THE INFINITE ATLAS PROCESS: CONVERGENCE TO EQUILIBRIUM

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Abstract. The semi-infinite Atlas process is a one-dimensional system of Brownian particles, where only the leftmost particle gets a unit drift to the right. Its particle spacing process has infinitely many stationary measures, with one distinguished translation invariant reversible measure. We show that the latter is attractive for a large class of initial configurations of slowly growing (or bounded) particle densities. Key to our proof is a new estimate on the rate of convergence to equilibrium for the particle spacing in a triangular array of finite, large size systems.

1. Introduction

Systems of competing Brownian particles interacting through their rank dependent drift and diffusion coefficient vectors have received much recent attention (for example, in stochastic portfolio theory, where they appear under the name first-order market model, see [9] and the references therein). For a fixed number of particles $m \in \mathbb{N}$, such system corresponds to the unique weak solution of

$$dY_i(t) = \sum_{j\geq 1} \gamma_j \, \mathbf{1}_{\{Y_i(t)=Y_{(j)}(t)\}} \, dt + \sum_{j\geq 1} \sigma_j \, \mathbf{1}_{\{Y_i(t)=Y_{(j)}(t)\}} dW_i(t) \,, \tag{1.1}$$

for $i = 1, \ldots, m$, where $\underline{\gamma} = (\gamma_1, \ldots, \gamma_m)$ and $\underline{\sigma} = (\sigma_1, \ldots, \sigma_m)$ are some constant drift and diffusion coefficient vectors and $(W_i(t), t \geq 0)$, $i \geq 1$ are independent standard Brownian motions. Here $Y_{(1)}(t) \leq Y_{(2)}(t) \leq \ldots \leq Y_{(m)}(t)$ are the ranked particles at time t and the $\mathbb{R}_+^{m-1}$-valued *spacings process* $\underline{Z}(t) = (Z_1(t), Z_2(t), \ldots, Z_{m-1}(t))$, $t \geq 0$, is given by

$$Z_k(t) := X_{k+1}(t) - X_k(t) := Y_{(k+1)}(t) - Y_{(k)}(t) \,, \qquad k \geq 1 \,. \tag{1.2}$$

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The variables $X_k(\cdot)$ and $Z_k(\cdot)$ correspond to the k-th ranked particle and k-th spacing, respectively, with $X_1 = \min_i Y_i$ denoting the leftmost particle. In particular, existence and uniqueness of such weak solution to (1.1) has been shown already in [3]. The corresponding ranked process $\underline{X}(t)$ solves the system

$$dX_j(t) = \gamma_j dt + \sigma_j dB_j(t) + dL_{j-1}(t) - dL_j(t)\,, \qquad j = 1, \dots, m \tag{1.3}$$

for independent standard Brownian motions $(B_j(t), t \geq 0)$, where $L_j(t)$ denotes the local time at zero of the non-negative semi-martingale $Z_j(\cdot)$, during $[0,t]$, with $L_0 \equiv 0$ and $L_m \equiv 0$. The spacing process $\underline{Z}(t)$ is thus a reflected Brownian motions (RBM) in a polyhedral domain. That is, the solution in $\mathbb{R}_+^{m-1}$ of

$$dZ_j = (\gamma_{j+1} - \gamma_j)dt + \sigma_{j+1} dB_{j+1} - \sigma_j dB_j + 2dL_j - dL_{j+1} - dL_{j-1}\,. \tag{1.4}$$

The general theory of such RBM (due to [11, 25], c.f. the survey [26]), characterizes those $(\underline{\gamma}, \underline{\sigma})$ for which the stationary distribution of $\underline{Z}(t)$ is a product of exponential random variables. Further utilizing this theory, [18] deduces verious stochastic comparison results, whereas [13] and the references therein, estimate the rate t_m, $m \gg 1$ of convergence in distribution of the spacing process $\underline{Z}(t)$. In particular, the *Atlas model* of m particles, denoted by $\text{ATLAS}_m(\gamma)$ (or ATLAS_m when $\gamma = 1$), corresponds to $\sigma_j \equiv 1$ and $\gamma_j = \gamma \mathbf{1}_{j=1}$. For $\text{ATLAS}_m(\gamma)$ it is shown in [15, Corollary 10] that the spacing process $\underline{Z}(t)$ has the unique invariant measure

$$\mu_\star^{(m,2\gamma)} := \bigotimes_{k=1}^{m-1} \mathbf{Exp}\big(2\gamma(1 - k/m)\big)\,, \quad m \in \mathbb{N}\,, \tag{1.5}$$

out of which [15, Theorem 1] deduces that

$$\mu_\star^{(\infty,2\gamma)} := \bigotimes_{k=1}^{\infty} \mathbf{Exp}(2\gamma), \tag{1.6}$$

is an invariant measure for the spacings process of $\text{ATLAS}_\infty(\gamma)$ (see also [16] for invariant measures of spacings when the particles follow linear Brownian motions which are repelled by their nearest neighbors through a potential). By time-space scaling we hereafter set $\gamma = 1$ WLOG and recall in passing that to rigorously construct the ATLAS_∞ we call $\underline{y} = (y_i)_{i \geq 1} \in \mathbb{R}^\infty$ *rankable* if there exists a bijective mapping to the ranked terms $y_{(1)} \leq y_{(2)} \leq y_{(3)} \leq \dots$ of $\underline{y}$. The solution of (1.1) starting at a fixed $\underline{y} \in \mathbb{R}^\infty$ is then well defined if a.s. the resulting process $\underline{Y} = (Y_1(t), Y_2(t), \dots)$ is rankable at all t (under some measurable ranking permutation). The ATLAS_∞ process is unique in law and well defined, when

$$\mathbb{P}\Big(\sum_{i \geq 1} e^{-\alpha Y_i(0)^2} < \infty\Big) = 1\,, \qquad \text{for any} \;\; \alpha > 0 \tag{1.7}$$

(see [23, Prop. 3.1]), and [14, Theorem 2] further constructs a strong solution of (1.1) in this setting (more generally, whenever $\sigma_0^2 = 0$, $j \mapsto \sigma_{j+1}^2$ is concave and eventually, both $\sigma_j^2 = 1$ and $\gamma_j = 0$). We focus here on ATLAS_∞, where WLOG all ATLAS_m, $m \in \mathbb{N} \cup \{\infty\}$, evolutions considered, start at a ranked configuration $\underline{Y}(0) = \underline{X}(0)$ having leftmost particle at zero (i.e. $X_1(0) = 0$), and which satisfies (1.7). For example, this applies when $\underline{Z}(0)$ is drawn from the product measure

$$\mu_\star^{(\infty,\lambda,a)} := \bigotimes_{k=1}^{\infty} \mathbf{Exp}(\lambda + ka)\,, \qquad \lambda > 0,\, a \geq 0\,. \tag{1.8}$$

The natural conjecture made in [15] that $\mu_\star^{(\infty,2)}$ is the only invariant measure for spacing of ATLAS_∞, has been refuted by [21] showing that $\{\mu_\star^{(\infty,2,a)},\, a \geq 0\}$, forms an *infinite family of such invariant measures* (similar invariant spacings measures appeared earlier, in the non-interacting discrete model studied by [17, 22]).

