communication game is *not* the diconvexification of a given set.

The paper is organized as follows. In the next section we present our leading example. Section 3 describes the model. Section 4 formulates the geometric characterizations of the equilibrium payoffs, illustrates them through examples, and provides a more detailed comparison with Aumann and Hart (2003). Formal proofs of Theorem 1 (one-shot, unilateral persuasion) and Theorem 2 (multistage, bilateral persuasion) are provided in Sections 5 and 6, respectively. We discuss extensions of the model in Section 7: mediated persuasion, unbounded number of talking stages, equilibrium refinement, and partial certifiability. The Appendix contains several additional examples.

2 An Example

In this section we study an example which motivates two aspects of our analysis. First, the example illustrates how by certifying their information players can reach equilibrium outcomes that cannot be achieved by any communication system with non-certifiable information. Second, the example shows that delayed information certification and multiple rounds of bilateral communication may be required to achieve some equilibrium payoffs, even if only one player has substantive information.

Consider two players, player 1 (the expert) and player 2 (the decisionmaker), who are playing a strategic form game which depends on the true state of Nature, k_1 or k_2, each of probability 1/2 (see Figure 1 on the following page). Player 1 knows the true state of Nature but player 2 does not know the actual game being played. Player 2 must choose action j_1, j_2, j_3, j_4 or j_5, and player 1 has no choice. The expected payoff of player 2, as a function of his action and his belief $p \in [0,1]$ about state k_1, is represented by Figure 2 on the next page (the thick lines denote his best-reply payoff).

Without communication possibilities (in the "silent game"), the only equilibrium payoff is $(0,7)$ since action j_3 yields the best expected payoff for player 2 given his prior belief $p = 1/2$. If, before player 2's decision, the players are able to talk to each other, but

	j_1	j_2	j_3	j_4	j_5
k_1	$5,0$	$3,4$	$0,7$	$4,9$	$2,10$
k_2	$1,10$	$3,9$	$0,7$	$5,4$	$6,0$

Figure 1: Introductory example.

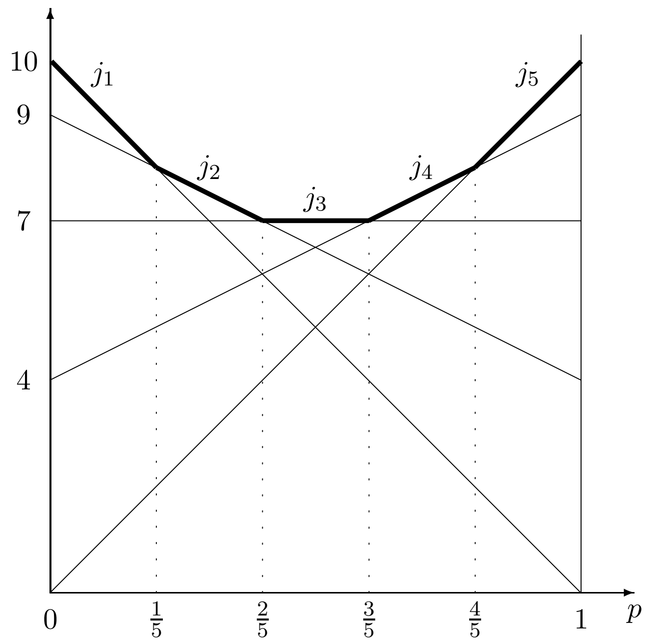

Figure 2: Player 2's expected payoffs (thin lines) and best-reply expected payoffs (thick lines) in the introductory example.

no information can be certified concerning the true state of Nature then, whatever the communication possibilities, the unique equilibrium payoff remains $(0,7)$. Information transmission is not possible here because if player 2 chooses his action conditionally on the messages sent by player 1 then, whatever the true state of Nature, player 1 has always an incentive to use the messages he should have sent at the other state. In other words, information which is transmitted to player 2 is never credible, even if in every state it is to the advantage of both players that player 1 tells the truth to player 2, and that the latter believes him. Notice that allowing unboundedly long communication, or even adding a mediator, cannot help here: one can check that the unique communication equilibrium outcome is the equilibrium j_3 of the silent game.

Assume now that player 1 can voluntarily certify his information concerning the real state of Nature. That is, his informational reports are assumed truthful (the making of false statements is prohibited), but he may withhold his information since he is not required to make positive disclosures. Assume first that player 1 can only send a single message and that player 2 cannot send any message. More precisely, assume that player 1 can choose between two types of reports: either he certifies his information (he sends message $m = c^1$ if the real state is k_1 and message $m = c^2$ if the real state is k_2), or he certifies no information (he sends message $m = \overline{m}$ which is available whatever the true state). It is easy to see that full revelation of information is now an equilibrium, denoted by FRE: player 2 chooses action j_5 if player 1 reveals that the true state is k_1, he chooses j_1 if player 1 reveals that the true state is k_2, and chooses j_3 if player 1 reveals nothing. In such a situation, player 1 has no incentive not to reveal his information because his payoff would be zero instead of 2 in state k_1 and 1 in state k_2. Obviously, player 2 also behaves rationally because he chooses the best action for him in each state of Nature.

As in usual cheap talk games, the non-revealing outcome is also an equilibrium, denoted by NRE, since player 2 can always ignore what player 1 says and choose action j_3.[1]

The two equilibrium outcomes described above are *not* the only equilibrium outcomes of the one-shot communication game with certifiable information. Indeed, if we allow player 1 to randomize, then there are two other *partially revealing* equilibria. One of them is better for player 1 than any of the previous pure strategy equilibria since it gives him a payoff of 2 whatever his type. In this equilibrium, denoted by PRE1, player 1 certifies his type (i.e., sends message c^1) with probability 1/3 and remains silent (i.e., sends message $\overline{m}$) with probability 2/3 in k_1, and he always remain silent in state k_2. Player 2's posterior beliefs are $\Pr(k_1 \mid \overline{m}) = \frac{\Pr(\overline{m}|k_1)\Pr(k_1)}{\Pr(\overline{m})} = \frac{2/6}{2/6+1/2} = 2/5$ and $\Pr(k_1 \mid c^1) = 1$, so he plays action j_5 when he receives message c^1 and is indifferent between j_2 and j_3 when he receives message $\overline{m}$. If he plays j_2 with probability 2/3 and j_3 with probability 1/3 after $\overline{m}$, and if he plays j_1 after the off-equilibrium message c^2 then player 1 has no incentive to deviate: in k_1 he gets a payoff of 2 if he sends message c^1 and also $(2/3) \times 3 + (1/3) \times 0 = 2$ if he sends message $\overline{m}$, so he is indifferent between the two messages; in k_2 he gets a payoff of 1 if he sends message c^2 and $(2/3) \times 3 + (1/3) \times 0 = 2$ if he sends message $\overline{m}$, so he strictly prefers to send message $\overline{m}$.

In the second partially revealing equilibrium with randomized certification, denoted by PRE2, player 1 always remains silent in state k_1; he certifies his type with probability 1/3 and remains silent with probability 2/3 in k_2. Player 2's posterior beliefs are $\Pr(k_1 \mid \overline{m}) = 3/5$ and $\Pr(k_1 \mid c^2) = 0$, so he plays action j_1 when he receives message c^2 and is indifferent

[1] However, notice that contrary to the fully revealing equilibrium, the non-revealing equilibrium is based on irrational choices off the equilibrium path since player 2 should not choose action j_3 when player 1 reveals him the true state of Nature (NRE is not subgame perfect). Restrictions to credible moves off the equilibrium path are investigated in Subsection 7.3.

between j_3 and j_4 when he receives message $\overline{m}$. If he plays j_3 with probability 4/5 and j_4 with probability 1/5 after message $\overline{m}$, and if he plays j_3 after the off-equilibrium message c^1 then it can be checked as before that player 1 has no incentive to deviate.[2]

Now, we show that if players are able to talk to each other during several bilateral communication rounds and to delay information certification, then player 1 can reach even a higher equilibrium payoff of 3 whatever his type. This equilibrium can be achieved in three communication stages. In the first two communication stages there is no information certification, and in the last communication stage player 1 will certify his information to player 2 conditionally on what both players said in the previous communication stages.

In the first communication stage player 1 partially reveals (without certifying) his information by using a random communication strategy which transmits the correct information with probability 3/4 so as to leave some doubt in player 2's mind. That is, he sends message $m = a$ with probability 3/4 if the real state is k_1 and with probability 1/4 if the real state if k_2. Symmetrically, he sends message $m = b$ with probability 3/4 if the real state is k_2 and with probability 1/4 if the real state if k_1 (the labeling of these two messages is irrelevant but both messages a and b are cheap talk messages: they must be available to player 1 whatever his type). From Bayes' rule, player 2 will believe state k_1 with probability 3/4 if he receives message a and with probability 1/4 if he receives message b. Hence, substantive but only partial information is conveyed, without any information certification. Communication cannot stop now since, as seen before, player 1 would have an incentive to deviate by *always* sending message a at k_1 and message b at k_2. Assume that player 2 chooses action j_2 whenever he receives message b. This choice is rational given his beliefs. Otherwise, when message a is sent, they agree on a jointly controlled $\frac{1}{2} - \frac{1}{2}$ lottery to reach the following compromise (this second communication stage conveys no substantive information, i.e., no information about the fundamentals of the game).[3] If head (H) occurs, then communication stops and thus player 1 chooses action j_4. On the contrary, if tail (T) occurs, then player 1 certifies his information in the last communication stage (he sends message c^k if the real state is k). Then, player 2 chooses action j_5 if c^1 is sent and action j_1 if c^2 is sent. Player 1 has no incentive to deviate if, for example, player 2 chooses action j_3 when player 1 deviates in the last communication stage

[2]Notice that contrary to the previous partially revealing equilibrium, this equilibrium is based on irrational choices off the equilibrium path since player 2 should not choose action j_3 when player 1 reveals him the true state of Nature (PRE2 is not subgame perfect). Again, see Subsection 7.3 for Nash equilibrium refinements.

[3]A jointly controlled lottery is a mechanism that generates a uniform probability distribution on any finite set from private random communication strategies so that a unilateral deviation does not change the probability distribution. For example, a $\frac{1}{2} - \frac{1}{2}$ lottery can be generated as follows: each player chooses a message in $\{a, b\}$ at random, both players announce their choices simultaneously and the outcome is head (H) if the messages coincide and tail (T) otherwise.

by remaining silent. The whole communication and decision process in this equilibrium is summarized by Figure 3 (where "JCL" stands for "jointly controlled lottery"). Player 2's expected payoff is $\frac{133}{16} = 8.3125$, and player 1's expected payoff is 3 whatever his type.

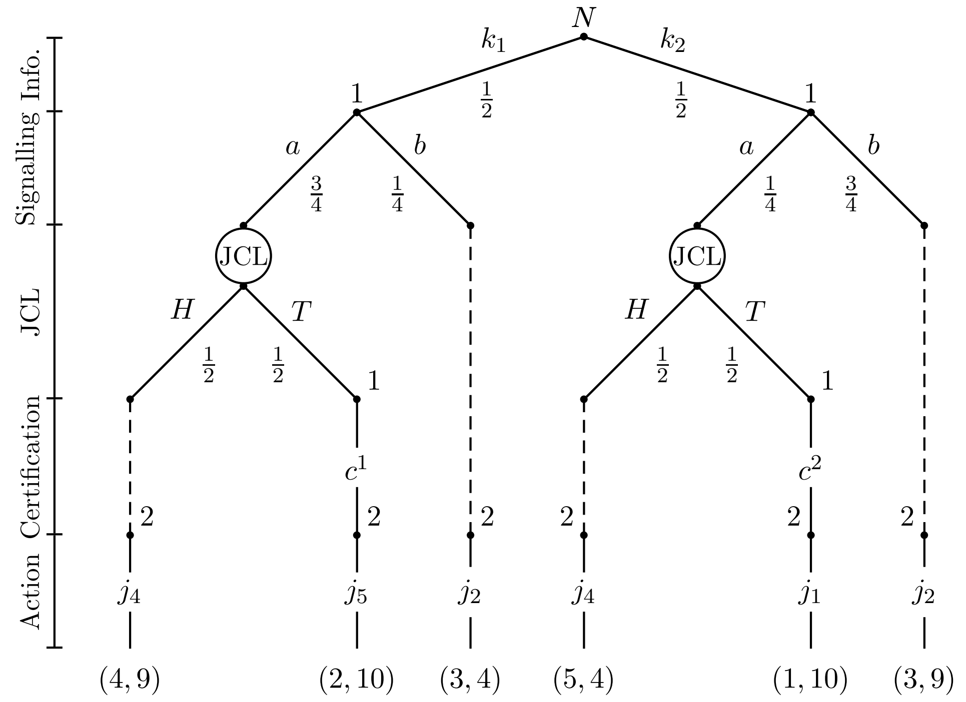

Figure 3: An equilibrium communication and decision tree for the introductory example.

In Section 4 we will provide geometric characterizations of all possible equilibrium payoffs of communication games with certifiable information. For example, the previous fully revealing equilibrium (FRE) and the two partially revealing equilibria (PRE1 and PRE2) of the unilateral persuasion game are simply characterized by the points FRE, PRE1 and PRE2 in Figure 7 on page 18. The non-existence of informative equilibrium in cheap talk games (of bounded length) in which information is not certifiable is simply characterized by the fact that the solid lines in Figure 7 never intercept. The geometric characterization of the equilibrium described above requiring information certification as well as multiple and bilateral communication stages is slightly more complex, and will be illustrated in Subsection 4.4.

3 Model

We consider two players: player 1 (the informed player, or expert) and player 2 (the uninformed decisionmaker (DM)). J ($|J| \geq 2$) is the finite action set of player 2 (player 1 has no action). K ($|K| \geq 2$) is the finite set of states (or types of player 1), with a common prior probability distribution $p = (p^1, \ldots, p^k, \ldots, p^K) \in \Delta(K)$. Let $\text{supp}[p] \equiv \{k \in K : p^k > 0\}$.[4] When player 2 chooses action $j \in J$ and the state is $k \in K$, the payoffs to player 1 and player 2 are $A^k(j)$ and $B^k(j)$, respectively.

3.1 Silent Game

The *silent game*, denoted by $\Gamma(p)$, consists of two phases. In the *information phase* a state $k \in K$ is picked at random according to the probability distribution p. Player 1 is perfectly informed about the true state k, while player 2 is not. In the *action phase*, player 2 chooses an action $j \in J$. Player 1 and player 2 receive payoffs $A^k(j)$ and $B^k(j)$, respectively.

A strategy of player 2 in the silent game $\Gamma(p)$ is a mixed action $y \in \Delta(J)$. We extend payoff functions linearly to mixed actions: $A^k(y) = \sum_{j \in J} y(j)\, A^k(j)$ and $B^k(y) = \sum_{j \in J} y(j)\, B^k(j)$. The set of (Bayesian) Nash equilibria of the silent game $\Gamma(p)$ is the set of optimal mixed actions for player 2 in the silent game $\Gamma(p)$. It is called the set of *non-revealing equilibrium outcomes* at p, and is denoted by:

$$Y(p) \equiv \arg \max_{y \in \Delta(J)} \underbrace{\sum_{k \in K} p^k\, B^k(y)}_{p\,B(y)} = \left\{ y \in \Delta(J) : \sum_{k \in K} p^k\, B^k(y) \geq \sum_{k \in K} p^k\, B^k(j),\ \forall\, j \in J \right\}.$$

Remark 1 A pure action is always sufficient to maximize the decisionmaker's payoff. So, for all $j,\, j' \in \text{supp}[Y(p)]$ and $y \in \Delta(J)$ we have $p\,B(j) = p\,B(j') \geq p\,B(y)$. However, mixed actions will become useful once the action phase will be preceded by communication: (i) on the equilibrium path, to make player 1 indifferent between several messages, and (ii) off the equilibrium path, to punish player 1.

