Management of a scarce resource and price rule: the case of sustainable fishing

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Abstract

In this paper, we investigated the problem of sustainable fishing. We explained how the fisheries maximize their profit according to the quantity of fish available in the sea and under the constraint of fines when the quota is exceeded. We showed the strategy issued from the resolution of the expected profit maximization is still better than naive strategies. Moreover, we define a level of fines which insures the double objective of the sustainable fishing.

1 Introduction

The economic importance of fishing in some regions leads many fisheries around the world to overfishing. This issue concerns FAO and the UN as it threatens food security and ecological balances. Cury et al. [1] have shown, in seven ecosystems (Atlantic, Pacific and southern oceans) that for 14 seabird species, the disastrous effects of food shortage, are induced by overfishing. The concept of sustainable fishing refers to methods designed not to over-exploit the resources, leads to the definition of measures and rules including fines delivered by authorities to avoid over-fishing. The theoretical problem is, therefore, to determine a cost rule for fines in order to allow the conservation of animal species that are exploited by humans according to a sustainable perspective. In this respect, the amount of fines is to ensure that the fish population does not fall below a certain threshold that guarantees its natural renewal. But it must also allow fisheries to make profits to prevent them from going bankrupt.
In our modeling, fisheries (called after fishermen) are controlled by fixed dates, while fishing is continuous depending on the quantity of fish available in the sea (the fish population evolves according to a logistic stochastic differential equation). They must pay a fine in case of exceeding their fishing quota. They are therefore seeking to maximize their profit, i.e., the quantity of fishes to fish given the fine to be paid if their quota is exceeded. Unlike the other articles dealing with this issue [2], the selling price of fish is not constant; it depends on the quantity (stock) of fishes remaining in the sea. Consequently, it is endogenous to the problem of sustainable fishing. This difference is not trivial for three reasons. First, the evolution of the price according to scarcity is a basic rule in economics. We can therefore question the relevance of fixed price models. Secondly, the increase in the price of fish with its scarcity encourages fishermen to violate quota rules. Indeed, any fines are offset by

the increase of their income, and it can lead to an increase in their profit if they are not strong enough. Third, considering a flexible price greatly complicates the resolution of this problem from a mathematical point of view. Given that, we will show how the resolution of this problem allows, on the one hand, to explain the behavior of the fisheries according to the amount of the fines, and, on the other hand, to fix a rule of price for the fines to guarantee a sustainable fishing.
The remainder of this article is structured as follows. Section 2 presents the problem formulation: the function of the expected profit for a fisherman and its two value functions. Section 3 characterizes the Hamilton Jacobi Bellman (HJB) equation linked with these value functions. Section 4 provides numerical results and interpretations. Concluding remarks are offered in Section 5.

2 Problem formulation

2.1 The model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We assume this space is equipped with two one-dimensional standard Brownian motions B and W. We denote by $\mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}$ the right continuous complete filtration generated by B and W where T is a constant. We assume that the correlation between the two Brownian motions is given by $\langle B, W\rangle_t = \rho t$.

We consider a fish population which evolves according to the classical logistic stochastic differential equation

$$dX_t \;=\; \eta X_t(\lambda - X_t)dt + \gamma X_t dB_t \;,$$

where η, λ and γ are three positive constants. We assume that we can fish this one and we denote by α_t the fishing rate at time t. For a given strategy $\alpha = (\alpha_t)_{0\leq t\leq T}$, we denote by X^α_t the associated population of fish, thus this one follows the stochastic differential equation

$$dX^\alpha_t \;=\; \eta X^\alpha_t(\lambda - X^\alpha_t)dt + \gamma X^\alpha_t dB_t - \alpha_t dt \;.$$

The fisherman sells his fish on the market at time t for the price P_t by unit, where the price P evolves with the following stochastic differential equation

$$dP_t \;=\; P_t(\mu(X^\alpha_t)dt + \sigma dW_t) \;,$$

where σ is a positive constant and μ is a positive Lipschitz function from $\mathbb{R}_+$ to $\mathbb{R}_+$ which is nonincreasing, that means the more is the quantity of fish the less is the price of the fish. Often we can find in the literature that the price P follows a Black and Scholes equation but it is more realistic if the drift of the equation depends on the quantity of fish.