As for the behavior of the leading particle of ATLAS_∞, [5] verifies [15, Conj. 3], that starting with spacing at the translation invariant equilibrium law $\mu_\star^{(\infty,2)}$ results with

$$t^{-1/4} X_1(t) \xrightarrow{d} \mathcal{N}(0, c)\,, \quad \text{when} \quad t \to \infty\,, \quad \text{some} \quad c \in (0, \infty)\,. \tag{1.9}$$

Similar asymptotic fluctuations at equilibrium were established in [8, 10] for a tagged particle in the *doubly-infinite* Harris system (the non-interacting model with $\gamma_j \equiv 0$, $\sigma_j \equiv 1$), for the symmetric exclusion process associated with the SRW on $\mathbb{Z}$ (starting with [2]), and for a discrete version of the Atlas model (see [12]). In contrast, initial spacing of law $\mu_\star^{(\infty,2,a)}$ for $a > 0$, induces a negative ballistic motion of the leading particle. Specifically, [24] and [21] show respectively, that $\{X_1(t) + at\}$ is then a tight collection, of zero-mean variables.

Little is known about the challenging out of equilibrium behavior of ATLAS_∞. From [15, Theorem 1] we learn that at critical spacing density $\lambda = 2$ the unit drift to the leftmost particle compensates the spreading of bulk particles to the left, thereby keeping the gaps at equilibrium. Such interplay between spacing density and drift is re-affirmed by [4], which shows that initial spacing law $\mu_\star^{(\infty,\lambda)}$ induces the a.s. convergence $t^{-1/2} X_1(t) \to \kappa$ with $\text{sign}(\kappa) = \text{sign}(2 - \lambda)$. From [19, Theorem 4.7] it follows that if ATLAS_∞ starts at spacing law ν_0 which stochastically dominates $\mu_\star^{(\infty,2)}$ (e.g. when $\nu_0 = \mu_\star^{(\infty,\lambda)}$ any $\lambda \leq 2$), then the finite dimensional distributions (FDD) of $\underline{Z}(t)$ converge to those of $\mu_\star^{(\infty,2)}$ as $t \to \infty$. However, nothing else is known about the domain of attraction of $\mu_\star^{(\infty,2)}$ (or about those of $\{\mu_\star^{(\infty,2,a)},\, a > 0\}$). For example, what happens when $\nu_0 = \mu_\star^{(\infty,\lambda)}$ with $\lambda > 2$?

Our main result, stated next, answers this question by drastically increasing the established domain of attraction of $\mu_\star^{(\infty,2)}$ spacing for ATLAS_∞.

Theorem 1.1. *Suppose the* ATLAS$_\infty$ *process starts at* $\underline{Z}(0) = (z_j)_{j\geq 1}$ *such that for eventually non-decreasing* $\theta(m)$ *with* $\inf_m\{\theta(m-1)/\theta(m)\} > 0$ *and* $\beta \in [1, 2)$,

$$\limsup_{m\to\infty} \frac{1}{m^\beta \theta(m)} \sum_{j=1}^m z_j \ < \infty\,, \tag{1.10}$$

$$\limsup_{m\to\infty} \frac{1}{m^\beta \theta(m)} \sum_{j=1}^m (\log z_j)_- \ < \infty\,, \tag{1.11}$$

$$\liminf_{m\to\infty} \frac{1}{m^{\beta'} \theta(m)} \sum_{j=1}^m z_j = \infty\,, \qquad \beta' := \beta^2/(1+\beta), \tag{1.12}$$

further assuming in case $\beta = 1$ *that* $\theta(m) \geq \log m$. *Then, the* FDD *of* $\underline{Z}(t)$ *for the* ATLAS$_\infty$ *converge as* $t \to \infty$ *to those of* $\mu_\star^{(\infty,2)}$.

For example, if $\lambda_j \in [c^{-1}, c]$ and $\lambda_j z_j$ are i.i.d. of finite mean such that $\mathbb{E}(\log z_1)_- < \infty$, then taking $\theta(m) \equiv 1$ and $\beta > 1$ small enough for $\beta' < 1$, it follows by the SLLN that (1.10)–(1.12) hold a.s. Namely, when ν_0 is any such product measure, $\underline{Z}(t)$ converges in FDD to $\mu_\star^{(\infty,2)}$. For independent $z_j \sim \mathbf{Exp}(\lambda_j)$ this applies even when $\lambda_j \uparrow \infty$ slow enough so $\sum_{j=1}^m \lambda_j^{-1}/(\sqrt{m}\log m)$ diverges (and hence (1.12) holds a.s.), or when $\lambda_j \downarrow 0$ slow enough to have $m^{-\beta}\sum_{j=1}^m \lambda_j^{-1}$ bounded (for some $\beta < 2$, so (1.10) holds).

Remark 1.2. *As matter of comparison, note that if* z_j *decays to zero slower than* $j^{-1/2}\log j$, *then* $\{z_j\}$ *satisfies the hypothesis of Theorem* 1.1 *(for* $\beta = 1$*), while for measures of the form* (1.8), *generically* z_j *decays like* j^{-1}.

The key to proving Theorem 1.1 is a novel control of the ATLAS$_m$ particle spacing distance from equilibrium at time t, in terms of the relative entropy distance of its initial law from equilibrium.

Proposition 1.3. *Start the* ATLAS$_m$ *system at initial spacing law* $\nu_0^{(m)}$ *of finite entropy* $H\big(\nu_0^{(m)}|\mu_\star^{(m,2)}\big)$ *and finite second moment* $\int \|\mathbf{z}\|^2 d\nu_0^{(m)}$. *Then, for any* $t > 0$ *the spacing law* $\nu_t^{(m)}$ *is absolutely continuous with respect to the marginal of* $\mu_\star^{(\infty,2)}$ *and the Radon-Nikodym derivative* g_t *satisfies*

$$\int \Big\{ \sum_{j=1}^{m-1} \big[(\partial_{z_{j-1}} - \partial_{z_j})\sqrt{g_t}\big]^2 \Big\} \prod_{j=1}^{m-1} 2e^{-2z_j} dz_j \leq \frac{1}{2t} H(\nu_0^{(m)}|\mu_\star^{(m,2)}) + \frac{1}{m}\,. \tag{1.13}$$

Combining Proposition 1.3 with Lyapunov functions for finite ATLAS systems (constructed for example in [7, 13]), yields the following information on convergence of the ATLAS$_m$ particle spacing FDD at times $t_m \to \infty$ fast enough.