The resulting equilibrium payoffs are the $(K+1)$-dimensional vectors (a, β), where $a = (a^1, \ldots, a^K)$, $a^k = A^k(y)$ is the payoff of player 1 of type k, which is only relevant if $k \in \text{supp}[p]$, and the scalar $\beta = p\,B(y)$ is player 2's expected payoff (expectation over k). Let $\mathcal{E}(p)$ be the set of equilibrium payoffs of $\Gamma(p)$, also called the set of *non-revealing*

[4] We could assume w.l.o.g. that $p^k > 0$ for all $k \in K$ but in order to capture the games corresponding to an updating of the prior over K, we allow $p^k = 0$ for some k's.

equilibrium payoffs at p.[5] That is,

$$\mathcal{E}(p) \equiv \{(a, \beta) \in \mathbb{R}^K \times \mathbb{R} : \exists\, y \in Y(p),\; a^k = A^k(y)\; \forall\, k \in \text{supp}[p],\; \beta = p\, B(y)\}.$$

3.2 Unilateral Persuasion Game

Here, we consider only direct (unmediated and noiseless) and unilateral communication, from player 1 to player 2. The set of messages available to player 1 is state-dependent and is denoted by $M(k)$ when his type is k. Let $M^1 = \bigcup_{k\in K} M(k)$ be the set of all messages that player 1 could send. The set $\bigcap_{k\in K} M(k)$ is the set of all cheap talk messages available to player 1, i.e., the set of all messages that player 1 can send whatever his type.

We assume that the set of cheap talk messages available to player 1 is nonempty. That is, there exists $\overline{m} \in M^1$ such that $M^{-1}(\overline{m}) = K$. This "right to remain silent" assumption will be needed for the "only if" part (from equilibrium to dimartingale) of Theorems 1 and 2. For the "if" part (from dimartingale to equilibrium), we will further assume that the message space and certifiability possibilities of the sender are sufficiently rich. That is, whatever his type k, and for each event $L \subseteq K$ containing k, player 1 can choose among a sufficiently large set of messages certifying that his real type is in L. Formally, we assume that

$$|\{m \in M^1 : M^{-1}(m) = L\}| \geq |L| + 1, \quad \text{for all } L \subseteq K.$$

Notice that this rich language and certifiability assumption implies the previous assumption that the set $\bigcap_{k\in K} M(k)$ is nonempty (simply take $L = K$). As we shall illustrate in Subsection 7.4, assuming full certifiability only for singleton events $L = \{k\}$ would *not* be sufficient for the "if" part of the theorems.

The signalling game determined by Γ and p, denoted by $\Gamma_S(p)$, is obtained by adding a one-shot talking phase to the silent game $\Gamma(p)$ before the action phase but after the information phase. Therefore, this game corresponds to a standard persuasion game (Milgrom, 1981; Shin, 1994; Seidmann and Winter, 1997) and has three phases (see Figure 4).

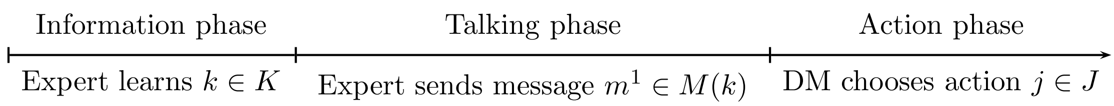

Figure 4: Unilateral persuasion (signalling) game $\Gamma_S(p)$.

[5]Our definition differs from Aumann and Hart's (2003) definition when the probability of some types vanishes. See Subsection 4.2 for a more detailed comparison.

The extensive form representation of the unilateral persuasion game with only two types, two cheap talk messages and one certificate for each type ($M(k) = \{a, b, c^k\}$, $k = k_1, k_2$) is given in Figure 5.

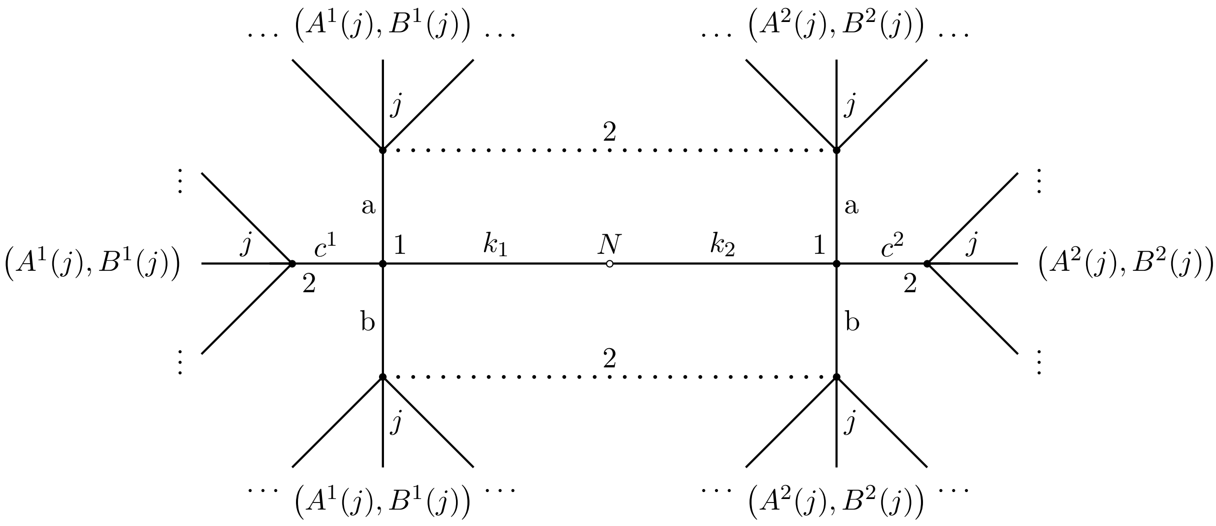

Figure 5: Extensive form of the unilateral persuasion game $\Gamma_S(p)$ with two types, two cheap talk messages and one certificate for each type ($M(k) = \{a, b, c^k\}$, $k = k_1, k_2$).

A strategy for player 1 in the unilateral persuasion game is a profile $\sigma = (\sigma^k)_{k \in K}$, with $\sigma^k \in \Delta(M(k))$ for all k. A strategy for player 2 is a function $\tau : M^1 \to \Delta(J)$. A pair of strategies (σ, τ) generates expected payoffs $(a^1_{\sigma,\tau}, \ldots, a^K_{\sigma,\tau})$ and $\beta_{\sigma,\tau}$ for player 1 and player 2, respectively. As usual, a (Bayesian) Nash equilibrium is a pair of strategies (σ, τ) satisfying

$$a^k_{\sigma,\tau} = \max_{\widetilde{\sigma}} a^k_{\widetilde{\sigma},\tau} \text{ for all } k \in \text{supp}[p]; \text{ and}$$

$$\beta_{\sigma,\tau} = \max_{\widetilde{\tau}} \beta_{\sigma,\widetilde{\tau}}.$$

Let $\mathcal{E}_S(p)$ be the set of Nash equilibrium payoffs of $\Gamma_S(p)$.

3.3 Multistage, Bilateral Persuasion Game

We consider an arbitrarily large but finite number $n \geq 1$ of communication rounds. In each communication round $t = 1, \ldots, n$ each player can directly send a message to the other. As in the unilateral persuasion game, the set of messages available to player 1 is denoted by $M(k)$ when his type is k, $M^1 = \bigcup_{k \in K} M(k)$ is the set of all messages that player 1 could send, and $\bigcap_{k \in K} M(k) \neq \emptyset$ is the set of all cheap talk messages available to player 1. The set of messages available to player 2 is denoted by M^2, with $|M^2| \geq 2$.

As in the unilateral persuasion game we assume that $|\{m \in M^1 : M^{-1}(m) = L\}| \geq |L|+1$ for all $L \subseteq K$. However, notice that in the multistage communication game it would be sufficient to have two cheap talk messages and that a combination of several certificates allows to certify any event $L \subseteq K$.[6] The above specific assumption on the richness of the message space is only for convenience.

The bilateral persuasion game with n communication stages, determined by Γ and p, is denoted by $\Gamma_n(p)$. It is obtained by adding a talking phase with n bilateral communication rounds to the silent game $\Gamma(p)$ before the action phase but after the information phase (see Figure 6). At each period $t = 1, \ldots, n$ of the *talking phase*, type $k \in K$ of player 1 sends a message $m_t^1 \in M(k)$ to player 2, and player 2 sends a message $m_t^2 \in M^2$ to player 1 (perfect monitoring). Messages are sent simultaneously.

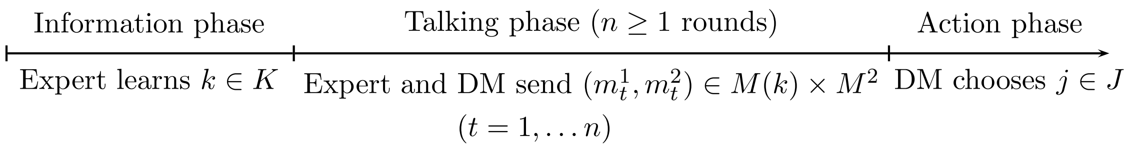

Figure 6: n-Stage bilateral persuasion game $\Gamma_n(p)$.

A t-period *history*, $t = 0, 1, \ldots, n$, is a sequence consisting of t pairs of messages,

$$h_t = (m_1^1, m_1^2, \ldots, m_t^1, m_t^2) \in (M^1 \times M^2)^t.$$

The set of all t-period histories is denoted by $M_t = (M^1 \times M^2)^t$. A strategy[7] σ of player 1 in the n-period communication game $\Gamma_n(p)$ consists of a sequence of functions $\sigma_1, \ldots, \sigma_n$, where $\sigma_t = (\sigma_t^1, \ldots, \sigma_t^K)$ and $\sigma_t^k : M_{t-1} \to \Delta(M(k))$ for $k \in K$ and $t = 1, \ldots, n$. A strategy τ of player 2 consists of a sequence of functions $\tau_1, \ldots, \tau_n$, and a function τ_{n+1}, where $\tau_t : M_{t-1} \to \Delta(M^2)$ for $t = 1, \ldots, n$, and $\tau_{n+1} : M_n \to \Delta(J)$.

A pair of strategies (σ, τ) generates expected payoffs $a_{\sigma,\tau} = (a_{\sigma,\tau}^1, \ldots, a_{\sigma,\tau}^K)$ and $\beta_{\sigma,\tau}$ for player 1 and player 2, respectively. The set of (Bayesian) Nash equilibrium of the persuasion game $\Gamma_n(p)$ is denoted by $\mathcal{E}_n(p)$. Notice that $\mathcal{E}_S(p) \subseteq \mathcal{E}_n(p) \subseteq \mathcal{E}_{n+1}(p)$ for all $n \geq 1$. Let $\mathcal{E}_B(p) = \bigcup_{n \geq 1} \mathcal{E}_n(p)$ be the set of Nash equilibrium payoffs of all multistage, bilateral persuasion games determined by Γ and p.

[6]That is, it would be sufficient to assume that $|\bigcap_{k \in K} M(k)| \geq 2$, and $\forall\, k$, $\forall\, k' \neq k$, $\exists\, m \in M(k)$, $M^{-1}(m) = K \backslash \{k'\}$.

[7]We focus on finite games with perfect recall. Hence, by Kuhn's (1953) theorem behavioral strategies are without loss of generality.

4 Characterization of Equilibrium Payoffs $\mathcal{E}_S(p)$ and $\mathcal{E}_B(p)$

4.1 Statement of the Results

Let H be the *graph* of the non-revealing equilibrium payoff correspondence, namely

$$H = \operatorname{gr}\mathcal{E} \equiv \{(a,\beta,p) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K) : (a,\beta) \in \mathcal{E}(p)\},$$

where $\mathcal{E}(p)$ has been defined in Subsection 3.1. Notice that the set $\mathcal{E}(p)$ is convex for all p. In other words, H is convex in (a,β) when p is kept constant. However, H need not be convex in (β,p) when a is kept constant.

For any (nonempty) set of types $L \subseteq K$, let

$$\text{INTIR}_L \equiv \{a \in \mathbb{R}^K : \exists\, \overline{y} \in \Delta(J),\ a^k \geq A^k(\overline{y})\ \forall\, k \in L\},$$

be the set of payoffs that are interim individually rational for player 1 when we restrict the individual rationality constraint to a subset L of player 1's set of types. Remark that $\text{INTIR}_L \subseteq \text{INTIR}_{L'}$ whenever $L' \subseteq L$. Let I be the graph of the payoffs that are interim individually rational for player 1 in the silent game $\Gamma(p)$:

$$I \equiv \{(a,\beta,p) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K) : a \in \text{INTIR}_{\text{supp}[p]}\}.$$

As H, I is convex in (a,β) when p is kept constant, but not in p when a is kept constant.[8] Obviously, every non-revealing equilibrium payoff is interim individually rational for player 1 so that $H \subseteq I$.

Let

$$H_1 \equiv \text{conv}_a(H) \cap I,$$

be the set of expected payoffs obtained from H by convexifying in (β,p) when the payoff of player 1, a, is kept constant and is interim individually rational for player 1. Even if H is included in I, payoffs in $\text{conv}_a(H)$ need not be interim individually rational for player 1, while this is clearly a necessary equilibrium condition. We thus have to require individual rationality explicitly in the definition of H_1.[9] It turns out that this requirement is also sufficient for the equilibrium characterization of the unilateral persuasion game.

Theorem 1 (Unilateral Persuasion) *The set $\mathcal{E}_S(p)$ of Nash equilibrium payoffs of the*

[8] For instance, in Example 1 in the appendix, $((0,0),\cdot,p) \in I$ for $p \in \{0,1\}$ but not for $p \in (0,1)$.

[9] The restriction to supp$[p]$ for individual rationality is irrelevant for the next theorem, but will be important in the multistage game.

unilateral persuasion game $\Gamma_S(p)$ *coincides with the* p*-section of* H_1*:*

$$\mathcal{E}_S(p) = H_1(p) \equiv \{(a, \beta) \in \mathbb{R}^K \times \mathbb{R} : (a, \beta, p) \in H_1\}.$$

In addition, any Nash equilibrium payoff of $\Gamma_S(p)$ *can be obtained with at most* $K + 1$ *messages.*

Proof. See Section 5. ■

From the proof of the "if" part of the theorem (the construction of the sender's strategy), the following proposition is immediate:

Proposition 1 *Let* $p^k > 0$ *for all* $k \in K$*. Every equilibrium of the unilateral persuasion game* $\Gamma_S(p)$ *is outcome equivalent (i.e., it induces the same probability distribution over player 2's decision conditionally on* k*) to a "canonical" equilibrium* (σ, τ) *with the following property:*

For all $m \in M^1$*, if* $\sigma^k(m) > 0$ *for some* $k \in K$*, then* $\sigma^{k'}(m) > 0$ *for all* $k' \in M^{-1}(m)$*.*

In particular, if a cheap talk message $\overline{m} \in \bigcap_{k \in K} M(k)$ is sent with strictly positive probability by player 1, then all types of player 1 send this message with strictly positive probability. More generally, the proposition says that in equilibrium we can assume without loss of generality that if player 2's posterior about a certain type k of player 1 is null after some message m sent with strictly positive probability, then $k \notin M^{-1}(m)$, i.e., message m certifies that k is not realized. In particular, all types have strictly positive posterior probability after a cheap talk message (sent with strictly positive probability in equilibrium). Without using the geometric characterization of Theorem 1, the intuition of the proposition is as follows. Assume that type k' does not send a message m but could have sent it (i.e., $m \in M(k')$). Then, the types who send message m could have sent another message instead of m that certifies that k' is not realized, without changing player 2's posteriors and so without changing the equilibrium outcome.

To get the equilibrium payoffs for persuasion games with several bilateral communication rounds, we first consider the payoffs obtained as convex combinations of elements in H_1 with p fixed which are interim individually rational for player 1: $H_1^* = \text{conv}_p(H_1) \cap I$. Since $H_1 \subseteq I$ and I is convex in (a, β) when p is fixed, $\text{conv}_p(H_1) \subseteq I$ so that $H_1^* = \text{conv}_p(H_1)$. We then proceed with H_1^* as we did above with H, namely convexifying in (p, β) keeping a constant and interim individually rational. This yields $H_{3/2} = \text{conv}_a(H_1^*) \cap I$. Next, by convexifying in (a, β) at p fixed, we get $H_2 = \text{conv}_p(H_{3/2}) = \text{conv}_p(H_{3/2}) \cap I$. The p-section of the set H_2 is the set of equilibrium payoffs of persuasion games with four communication rounds: a jointly controlled lottery, a step of signalling, a second jointly controlled lottery, and a second step of signalling. Next, let H_3 be the set obtained from H_2 by convexifying in (β, p) when player 1's payoff a is fixed, and then by convexifying in (a, β) when player 2's belief p is fixed, with again the restriction that the payoff of player 1 is interim individually rational for the types with a strictly positive posterior. The p-section of the set H_3 is the set of equilibrium payoffs of persuasion games with six communication rounds. The set H_n, $n \geq 2$, thus corresponds to $2n$ stages of "canonical" communication, in which signalling and jointly controlled lotteries alternate. We introduce a slight disymmetry in the definition of H_1, which captures a single stage of signalling for player 1. The limit of the increasing sequence H_1, H_2, ... constructed in this way is denoted by $\text{di-co}^{\text{IR}}(H) \equiv \bigcup_{l \geq 1} H_l$ to recall the process of diconvexification used in the construction. Observe that, since I is not a di-convex set, $\text{di-co}^{\text{IR}}(H)$ need not be di-convex (see the comparison with Aumann and Hart, 2003 in the next subsection). Points in $\text{di-co}^{\text{IR}}(H)$ correspond to all equilibrium payoffs of bilateral persuasion games of bounded length. In the next theorem, the set $\text{di-co}^{\text{IR}}(H)$ is expressed more elegantly as the set of starting points of particular martingales that converge to H.