We consider a positive increasing sequence $(T_i)_{1\leq i\leq N}$ where each T_i is the time when the regulatory body checks the quantity of fish X^α, and $T_N = T$. We assume that each T_i is a constant. If $X^\alpha_{T_i} > \Gamma$ then the fisherman can continue to fish, if $X^\alpha_{T_i} \leq \Gamma$ then the fisherman can no more fish until $\tau^\alpha_i := \inf\{T_k\,,\ k \geq i\ :\ X^\alpha_{T_k} > \Gamma\}$.

We define the set $\mathcal{A}$ of admissible controls as the set of strategies α such that α is an $\mathbb{F}$-adapted process defined in $[0, \bar{a}]$, X^α is nonnegative and α is null on $[T_i, \tau^\alpha_i)$ for any $1 \leq i \leq N$, $\bar{a}$ is a constant such that the rate of fish is upper bounded.

The objective of the fisherman is to optimize the expected profit that can be extracted from fishing over a finite horizon T

$$V_0(x) \ := \ \sup_{\alpha\in\mathcal{A}} \mathbb{E}\Big[\int_0^T e^{-\beta t}(P_t\alpha_t - C(\alpha_t))dt - e^{-\beta T} f((\Gamma - X_T^\alpha)^+, P_T^\alpha)\Big] , \qquad (2.1)$$

and finding a strategy $\alpha^* \in \mathcal{A}$ such that

$$V_0(x) \ = \ \mathbb{E}\Big[\int_0^T e^{-\beta t}(P_t\alpha_t^* - C(\alpha_t^*))dt - e^{-\beta T} f((\Gamma - X_T^{\alpha^*})^+, P_T^{\alpha^*})\Big] ,$$

where β is a positive constant, $(.)^+$ denotes the positive part, C is a positive increasing convex function representing the cost of harvesting and f is a map from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}_+$ which corresponds to a tax that the fisherman must pay if at time T the quantity of fish X_T^α is lower than the level Γ.

(**H**f) $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing and Lipschitz function: there exist a positive constant L such that

$$|f(x,y) - f(x',y')| \ \leq \ L(|x-x'| + |y-y'|) ,$$

for all $(x,y),(x',y') \in \mathbb{R}_+ \times \mathbb{R}_+$, and $f(0,y) = 0$ for any $y \in \mathbb{R}_+$.

2.2 The value function

In order to provide an analytic characterization of the value function V_0 defined by (2.1), we need to extend the definition of this control problem to general initial conditions.

Unfortunately, the considered controlled system is not Markovian. Indeed, the control process α is subject to the constraint that is fixed only at each time T_i but holds over $[T_i, T_{i+1})$. Thus we need to keep in mind the constraint and we therefore consider two cases (we can fish or we can not) and two value functions. This approach is inspired by that of Bruder and Pham [3] who consider a delayed controlled system. They enlarge the controlled system to make it Markovian. Similarly, we enlarge our system by adding a parameter which indicates whether the agent is allowed to fish or not on the considered period $[T_i, T_{i+1})$. However, we notice that our resulting PDE is different from their since we get a coupled system whereas they get a recursive one.

For any $t \in [0,T]$, $x \geq 0$ and $i \in \{0,1\}$ we denote $\mathcal{A}_{t,i}(x)$ the set

$$\begin{aligned} \mathcal{A}_{t,i}(x) \ := \ \Big\{ & \alpha = (\alpha_s)_{t\leq s\leq T} \ , \ \alpha_s \text{ is } \mathcal{F}_s-\text{measurable and valued in } [0,\bar{a}] \ , \\ & \alpha_s = 0 \text{ on } [t, \tau_{p(t)}^\alpha) \text{ if } i = 0 \ , \\ & \alpha_s = 0 \text{ on } [T_k, \tau_k^\alpha) \text{ for any } p(t)+1 \leq k \leq N \Big\} , \end{aligned}$$

where $p(t) := \sup\{T_i \ , T_i \leq t\}$.
Let $\mathcal{Z} := \mathbb{R}_+ \times \mathbb{R}_+^* \times \{0,1\}$. For $z = (x,p,i) \in \mathcal{Z}$ and $\alpha \in \mathcal{A}_{t,i}(x)$, we denote by $Z^{t,z,\alpha} :=$