Corollary 1.4. *Starting the* ATLAS$_m$ *system at initial spacing law* $\nu_0^{(m)}$ *of finite second moment, for any fixed* $k \ge 1$, *the joint density of* $(Z_1(t_m), \dots, Z_k(t_m))$ *with respect to the corresponding marginal of* $\mu_\star^{(\infty,2)}$, *converges to one, provided* t_m *is large enough so both* $t_m^{-1} H\big(\nu_0^{(m)} | \mu_\star^{(m,2)}\big) \to 0$, *and* $t_m^{-1} \sum_{j=1}^k Z_j(0) \to 0$ *(in* $\nu_0^{(m)}$*-probability), as* $m \to \infty$.

Remark 1.5. *For concreteness we focused on the* ATLAS$_\infty$ *process, but a similar proof applies for systems of competing Brownian particles where* $\sigma_j^2 \equiv 1$, $\gamma_1 > 0$ *and* $j \mapsto \gamma_j$ *is non-increasing and eventually zero. We further expect this to extend to some of the two-sided infinite systems considered in* [20, Sec. 3], *and that such an approach may help in proving the attractivity of* $\mu_\star^{(\infty,2,a)}$ *in the* ATLAS$_\infty$ *system.*

In Section 2 we prove Proposition 1.3 and Corollary 1.4, whereas in Section 3, we deduce Theorem 1.1 from Corollary 1.4 by a suitable coupling of the ATLAS$_m$ system and the left-most k particles of ATLAS$_\infty$ up to time t_m.

2. Entropy control for ATLAS$_m$: Proposition 1.3 and Corollary 1.4

Recall (1.3), which for ATLAS$_m$ is

$$X_j(t) = X_j(0) + \mathbf{1}_{\{j=1\}} t + B_j(t) + L_{j-1}(t) - L_j(t) \qquad j = 1, \dots, m\,, \tag{2.1}$$

where $L_j(t)$ denotes the local time on $\{s \in [0,t] : Z_j(s) = 0\}$ for $1 \le j < m$, with $L_0(t) = L_m(t) \equiv 0$. Let $\mathbb{X}_m := \{\mathbf{x} : x_1 \le x_2 \dots \le x_m\} \subset \mathbb{R}^m$. The generator of the $\mathbb{X}_m$-valued Markov process $\underline{X}(t)$ is then

$$(\widehat{\mathcal{L}}_m g)(\mathbf{x}) := \frac{1}{2} \sum_{j=1}^m \partial_{x_j}^2 + \partial_{x_1} \tag{2.2}$$

defined on the core of smooth bounded functions $g(\cdot)$ on $\mathbb{X}_m$ satisfying the Neumann boundary conditions

$$\big(\partial_{x_j} - \partial_{x_{j+1}}\big)\, g\big|_{x_j = x_{j+1}} = 0, \qquad j = 1, \dots, m-1\,.$$

Specializing (1.4) the corresponding $\mathbb{R}_+^{m-1}$-valued spacings $Z_j(t) = X_{j+1}(t) - X_j(t)$ are then such that for $1 \le j \le m-1$,

$$Z_j(t) = Z_j(0) - \mathbf{1}_{\{j=1\}} t + B_{j+1}(t) - B_j(t) + 2L_j(t) - L_{j+1}(j) - L_{j-1}(t)\,. \tag{2.3}$$

Let $\Delta^{(m)}$ denote the discrete Laplacian with Dirichlet boundary conditions at 0 and m. Hence, using hereafter the convention of $\partial_{z_0} = \partial_{z_m} \equiv 0$,

$$\Delta^{(m)} \partial_{z_j} := \partial_{z_{j-1}} - 2\partial_{z_j} + \partial_{z_{j+1}}\,, \qquad j = 1, \dots, m-1\,. \tag{2.4}$$

Following this convention, in combination with the rule

$$\partial_{x_j} = \partial_{z_{j-1}} - \partial_{z_j}\,, \qquad j = 1, \dots, m\,,$$

the generator of the $\mathbb{R}_+^{m-1}$-valued Markov process $\underline{Z}(t)$ is thus

$$\mathcal{L}_m = \frac{1}{2}\sum_{j=1}^{m}\big(\partial_{z_{j-1}} - \partial_{z_j}\big)^2 - \partial_{z_1} = -\frac{1}{2}\sum_{j=1}^{m-1}\partial_{z_j}(\Delta^{(m)}\partial_{z_j}) - \partial_{z_1} \tag{2.5}$$

defined on the core $\mathcal{C}_m$ of local, smooth functions $h(\mathbf{z})$ such that

$$(\Delta^{(m)}\partial_{z_j})h\Big|_{z_j=0} = 0, \qquad j = 1, \dots, m-1\,. \tag{2.6}$$

Recall that $\mu_\star^{(m,2)}(\cdot)$ is the (unique) stationary law of $\underline{Z}(t)$ for ATLAS_m. In fact, for the density on $\mathbb{R}_+^{m-1}$ of $\mu_\star^{(m,2)}(\cdot)$,

$$p_m(\mathbf{z}) := \prod_{j=1}^{m-1}\alpha_j e^{-\alpha_j z_j}\,, \qquad\qquad \alpha_j := 2(1-j/m)\,, \tag{2.7}$$

a direct calculation shows that

$$\frac{1}{2}\sum_{j=1}^{m-1}\alpha_j\Delta^{(m)}\partial_{z_j} = -\partial_{z_1}\,. \tag{2.8}$$

Combining the LHS of (2.5) with (2.8) yields the symmetric form of the generator

$$\mathcal{L}_m = -\frac{1}{2p_m}\sum_{j=1}^{m-1}\partial_{z_j}\big(p_m\Delta^{(m)}\partial_{z_j}\big) \tag{2.9}$$

Using (2.9) and integration by parts, we have for bounded, smooth g, h satisfying (2.6)

$$\begin{aligned}\int g(-\mathcal{L}_m h)d\mu_\star^{(m,2)} &= \int h(-\mathcal{L}_m g)d\mu_\star^{(m,2)} \\ &= \frac{1}{2}\int\Big\{\sum_{j=1}^{m}[(\partial_{z_{j-1}} - \partial_{z_j})g][(\partial_{z_{j-1}} - \partial_{z_j})h]\Big\}d\mu_\star^{(m,2)} := \mathcal{D}_m(g,h)\,.\end{aligned} \tag{2.10}$$

We see that $d\mu_\star^{(m,2)} = p_m(\mathbf{z})d\mathbf{z}$ is reversible for this dynamic, and the corresponding Dirichlet form $\mathcal{D}_m(h) := \mathcal{D}_m(h,h)$, extends from $\mathcal{C}_m$, now only as

$$\mathcal{D}_m(h) = \frac{1}{2}\int\Big\{\sum_{j=1}^{m}\big[\big(\partial_{z_{j-1}} - \partial_{z_j}\big)\,h\big]^2\Big\}p_m(\mathbf{z})d\mathbf{z}\,, \tag{2.11}$$

to the Sobolev space $W^{1,2}(\mu_\star^{(2,m)})$ of functions on $\mathbb{R}_+^{m-1}$ with $L^2(\mu_\star^{(2,m)})$-derivatives.