Theorem 2 (Multistage, Bilateral Persuasion) *The set $\mathcal{E}_B(p)$ of all Nash equilibrium payoffs from bilateral persuasion games $\Gamma_n(p)$, $n \geq 1$, coincides with the p-section of* $\text{di-co}^{\text{IR}}(H)$*:*

$$\mathcal{E}_B(p) = H_B(p) \equiv \{(a, \beta) \in \mathbb{R}^K \times \mathbb{R} : (a, \beta, p) \in \text{di-co}^{\text{IR}}(H)\}.$$

Equivalently, $(a, \beta) \in \mathcal{E}_B(p)$ if and only if there exists a martingale $\boldsymbol{z} = (\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_N)$, with $\boldsymbol{z}_s = (\boldsymbol{a}_s, \boldsymbol{\beta}_s, \boldsymbol{p}_s) \in I$ for all $s = 0, 1, \ldots, N$, satisfying the following properties:

(D1) $\boldsymbol{z}_0 = (a, \beta, p)$. That is, the starting point (and expectation) of the martingale is the Nash equilibrium payoff under consideration.

(D2) $\boldsymbol{z}_N \in H$. That is, the martingale converges to the set of non-revealing equilibrium payoffs: $(\boldsymbol{a}_N, \boldsymbol{\beta}_N) \in \mathcal{E}(\boldsymbol{p}_N)$.

(D3) $\boldsymbol{a}_{s+1} = \boldsymbol{a}_s$ for all even s and $\boldsymbol{p}_{s+1} = \boldsymbol{p}_s$ for all odd s. That is, the martingale is a dimartingale.[10]

Proof. See Section 6. ■

[10] All statements involving random variables should be understood to hold for all states occurring with strictly positive probability.

Remark 2 Requiring $\boldsymbol{a}_N \in \text{INTIR}_K$ guarantees $\boldsymbol{a}_s \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_s]}$ for all s, but is a much too strong condition: it is easy to construct an example with an equilibrium payoff $(a, \beta) \in \mathcal{E}_B(p)$ but $\boldsymbol{a}_N \notin \text{INTIR}_K$, $K \neq \text{supp}[\boldsymbol{p}_N]$. On the other hand, requiring $\boldsymbol{a}_0 \in \text{INTIR}_K$ is not sufficient. Indeed, one can easily construct a dimartingale with $a_0 \in \text{INTIR}_K$, $(\boldsymbol{a}_N, \boldsymbol{\beta}_N, \boldsymbol{p}_N) \in H$, but $(a, \beta) \notin \mathcal{E}_B(p)$ ($\boldsymbol{a}_s \notin \text{INTIR}_{\text{supp}[\boldsymbol{p}_s]}$ for some history at s). More generally, the condition $\boldsymbol{z}_s \in I$ is redundant at some stages s but not at all of them. For instance, if s is even, $\boldsymbol{a}_{s+1} = \boldsymbol{a}_s$, $\boldsymbol{a}_s \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_s]}$, and the fact that $\text{supp}[\boldsymbol{p}_{s+1}] \subseteq \text{supp}[\boldsymbol{p}_s]$ imply $\boldsymbol{a}_{s+1} \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_{s+1}]}$. But the converse is not true: one may have $\boldsymbol{a}_{s+1} \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_{s+1}]}$ without having $\boldsymbol{a}_s = \boldsymbol{a}_{s+1} \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_s]}$. If s is odd, $\boldsymbol{p}_{s+1} = \boldsymbol{p}_s$, $\boldsymbol{a}_{s+1} \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_{s+1}]}$ and the martingale property imply that $\boldsymbol{a}_s \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_s]}$. Again, the converse is not true. These properties explain why, starting from the end of the process in order to construct $\text{di-co}^{\,\text{IR}}(H)$, one had to intercept with I only when convexifying at a fixed.

Remark 3 If there exists a worst outcome for player 1 (i.e., an action $j_w \in J$ such that $A^k(j_w) \leq A^k(j)$ for all $k \in K$ and $j \in J$), then the individual rationality conditions are automatically satisfied.

4.2 Comparison with Aumann and Hart (2003)

When some coordinates of p vanish, Aumann and Hart (2003) consider the *modified* equilibrium payoffs $\mathcal{E}^+(p)$ of the silent game $\Gamma(p)$, which is the same as $\mathcal{E}(p)$ except that when the probability of one of player 1's type vanishes, then the corresponding type of player 1 can only get *more* than his equilibrium payoff. That is, the set of modified non-revealing equilibrium payoffs is the set of all payoffs (a, β) such that there exits an equilibrium $y \in Y(p)$ of the silent game $\Gamma(p)$ satisfying

(i) $a^k \geq A^k(y)$, for all $k \in K$;

(ii) $a^k = A^k(y)$ if $p^k \neq 0$;

(iii) $\beta = \sum_{k \in K} p^k \, B^k(y)$.

The graph of the modified non-revealing equilibrium payoff correspondence is

$$G \equiv \text{gr}\, \mathcal{E}^+ \equiv \{(a, \beta, p) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K) : (a, \beta) \in \mathcal{E}^+(p)\}.$$

Here, we consider the more natural set of non-revealing equilibrium payoffs, $\mathcal{E}(p)$, in which it is understood that the types of player 1 which have probability zero can get *any* payoff (only conditions (ii) and (iii) above must be satisfied) Clearly, $\mathcal{E}^+(p) \subseteq \mathcal{E}(p)$ and if p has full support, both sets coincide.

Let di-co (G) be the smallest set which contains G and is convex in (a, β) (respectively (β, p)) when p (respectively a) is fixed. Aumann and Hart (2003) characterize the set of all equilibrium payoffs achieved with finitely many stages of bilateral cheap talk as the p-section of di-co (G). This extremely elegant characterization relies on the identification of the modified set of non-revealing equilibrium payoffs $\mathcal{E}^+(p)$ for every non interior p, which ensures that *all* equilibrium conditions of player 1 can be written as equalities, namely captured by a dimartingale property. In this framework, player 1's expected payoff remains fully interim individually rational (in INTIR_K) all along the communication process.

Our starting set H corresponds to the *non*-modified graph of the non-revealing equilibrium payoff correspondence in the sense that we do not impose any condition on player 1's payoff when his type has zero probability. The geometric properties of our final graph of equilibrium payoffs are not so transparent since, as observed above, di-co $^{\text{IR}}(H)$ is not necessarily convex in (β, p) when a is fixed. Obviously, this set is convex in (a, β) when p is fixed since the players can perform jointly controlled lotteries. If player 1 can send certificates in addition to cheap talk messages, some states of nature may be eliminated forever. Player 1's individual rationality conditions must thus be expressed relatively to the remaining possible states. These individual rationality conditions are more important than in the case of pure cheap talk because player 2 can punish player 1 if he does not send a sufficiently precise message.

4.3 Illustration of Theorem 1 (Unilateral Persuasion)

For the introductory example, the graph of the modified non-revealing equilibrium payoff correspondence, $G = \text{gr}\,\mathcal{E}^+$, is represented on the (a^1, a^2)-coordinates by solid lines in Figure 7 on the next page. The graph of the non-revealing equilibrium payoff correspondence, $H = \text{gr}\,\mathcal{E}$, is represented in the same figure by the solid and dashed lines. The sets G and H are also described in the second and third columns of Table 1 on page 20. Since all points at the north-east of $(0, 0)$ are interim individually rational for player 1, convexifying the set H by keeping a constant and interim individually rational yields three new points at $p = 1/2$: FRE, PRE1 and PRE2, which are exactly the three Nash equilibrium payoffs found in Section 2, in addition to the non-revealing equilibrium (NRE). Indeed, each of these points corresponds to two non-revealing equilibrium payoffs, at two different p's forming an interval that includes $p = 1/2$, giving the same payoff to player 1. Notice that, for example, the point PRE3 is not an equilibrium payoff for $p = 1/2$ because $1/2$

lies outside the interval $[3/5, 1]$.

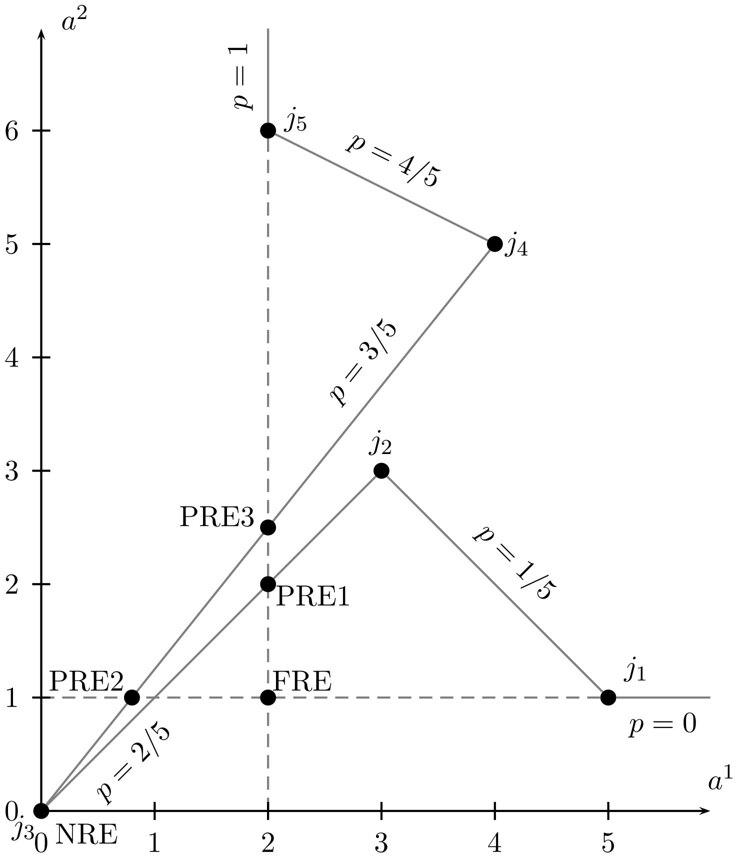

Figure 7: Modified non-revealing equilibrium payoffs (solid lines) and interim individually rational non-revealing equilibrium payoffs (solid and dashed lines) of the expert in the introductory example.

4.4 Illustration of Theorem 2 (Multistage, Bilateral Persuasion)

The dimartingale corresponding to the equilibrium with three talking stages of the introductory example (see Figure 3 on page 8) is represented by Figure 8 on the following page, where the two numbers in parentheses ((1) and (2)) correspond to non-revealing equilibrium payoffs ensuring the dimartingale property (D3) of Theorem 2. It leads to the point j_2 at $p = 1/2$ in Figure 7, which is not achievable at $p = 1/2$ with only one step of diconvexification.

Adding a jointly controlled lottery before a signalling stage allows a convexification by keeping p fixed. This leads to the graph $H_1^* = \text{conv}_p(H_1)$ described on the a-coordinates in the fourth column of Table 1. For example, adding a jointly controlled lottery before a signalling stage at $p = 1/2$ leads to all convex combinations of equilibrium payoffs of

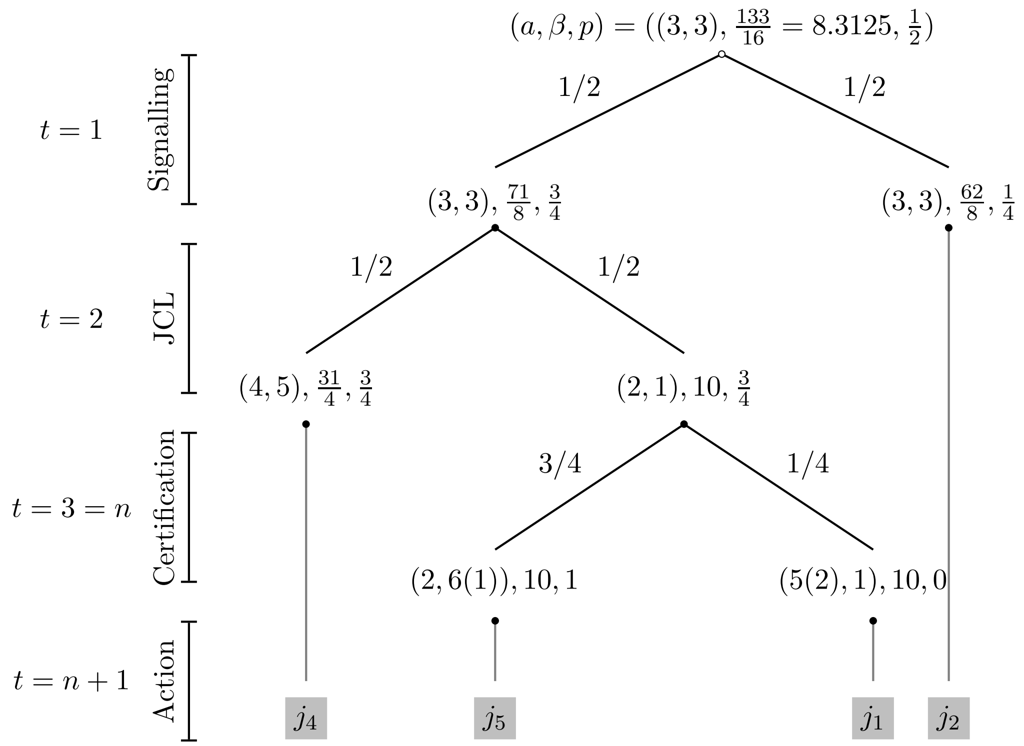

Figure 8: Dimartingale/diconvexification corresponding to the equilibrium with three talking stages in the introductory example.

the unilateral persuasion game, $[j_3, \text{FRE,PRE1,PRE2}]$. Adding a second signalling stage allows a second convexification by keeping a fixed. One can check that this does not yield new equilibrium payoffs, except for $p \in (2/5, 3/5)$. Indeed, for $p \in (2/5, 3/5)$ one can combine the sets $H_1^*(p') = [j_2, \text{PRE2}, \text{FRE}]$, $p' \in (1/5, 2/5)$, and $H_1^*(p'') = [j_4, \text{PRE3,FRE}]$, $p'' \in (3/5, 4/5)$, which leads to the payoffs in the triangle $[j_2, \text{PRE1,FRE}]$, which were not achievable at $p \in (2/5, 3/5)$ with only 2 communication stages. Hence, for $p \in (2/5, 3/5)$, $H_2(p) = H_1^*(p) \cup [j_2, \text{PRE1,FRE}] = [j_3, \text{PRE2}, j_2, \text{FRE}]$. It is easy to verify that one cannot get new points after two steps of diconvexification in both directions, so $H_2 = H_n$ for all $n \geq 2$.