$(X^{t,x,\alpha}, P^{t,z,\alpha}, I^{t,z,\alpha})$ the triple of processes defined by

$$
\begin{aligned}
X_s^{t,x,\alpha} &= x + \int_t^s \eta X_u^{t,x,\alpha}(\lambda - X_u^{t,x,\alpha})du + \int_t^s \gamma X_u^{t,x,\alpha} dB_u - \int_t^s \alpha_u du \;, \\
P_s^{t,z,\alpha} &= p + \int_t^s \mu(X_u^{t,x,\alpha}) P_u^{t,z,\alpha} du + \int_t^s \sigma P_u^{t,z,\alpha} dW_u \;, \\
I_s^{t,z,\alpha} &= i \mathbb{1}_{t \le s < T_{p(t)+1}} + \sum_{i=p(t)+1}^{N-1} \mathbb{1}_{X_{T_i}^{t,x,\alpha} > \Gamma} \mathbb{1}_{T_i \le s < T_{i+1}} \;.
\end{aligned}
$$

For any $t \in [0,T]$ and $z \in \mathcal{Z}$, we consider the value function v defined by

$$
v(t,z) \;\; := \sup_{\alpha \in \mathcal{A}_{t,i}(x)} \mathbb{E}\Big[\int_t^T e^{-\beta(s-t)} (P_s^{t,z,\alpha} \alpha_s - C(\alpha_s)) ds - e^{-\beta(T-t)} f\big((\Gamma - X_T^{t,x,\alpha})^+, P_T^{t,p,\alpha}\big) \Big] \;.
$$

We also consider the two value functions v_0 and v_1 defined on $[0,T] \times \mathbb{R}_+ \times \mathbb{R}_+^*$ by

$$
v(t,z) \;\; = \;\; v_0(t,x,p)\mathbb{1}_{i=0} + v_1(t,x,p)\mathbb{1}_{i=1} \;.
$$

The value function v_0 corresponds to the case where at time t the fisherman can not fish until the next checking time, while the value function v_1 corresponds to the case where at time t the fisherman can fish.

3 HJB characterization

The HJB equations related to the value functions v_0 and v_1 are

$$
\begin{cases}
-\partial_t v_0(t,x,p) - \mathcal{L}^0 v_0(t,x,p) = 0 & (t,x,p) \in [0,T] - \{T_j\}_{1 \le j \le N} \times \mathbb{R}_+ \times \mathbb{R}_+^* \\
v_0(T_j^-, x, p) = v_0(T_j,x,p)\mathbb{1}_{x \le \Gamma} + v_1(T_j,x,p)\mathbb{1}_{x > \Gamma} & (j,x,p) \in \{1,\ldots,N-1\} \times \mathbb{R}_+ \times \mathbb{R}_+^* \\
v_0(T_N^-, x, p) = -f((\Gamma - x)^+, p) & (x,p) \in \mathbb{R}_+ \times \mathbb{R}_+^*
\end{cases}
\tag{3.2}
$$

and

$$
\begin{cases}
-\partial_t v_1(t,x,p) - \sup_{0 \le a \le \bar{a}} \big\{ \mathcal{L}^a v_1(t,x,p) + pa - C(a) \big\} = 0 & (t,x,p) \in [0,T] - \{T_j\}_{1 \le j \le N} \times \mathbb{R}_+ \times \mathbb{R}_+^* \\
v_1(T_j^-, x, p) = v_0(T_j,x,p)\mathbb{1}_{x \le \Gamma} + v_1(T_j,x,p)\mathbb{1}_{x > \Gamma} & (j,x,p) \in \{1,\ldots,N-1\} \times \mathbb{R}_+ \times \mathbb{R}_+^* \\
v_1(T_N^-, x, p) = -f((\Gamma - x)^+, p) & (x,p) \in \mathbb{R}_+ \times \mathbb{R}_+^*
\end{cases}
\tag{3.3}
$$

where $\mathcal{L}^a$ is the operator associated to the diffusions

$$
\mathcal{L}^a \varphi \;\; = \;\; -\beta\varphi + \eta x(\lambda - x)\partial_x \varphi + \frac{|\gamma x|^2}{2} \partial_x^2 \varphi + \mu(x) p \partial_p \varphi + \frac{|\sigma p|^2}{2} \partial_p^2 \varphi + \rho\sigma\gamma p x \partial_{px}^2 \varphi - a \partial_x \varphi \;.
$$