We also consider the Markov dynamics on $\mathbb{R}_+^{m-1}$ for the spacing process of an ATLAS$_m$ whose m-th particle $X_m(s) = X_m(0)$ is frozen. Under the same convention as before, the generator for this, right-anchored dynamics, is

$$\widetilde{\mathcal{L}}_m = \frac{1}{2}\sum_{j=1}^{m-1}\left(\partial_{z_{j-1}} - \partial_{z_j}\right)^2 - \partial_{z_1} \tag{2.12}$$

for the core $\widetilde{\mathcal{C}}_m$ of local, smooth functions $h(\mathbf{z})$ such that

$$(\widetilde{\Delta}^{(m)}\partial_{z_j})h\Big|_{z_j=0} = 0, \qquad j = 1,\dots,m-1\,, \tag{2.13}$$

where $\widetilde{\Delta}^{(m)}$ is the discrete Laplacian with mixed boundary conditions. Specifically,

$$\widetilde{\Delta}^{(m)}\partial_{z_j} = \partial_{z_{j-1}} - 2\partial_{z_j} + \partial_{z_{j+1}}\,, \quad 1 \le j \le m-2\,, \quad \widetilde{\Delta}^{(m)}\partial_{z_{m-1}} = \partial_{z_{m-2}} - \partial_{z_{m-1}} \tag{2.14}$$

For the remainder of this section we identify $\mu_\star^{(\infty,2)}$ with its marginal on $\mathbf{z} = (z_1,\dots,z_{m-1})$, whose density on $\mathbb{R}_+^{m-1}$ is

$$q_m(\mathbf{z}) := \prod_{j=1}^{m-1} 2e^{-2z_j}\,. \tag{2.15}$$

Analogously to (2.9) we find that

$$\widetilde{\mathcal{L}}_m = -\frac{1}{2q_m}\sum_{j=1}^{m-1}\partial_{z_j}\left(q_m\widetilde{\Delta}^{(m)}\partial_{z_j}\right). \tag{2.16}$$

This (marginal of) $\mu_\star^{(\infty,2)}$ is thus reversible (stationary and ergodic) for the right-anchored dynamics, and similarly to (2.10)–(2.11) for bounded, smooth h satisfying (2.13), the associated Dirichlet form is given (on $W^{1,2}(q_m d\mathbf{z})$) by

$$\widetilde{\mathcal{D}}_m(h) = \frac{1}{2}\int\Big\{\sum_{j=1}^{m-1}\left[(\partial_{z_{j-1}} - \partial_{z_j})h\right]^2\Big\}q_m(\mathbf{z})d\mathbf{z}\,. \tag{2.17}$$

Indeed, this reversible measure corresponds to starting the right-anchored dynamics with $X_1(0) = 0$ and a Gamma$(2, m-1)$ law for the frozen X_m, with the remaining $m-2$ initial particle positions chosen independently and uniformly on $[0, X_m]$.

Proof of Proposition 1.3. Fixing $m \ge 2$, we start the finite particle dynamics of generator $\widetilde{\mathcal{L}}_m$ of (2.9), with initial law $\nu_0^{(m)}$ on $\mathbb{R}_+^{m-1}$ whose density

$$f_0 := \frac{d\nu_0^{(m)}}{d\mu_\star^{(m,2)}}\,, \tag{2.18}$$

has the finite entropy

$$H(\nu_0^{(m)}|\mu_\star^{(m,2)}) = \int [f_0 \log f_0](\mathbf{z}) p_m(\mathbf{z}) d\mathbf{z} =: H_m(f_0)\,. \tag{2.19}$$

Recall [1] that a Wasserstein solution of the Fokker-Planck equation

$$\partial_t f_t = \mathcal{L}_m f_t\,, \tag{2.20}$$

starting at f_0, is a continuous (in the topology of weak convergence) collection of probability measures $t \mapsto f_t \mu_\star^{(2,m)}$ such that for any s, the derivatives $(\partial_{z_{j-1}} - \partial_{z_j})\sqrt{f_s}$ exist a.e. in $\mathbb{R}_+^{m-1}$, with

$$\int_0^t \mathcal{D}_m(\sqrt{f_s}) ds < \infty \qquad \forall t < \infty \tag{2.21}$$

and moreover for any fixed compactly supported smooth function $\zeta(t, \mathbf{z})$ on $\mathbb{R}_+ \times \mathbb{R}_+^{m-1}$,

$$\int_0^\infty \Big\{ \mathcal{D}_m(f_t, \zeta(t,\cdot)) - \int \partial_t \zeta(t,\mathbf{z}) f_t(\mathbf{z}) p_m(\mathbf{z}) d\mathbf{z} \Big\}\, dt \;=\; 0\,. \tag{2.22}$$

By [1, Theorem 6.6], the law $\nu_t^{(m)}$ that corresponds to a starting measure $\nu_0^{(m)}$ of finite entropy and finite second moment, is a Wasserstein solution $\nu_t^{(m)} = f_t \mu_\star^{(m,2)}$ of (2.20).[1] As $p_m(\cdot)$ is log-concave, from [1, Theorem 6.6] we further have that then $\sqrt{f_t} \in W^{1,2}(\mu_\star^{(2,m)})$ and

$$\mathcal{D}_m\left(\sqrt{f_t}\right) < \infty \qquad \forall t > 0\,, \tag{2.23}$$

with $t \mapsto \mathcal{D}_m\left(\sqrt{f_t}\right)$ non-increasing and

$$H_m(f_t) - H_m(f_0) = -4 \int_0^t \mathcal{D}_m\left(\sqrt{f_s}\right)\, ds\,. \tag{2.24}$$

Consequently, for any $t \geq 0$,

$$4t\, \mathcal{D}_m\left(\sqrt{f_t}\right) \leq 4 \int_0^t \mathcal{D}_m\left(\sqrt{f_s}\right) ds = H_m(f_0) - H_m(f_t) \leq H_m(f_0)\,. \tag{2.25}$$

Next, comparing (2.7) with (2.15), notice that $q_m = p_m h_m$ for the strictly positive

$$h_m(\mathbf{z}) := \prod_{j=1}^{m-1} \frac{2}{\alpha_j} e^{-\frac{2j}{m} z_j}\,,$$

such that

$$(\partial_{z_{j-1}} - \partial_{z_j})\sqrt{h_m} = \left(m^{-1} - \mathbf{1}_{\{j=m\}}\right)\sqrt{h_m}\,. \tag{2.26}$$

[1] Though we could not find a reference for it, we expect f_t to be also a strong solution of (2.20) which satisfies the boundary conditions of (2.6).