5 Proof of Theorem 1

We assume w.l.o.g. that $\text{supp}[p] = K$, so that $\mathcal{E}_S(p)$ can be characterized equivalently as the p-section of $\text{conv}_a(H) \cap \{(a, \beta, p) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K) : a \in \text{INTIR}_K\}$.

p	G	H	$H_1^* = \mathrm{conv}_p(H_1)$	H_2
0	$(a^1 \geq 5, 1)$	$(a^1, 1)$	$\cdots$	$\cdots$
$(0, \frac{1}{5})$	j_1	j_1	$[j_1, \mathrm{PRE2}]$	$\cdots$
$\frac{1}{5}$	$[j_1, j_2]$	$[j_1, j_2]$	$[j_1, j_2, \mathrm{PRE2}]$	$\cdots$
$(\frac{1}{5}, \frac{2}{5})$	j_2	j_2	$[j_2, \mathrm{PRE2}, \mathrm{FRE}]$	$\cdots$
$\frac{2}{5}$	$[j_2, j_3]$	$[j_2, j_3]$	$[j_2, \mathrm{PRE2}, j_3, \mathrm{FRE}]$	$\cdots$
$(\frac{2}{5}, \frac{3}{5})$	j_3	j_3	$[j_3, \mathrm{FRE,PRE1,PRE2}]$	$[j_3, \mathrm{PRE2}, j_2, \mathrm{FRE}]$
$\frac{3}{5}$	$[j_3, j_4]$	$[j_3, j_4]$	$[j_3, j_4, \mathrm{FRE}]$	$\cdots$
$(\frac{3}{5}, \frac{4}{5})$	j_4	j_4	$[j_4, \mathrm{PRE3,FRE}]$	$\cdots$
$\frac{4}{5}$	$[j_4, j_5]$	$[j_4, j_5]$	$[j_4, j_5, \mathrm{FRE}]$	$\cdots$
$(\frac{4}{5}, 1)$	j_5	j_5	$[j_5, \mathrm{FRE}]$	$\cdots$
1	$(2, a^2 \geq 6)$	$(2, a^2)$	$\cdots$	$\cdots$

Table 1: Diconvexification of the non-revealing equilibrium payoffs of the introductory example. "$\cdots$" means "as in the previous column".

5.1 From equilibrium to constrained convexification: $\mathcal{E}_S(p) \subseteq H_1(p)$

Let (σ, τ) be any Nash equilibrium of the unilateral persuasion game $\Gamma_S(p)$, where $p^k > 0$ for all $k \in K$, and let $(a, \beta) \in \mathcal{E}_S(p)$ be the associated equilibrium payoffs. We must show that (a, β, p) is in H_1, i.e., (a, β, p) can be obtained as a convex combination of points in $H = \mathrm{gr}\,\mathcal{E}$ by keeping a constant and interim individually rational ($a \in \mathrm{INTIR}_K$). Let $P = P_{\sigma,\tau,p}$ be the probability distribution on $\Omega = K \times M^1 \times J$ generated by players' strategies and the priors. So,

$$P(m) = \sum_{k \in K} p^k \, \sigma^k(m),$$

is the (ex ante) probability that player 1 sends message $m \in M^1$. Let $M^* = \{m \in M^1 : P(m) > 0\}$. For all $m \in M^*$, let

$$p_m^k = P(k \mid m) = \frac{p^k \, \sigma^k(m)}{P(m)},$$

be player 2's posterior about player 1's type after receiving message m, let $p_m = (p_m^k)_{k \in K}$, and let

$$\beta_m = \sum_{k \in K} p_m^k \, B^k(\tau(m)),$$

be the resulting expected payoff for player 2 when m is reached. Since $p^k = \sum_{m \in M^*} P(m)\, p_m^k$ for all $k \in K$ and $\beta = \sum_{m \in M^*} P(m)\, \beta_m$, we have

$$(a, \beta, p) = \sum_{m \in M^*} P(m)\, (a, \beta_m, p_m).$$

So, to show that (a, β, p) is a convex combination of points in H be keeping a constant it suffices to show that $(a, \beta_m, p_m) \in H$ for all $m \in M^*$, i.e., $(a, \beta_m) \in \mathcal{E}(p_m)$ for all $m \in M^*$. Player 2's equilibrium condition implies that $\tau(m) \in Y(p_m)$ for all $m \in M^*$, so condition (iii) in the definition of $\mathcal{E}(p_m)$ (see page 16) is satisfied for all $m \in M^*$. Player 1's equilibrium condition implies that $A^k(\tau(m)) = A^k(\tau(m'))$ whenever $\sigma^k(m) > 0$ and $\sigma^k(m') > 0$ (player 1 of type k should be indifferent between all messages that he sends with strictly positive probability), so

$$a^k = \sum_{m \in M^*} \sigma^k(m)\, A^k(\tau(m)) = A^k(\tau(m)),$$

for all m such that $\sigma^k(m) > 0$ (which is equivalent to $p_m^k > 0$ because $p^k > 0$), so condition (ii) in the definition of $\mathcal{E}(p_m)$ is also satisfied for all $m \in M^*$.

Remark 4 Notice that when $p_m^k = 0$ we may have $a^k < A^k(\tau(m))$ (because type k *cannot* send message m when $m \notin M(k)$), so when some coordinates of p_m vanish it is possible that $(a, \beta_m, p_m) \notin G \equiv \operatorname{gr} \mathcal{E}^+$, contrary to the case of cheap talk (Aumann and Hart, 2003).

It remains to show that $a \in \mathrm{INTIR}_K$. Consider a message $\overline{m} \in \bigcap_{k \in K} M(k)$ (which exists by the "right to remain silent" assumption), and let $\overline{y} = \tau(\overline{m})$ ($\overline{m}$ may or may not be a message sent by player 1 with positive probability, so there may be no rationality condition on $\overline{y}$ for player 2 as long as no equilibrium refinement is introduced). By player 1's equilibrium condition, for all $k \in K$ and m such that $\sigma^k(m) > 0$ we have $a^k = A^k(\tau(m)) \geq A^k(\overline{y})$, which proves that $a \in \mathrm{INTIR}_K$. ■

5.2 From constrained convexification to equilibrium: $H_1(p) \subseteq \mathcal{E}_S(p)$

We start from (a, β, p), a convex combination of points in H by keeping a constant, with $a \in \mathrm{INTIR}_K$ and $p^k > 0$ for all $k \in K$, and we construct an equilibrium (σ, τ) of the unilateral persuasion game $\Gamma_S(p)$ with expected payoffs (a, β). Since $(a, \beta, p) \in \mathrm{conv}_a(H)$, we can write

$$(a, \beta, p) = \sum_{w \in W} \pi(w)\, (a, \beta_w, p_w),$$

with $\pi \in \Delta(W)$ and $(a, \beta_w, p_w) \in H$ for all $w \in W$. Without loss of generality we assume that π has full support. In addition, from Carathéodory's theorem we can let $|W| \leq K+1$ since the dimension of $(\beta, p) \in \mathbb{R} \times \Delta(K)$ is equal to K. For all $w \in W$, we associate a set of types $\text{supp}[p_w] \equiv \{k \in K : p_w^k > 0\}$ and a message $m_w \in M^1$ with $m_w \neq m_{w'}$ for $w \neq w'$, and $M^{-1}(m_w) = \text{supp}[p_w]$. This is possible given our rich language and certifiability assumption.

Player 1's strategy σ. For all $k \in K$ and $w \in W$ define

$$\sigma^k(m_w) = \frac{\pi(w)\, p_w^k}{p^k} \quad \text{(and } \sigma^k(m) = 0 \text{ if } m \neq m_w \text{ for all } w \in W\text{).}$$

Player 2's strategy τ. Since by assumption $(a, \beta_w) \in \mathcal{E}(p_w)$, for all $w \in W$ we can define (see condition (ii) and (iii) of $\mathcal{E}(p_w)$),

$$y_w = \tau(m_w) \in Y(p_w) \;\text{ such that }\; \begin{cases} a^k = A^k(\tau(m_w)) \text{ if } p_w^k > 0 \\ \beta_w = \sum_{k \in K} p_w^k\, B^k(\tau(m_w)). \end{cases}$$

For the other messages $m \neq m_w$, $w \in W$, since by definition $a \in \text{INTIR}_K$, we can define

$$\tau(m) = \overline{y} \;\text{ such that }\; a^k \geq A^k(\overline{y}) \text{ for all } k \in K.$$

Payoffs. We first verify that (a, β) is the payoff generated by the strategy profile (σ, τ) defined just before. Let $P = P_{\sigma,\tau,p}$ be the probability distribution on $\Omega = K \times M^1 \times J$ generated by those strategies and the prior, and let $E = E_{\sigma,\tau,p}$ be the associated expectation operator. First, we check that $P(m_w) = \pi(w)$ for all $w \in W$:

$$P(m_w) = \sum_{k \in K} p^k\, \sigma^k(m_w) = \sum_{k \in K} p^k \frac{\pi(w)\, p_w^k}{p^k} = \sum_{k \in K} \pi(w)\, p_w^k = \pi(w) \sum_{k \in K} p_w^k = \pi(w).$$

By construction, player 1's expected payoff when his type is k is given by

$$\begin{aligned} E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] &= \sum_{w \in W} P[\boldsymbol{m} = m_w \mid \boldsymbol{k} = k] E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k, \boldsymbol{m} = m_w] \\ &= \sum_{w \in W} \sigma^k(m_w) \sum_{j \in J} \tau(m_w)(j) A^k(j) = \sum_{w \in W} \sigma^k(m_w)\, A^k(\tau(m_w)) = a^k, \end{aligned}$$

the last equality following from the construction of player 2's strategy: $A^k(\tau(m_w)) = a^k$

whenever $\sigma^k(m_w) > 0$ ($\Leftrightarrow p_w^k > 0$ because $p^k > 0$). Finally, player 2's expected payoff is

$$\begin{aligned} E[B^{\boldsymbol{k}}(\boldsymbol{j})] &= \sum_{k\in K} p^k E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] \\ &= \sum_{k\in K} p^k \sum_{w\in W} P[\boldsymbol{m} = m_w \mid \boldsymbol{k} = k] E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k, \boldsymbol{m} = m_w] \\ &= \sum_{k\in K} p^k \sum_{w\in W} \sigma^k(m_w) \sum_{j\in J} \tau(m_w)(j) B^k(j) = \sum_{k\in K} p^k \sum_{w\in W} \frac{\pi(w) p_w^k}{p^k} B^k(\tau(m_w)) \\ &= \sum_{w\in W} \pi(w) \sum_{k\in K} p_w^k \, B^k(\tau(m_w)) = \sum_{w\in W} \pi(w)\, \beta_w = \beta. \end{aligned}$$

Equilibrium condition for player 2. Next, we verify that τ is a best reply for player 2 to player 1's strategy σ. Since we have defined $\tau(m_w) \in Y(p_w)$ for all $w \in W$, and since the messages $(m_w)_{w\in W}$ are the only messages sent with strictly positive probability by player 1, it suffices to verify that p_w is the correct posterior belief of player 2 when he receives message m_w. This is immediately obtained by Bayes's rule given the definition of the strategy σ of player 1:

$$P[\boldsymbol{k} = k \mid \boldsymbol{m} = m_w] = \frac{P[\boldsymbol{m} = m_w \mid \boldsymbol{k} = k] P[\boldsymbol{k} = k]}{P[\boldsymbol{m} = m_w]} = \frac{\sigma^k(m_w) p^k}{\pi(w)} = p_w^k.$$

Equilibrium condition for player 1. Finally, we verify that σ^k is a best reply for player 1 of type k to player 2's strategy τ. Player 1 of type k sends each message m_w, $w \in W$, satisfying $p_w^k > 0$ ($\Leftrightarrow \sigma^k(m_w) > 0$ because $p^k > 0$) with strictly positive probability. By construction of player 2's strategy we have $A^k(\tau(m_w)) = a^k$ (see the previous paragraph "payoffs") for all such messages, so type k is indeed indifferent between all these messages. Next, remark that type k *cannot* send the other messages m_w satisfying $p_w^k = 0$ because such messages are such that $M^{-1}(m_w) = \text{supp}[p_w]$, with $k \notin \text{supp}[p_w]$ (by the definition of $\text{supp}[p_w]$ since $p_w^k = 0$), so $m_w \notin M(k)$. Finally, if player 1 sends a message off the equilibrium path, $\overline{m} \neq m_w$ for all $w \in W$ (so $P(\overline{m}) = 0$), then he gets $A^k(\tau(\overline{m})) = A^k(\overline{y}) \leq a^k = A^k(\tau(m_w))$ for $\sigma^k(m_w) > 0$, so he does not deviate. ∎

6 Proof of Theorem 2

As in the proof of Theorem 1, we assume w.l.o.g. that $\text{supp}[p] = K$.

6.1 From equilibrium to constrained dimartingale: $\mathcal{E}_B(p) \subseteq H_B(p)$

Except for the construction of player 1's sequence of virtual payoffs and the fact that we consider martingales that are bounded in length, this part of the proof is similar to the proof of Hart (1985) and Aumann and Hart (2003). Let (σ, τ) be any Nash equilibrium of the communication game $\Gamma_n(p)$ for some finite $n \geq 1$, where $p^k > 0$ for all $k \in K$, with payoffs $a = (a^1, \ldots, a^K) \in \mathbb{R}^K$ for player 1 and $\beta \in \mathbb{R}$ for player 2. We construct a sequence of random variables $\boldsymbol{z} = (\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_N)$, with $N = 2n$, satisfying properties (D1) to (D3) of Theorem 2, the interim individual rationality conditions $\boldsymbol{z}_s \in I$ for all s, and the martingale property: $E[\boldsymbol{z}_{s+1} \mid \boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_s] = \boldsymbol{z}_s, \quad s = 0, 1, \ldots, N$. We work on the probability space $\Omega = K \times M_n \times J$, where $M_n = (M^1 \times M^2)^n$. A realization $\omega = (k, m_1^1, m_1^2, \ldots, m_t^1, m_t^2, \ldots, m_n^1, m_n^2, j) \in \Omega$ consists in a type for player 1, a final communication history, and an action for player 2. All random variables (denoted in bold letters when there may be a risk of confusion) are defined on Ω. Let $P = P_{\sigma,\tau,p}$ be the probability distribution on Ω generated by players' strategies and the prior probability distribution on player 1's set of types, and let $E = E_{\sigma,\tau,p}$ be the corresponding expectation operator. For example, $P[\boldsymbol{k} = k] = p^k$ and $P[\boldsymbol{m}_t^1 = m \mid \boldsymbol{h}_{t-1} = h_{t-1}, \boldsymbol{k} = k] = \sigma_t^k(h_{t-1})(m)$.

For $s = 0, \ldots, N$ we construct a new "half-steps" random variable on Ω, $\boldsymbol{g}_s$, that corresponds to every history of talk, plus every history of talk followed by player 1's message in the next period. Formally,

$$g_s \equiv \begin{cases} h_t = (m_1^1, m_1^2, \ldots, m_t^1, m_t^2), & \text{if } s = 2t \text{ is even, } t = 0, \ldots, n \\ (h_t, m_{t+1}^1), & \text{if } s = 2t+1 \text{ is odd, } t = 0, \ldots, n-1. \end{cases}$$

So, $g_0 = h_0 = \emptyset$, $g_N = g_{2n} = h_n$, when s is even the last message in g_s is from player 2, and when s is odd the last message in g_s is from player 1. We consider this new random variable in order to have the dimartingale property (D3).

Sequence of posteriors $(\boldsymbol{p}_s)_{s=0,1,\ldots,N}$. For each $k \in K$ and $s = 0, \ldots, N$, define

$$\boldsymbol{p}_s^k \equiv P[\boldsymbol{k} = k \mid \boldsymbol{g}_s],$$

and $\boldsymbol{p}_s = (\boldsymbol{p}_s^k)_{k \in K} \in \Delta(K)$.

Lemma 1 *The sequence $(\boldsymbol{p}_s^k)_{s=0,\ldots,N}$ is a (bounded) martingale satisfying*

(i) $\boldsymbol{p}_0 = p$;

(ii) $\boldsymbol{p}_{s+1} = \boldsymbol{p}_s$ *for all odd* s.

Proof. The martingale property is simply due to the fact that $(\boldsymbol{p}_s^k)_{s=0,\ldots,N}$ is a sequence of posteriors by conditioning on more and more information (it is adapted to the sequence of fields $(\mathcal{G}_s)_{s=0,\ldots,N}$ generated by $(\boldsymbol{g}_s)_{s=0,\ldots,N}$). (i) is immediate: $\boldsymbol{p}_0^k = P[\boldsymbol{k} = k \mid \boldsymbol{g}_0] = P[\boldsymbol{k} = k] = p^k$. To prove (ii), let $s = 2t+1$ be an odd number. For each $k \in K$ we have

$$\boldsymbol{p}_{s+1}^k = P[\boldsymbol{k} = k \mid \boldsymbol{g}_{s+1}] = P[\boldsymbol{k} = k \mid \boldsymbol{h}_t, \boldsymbol{m}_{t+1}^1, \boldsymbol{m}_{t+1}^2] = P[\boldsymbol{k} = k \mid \boldsymbol{h}_t, \boldsymbol{m}_{t+1}^1] = \boldsymbol{p}_s^k,$$

the last but one equality following from the fact that, conditional on $(\boldsymbol{h}_t, \boldsymbol{m}_{t+1}^1)$, $\boldsymbol{m}_{t+1}^2$ and $\boldsymbol{k}$ are independent. ∎

Sequence of player 2's payoff $(\boldsymbol{\beta}_s)_{s=0,1,\ldots,N}$. For each $s = 0, \ldots, N$, define

$$\boldsymbol{\beta}_s \equiv E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s],$$

and let $\boldsymbol{y} = \tau_{n+1}(\boldsymbol{g}_N)$.

Lemma 2 *The sequence $(\boldsymbol{\beta}_s)_{s=0,\ldots,N}$ is a (bounded) martingale satisfying*

(i) $\boldsymbol{\beta}_0 = \beta$;

(ii) $\boldsymbol{\beta}_N = \sum_{k\in K} \boldsymbol{p}_N^k B^k(\boldsymbol{y})$, *with* $\boldsymbol{y} \in Y(\boldsymbol{p}_N)$.

Proof. The martingale property is due to the fact that $(\boldsymbol{\beta}_s)_{s=0,\ldots,N}$ is a sequence of conditional expectations of a fixed random variable by conditioning on more and more information. (i) is immediate by the definition of β: $\boldsymbol{\beta}_0 = E[B^{\boldsymbol{k}}(\boldsymbol{j})] = E\left[E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k}]\right] = \sum_{k\in K} p^k E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] = \beta$. Next, we have

$$\begin{aligned}\boldsymbol{\beta}_N \equiv E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_N] = E\left[E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_N, \boldsymbol{k}]\right] &= \sum_{k\in K} P[\boldsymbol{k} = k \mid \boldsymbol{g}_N] E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_N, \boldsymbol{k} = k] \\ = \sum_{k\in K} \boldsymbol{p}_N^k E[B^k(\boldsymbol{j}) \mid \boldsymbol{g}_N] = \sum_{k\in K} \boldsymbol{p}_N^k B^k(\tau_{n+1}(\boldsymbol{g}_N)),\end{aligned}$$

the last but one equality following from the fact that, conditional on $\boldsymbol{g}_N$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are independent.[11] The equilibrium condition of player 2 implies that $\boldsymbol{y} = \tau_{n+1}(\boldsymbol{g}_N) \in Y(\boldsymbol{p}_N)$. This completes the proof of the lemma. ∎

At this stage, we have constructed $(\boldsymbol{p}_s)_{s=0,1,\ldots,N}$ and $(\boldsymbol{\beta}_s)_{s=0,1,\ldots,N}$ that have all the properties required by the theorem. It remains to construct an appropriate sequence of player 1's payoffs, which is more delicate.

[11] For the last equality, remember that we have extended B^k linearly to mixed actions.

Sequence of player 1's vector payoff $(\boldsymbol{a}_s^k)_{s=0,1,\ldots,N}$, $k \in K$. A first definition that could come to mind for the characterization of the sequence of player 1's payoffs is to simply take

$$E[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_s],$$

which is always well defined. However, it is not relevant, in general, for type k (except when $s = N$). To see this, consider a very simple example with one unilateral communication period ($N = 1$), two types of equal probability ($K = \{k_1, k_2\}$, $p^1 = p^2 = 1/2$), and assume that in the first talking period type k_1 sends message m with probability one and type k_2 sends message m' with probability one. After message m, player 2 chooses action j_1, and after message m' he chooses action j_2. Then, we would have $E[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_0] = (1/2)A^k(j_1) + (1/2)A^k(j_2)$, which is not meaningful for any type k.

A more meaningful definition of k's expected payoff is

$$E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k].$$

Unfortunately, it is not well defined when $P[\boldsymbol{g}_s = g_s \mid \boldsymbol{k} = k] = 0$, and this can happen even when $P[\boldsymbol{g}_s = g_s] > 0$. This can be seen easily in the previous example, where $E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_1 = m', \boldsymbol{k} = k_1]$ is not well defined albeit $P[\boldsymbol{g}_1 = m'] = 1/2 > 0$.

Finally, it is worth noticing that the definition used by Aumann and Hart (2003) does not work in our setup. Indeed, they define the (highest) payoff that player 1 of type k can achieve against player 2's strategy τ after the history $\boldsymbol{g}_s$ as

$$\sup_{\tilde{\sigma}} E_{\tilde{\sigma},\tau,p}[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_s],$$

where the supremum is over all strategies $\tilde{\sigma}$ of player 1 such that $P_{\tilde{\sigma},\tau,p}[\boldsymbol{g}_s \mid \boldsymbol{k} = k] > 0$. But this is not necessarily well defined in our setup even when $P[\boldsymbol{g}_s = g_s] > 0$ because a history g_s may contain a message (certificate) that *cannot* be sent by type k (for example, $g_1 = m \notin M(k)$).

Hence, we follow a different, and somehow simpler, approach. For each $k \in K$, we construct the sequence of type k's (virtual) payoff $(\boldsymbol{a}_s^k)_{s=0,1,\ldots,N}$ as follows. Let $\boldsymbol{a}_s^k = a_s^k(\boldsymbol{g}_s)$. When $P[\boldsymbol{g}_s = g_s \mid \boldsymbol{k} = k] > 0$, we define

$$a_s^k(g_s) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{k} = k],$$

which is unambiguously type k's expected payoff given the history g_s (and k). Clearly, for $s = 0$, $a_s^k(g_s)$ is always well defined: $a_0^k(g_0) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] = a^k$. More generally, assume inductively that $a_s^k(g_s)$ is well defined, i.e., assume that $P[\boldsymbol{g}_s = g_s \mid \boldsymbol{k} = k] > 0$. If $s = 2t - 1$ is odd, then $g_{s+1} = (g_s, m_t^2)$, so $P[\boldsymbol{g}_{s+1} = g_{s+1} \mid \boldsymbol{k} = k] > 0$ when $P[\boldsymbol{m}_t^2 = m_t^2 \mid \boldsymbol{g}_s = g_s] > 0$, which implies that $a_{s+1}^k(g_{s+1})$ remains well defined. If $s = 2t$ is even, then we may have a problem to define $a_{s+1}^k(g_{s+1})$ because now it is player 1's message that is added to the history: $g_{s+1} = (g_s, m_{t+1}^1)$. Indeed, we may have $P[\boldsymbol{m}_{t+1}^1 = m_{t+1}^1 \mid \boldsymbol{g}_s = g_s, \boldsymbol{k} = k] = \sigma_{t+1}^k(m_{t+1}^1 \mid h_t) = 0$ (even when $P[\boldsymbol{m}_{t+1}^1 = m_{t+1}^1 \mid \boldsymbol{g}_s = g_s] > 0$), so $P[\boldsymbol{g}_{s+1} = g_{s+1} \mid \boldsymbol{k} = k] = 0$. It that situation, we let

$$a_{s+1}^k(g_s, m_{t+1}^1) = a_s^k(g_s).$$

First, notice that the equilibrium condition of player 1 implies $a_s^k(g_s) = a_{s+1}^k(g_s, m)$ for all m such that $\sigma_{t+1}^k(m \mid g_s) > 0$. Second notice that we will have the same problem in all histories following (g_s, m_{t+1}^1) (they have probability 0 conditional on k), so we fix more generally k's payoff for all these histories: $a_{s+l}^k(g_s, m_{t+1}^1, \ldots) = a_s^k(g_s), \quad l = 1, 2 \ldots$. All this construction can be summarized formally as follows. For each $s = 0, \ldots, N$ and $k \in K$ define the random variable $\boldsymbol{f}_s^k$ as the longest subhistory of $\boldsymbol{g}_s$ satisfying $P[\boldsymbol{f}_s^k \mid \boldsymbol{k} = k] > 0$ (notice that this history necessarily ends with player 2's message, or is equal to $\boldsymbol{g}_s$), and let

$$\boldsymbol{a}_s^k = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{f}_s^k, \boldsymbol{k} = k].$$

This definition is equivalent to,

$$\boldsymbol{a}_s^k = \begin{cases} E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k], & \text{if } \boldsymbol{p}_s^k > 0 \\ \boldsymbol{a}_{\boldsymbol{r}}^k, & \text{if } \boldsymbol{p}_s^k = 0, \end{cases}$$

where $\boldsymbol{r}$ is a random variable (stopping time) which is equal to the largest r such that $\boldsymbol{p}_r^k > 0$.

Lemma 3 *For every $k \in K$, the sequence $(\boldsymbol{a}_s^k)_{s=0,\ldots,N}$ is a (bounded) martingale satisfying*

(i) $\boldsymbol{a}_0^k = a^k$*;*

(ii) $\boldsymbol{a}_{s+1}^k = \boldsymbol{a}_s^k$ *for all even s;*

(iii) If $\boldsymbol{p}_N^k > 0$, then $\boldsymbol{a}_N^k = A^k(\boldsymbol{y})$, with $\boldsymbol{y} \in Y(\boldsymbol{p}_N)$.

Proof. To prove the martingale property we must show that $E[\boldsymbol{a}_{s+1}^k \mid \boldsymbol{g}_s] = \boldsymbol{a}_s^k$, for all $s = 0, 1, \ldots, N$. If $\boldsymbol{p}_{s+1}^k = 0$, then this property is immediate because by construction we have $\boldsymbol{a}_{s+1}^k = \boldsymbol{a}_s^k = \boldsymbol{a}_r^k$, where $r \leq s$ is the largest number such that $\boldsymbol{p}_r^k > 0$. Now, consider the case $\boldsymbol{p}^k_{s+1} > 0$, and let $s = 2t-1$ be odd (when s is even, the martingale property will follow from (ii)). Thus, $\boldsymbol{p}^k_s > 0$ and $\boldsymbol{g}_{s+1} = (\boldsymbol{g}_s, \boldsymbol{m}^2_t)$, which implies

$$\begin{cases} \boldsymbol{a}^k_{s+1} = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}^2_t, \boldsymbol{k} = k] \\ \boldsymbol{a}^k_s = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k]. \end{cases}$$

So,

$$\begin{aligned} E[\boldsymbol{a}^k_{s+1} \mid \boldsymbol{g}_s] &= \sum_{m \in \text{supp}[\tau_t(\boldsymbol{g}_s)]} P[\boldsymbol{m}^2_t = m \mid \boldsymbol{g}_s] E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}^2_t = m, \boldsymbol{k} = k] \\ &= \sum_{m \in \text{supp}[\tau_t(\boldsymbol{g}_s)]} P[\boldsymbol{m}^2_t = m \mid \boldsymbol{g}_s, \boldsymbol{k} = k] E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}^2_t = m, \boldsymbol{k} = k] \\ &= E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k] = \boldsymbol{a}^k_s, \end{aligned}$$

the second equality following from the fact that $\boldsymbol{m}^2_t$ and $\boldsymbol{k}$ are independent conditional on $\boldsymbol{g}_s$. This proves the martingale property for all odd s. Property (i) is immediate: $\boldsymbol{a}^k_0 = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] = a^k$ by the definition of a^k. To prove (ii) let $s = 2t$ be even, so $\boldsymbol{g}_{s+1} = (\boldsymbol{g}_s, \boldsymbol{m}^1_{t+1})$. As before, when $\boldsymbol{p}^k_{s+1} = 0$ the property is immediate because $\boldsymbol{a}^k_{s+1} = \boldsymbol{a}^k_s = \boldsymbol{a}^k_r$, with $r \leq s$. When $\boldsymbol{p}^k_{s+1} > 0$, then $\boldsymbol{p}^k_s > 0$ and $\boldsymbol{g}_{s+1} = (\boldsymbol{g}_s, \boldsymbol{m}^1_{t+1})$, so

$$\begin{cases} \boldsymbol{a}^k_{s+1} = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}^1_{t+1}, \boldsymbol{k} = k] \\ \boldsymbol{a}^k_s = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k]. \end{cases}$$

In such a situation these two terms are equal by the equilibrium condition of player 1 since every message m^1_{t+1} player 1 of type k sends with strictly positive probability given $\boldsymbol{g}_s$ (and $\boldsymbol{k} = k$) should yield the same expected payoff to player 1 of type k:

$$\begin{aligned} \boldsymbol{a}^k_s &= \sum_{m \in \text{supp}[\sigma^k_{t+1}(\boldsymbol{g}_s)]} P[\boldsymbol{m}^1_{t+1} = m \mid \boldsymbol{g}_s, \boldsymbol{k} = k] E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}^1_{t+1} = m, \boldsymbol{k} = k] \\ &= E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}^1_{t+1} = m, \boldsymbol{k} = k], \quad \text{for all } m \in \text{supp}[\sigma^k_{t+1}(\boldsymbol{g}_s)] \\ &= \boldsymbol{a}^k_{s+1}. \end{aligned}$$

Finally, to prove (iii), assume that $\boldsymbol{p}^k_N > 0$, so

$$\begin{aligned} \boldsymbol{a}^k_N &= E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_N, \boldsymbol{k} = k] = E[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_N] \\ &= A^k(\tau_{n+1}(\boldsymbol{g}_N)) = A^k(\boldsymbol{y}), \text{ with } \boldsymbol{y} = \tau_{n+1}(\boldsymbol{g}_N) \in Y(\boldsymbol{p}_N), \end{aligned}$$

the second equality following from the fact that $\boldsymbol{j}$ and $\boldsymbol{k}$ are independent conditional on

$\boldsymbol{g}_N$, and the last from the equilibrium condition of player 2. ■

Lemma 4 *For every $s = 0, 1, \ldots, N$ we have $\boldsymbol{a}_s \in \mathrm{INTIR}_{\mathrm{supp}[\boldsymbol{p}_s]}$.*

Proof. Let us fix a history g_s such that $P[\boldsymbol{g}_s = g_s] > 0$ and let $\mathrm{supp}[p_s] \subseteq K$, $\mathrm{supp}[p_s] \neq \emptyset$, be the set of types with a strictly positive posterior probability: $p_s^k = P[\boldsymbol{k} = k \mid \boldsymbol{g}_s = g_s] > 0$ for all $k \in \mathrm{supp}[p_s]$. We must show that there exists $\overline{y} \in \Delta(J)$ such that

$$E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{k} = k] \geq A^k(\overline{y}), \text{ for all } k \in \mathrm{supp}[p_s].$$

Player 1's equilibrium condition implies (in particular) that, whatever his type $k \in \mathrm{supp}[p_s]$, if he sends the same message $\overline{m} \in \bigcap_{k \in K} M(k)$ in all upcoming periods $t' \geq \overline{t}$ (where $\overline{t} = (s+2)/2$ if s is even, and $\overline{t} = (s+3)/2$ if s is odd), then his expected payoff in the current period ($s/2$ if s is even, and $(s+1)/2$ if s is odd) is not increased, so

$$E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{k} = k] \geq E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}_{t'}^1 = \overline{m} \,\forall\, t' \geq \overline{t}, \boldsymbol{k} = k], \text{ for all } k \in \mathrm{supp}[p_s].$$

The right hand side only depends on player 2's strategy and is thus well defined. As a consequence, given $\boldsymbol{g}_s = g_s$ and $\boldsymbol{m}_{t'}^1 = \overline{m} \;\forall\; t' \geq \overline{t}$, which specifies the sequence of all player 1's messages in the talking phase, $\boldsymbol{j}$ and $\boldsymbol{k}$ are independent. This implies

$$\begin{aligned} E\left[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}_{t'}^1 = \overline{m} \,\forall\, t' \geq \overline{t}, \boldsymbol{k} = k\right] &= E\left[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}_{t'}^1 = \overline{m} \,\forall\, t' \geq \overline{t}\right] \\ &= A^k\left(E\left[\tau_{n+1}(\boldsymbol{g}_N) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}_{t'}^1 = \overline{m} \,\forall\, t' \geq \overline{t}\right]\right). \end{aligned}$$

(Remember that we have extended linearly A^k to mixed actions.) Hence, by letting

$$\overline{y} = E\left[\tau_{n+1}(\boldsymbol{g}_N) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}_{t'}^1 = \overline{m} \,\forall\, t' \geq \overline{t}\right],$$

which does not depend on k (it only depends on g_s and $\overline{m}$), we have completed the proof of the lemma. ■

As we have already mentioned, $(\boldsymbol{p}_s)_{s=0,1,\ldots,N}$ and $(\boldsymbol{\beta}_s)_{s=0,1,\ldots,N}$ have all the properties required by Theorem 2 by Lemma 1 and Lemma 2. By Lemma 3 and Lemma 4, the sequence $(\boldsymbol{a}_s)_{s=0,1,\ldots,N}$ also satisfies all the properties of the theorem. This completes the proof of the "only if" part of Theorem 2. ■

6.2 From constrained dimartingale to equilibrium: $H_B(p) \subseteq \mathcal{E}_B(p)$

Let $\boldsymbol{z} = (\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_N)$ be a martingale over some probability space $(F, \mathcal{F}, \pi)$ and (finite) sub σ-fields $(\mathcal{F}_t)_{t=1,\ldots,N}$, satisfying the properties of Theorem 2, with $p^k > 0$ for all $k \in K$, and $N = n$. We construct a Nash equilibrium (σ, τ) of the n-stage communication game $\Gamma_n(p)$ with expected payoffs (a, β). First, for convenience we define the martingale $\boldsymbol{z}$ on the nodes of a probability tree. We introduce a set W with $K+1$ elements, write F as W^N, and the atoms of $\mathcal{F}_t$ as elements g_t of W^t. We thus describe the martingale $\boldsymbol{z}$ as

$$\boldsymbol{z} = (z_t(\boldsymbol{g}_t))_{t=0,1,\ldots,n},$$

where for each $t = 0, 1, \ldots, n$, $\boldsymbol{g}_t \in W^t$, and

$$z_t(g_t) = (a_t(g_t), \beta_t(g_t), p_t(g_t)) = \sum_{w \in \operatorname{supp}[\pi(\cdot \mid g_t)]} \pi(w \mid g_t)\, z_{t+1}(g_t, w),$$

for all $g_t \in W^t$ satisfying $\pi(g_t) > 0$ (this is the martingale property). Notice that this implies $E[\boldsymbol{z}_t] = E[z_t(\boldsymbol{g}_t)] = \sum_{g_t \in W^t} \pi(g_t)\, z_t(g_t) = z_0$, $t = 0, 1, \ldots, n$. The properties of the martingale in Theorem 2 can be restated as follows:

(D1) $z_0(g_0) = z_0 = (a, \beta, p)$.