Hence, for $f_t = h_m g_t$, using that $\sum_{j=1}^m \big(\partial_{z_{j-1}} - \partial_{z_j}\big) = 0$ and $\int g_t q_m d\mathbf{z} = 1$, we arrive at

$$2\mathcal{D}_m(\sqrt{f_t}) = \int \Big\{ \sum_{j=1}^m \Big[\big(\partial_{z_{j-1}} - \partial_{z_j}\big)\sqrt{g_t} + \Big(m^{-1} - \mathbf{1}_{\{j=m\}}\Big)\sqrt{g_t}\Big]^2 \Big\} q_m(\mathbf{z}) d\mathbf{z}$$
$$= 2\widetilde{\mathcal{D}}_m(\sqrt{g_t}) - m^{-1} + \int \big[\sqrt{g_t} - \partial_{z_{m-1}}\sqrt{g_t}\big]^2 q_m(\mathbf{z}) d\mathbf{z}\,. \tag{2.27}$$

Combining (2.25) and (2.27) we see that for any $t > 0$,

$$2\widetilde{\mathcal{D}}_m(\sqrt{g_t}) \le \frac{1}{2t} H_m(f_0) + m^{-1}\,, \tag{2.28}$$

where g_t is precisely the density of $\nu_t^{(m)}$ with respect to the marginal of $\mu_\star^{(\infty,2)}$. In view of the definitions (2.17) and (2.19), the preceding bound matches our claim (1.13). □

With $\widetilde{D}_{k+1}(\sqrt{g})$ invariant to mass-shifts $g(\mathbf{z}) \mapsto \exp(2\sum_{j=1}^k \theta_j) g(\mathbf{z} - \boldsymbol{\theta})$, having $\widetilde{D}_{k+1}(\sqrt{g_m}) \to 0$ does not imply (uniform) tightness of the collection of probability measures $\{g_m q_{k+1} d\mathbf{z}\}_{m\in\mathbb{N}}$. Instead, when proving Corollary 1.4, we rely for tightness on the following direct consequence of [13, Sec. 3].

Lemma 2.1. *For* ATLAS$_{k+1}$, *some $c_1 = c_1(k)$ finite and $D(t) := \sum_{j=1}^k Z_j(t)$,*

$$\lim_{x\to\infty} \sup_{t \ge c_1 D(0)} \{\, \mathbb{P}(D(t) \ge x)\,\} = 0\,, \tag{2.29}$$

where the supremum is also over all initial configurations.

Proof. Building on the construction in [7, Sec. 3] of Lyapunov functions for RBM in polyhedral domains, while proving [13, Thm. 3] the authors show that for the ATLAS$_{k+1}$ (and more generally, for the spacing associated with (1.1), whenever $j \mapsto \gamma_j$ is non-increasing and $j \mapsto \sigma_j^2$ forms an arithmetic progression), one has

$$\mathbb{E}[V(\mathbf{Z}(t))] \le e^{-t}[V(\mathbf{Z}(0))] + c_2\,, \qquad \forall t \ge 0\,, \tag{2.30}$$

where $V(\mathbf{z}) = e^{\langle \mathbf{v},\mathbf{z}\rangle}$ for some strictly positive $\mathbf{v}$ and $c_2 < \infty$ (see [13, inequality (51)]). Noting that $\langle \mathbf{v}, \mathbf{Z}(t)\rangle / D(t) \in [c_1^{-1}, c_1]$ (with $c_1 := \max_j\{v_j \vee v_j^{-1}\}$), we get upon combining (2.30) with Markov's inequality that for any initial configuration $\mathbf{Z}(0)$,

$$\mathbb{P}(D(t) \ge x) \le \mathbb{P}(\langle \mathbf{v}, \mathbf{Z}(t)\rangle \ge x/c_1) \le e^{-x/c_1}[e^{-t} e^{c_1 D(0)} + c_2]\,. \tag{2.31}$$

For $t \ge c_1 D(0)$ the RHS of (2.31) is at most $e^{-x/c_1}(1 + c_2)$, yielding (2.29). □

Proof of Corollary 1.4. Fix probability densities $h_0 \neq h_1$ WRT the law $q_m d\mathbf{z}$ on $\mathbb{R}_+^{m-1}$, such that $\sqrt{h_0}$, $\sqrt{h_1} \in W^{1,2}(q_m d\mathbf{z})$. Both properties then apply for $h_\lambda := \lambda h_1 + (1-\lambda)h_0$, any $\lambda \in (0,1)$, and it is not hard to verify that

$$\frac{d^2\widetilde{\mathcal{D}}_m(\sqrt{h_\lambda})}{d^2\lambda} = \int \Big\{ \sum_{j=1}^{m-1} \big[\sqrt{\alpha_0}(\partial_{z_{j-1}} - \partial_{z_j})\sqrt{h_1} - \sqrt{\alpha_1}(\partial_{z_{j-1}} - \partial_{z_j})\sqrt{h_0}\big]^2 \Big\} \alpha_0\alpha_1 q_m d\mathbf{z} \,,$$

where the non-negative $\alpha_0 := h_0/h_\lambda$, $\alpha_1 := h_1/h_\lambda$ are uniformly bounded (per λ). Consequently, the map $h \mapsto \widetilde{\mathcal{D}}_m(\sqrt{h})$ is convex on the set of probability densities h with respect to the product law $q_m d\mathbf{z}$ on $\mathbb{R}_+^{m-1}$.

The marginal density on $(z_1, \dots, z_k)$ (WRT the k-th marginal of $\mu_\star^{(\infty,2)}$), is given for $\nu_t^{(m)}(d\mathbf{z}) = g_t q_m d\mathbf{z}$ and $1 \le k < m$, by

$$g_{t,k}(z_1, \dots, z_k) := \int g_t(\mathbf{z}) \prod_{j=k+1}^{m-1} 2e^{-2z_j} dz_j \,. \tag{2.32}$$

Thus, by the convexity of $\widetilde{\mathcal{D}}_m(\sqrt{\cdot})$ and the formula (2.17), we have that

$$\widetilde{\mathcal{D}}_m(\sqrt{g_t}) = \int \widetilde{\mathcal{D}}_m(\sqrt{g_t}) \prod_{j=k+1}^{m-1} 2e^{-2z_j} dz_j \ge \widetilde{\mathcal{D}}_m(\sqrt{g_{t,k}}) \ge \widetilde{\mathcal{D}}_{k+1}(\sqrt{g_{t,k}}) \,. \tag{2.33}$$

In particular, fixing $k \ge 1$ and choosing $t_m \to \infty$ as in the statement of the corollary, we deduce from (1.13) and (2.33) that

$$\lim_{m\to\infty} \widetilde{\mathcal{D}}_{k+1}\left(\sqrt{g_{t_m,k}}\right) = 0 \,. \tag{2.34}$$