(D2) If $\pi(g_n) > 0$, then $(a_n(g_n), \beta_n(g_n)) \in \mathcal{E}(p_n(g_n))$.

(D3) $a_{t+1}(g_{t+1}) = a_t(g_t)$ for all even t and $p_{t+1}(g_{t+1}) = p_t(g_t)$ for all odd t, if $\pi(g_{t+1}) > 0$.

The interim individual rationality conditions for player 1 are restated as: for all $t = 0, 1, \ldots, n$, if $\pi(g_t) > 0$, then $a_t(g_t) \in \mathrm{INTIR}_{\operatorname{supp}[p_t(g_t)]}$.

In odd periods t, w_t is associated to a message $m_t^1 \in M^1$ of player 1 (player 2's message does not affect players' decisions at these periods), and in even periods t, w_t is directly associated to a jointly controlled lottery (possibly a series of jointly controlled lotteries), which is not explicitly formalized here.[12] Therefore, a history of messages h_n consists, with some abuse of notation, in a message $m_t^1 \in M^1$ of player 1 in each odd period t, and in a realization $w_t \in W$ of one or several jointly controlled lotteries in each even period t. Accordingly, in the remaining of the proof we only construct explicitly player 1's strategy σ_{t+1}^k, $k \in K$, when t is even, and player 2's strategy in the action phase, τ_{n+1}. The set of

[12]The technique is standard; see, e.g., Aumann and Maschler (1995) and Aumann and Hart (2003). Note that irrational probabilities might lead to infinitely many jointly controlled lotteries (see Subsection 7.2). For simplicity, the reader may simply consider w_t as a signal publicly observed in even periods.

histories of the talking phase up to period t is

$$M_t = \begin{cases} (M^1 \times W)^{t/2} & \text{if } t \text{ is even,} \\ (M^1 \times W)^{(t-1)/2} \times W & \text{if } t \text{ is odd.} \end{cases}$$

To each sequence $g_t = (w_1, \ldots, w_t) \in W^t$ we associate a history $\phi_t(g_t) \in M_t$, with $\phi_t(g_t) \neq \phi_t(g_t')$ whenever $g_t \neq g_t'$, as follows:

$$\begin{aligned} \phi_t(g_t) &= \phi_t(w_1, w_2, w_3, w_4 \ldots, w_t) \\ &= (m_1(w_1), w_2, m_3(g_3), w_4, \ldots), \end{aligned}$$

where $g_r = (w_1, \ldots, w_r)$, $r < t$, is a subsequence of g_t, and for all odd t, $m_t(g_t) \in M^1$, $m_t(g_{t-1}, w_t) \neq m_t(g_{t-1}, w_t')$ whenever $w_t \neq w_t'$, and

$$M^{-1}(m_t(g_t)) = \text{supp}[p_t(g_t)].$$

Player 1's strategy σ. For each even period $t = 0, 2, 4, \ldots$, each sequence $g_t \in W^t$ with strictly positive probability and each type $k \in \text{supp}[p_t(g_t)]$ we construct player 1's local strategy $\sigma_{t+1}^k(\phi_t(g_t))$. For each $w \in \text{supp}[\pi(\cdot \mid g_t)]$, define

$$\sigma_{t+1}^k(m_{t+1}(g_t, w) \mid \phi_t(g_t)) = \frac{\pi(w \mid g_t)\, p_{t+1}^k(g_t, w)}{p_t^k(g_t)},$$

and $\sigma_{t+1}^k(m \mid \phi_t(g_t)) = 0$ if $m \neq m_{t+1}(g_t, w)$ for all $w \in W$.

Player 2's strategy τ. We construct the local strategy $\tau_{n+1}(h_n)$ of player 2 for each final history of talk $h_n \in M_n$, with *and* without strictly positive probability (players' strategies in the talking phase are irrelevant off the equilibrium path, but player 2's strategy in the action phase is very important even after 0-probability histories).

If $\pi(g_n) > 0$ for $g_n \in W^n$, then by the second property of the martingale assumed in the theorem, $(a_n(g_n), \beta_n(g_n)) \in \mathcal{E}(p_n(g_n))$, so we can define,

$$y(g_n) = \tau_{n+1}(\phi_n(g_n)) \in Y(p_n(g_n)) \text{ such that } \begin{cases} a_n^k(g_n) = A^k(y(g_n)) \text{ if } p_n^k(g_n) > 0 \\ \beta_n(g_n) = \sum_{k \in K} p_n^k(g_n)\, B^k(y(g_n)). \end{cases}$$

If $\pi(g_n) = 0$ for $g_n \in W^n$, then consider the shortest subsequence $g_t = (w_1, w_2, \ldots, w_t)$

of $g_n = (w_1, w_2, \ldots, w_n)$ (note: t may be 0) such that $\pi(g_t) > 0$ and define

$$\tau_{n+1}(\phi_n(g_n)) = \overline{y} \ \text{ such that } \ a_t^k(g_t) \geq A^k(\overline{y}) \ \text{ for all } k \in \operatorname{supp}[p_t(g_t)].$$

This is possible by the individual rationality conditions of the martingale.

The strategy profile (σ, τ) of the communication game $\Gamma_n(p)$ is now completely defined (except, as explained above, for the JCL). We next check that it generates the appropriate expected payoffs and that it constitutes a Nash equilibrium of $\Gamma_n(p)$. Let $P = P_{\sigma,\tau,p}$ be the probability distribution on $\Omega = K \times M_n \times J$ induced by (σ, τ) and p, and let $E = E_{\sigma,\tau,p}$ be the corresponding expectation operator.[13]

Lemma 5 *For all $t = 0, 1, \ldots, n$ and $g_t \in W^t$ we have:*

(i) $P[\boldsymbol{h}_t = \phi_t(g_t)] = \pi(g_t)$;

(ii) $P[\boldsymbol{k} = k \mid \boldsymbol{h}_t = \phi_t(g_t)] = p_t^k(g_t)$ for all $k \in K$, $\pi(g_t) > 0$.

Proof. By induction on t. For $t = 0$ property (ii) is immediate: $P[\boldsymbol{k} = k] = p^k = p_0^k(g_0)$. For $t = 1$:

(i) We have:

$$\begin{aligned} P[\boldsymbol{h}_1 = \phi_1(g_1)] &= \sum_{k \in K} p^k P[\boldsymbol{h}_1 = \phi_1(g_1) \mid \boldsymbol{k} = k] \\ &= \sum_{k \in K} p^k \sigma_1^k(\phi_1(g_1)) = \sum_{k \in K} p^k \sigma_1^k(m_1(g_1)) \\ &= \sum_{k \in K} p^k \frac{\pi(g_1)\, p_1^k(g_1)}{p^k} = \pi(g_1) \sum_{k \in K} p_1^k(g_1) = \pi(g_1). \end{aligned}$$

(ii) We have:

$$\begin{aligned} P[\boldsymbol{k} = k \mid \boldsymbol{h}_1 = \phi_1(g_1)] &= \frac{P[\boldsymbol{h}_1 = \phi_1(g_1) \mid \boldsymbol{k} = k] P[\boldsymbol{k} = k]}{P[\boldsymbol{h}_1 = \phi_1(g_1)]} \\ &= \frac{\sigma_1^k(m_1(g_1))\, p^k}{P[\boldsymbol{h}_1 = \phi_1(g_1)]} = \frac{\sigma_1^k(m_1(g_1))\, p^k}{\pi(g_1)} \text{ by (i) just above} \\ &= \frac{\pi(g_1)\, p_1^k(g_1)}{p_0^k} \frac{p^k}{\pi(g_1)} = p_1^k(g_1). \end{aligned}$$

Now assume that properties (i) and (ii) are satisfied at t, and let us check them at $t + 1$. We distinguish two cases: (a) t is odd, i.e., a JCL is added in $t + 1$; (b) t is

[13]Since JCL are not formalized, P and E also depend on π for the realizations $w_t \in W$ of JCL (public signals) in even periods.

even, i.e., player 1's signal is added in $t+1$. Case (a) is simpler because we can exploit the fact that the JCL does not depend on k. In the rest of the proof of the lemma, let $g_{t+1} = (g_t, w_{t+1}) \in W^{t+1}$.

(a) (i) Since $t+1$ is even we have:

$$\begin{aligned}
P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})] &= P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), w_{t+1})] \\
&= P[\boldsymbol{h}_t = \phi_t(g_t)]\, P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), w_{t+1}) \mid \boldsymbol{h}_t = \phi_t(g_t)] \\
&= \pi(g_t)\, \pi(w_{t+1} \mid g_t), \quad \text{by property (i) at } t \\
&= \pi(g_t, w_{t+1}) = \pi(g_{t+1}).
\end{aligned}$$

(a) (ii) Since $t+1$ is even we have:

$$\begin{aligned}
P[\boldsymbol{k} = k \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})] &= P[\boldsymbol{k} = k \mid \boldsymbol{h}_{t+1} = (\phi_t(g_t), w_{t+1})] \\
&= P[\boldsymbol{k} = k \mid \boldsymbol{h}_t = \phi_t(g_t)] \text{ because } w_{t+1} \text{ and } k \text{ are independent} \\
&= p_t^k(g_t) \text{ by property (ii) at } t \\
&= p_{t+1}^k(g_{t+1}) \text{ by the third property of the martingale.}
\end{aligned}$$

(b) (i) Since $t+1$ is odd we have:

$$\begin{aligned}
P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})] &= P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), m_{t+1}(g_{t+1})] \\
&= P[\boldsymbol{h}_t = \phi_t(g_t)]\, P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), m_{t+1}(g_{t+1})) \mid \boldsymbol{h}_t = \phi_t(g_t)] \\
&= \pi(g_t)\, P[\boldsymbol{m}_{t+1} = m_{t+1}(g_{t+1}) \mid \boldsymbol{h}_t = \phi_t(g_t)], \quad \text{by property (i) at } t \\
&= \pi(g_t) \sum_{k \in K} p_t^k(g_t)\, \sigma_{t+1}^k(m_{t+1}(g_{t+1}) \mid \phi_t(g_t)) \\
&= \pi(g_t) \sum_{k \in K} p_t^k(g_t) \frac{\pi(w_{t+1} \mid g_t)\, p_{t+1}^k(g_{t+1})}{p_t^k(g_t)} \\
&= \pi(g_t)\, \pi(w_{t+1} \mid g_t) \sum_{k \in K} p_{t+1}^k(g_{t+1}) \\
&= \pi(g_t)\, \pi(w_{t+1} \mid g_t) = \pi(g_t, w_{t+1}) = \pi(g_{t+1}).
\end{aligned}$$

(b) (ii) Since $t+1$ is odd we have:

$$
\begin{aligned}
P[\boldsymbol{k}=k \mid \boldsymbol{h}_{t+1}=\phi_{t+1}(g_{t+1})] &= \frac{P[\boldsymbol{h}_{t+1}=\phi_{t+1}(g_{t+1}) \mid \boldsymbol{k}=k]P[\boldsymbol{k}=k]}{P[\boldsymbol{h}_{t+1}=\phi_{t+1}(g_{t+1})]} \\
&= \frac{P[\boldsymbol{h}_{t+1}=\phi_{t+1}(g_{t+1}) \mid \boldsymbol{h}_t=\phi_t(g_t), \boldsymbol{k}=k]P[\boldsymbol{h}_t=\phi_t(g_t) \mid \boldsymbol{k}=k]P[\boldsymbol{k}=k]}{P[\boldsymbol{h}_{t+1}=\phi_{t+1}(g_{t+1})]} \\
&= \frac{P[\boldsymbol{m}_{t+1}=m_{t+1}(g_{t+1}) \mid \boldsymbol{h}_t=\phi_t(g_t), \boldsymbol{k}=k]P[\boldsymbol{h}_t=\phi_t(g_t) \mid \boldsymbol{k}=k]P[\boldsymbol{k}=k]}{\pi(g_{t+1})} \\
&= \frac{\sigma^k_{t+1}(m_{t+1}(g_{t+1}) \mid \phi_t(g_t))P[\boldsymbol{h}_t=\phi_t(g_t)]P[\boldsymbol{k}=k \mid \boldsymbol{h}_t=\phi_t(g_t)]}{\pi(g_{t+1})},
\end{aligned}
$$

the last but one equality following from property (i) at $t+1$, which has been checked just before. By properties (i) and (ii) at t this yields:

$$
\begin{aligned}
P[\boldsymbol{k}=k \mid \boldsymbol{h}_{t+1}=\phi_{t+1}(g_{t+1})] &= \frac{\sigma^k_{t+1}(m_{t+1}(g_{t+1}) \mid \phi_t(g_t))\pi(g_t)p^k_t(g_t)}{\pi(g_{t+1})} \\
&= \frac{\pi(w_{t+1} \mid g_t)\, p^k_{t+1}(g_{t+1})}{p^k_t(g_t)} \frac{p^k_t(g_t)\pi(g_t)}{\pi(g_{t+1})} = p^k_{t+1}(g_{t+1}).
\end{aligned}
$$

This completes the proof of Lemma 5. ■

Lemma 6 *We have:*

(i) $E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k}=k] = a^k$ *for all* $k \in K$*;*

(ii) $E[B^{\boldsymbol{k}}(\boldsymbol{j})] = \beta$.

Proof. (i) We show by induction on t (starting from $t=n$) that, for $t=0,1,\ldots,n$,

$$a^k_t(g_t) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_t=\phi_t(g_t), \boldsymbol{k}=k], \quad \forall\, k \in \operatorname{supp}[p_t(g_t)]. \tag{1}$$

In particular, for $t=0$, this will lead to what we are required to prove:

$$a^k = a^k_0(g_0) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_0=\phi_0(g_0), \boldsymbol{k}=k] = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k}=k].$$

Let $t=n$. If $k \in \operatorname{supp}[p_n(g_n)]$, then, by the construction of player 2's strategy,

$$
\begin{aligned}
a^k_n(g_n) &= A^k(\tau_{n+1}(\phi_n(g_n))) \\
&= E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_n=\phi_n(g_n), \boldsymbol{k}=k],
\end{aligned}
$$

so property (1) is satisfied for $t=n$. Now assume that the property is satisfied at $t+1$

and let us check it at t. Let $k \in \mathrm{supp}[p_t(g_t)]$. By the martingale property, we have

$$a_t^k(g_t) = \sum_{w \in \mathrm{supp}[\pi(\cdot|g_t)]} \pi(w \mid g_t)\, a_{t+1}^k(g_t, w).$$

We distinguish two cases: when t is odd and when t is even.