For $r \ge 2$ and the Markov generator $\widetilde{\mathcal{L}}_r$ of (2.12) consider the functional on the collection of probability measures ν on $\mathbb{R}_+^{r-1}$ defined by

$$\widetilde{I}_r(\nu) := \sup_{h \gg 0} \Big\{ \int h^{-1}(-\widetilde{\mathcal{L}}_r h)\, d\nu \Big\} \,, \tag{2.35}$$

where the supremum is taken over all bounded away from zero, twice continuously differentiable functions having the boundary conditions (2.13) at $m = r$. With $h^{-1}(-\widetilde{\mathcal{L}}_r h)$ then continuous and bounded, clearly $\widetilde{I}_r(\cdot)$ is l.s.c. in the weak topology on probability measures in $\mathbb{R}_+^{r-1}$. Further, recall from [6, Thm. 5] that $\widetilde{I}_r(\nu) = \infty$ unless $\nu = g q_r d\mathbf{z}$ for a probability density g such that $\sqrt{g} \in W^{1,2}(q_r d\mathbf{z})$, or equivalently $\widetilde{\mathcal{D}}_r(\sqrt{g}) < \infty$, in which case $\widetilde{I}_r(\nu) = \widetilde{\mathcal{D}}_r(\sqrt{g})$. Hence, (2.34) amounts to $\widetilde{I}_{k+1}(\nu_{t_m}^{(m,k)}) \to 0$ for the joint law $\nu_{t_m}^{(m,k)}$ of $(Z_1(t_m), \dots, Z_k(t_m))$ and any weak limit point of these laws must have a density g WRT $q_{k+1} d\mathbf{z}$ such that $\widetilde{\mathcal{D}}_{k+1}(\sqrt{g}) = 0$. From (2.17) it is thus necessarily that throughout $\mathbb{R}_+^k$,

$$\partial_{z_1}\sqrt{g} = 0, \qquad \left(\partial_{z_{j-1}} - \partial_{z_j}\right)\sqrt{g} = 0, \quad j = 2, \dots, k \,,$$

so as claimed, the only possible limit point is $g \equiv 1$. By Prohorov's theorem, it remains to verify that $\{\nu_{t_m}^{(m,k)}\}$ are uniformly tight, namely, to provide a uniform in m tail-decay for $\sum_{j=1}^k Z_j(t_m)$ in the corresponding ATLAS$_m$ system. To this end, recall [19, Cor. 3.10(ii)] that under the same driving Brownian motions $\{B_j(s)\}$ and initial configuration, the first k spacings increase when all particles to the right of $X_{k+1}(0)$ are removed. Consequently, it suffices to provide a uniform in m tail decay for the diameter $D(t_m)$ of an ATLAS$_{k+1}$ system of initial spacing distribution $\nu_0^{(m,k)}$. Fixing $\epsilon > 0$ we have from (2.29) the existence of finite $c_1 = c_1(k)$ and $x = x(\epsilon)$ such that for a given initial configuration, if $t \geq c_1 D(0)$ then $\mathbb{P}(D(t) \geq x) \leq \epsilon$. By our assumption that $t_m^{-1}\rho_m \to 0$ in $\nu_0^{(m,k)}$-probability, for the (random) initial diameter $\rho_m := X_{k+1}(0) - X_1(0)$, we have that $\mathbb{P}(c_1\rho_m \geq t_m) \leq \epsilon$ for all $m \geq m_0(\epsilon)$, in which case

$$\mathbb{P}(D(t_m) \geq x) \leq \mathbb{P}(D(t_m) \geq x, t_m \geq c_1\rho_m) + \mathbb{P}(c_1\rho_m \geq t_m) \leq 2\epsilon\,.$$

With $\epsilon > 0$ arbitrarily small and $x = x(\epsilon)$ independent of m, we have thus established the required uniform tightness. □

Remark 2.2. *The proof of Proposition* 1.3 *is easily adapted to deal with the right-anchored dynamic (of the generator* $\widetilde{\mathcal{L}}_m$ *given in* (2.12)*). It yields for the latter dynamic the bound of* (1.13)*, now with* $(2t)^{-1}H(\nu_0^{(m)}|\otimes_{k=1}^{m-1}\mathbf{Exp}(2))$ *in the* RHS. *The proof of Lemma* 2.1 *also adapts to right-anchored dynamics, hence the conclusion of Corollary* 1.4 *applies for sequences of right-anchored dynamics when the latter expression decays to zero at* $t = t_m \to \infty$ *such that* $t_m^{-1}\sum_{j=1}^k Z_j(0) \to 0$.

3. Coupling to ATLAS$_\infty$: Proof of Theorem 1.1

Let $\mathcal{G}(a) = (2\pi)^{-1/2}\int_a^\infty e^{-x^2/2}dx$ and consider the ATLAS$_\infty$ process $\underline{Y}(t) = \{Y_i(t)\}$, denoting by $\underline{X}(t) = \{X_j(t)\}$ the corresponding ranked configuration. We first provide three elementary bounds for this process that are key to the proof of Theorem 1.1.

Lemma 3.1. *For any initial condition* $\underline{X}(0)$*,* $\ell \geq 1$ *and* $t, \Gamma > 0$*,*

$$\mathbb{P}\Big(\sup_{s\in[0,t]}\{X_1(s)\} \geq \Gamma\Big) \leq 2\mathcal{G}\Big(\frac{\ell\,\Gamma - t - \sum_{j=1}^{\ell} X_j(0)}{\sqrt{\ell t}}\Big)\,. \tag{3.1}$$

Proof. Starting WLOG at $\underline{Y}(0) = \underline{X}(0)$, we have that for any $s \geq 0$,

$$X_1(s) \leq \frac{1}{\ell}\sum_{i=1}^{\ell} Y_i(s) \leq \frac{s}{\ell} + \frac{1}{\ell}\sum_{j=1}^{\ell} X_j(0) + \frac{1}{\sqrt{\ell}}\widetilde{W}(s)\,,$$

where $\widetilde{W}(s) := \ell^{-1/2}\sum_{i=1}^{\ell} W_i(s)$ is a standard Brownian motion. Thus, by the reflection principle,

$$\mathbb{P}\Big(\sup_{s\in[0,t]}\{X_1(s)\}\geq\Gamma\Big)\leq\mathbb{P}\Big(\sup_{s\in[0,t]}\{\frac{1}{\sqrt{\ell}}\widetilde{W}(s)\}\geq\Gamma-\frac{t}{\ell}-\frac{1}{\ell}\sum_{j=1}^{\ell}X_j(0)\Big)$$
$$=2\mathbb{P}\Big(\widetilde{W}(t)\geq\sqrt{\ell}\,\Gamma-\frac{t}{\sqrt{\ell}}-\frac{1}{\sqrt{\ell}}\sum_{j=1}^{\ell}X_j(0)\Big)\,,$$

which upon Brownian scaling yields the stated bound of (3.1). □

Lemma 3.2. *For $X_1(0)\geq 0$, Γ and $k\geq 2$ such that $\Gamma^{(k)} := (\Gamma - X_k(0))/3 > 0$, any $\ell\geq 1$ and $t>0$, we have that*

$$\mathbb{P}\Big(\sup_{s\in[0,t]}\{X_k(s)\}\geq\Gamma\Big)\leq 2\mathcal{G}\Big(\frac{\ell\,\Gamma^{(k)}-t-\sum_{j=1}^{\ell}X_j(0)}{\sqrt{\ell t}}\Big)+4k\mathcal{G}\Big(\frac{\Gamma^{(k)}}{\sqrt{t}}\Big)\,. \tag{3.2}$$

Proof. Recall [19, Cor. 3.12(ii)] that keeping the same driving Brownian motions $\{B_j(s)\}$ and initial configuration $\underline{X}(0)$, the spacing vector $\underline{Z}(t)$ is pointwise decreasing in γ. Further, by [19, Cor. 3.10(ii)], the first $k-1$ spacings increase when all particles to the right of $X_k(0)$ are removed. Consequently, that value of $X_k(t)-X_1(t)$ at the drift $\gamma=1$ of ATLAS$_\infty$ is bounded by its value for a k-particle Harris system (of $\gamma=0$), starting at same positions as the original ATLAS$_\infty$ process left-most k particles. In the latter case, letting $V_k(s) := \max_{j=1}^{k}\{B_j(s)\}$ and the identically distributed $V'_k(s) := \max_{j=1}^{k}\{-B_j(s)\}$, our assumption that $X_1(0)\geq 0$ results with

$$X_k(s)-X_1(s)\leq X_k(0)+V_k(s)+V'_k(s)\,.$$

Thus, with $\Gamma^{(k)} = (\Gamma - X_k(0))/3$ and $\{\widetilde{B}(s)\}$ denoting a standard Brownian motion, we get by the union bound that

$$\mathbb{P}\Big(\sup_{s\in[0,t]}\{X_k(s)\}\geq\Gamma\Big)\leq\mathbb{P}\Big(\sup_{s\in[0,t]}\{X_1(s)+V_k(s)+V'_k(s)\}\geq\Gamma-X_k(0)\Big)$$
$$\leq\mathbb{P}\Big(\sup_{s\in[0,t]}\{X_1(s)\}\geq\Gamma^{(k)}\Big)+2k\mathbb{P}\Big(\sup_{s\in[0,t]}\{\widetilde{B}(s)\}\geq\Gamma^{(k)}\Big)\,.$$

Consequently, by (3.1) and the reflection principle,

$$\mathbb{P}\Big(\sup_{s\in[0,t]}\{X_k(s)\}\geq\Gamma\Big)\leq 2\mathcal{G}\Big(\frac{\ell\,\Gamma^{(k)}-t-\sum_{j=1}^{\ell}X_j(0)}{\sqrt{\ell t}}\Big)+4k\mathcal{G}\Big(\frac{\Gamma^{(k)}}{\sqrt{t}}\Big)\,,$$

as claimed. □

Lemma 3.3. *For any $m \geq 0$, $t, \Gamma > 0$ and initial configuration $\underline{Y}(0) = \underline{X}(0)$,*

$$\mathbb{P}\Big(\inf_{s\in[0,t]} \inf_{i>m} \{Y_i(s)\} \leq \Gamma\Big) \leq 2 \sum_{i>m} \mathcal{G}\Big(\frac{X_i(0)-\Gamma}{\sqrt{t}}\Big) . \tag{3.3}$$

Proof. Removing the drift in the ATLAS model decreases all coordinates of $\underline{Y}(s)$ and correspondingly increases the LHS of (3.3). Thus,

$$\begin{aligned}\mathbb{P}\Big(\inf_{i>m} \inf_{s\in[0,t]} \{Y_i(s)\} \leq \Gamma\Big) &\leq \sum_{i>m} \mathbb{P}\Big(\inf_{s\in[0,t]} \{W_i(s)\} \leq \Gamma - X_i(0)\Big) \\ &= 2\sum_{i>m} \mathbb{P}\Big(W(1) \leq \frac{\Gamma - X_i(0)}{\sqrt{t}}\Big) = 2\sum_{i>m} \Big(\frac{X_i(0)-\Gamma}{\sqrt{t}}\Big) ,\end{aligned}$$

as claimed. □

Proof of Theorem 1.1. Given initial spacing configuration $\underline{z}$ that satisfies (1.10) and (1.11), consider the following two sequences of initial distributions for the finite increment vectors $\underline{Z}_m(0) := (Z_1(0), \dots, Z_{m-1}(0))$ of the ATLAS$_m$ process, $m \geq 2$. Starting at the measure $\nu_0^{(m,-)}(\cdot) = \mu_\star^{(m,2)}(\cdot|\underline{Z}_m(0) \leq \underline{z}_m)$ for the given $\underline{z}_m = (z_1, \dots, z_{m-1})$ yields for same driving Brownian motions an ATLAS$_m$ spacing process $\underline{Z}_m^-(s)$ which is dominated at all times $s \geq 0$ by the corresponding process that started at spacing $\underline{z}_m$, whereas $\nu_0^{(m,+)}(\cdot) = \mu_\star^{(m,2)}(\cdot|\underline{Z}_m(0) \geq \underline{z}_m)$ similarly yields a spacing process $\underline{Z}_m^+(s)$ that dominates the spacing for the original process which started at $\underline{z}_m$. The corresponding relative entropies are then

$$H_m^+ := H(\nu_0^{(m,+)}|\mu_\star^{(m,2)}) = -\log \mu_\star^{(m,2)}(\{\prod_{j=1}^{m-1}[z_j,\infty)\}) = \sum_{j=1}^{m-1} \alpha_j z_j \leq 2\sum_{j=1}^{m-1} z_j \tag{3.4}$$

$$\begin{aligned}H_m^- := H(\nu_0^{(m,-)}|\mu_\star^{(m,2)}) &= -\log \mu_\star^{(m,2)}(\{\prod_{j=1}^{m-1}[0,z_j]\}) = \sum_{j=1}^{m-1} -\log(1-e^{-\alpha_j z_j}) \\ &\leq \sum_{j=1}^{m-1} \big[1 + (\log \alpha_j z_j)_-\big] \leq 2m\log m + \sum_{j=1}^{m-1} (\log z_j)_- ,\end{aligned} \tag{3.5}$$

since $-\log(1-e^{-u}) \leq 1 + (\log u)_-$ for all $u \geq 0$, while $\alpha_j \geq 2/m$ (see (2.7)), hence $\log(e/\alpha_j) \leq 2\log m$. By (1.10) and (3.4),

$$\limsup_{m\to\infty} \frac{H(\nu_0^{(m,+)}|\mu_\star^{(m,2)})}{m^\beta \theta(m)} < \infty .$$

With $\theta(m) \ge \log m$ in case $\beta = 1$, we similarly deduce from (1.11) and (3.5) that

$$\limsup_{m\to\infty} \frac{H(\nu_0^{(m,-)}|\mu_\star^{(m,2)})}{m^\beta \theta(m)} < \infty\,.$$