If t is odd. Then, $p_{t+1}(g_t, w) = p_t(g_t)$ for all $w \in \mathrm{supp}[\pi(\cdot \mid g_t)]$, which implies $\mathrm{supp}[p_{t+1}(g_t, w)] = \mathrm{supp}[p_t(g_t)]$, so $k \in \mathrm{supp}[p_{t+1}(g_t, w)]$ for all $w \in \mathrm{supp}[\pi(\cdot \mid g_t)]$. Therefore, by the induction hypothesis, for all $w \in \mathrm{supp}[\pi(\cdot \mid g_t)]$ we have

$$a_{t+1}^k(g_t, w) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k],$$

so

$$\begin{aligned}
a_t^k(g_t) &= \sum_{w \in \mathrm{supp}[\pi(\cdot|g_t)]} \pi(w \mid g_t)\, E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k] \\
&= \sum_{w \in \mathrm{supp}[\pi(\cdot|g_t)]} P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), w) \mid \boldsymbol{h}_t = \phi_t(g_t)]\; E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k] \\
&= \sum_{w \in \mathrm{supp}[\pi(\cdot|g_t)]} P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), w) \mid \boldsymbol{h}_t = \phi_t(g_t), \boldsymbol{k} = k]\; E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k] \\
&= E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_t = \phi_t(g_t), \boldsymbol{k} = k].
\end{aligned}$$

If t is even. Then, $a_{t+1}^k(g_t, w) = a_t^k(g_t)$ for all $w \in \mathrm{supp}[\pi(\cdot \mid g_t)]$, which implies, by the induction hypothesis,

$$a_t^k(g_t) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k],$$

for all w such that $p_{t+1}^k(g_t, w) > 0$. Hence, $a_t^k(g_t)$ is also equal to any average of the previous value, so we get property (1) at t.

(ii) Player 2's expected payoff is

$$\begin{aligned}
E[B^{\boldsymbol{k}}(\boldsymbol{j})] &= \sum_{k\in K} p^k E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k}=k] \\
&= \sum_{k\in K} p^k \sum_{h_n\in M_n} P[\boldsymbol{h}_n = h_n \mid \boldsymbol{k}=k]\, E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k}=k, \boldsymbol{h}_n = h_n] \\
&= \sum_{k\in K} p^k \sum_{h_n\in M_n} P[\boldsymbol{h}_n = h_n \mid \boldsymbol{k}=k] \sum_{j\in J} \tau_{n+1}(h_n)(j) B^k(j) \\
&= \sum_{k\in K} p^k \sum_{h_n\in M_n} P[\boldsymbol{h}_n = h_n \mid \boldsymbol{k}=k]\, B^k(\tau_{n+1}(h_n)) \\
&= \sum_{h_n\in M_n} P[\boldsymbol{h}_n = h_n] \sum_{k\in K} P[\boldsymbol{k}=k \mid \boldsymbol{h}_n = h_n]\, B^k(\tau_{n+1}(h_n)) \\
&= \sum_{g_n\in W^n} \pi(g_n) \sum_{k\in K} p_n^k(g_n)\, B^k(\tau_{n+1}(\phi_n(g_n))), \text{ by Lemma 5} \\
&= \sum_{g_n\in W^n} \pi(g_n)\beta_n(g_n), \text{ by the construction of player 2's strategy} \\
&= E[\boldsymbol{\beta}_n] = \beta_0 = \beta.
\end{aligned}$$

This completes the proof of Lemma 6. ■

Lemma 7 *The strategy τ of player 2 is a best reply to the strategy σ of player 1 in the n-stage communication game $\Gamma_n(p)$.*

Proof. Since $\tau_{n+1}(\phi_n(g_n)) \in Y(p_n(g_n))$ for $\pi(g_n) > 0$ it suffices to check that $p_n^k(g_n) = P[\boldsymbol{k} = k \mid \boldsymbol{h}_n = \phi_n(g_n)]$ for all $k \in K$. This as been proved in Lemma 5 (property (ii) with $t = n$). ■

Lemma 8 *The strategy σ of player 1 is a best reply to the strategy τ of player 2 in the n-stage communication game $\Gamma_n(p)$.*

Proof. Fix t even, g_t such that $\pi(g_t) > 0$ and w such that $\pi(w \mid g_t) > 0$. Assume that player 1's type k is such that $p_t^k(g_t) > 0$. The strategy σ prescribes to send message $m_{t+1}(g_t, w)$ with probability $\sigma_{t+1}^k(m_{t+1}(g_t, w) \mid \phi_t(g_t)) > 0$ and any message which is not of the form $m_{t+1}(g_t, w)$ with probability 0. By construction, player 1 of type k is not able to send a message m of the form $m_{t+1}(g_t, w')$ with $p_{t+1}^k(g_t, w') = 0$, namely a message m that is sent along the equilibrium path but is not sent by type k. Furthermore, by the interim individually rational condition, player 1 cannot profit from sending a message m off the equilibrium path, namely a message m not of the form $m_{t+1}(g_t, w')$. Finally, if from stage $t + 2$ on, player 1 follows the prescribed strategy σ, he cannot gain at stage

$t+1$ by sending $m_{t+1}(g_t, w)$ with a probability different from $\sigma^k_{t+1}(m_{t+1}(g_t, w) \mid \phi_t(g_t))$. Indeed, by the dimartingale property (D3) on page 30 and property (1) on page 34, he is indifferent between all the allowed messages. Hence, by an induction argument, player 1 cannot gain by manipulating the probabilities of allowed messages. ■

By Lemmas 6, 7 and 8, we have constructed the appropriate strategy profile. This completes the proof of Theorem 2. ■

7 Discussion and Extensions

7.1 Mediated Persuasion

In this paper we assumed that communication between the expert and the decisionmaker takes place face-to-face. This excludes correlated extraneous signals and private recommendations. In particular, there is no uncertainty on the messages received by each party during the talking phase. If a mediator were available and if any form of costless communication were possible between the players, then the resulting set of equilibrium outcomes would be the set of *certification equilibrium* outcomes introduced by Forges and Koessler (2005). Under the assumption of full certifiability made it the current paper, a single stage of mediated certification is sufficient and the set of certification equilibrium outcomes has a canonical representation characterized by a transition probability $\mu : K \to \Delta(J)$ and a punishment strategy $\overline{y} \in \Delta(J)$ satisfying the informational incentive constraint

$$A^k(\mu(\cdot \mid k)) \equiv \sum_{j \in J} \mu(j \mid k) A^k(j) \geq A^k(\overline{y}) \text{ for all } k \in K, \tag{2}$$

and the strategic incentive constraint

$$\begin{aligned} &\sum_{k \in K} p^k \sum_{j \in J} \mu(j \mid k) B^k(j) \geq \sum_{k \in K} p^k \sum_{j \in J} \mu(j \mid k) B^k(d(j)), \quad \forall\, d : J \to J \\ \Leftrightarrow \quad &\sum_{k \in K} \Pr_\mu(k \mid j) B^k(j) \geq \sum_{k \in K} \Pr_\mu(k \mid j) B^k(j'), \quad \forall\, j \in \operatorname{supp}[\mu],\ j' \in J. \end{aligned} \tag{3}$$

(The proof of this claim can be found in Forges and Koessler, 2005). Let $\mathcal{E}_M(p) \subseteq \mathbb{R}^K \times \mathbb{R}$ be the resulting set of mediated certification equilibrium payoffs. This set includes the set of equilibrium payoffs achieved with face-to-face communication, so $\mathcal{E}(p) \subseteq \mathcal{E}_S(p) \subseteq \mathcal{E}_B(p) \subseteq \mathcal{E}_M(p)$, and all these inclusions may be strict.

The set of *communication equilibrium* outcomes (Myerson, 1982; Forges, 1986) is characterized by recommendations satisfying (3) and (4):

$$\sum_{j\in J} \mu(j \mid k) A^k(j) \geq \sum_{j\in J} \mu(j \mid k') A^k(j) \text{ for all } k,\, k' \in K. \tag{4}$$

Since condition (4) is a stronger requirement than (2), the set of certification equilibrium outcomes also includes the set of communication equilibrium outcomes.

The analysis is much more tractable when a mediator is available to help the players to communicate and to certify their information.[14] For example, the equilibrium outcome with three talking stages of the introductory example (see Figure 3 on page 8) can easily be implemented with the help of a mediator as follows. First, player 1 chooses whether to make a certifiable report to the mediator concerning the true state of the world. When there are only two types, player 1 has two possible reports in every state k: either he certifies his information by sending message c^k or he certifies nothing. Afterwards, the mediator gives a (random) recommendation of action to player 2 conditionally on the report of player 1. Denote respectively by $\mu(j \mid k)$ and $\overline{y}(j)$ the probabilities that the mediator recommends action j to player 2 when player 1 sends message c^k and $m \neq c^1$, c^2, respectively. The following recommendations mimic the equilibrium outcome:

$$\begin{aligned} &\mu(j_4 \mid k_1) = \mu(j_5 \mid k_1) = 3/8 && \mu(j_2 \mid k_1) = 1/4 \\ &\mu(j_1 \mid k_2) = \mu(j_4 \mid k_2) = 1/8 && \mu(j_2 \mid k_2) = 3/4 \\ &\overline{y}(j_3) = 1. \end{aligned}$$

If player 1 completely certifies his information and player 2 follows the recommendation of the mediator, then no player has an incentive to deviate. Indeed, player 1 never deviates since by certifying his information his payoff is always strictly positive, whereas by not certifying his information his payoff would be zero. From Bayes' rule, player 2's beliefs about the state of Nature given the recommendations of the mediator are $\Pr_\mu(k_1 \mid j_5) = 1$, $\Pr_\mu(k_1 \mid j_4) = 3/4$, $\Pr_\mu(k_1 \mid j_2) = 1/4$ and $\Pr_\mu(k_1 \mid j_1) = 0$, so the recommendations are optimal for him given his beliefs.

7.2 Persuasion without a Deadline

Throughout this paper, we assumed that an arbitrarily large maximum number of communication stages was fixed in advance, namely that players were constrained by a deadline.

[14] In particular, certification equilibrium outcomes can be characterized in a canonical way for Bayesian games with any number of players, any information structure, and any assumption on certifiability possibilities.

In the case of cheap talk, Forges (1984, 1990a) shows that new equilibrium outcomes can be reached if no deadline is imposed to the (almost surely finite) players' conversations, namely if the length of this conversation is endogenously determined by the equilibrium strategies. The same phenomenon obviously occurs in the more general model of this paper and our results are easily adapted so as to cover almost surely finite, long persuasion without a deadline. One simply has to consider dimartingales which converge almost surely in a finite, but not uniformly bounded, number of stages. Aumann and Hart (2003) go further by considering *any* dimartingales, in particular those which do not almost surely converge in finitely many stages. In Aumann and Hart's (2003) model, time has order $\omega + 1$; that is, there is an infinite sequence of time periods, with an additional period *after* the whole sequence. This approach, which entails conceptual and technical difficulties (see Aumann and Hart, 2003, Sections 4.2 and 8), is not, at least today, sustained by any game-theoretical example (see however Aumann and Hart, 1986, for a mathematical example). In the cheap talk case, Krishna (2005) provides sufficient conditions for the set of equilibrium payoffs from infinite conversations to be the same as the set of equilibrium payoffs from conversations which are finite with probability one.

7.3 Sequential Rationality

It is well known that in usual (one-shot, unilateral) cheap talk games, standard equilibrium refinements do not eliminate any Nash equilibrium outcome. In particular, the non-revealing equilibrium outcome is always a sequential equilibrium outcome. In long cheap talk games, we are not aware of any example with a Nash equilibrium outcome that cannot be sustained by sequentially rational strategies. But the characterization of the set of sequential equilibrium payoffs in long cheap talk games is still an open problem.

When information is certifiable, it is very easy to construct games where the set of sequential equilibrium outcomes is strictly included in the set of Nash equilibrium outcomes. For instance, player 2 is not sequentially rational off the equilibrium path when player 1 fully certifies his type in the non-revealing equilibrium of the introductory example for $p \in (2/5, 3/5)$, in the second partially revealing equilibrium (PRE2) of the introductory example for $p \in (0, 3/5)$, in the non-revealing equilibrium of Example 1 in the appendix for $p \in (0, 1)$, and in the non-revealing equilibrium of Example 2 in the appendix for $p \in (0, 2/3)$. Likewise, in Example 3 in the appendix, if we add a third action (a worst outcome) yielding a negative payoff to both players in both states, then there would be a fully revealing Nash equilibrium payoff which cannot be sustained by any sequentially rational strategy for player 2.

If we want player 2's strategy to be sequentially rational in the action phase, then we have to strengthen player 1's interim individual rationality condition. That is, the punishment strategy which is used by player 2 off the equilibrium path must be optimal for player 2 for at least one belief over K consistent with the history of certificates sent by player 1. More precisely, in our geometric characterizations, we should replace INTIR_L by

$$\mathrm{INTIR}^*_L \equiv \{a \in \mathbb{R}^K : \forall\, M \subseteq L,\, \exists\, p_M \in \Delta(M) \text{ and } \overline{y}_M \in Y(p_M),\; a^k \geq A^k(\overline{y}_M)\; \forall\, k \in M\}.$$

Notice that subgame perfection is obtained as a special case when events M in the previous equation are reduced to singletons. It is also interesting to remark that with this modification, the set of equilibrium payoffs that we obtain does not include, in general, Aumann and Hart's (2003) set anymore. For example, as we noticed above, non-revealing equilibrium payoffs do not always belong to the former set.

7.4 Partial Certifiability

As we noticed in Footnote 6, our results do not require that all events are certifiable with a single message when multiple stages of communication are allowed. However, if the condition of the message correspondence M in Footnote 6 is not satisfied, partial certifiability may significantly complicate the analysis and restrict the set of equilibrium outcomes. This is true even in long persuasion games in which every single type is fully certifiable. For example, in the silent game of Figure 9, the equilibrium payoff $((0,1,1),1) \in \mathcal{E}_S(p)$ can be obtained in the persuasion game when player 1 certifies that his type belongs to $\{k_2, k_3\}$, but is not an equilibrium payoff of the persuasion game if there is no message m such that $M^{-1}(m) = \{k_2, k_3\}$. The general geometric characterization of equilibrium payoffs of long persuasion games with partially certifiable types is left for further research.

	j_1	j_2
k_1	$0, 1$	$1, -2$
k_2	$0, 2$	$1, 1$
k_3	$0, -2$	$1, 1$

Figure 9:

A Appendix: Simple Examples

Example 1 (Full revelation without certification) In the silent game of Figure 10, the non-revealing equilibria are

$$Y(p) = \begin{cases} \{j_1\} & \text{if } p > 3/4, \\ \{j_2\} & \text{if } p < 3/4, \\ \Delta(J) & \text{if } p = 3/4. \end{cases}$$

The corresponding interim individually rational equilibrium payoffs of the expert are represented by Figure 11 in solid lines. They coincide with Aumann and Hart's (2003) modified non-revealing equilibrium payoffs, so a fully revealing equilibrium (FRE) exists in the communication game whether or not the expert's types are certifiable.

	j_1	j_2	
k_1	$1,1$	$0,0$	p
k_2	$0,0$	$3,3$	$(1-p)$

Figure 10: Silent game of Example 1.

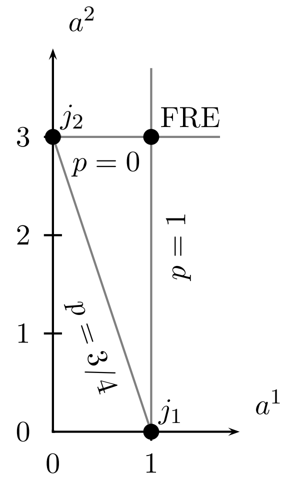

Figure 11: Modified and interim individually rational non-revealing equilibrium payoffs of the expert in Example 1.

Example 2 (Full revelation with certification) In the silent game of Figure 12, the expert always wants the decisionmaker to choose the same action whatever his type. The interim individually rational non-revealing equilibrium payoffs of the expert are represented by Figure 13 in solid and dashed lines. Here, information transmission is not possible with cheap talk only (the solid lines never intercept), while a FRE exists when the expert's types are certifiable.

	j_1	j_2	
k_1	$3, 2$	$1, 0$	p
k_2	$3, 0$	$1, 4$	$(1-p)$

Figure 12: Silent game of Example 2.