Fixing $k \ge 2$, the uniform over $m \ge 2k$ first moment bound,

$$\nu_0^{(m,-)}\big[\sum_{j=1}^k Z_j(0)\big] \le \nu_0^{(m,+)}\big[\sum_{j=1}^k Z_j(0)\big] = \sum_{j=1}^k (z_j + \alpha_j^{-1}) \le \sum_{j=1}^k (z_j+1)\,,$$

yields by Markov's inequality that $t_m^{-1}\sum_{j=1}^k Z_j(0) \to 0$ in $\nu_0^{(m,\pm)}$-probability, for any $t_m \to \infty$. Further, the second moment of $\nu_0^{(m,-)}$ is finite (being at most $\|\underline{z}\|^2$), as is the second moment of $\nu_0^{(m,+)}$ (being at most the product of $e^{H_m^+}$ and the finite second moment of $\mu_\star^{(m,2)}$). Hereafter, let

$$t_m := 2m^\beta \theta(m)\psi(m)\,, \tag{3.6}$$

for some slowly growing $\psi(m) \uparrow \infty$ such that $\inf_m\{\psi(m-1)/\psi(m)\} > 0$. Setting $m_n = m_n(\mathbf{s}) := \inf\{m \ge 2 : t_m \ge s_n\}$, our constraints on $\theta(\cdot)$ and $\psi(\cdot)$ yields that for t_m of (3.6) and *any* $s_n \uparrow \infty$,

$$\inf_{n\ge1}\Big\{\frac{s_n}{t_{m_n}}\Big\} \ge \inf_{m\ge2}\Big\{\frac{t_{m-1}}{t_m}\Big\} > 0\,.$$

Thus, by the preceding, upon applying Corollary 1.4 to the ATLAS$_m$ model initialized at $\nu_0^{(m,\pm)}$ we have that the joint law of the first k coordinates of $\underline{Z}^\pm_{m_n}(s_n)$, converges as $n \to \infty$ to the corresponding marginal of $\mu_\star^{(\infty,2)}$. The same limit in distribution then applies for the spacing $(Z_1(s_n),\dots,Z_k(s_n))$ of ATLAS$_{m_n}$ started at $\underline{z}_{m_n}$ (which is sandwiched between the corresponding marginals of $\underline{Z}^-_{m_n}(s_n)$ and $\underline{Z}^+_{m_n}(s_n)$). Assuming further that $\sup_m\{\psi(m)/m\} \le 1$, we claim that for $X_1(0) = 0$ and the given initial spacing $\underline{Z}(0) = \underline{z}$, the RHS of (3.2) is summable over m, at $t = t_m$ of (3.6) and

$$\Gamma_m := 36 m^{\beta'}\theta(m)\psi(m)^{\beta/(1+\beta)}\,, \quad \ell_m := [m^{\beta/(1+\beta)}\psi(m)^{1/(1+\beta)}]\,. \tag{3.7}$$

Indeed, since $\theta(\cdot)$ is eventually non-decreasing, we have then that

$$\frac{1}{12}\Gamma_m \ell_m \ge t_m \ge \ell_m^{1+\beta}\theta(\ell_m)\,, \qquad \forall m \ge m_\star \tag{3.8}$$

Further, with k fixed and $\Gamma_m \uparrow \infty$, necessarily $X_k(0) \le \Gamma_m/8$ for all $m \ge m_\star$ large enough, in which case from (3.8), the RHS of (3.2) is bounded above for such m, t_m, Γ_m and ℓ_m, by

$$2\mathcal{G}\Big(\sqrt{\ell_m^\beta \theta(\ell_m)}\Big) + 4k\mathcal{G}\Big(3\sqrt{\ell_m^{\beta-1}\theta(\ell_m)}\Big)$$

and recalling that $\theta(\ell_m) \geq \frac{1}{2}\log m$ when $\beta = 1$, it is easy to verify that the preceding bound is summable in m. Next, utilizing (1.12), we can further make sure that $\psi(m) \uparrow \infty$ slowly enough so that for any fixed $\kappa < \infty$,

$$X_m(0) \geq (\kappa+1)\Gamma_m\,, \qquad \forall m \geq m_\kappa \tag{3.9}$$

so that for $m \geq m_\kappa$ the RHS of (3.3) is bounded above, at $t = t_m$ and $\Gamma = \Gamma_m$, by

$$2\sum_{j=1}^{\infty} \mathcal{G}\Big(\kappa\Gamma_{m+j}/\sqrt{t_m}\Big)\,. \tag{3.10}$$

Note that $\beta' \geq 1/2$ and $\delta := \beta/(1+\beta) - 1/2 \geq 0$ is strictly positive when $\beta > 1$. Thus, $\beta' - \beta/2 = \delta\beta \geq 0$ and setting $\kappa' := (18\kappa)^2$, we deduce from (3.6) and (3.7) that

$$\kappa\frac{\Gamma_m}{\sqrt{t_m}}\frac{\Gamma_{m+j}}{\Gamma_m} \geq m^{\beta\delta}\theta(m)^{1/2}\sqrt{2\kappa'(1+j/m)}\,.$$

Increasing m_κ if needed, we have that $m^{2\beta\delta}\theta(m) \geq \log m$ for all $m \geq m_\kappa$, in which case for $b_m := m^{-1}\log m$, the expression (3.10) is further bounded by

$$2\sum_{j=1}^{\infty}\mathcal{G}\Big(\sqrt{2\kappa'(1+j/m)\log m}\Big) \leq 2m^{-\kappa'}\sum_{j=1}^{\infty} e^{-\kappa' j b_m} \leq \frac{2}{\kappa' b_m} m^{-\kappa'}$$

(recall the elementary bound $\mathcal{G}(x) \leq e^{-x^2/2}$ for $x \geq 0$). Thus, for any $\kappa' > 2$, such choices of t_m and Γ_m guarantee that the RHS of (3.3) is also summable over m. Combining Lemma 3.2, Lemma 3.3 and the Borel-Cantelli lemma we deduce that almost surely, the events

$$\mathcal{A}_m := \Big\{ \sup_{s\in[0,t_m]}\{X_k(s)\} < \Gamma_m \leq \inf_{s\in[0,t_m],i>m}\{Y_i(s)\}\Big\}\,,$$

occur for all m large enough. Note that $\mathcal{A}_m$ implies that throughout $[0,t_m]$ the k left-most particles of the ATLAS$_\infty$ process are from among the initially left-most m particles. From this it follows that under $\mathcal{A}_{m_n}$ the spacing $(Z_1(s_n),\ldots,Z_{k-1}(s_n))$ for the ATLAS$_{m_n}$ coincide with those for the ATLAS$_\infty$, when using the same initial configuration $\underline{z}$ and driving Brownian motions $\{W_i(s)\}$. Having proved already the convergence in distribution when $n \to \infty$, of the first $k-1$ spacing for ATLAS$_{m_n}$ at time s_n and that the events $\mathcal{A}_m$ occur for all m large enough, we conclude that the FDD of spacing for ATLAS$_\infty$ converge to those of $\mu_\star^{(\infty,2)}$, along any (non-random) sequence $s_n \uparrow \infty$, as claimed. □

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