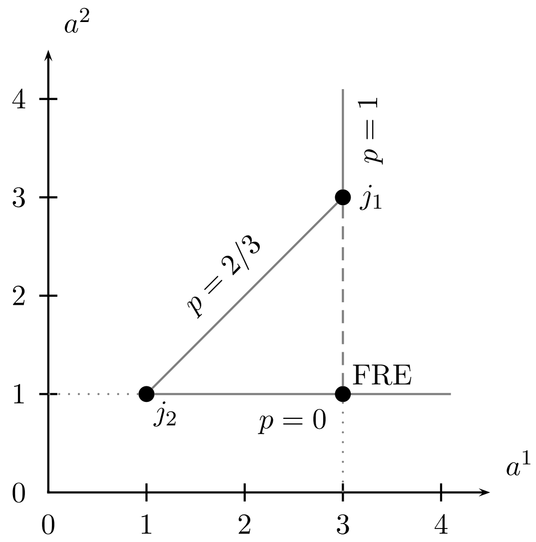

Figure 13: Modified (solid lines) and interim individually rational (solid and dashed lines) non-revealing equilibrium payoffs of the expert in Example 2.

Example 3 (No revelation) In the silent game of Figure 14, cheap talk and information certification cannot matter. The optimal actions of the decisionmaker are the same as in Example 2. The corresponding interim individually rational non-revealing equilibrium payoffs of the expert are represented by Figure 15 in solid lines. The dotted lines do not belong to the set of interim individually rational payoffs, so the persuasion game does not admit a fully revealing equilibrium.

Example 4 (Partial revelation without certification) In the silent game of Figure 16, the interim individually rational non-revealing equilibrium payoffs of the expert are rep-

	j_1	j_2	
k_1	$3,2$	$4,0$	p
k_2	$3,0$	$1,4$	$(1-p)$

Figure 14: Silent game of Example 3.

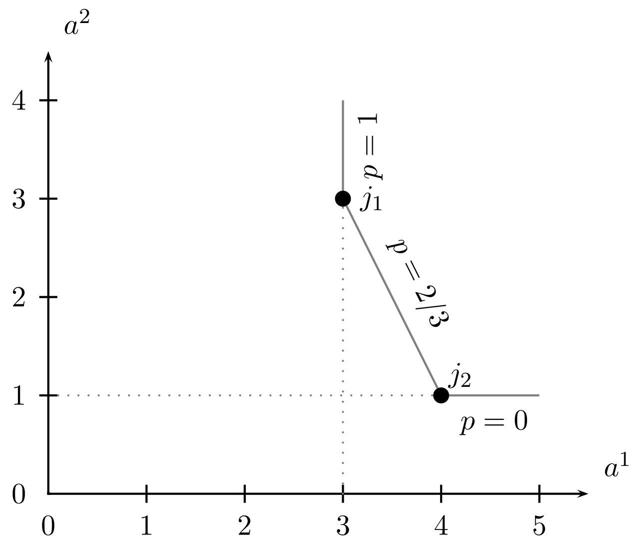

Figure 15: Modified and interim individually rational non-revealing equilibrium payoffs of the expert (solid lines) in Example 3.

resented by Figure 17 in solid lines. As in Example 3, this game does not admit a fully revealing equilibrium (the dotted lines are not interim individually rational), but it has a partially revealing equilibrium for $p \in (3/10, 4/5)$.

	j_1	j_2	j_3	j_4	
k_1	$4, 0$	$2, 7$	$5, 9$	$1, 10$	p
k_2	$1, 10$	$4, 7$	$4, 4$	$2, 0$	$1 - p$

Figure 16: Silent game of Example 4.

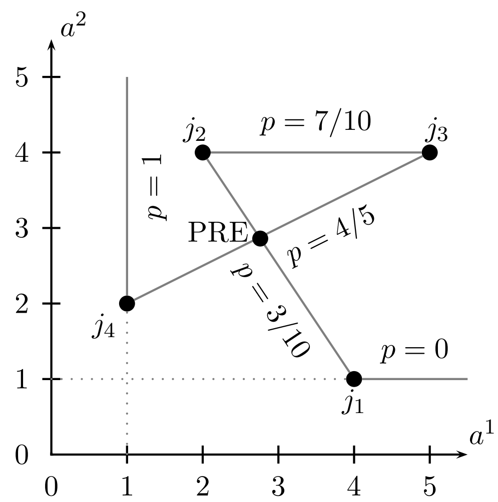

Figure 17: Modified and interim individually rational non-revealing equilibrium payoffs of the expert (solid lines) in Example 4.

References

AUMANN, R. J. AND S. HART (1986): "Bi-Convexity and Bi-Martingales," *Israel Journal of Mathematics*, 54, 159–180.

——— (2003): "Long Cheap Talk," *Econometrica*, 71, 1619–1660.

AUMANN, R. J. AND M. B. MASCHLER (1995): *Repeated Games of Incomplete Information*, Cambridge, Massachusetts: MIT Press.

BEN-PORATH, E. (2003): "Cheap Talk in Games with Incomplete Information," *Journal of Economic Theory*, 108, 45–71.

CRAWFORD, V. P. AND J. SOBEL (1982): "Strategic Information Transmission," *Econometrica*, 50, 1431–1451.

FORGES, F. (1984): "Note on Nash equilibria in repeated games with incomplete information," *International Journal of Game Theory*, 13, 179–187.

——— (1986): "An Approach to Communication Equilibria," *Econometrica*, 54, 1375–1385.

——— (1990a): "Equilibria with Communication in a Job Market Example," *Quarterly Journal of Economics*, 105, 375–398.

——— (1990b): "Universal Mechanisms," *Econometrica*, 58, 1341–1364.

FORGES, F. AND F. KOESSLER (2005): "Communication Equilibria with Partially Verifiable Types," *Journal of Mathematical Economics*, 41, 793–811.

GERARDI, D. (2004): "Unmediated Communication in Games with Complete and Incomplete Information," *Journal of Economic Theory*, 114, 104–131.

HART, S. (1985): "Nonzero-Sum Two-Person Repeated Games with Incomplete Information," *Mathematics of Operations Research*, 10, 117–153.

KRISHNA, V. AND J. MORGAN (2004): "The Art of Conversation: Eliciting Information from Experts through Multi-Stage Communication," *Journal of Economic Theory*, 117, 147–179.

KRISHNA, V. R. (2005): "Extended Conversations in Sender-Receiver Games," mimeo.

KUHN, H. W. (1953): "Extensive Games and the Problem of Information," in *Contributions to the Theory of Games*, ed. by H. W. Kuhn and A. W. Tucker, Princeton: Princeton University Press, vol. 2.

MILGROM, P. (1981): "Good News and Bad News: Representation Theorems and Applications," *Bell Journal of Economics*, 12, 380–391.

MILGROM, P. AND J. ROBERTS (1986): "Relying on the Information of Interested Parties," *Rand Journal of Economics*, 17, 18–32.

Myerson, R. B. (1982): "Optimal Coordination Mechanisms in Generalized Principal-Agent Problems," *Journal of Mathematical Economics*, 10, 67–81.

——— (1986): "Multistage Games with Communication," *Econometrica*, 54, 323–358.

Seidmann, D. J. and E. Winter (1997): "Strategic Information Transmission with Verifiable Messages," *Econometrica*, 65, 163–169.

Shin, H. S. (1994): "The Burden of Proof in a Game of Persuasion," *Journal of Economic Theory*, 64, 253–264.

Simon, R. S. (2002): "Separation of Joint Plan Equilibrium Payoffs from the Min-Max Functions," *Games and Economic Behavior*, 41, 79–102.

CESifo Working Paper Series

(for full list see www.cesifo-group.de)

1605 Kathleen M. Day and Stanley L. Winer, Policy-induced Internal Migration: An Empirical Investigation of the Canadian Case, November 2005

1606 Paul De Grauwe and Cláudia Costa Storti, Is Monetary Policy in the Eurozone less Effective than in the US?, November 2005

1607 Per Engström and Bertil Holmlund, Worker Absenteeism in Search Equilibrium, November 2005

1608 Daniele Checchi and Cecilia García-Peñalosa, Labour Market Institutions and the Personal Distribution of Income in the OECD, November 2005

1609 Kai A. Konrad and Wolfgang Leininger, The Generalized Stackelberg Equilibrium of the All-Pay Auction with Complete Information, November 2005

1610 Monika Buetler and Federica Teppa, Should you Take a Lump-Sum or Annuitize? Results from Swiss Pension Funds, November 2005

1611 Alexander W. Cappelen, Astri D. Hole, Erik Ø. Sørensen and Bertil Tungodden, The Pluralism of Fairness Ideals: An Experimental Approach, December 2005

1612 Jack Mintz and Alfons J. Weichenrieder, Taxation and the Financial Structure of German Outbound FDI, December 2005

1613 Rosanne Altshuler and Harry Grubert, The Three Parties in the Race to the Bottom: Host Governments, Home Governments and Multinational Companies, December 2005

1614 Chi-Yung (Eric) Ng and John Whalley, Visas and Work Permits: Possible Global Negotiating Initiatives, December 2005

1615 Jon H. Fiva, New Evidence on Fiscal Decentralization and the Size of Government, December 2005

1616 Andzelika Lorentowicz, Dalia Marin and Alexander Raubold, Is Human Capital Losing from Outsourcing? Evidence for Austria and Poland, December 2005

1617 Aleksander Berentsen, Gabriele Camera and Christopher Waller, Money, Credit and Banking, December 2005

1618 Egil Matsen, Tommy Sveen and Ragnar Torvik, Savers, Spenders and Fiscal Policy in a Small Open Economy, December 2005

1619 Laszlo Goerke and Markus Pannenberg, Severance Pay and the Shadow of the Law: Evidence for West Germany, December 2005

1620 Michael Hoel, Concerns for Equity and the Optimal Co-Payments for Publicly Provided Health Care, December 2005

1621 Edward Castronova, On the Research Value of Large Games: Natural Experiments in Norrath and Camelot, December 2005

1622 Annette Alstadsæter, Ann-Sofie Kolm and Birthe Larsen, Tax Effects, Search Unemployment, and the Choice of Educational Type, December 2005

1623 Vesa Kanniainen, Seppo Kari and Jouko Ylä-Liedenpohja, Nordic Dual Income Taxation of Entrepreneurs, December 2005

1624 Lars-Erik Borge and Linn Renée Naper, Efficiency Potential and Efficiency Variation in Norwegian Lower Secondary Schools, December 2005

1625 Sam Bucovetsky and Andreas Haufler, Tax Competition when Firms Choose their Organizational Form: Should Tax Loopholes for Multinationals be Closed?, December 2005

1626 Silke Uebelmesser, To go or not to go: Emigration from Germany, December 2005

1627 Geir Haakon Bjertnæs, Income Taxation, Tuition Subsidies, and Choice of Occupation: Implications for Production Efficiency, December 2005

1628 Justina A. V. Fischer, Do Institutions of Direct Democracy Tame the Leviathan? Swiss Evidence on the Structure of Expenditure for Public Education, December 2005

1629 Torberg Falch and Bjarne Strøm, Wage Bargaining and Political Strength in the Public Sector, December 2005

1630 Hartmut Egger, Peter Egger, Josef Falkinger and Volker Grossmann, International Capital Market Integration, Educational Choice and Economic Growth, December 2005

1631 Alexander Haupt, The Evolution of Public Spending on Higher Education in a Democracy, December 2005

1632 Alessandro Cigno, The Political Economy of Intergenerational Cooperation, December 2005

1633 Michiel Evers, Ruud A. de Mooij and Daniel J. van Vuuren, What Explains the Variation in Estimates of Labour Supply Elasticities?, December 2005

1634 Matthias Wrede, Health Values, Preference Inconsistency, and Insurance Demand, December 2005

1635 Hans Jarle Kind, Marko Koethenbuerger and Guttorm Schjelderup, Do Consumers Buy Less of a Taxed Good?, December 2005

1636 Michael McBride and Stergios Skaperdas, Explaining Conflict in Low-Income Countries: Incomplete Contracting in the Shadow of the Future, December 2005

1637 Alfons J. Weichenrieder and Oliver Busch, Artificial Time Inconsistency as a Remedy for the Race to the Bottom, December 2005

1638 Aleksander Berentsen and Christopher Waller, Optimal Stabilization Policy with Flexible Prices, December 2005

1639 Panu Poutvaara and Mikael Priks, Violent Groups and Police Tactics: Should Tear Gas Make Crime Preventers Cry?, December 2005

1640 Yin-Wong Cheung and Kon S. Lai, A Reappraisal of the Border Effect on Relative Price Volatility, January 2006

1641 Stefan Bach, Giacomo Corneo and Viktor Steiner, Top Incomes and Top Taxes in Germany, January 2006

1642 Johann K. Brunner and Susanne Pech, Optimum Taxation of Life Annuities, January 2006

1643 Naércio Aquino Menezes Filho, Marc-Andreas Muendler and Garey Ramey, The Structure of Worker Compensation in Brazil, with a Comparison to France and the United States, January 2006

1644 Konstantinos Angelopoulos, Apostolis Philippopoulos and Vanghelis Vassilatos, Rent-Seeking Competition from State Coffers: A Calibrated DSGE Model of the Euro Area, January 2006

1645 Burkhard Heer and Bernd Suessmuth, The Savings-Inflation Puzzle, January 2006

1646 J. Stephen Ferris, Soo-Bin Park and Stanley L. Winer, Political Competition and Convergence to Fundamentals: With Application to the Political Business Cycle and the Size of Government, January 2006

1647 Yu-Fu Chen, Michael Funke and Kadri Männasoo, Extracting Leading Indicators of Bank Fragility from Market Prices – Estonia Focus, January 2006

1648 Panu Poutvaara, On Human Capital Formation with Exit Options: Comment and New Results, January 2006

1649 Anders Forslund, Nils Gottfries and Andreas Westermark, Real and Nominal Wage Adjustment in Open Economies, January 2006

1650 M. Hashem Pesaran, Davide Pettenuzzo and Allan G. Timmermann, Learning, Structural Instability and Present Value Calculations, January 2006

1651 Markku Lanne and Helmut Luetkepohl, Structural Vector Autoregressions with Nonnormal Residuals, January 2006

1652 Helge Berger, Jakob de Haan and Jan-Egbert Sturm, Does Money Matter in the ECB Strategy? New Evidence Based on ECB Communication, January 2006

1653 Axel Dreher and Friedrich Schneider, Corruption and the Shadow Economy: An Empirical Analysis, January 2006

1654 Stefan Brandauer and Florian Englmaier, A Model of Strategic Delegation in Contests between Groups, January 2006

1655 Jan Zápal and Ondřej Schneider, What are their Words Worth? Political Plans and Economic Pains of Fiscal Consolidations in New EU Member States, January 2006

1656 Thiess Buettner, Sebastian Hauptmeier and Robert Schwager, Efficient Revenue Sharing and Upper Level Governments: Theory and Application to Germany, January 2006

1657 Daniel Haile, Abdolkarim Sadrieh and Harrie A. A. Verbon, Cross-Racial Envy and Underinvestment in South Africa, February 2006

1658 Frode Meland and Odd Rune Straume, Outsourcing in Contests, February 2006

1659 M. Hashem Pesaran and Ron Smith, Macroeconometric Modelling with a Global Perspective, February 2006

1660 Alexander F. Wagner and Friedrich Schneider, Satisfaction with Democracy and the Environment in Western Europe – a Panel Analysis, February 2006

1661 Ben J. Heijdra and Jenny E. Ligthart, Fiscal Policy, Monopolistic Competition, and Finite Lives, February 2006

1662 Ludger Woessmann, Public-Private Partnership and Schooling Outcomes across Countries, February 2006

1663 Topi Miettinen and Panu Poutvaara, Political Parties and Network Formation, February 2006

1664 Alessandro Cigno and Annalisa Luporini, Optimal Policy Towards Families with Different Amounts of Social Capital, in the Presence of Asymmetric Information and Stochastic Fertility, February 2006

1665 Samuel Muehlemann and Stefan C. Wolter, Regional Effects on Employer Provided Training: Evidence from Apprenticeship Training in Switzerland, February 2006

1666 Laszlo Goerke, Bureaucratic Corruption and Profit Tax Evasion, February 2006

1667 Ivo J. M. Arnold and Jan J. G. Lemmen, Inflation Expectations and Inflation Uncertainty in the Eurozone: Evidence from Survey Data, February 2006

1668 Hans Gersbach and Hans Haller, Voice and Bargaining Power, February 2006

1669 Françoise Forges and Frédéric Koessler, Long Persuasion Games, February 2006