Proposition 3.1. *Let w^0 and w^1 be two functions in $C^{1,2}([T_i, T_{i+1}) \times \mathbb{R}_+ \times \mathbb{R}_+^*) \cap C^0([T_i, T_{i+1}] \times \mathbb{R}_+ \times \mathbb{R}_+^*)$ for any $i \in \{0,\ldots,N-1\}$, with $T_0 = 0$, and satisfying a quadratic growth condition, i.e. there exists a positive constant C such that*

$$
|w^0(t,x,p)| + |w^1(t,x,p)| \;\; \le \;\; C(1 + |x|^2 + |p|^2) \;, \quad \forall\, (t,x,p) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+^* \;.
$$

(i) Suppose that

$$\begin{cases} -\partial_t w_0(t,x,p) - \mathcal{L}^0 w_0(t,x,p) \geq 0 & (t,x,p) \in [0,T] - \{T_j\}_{1\leq j\leq N} \times \mathbb{R}_+ \times \mathbb{R}_+^* \\ w_0(T_j^-,x,p) \geq w_0(T_j,x,p)\mathbb{1}_{x\leq\Gamma} + w_1(T_j,x,p)\mathbb{1}_{x>\Gamma} & (j,x,p) \in \{1,\dots N-1\} \times \mathbb{R}_+ \times \mathbb{R}_+^* \\ w_0(T^-,x,p) \geq -f((\Gamma-x)^+,p) & (x,p) \in \mathbb{R}_+ \times \mathbb{R}_+^* \end{cases} \quad (3.4)$$

and

$$\begin{cases} -\partial_t w_1(t,x,p) - \sup_{0\leq a\leq\bar{a}} \{\mathcal{L}^a w_1(t,x,p) + pa - C(a)\} \geq 0 & (t,x,p) \in [0,T] - \{T_j\}_{1\leq i\leq N} \times \mathbb{R}_+ \times \mathbb{R}_+^* \\ w_1(T_j^-,x,p) \geq w_0(T_j,x,p)\mathbb{1}_{x\leq\Gamma} + w_1(T_j,x,p)\mathbb{1}_{x>\Gamma} & (j,x,p) \in \{1,\dots N-1\} \times \mathbb{R}_+ \times \mathbb{R}_+^* \\ w_1(T^-,x,p) \geq -f((\Gamma-x)^+,p) & (x,p) \in \mathbb{R}_+ \times \mathbb{R}_+^* . \end{cases} \quad (3.5)$$

Then the function w defined by $w(t,z) := w_0(t,x,p)\mathbb{1}_{i=0} + w_1(t,x,p)\mathbb{1}_{i=1}$ satisfies $w(t,z) \geq v(t,z)$ on $[0,T] \times \mathcal{Z}$.

(ii) Suppose further that for any $z \in \mathcal{Z}$, there exists a measurable function $\hat{\alpha}(t,z)$ valued in $[0,\bar{a}]$ such that we have :
if $i = 0$

$$-\partial_t w(t,z) - \mathcal{L}^0 w(t,z) \quad = \quad 0$$

and if $i = 1$

$$-\partial_t w(t,z) - \sup_{a\in[0,\bar{a}]} \left[\mathcal{L}^a w(t,z) + pa - C(a)\right] \quad = \quad -\partial_t w(t,z) - \mathcal{L}^{\hat{\alpha}(t,z)} w(t,z) - p\hat{\alpha}(t,z) + C(\hat{\alpha}(t,z)) = 0$$

and

$$w(T_j^-,z) \quad = \quad w(T_j,x,p,0)\mathbb{1}_{x\leq\Gamma} + w(T_j,x,p,1)\mathbb{1}_{x>\Gamma} \quad (j,z) \in \{1,\dots,N-1\} \times \mathcal{Z}$$

and

$$w(T^-,z) = -f((\Gamma-x)^+,p)$$

the SDE

$$\begin{aligned} X_s^{t,x,\hat{\alpha}} &= x + \int_t^s \eta X_u^{t,x,\hat{\alpha}}(\lambda - X_u^{t,x,\hat{\alpha}})du + \int_t^s \gamma X_u^{t,x,\hat{\alpha}} dB_u - \int_t^s \hat{\alpha}_u du \\ P_s^{t,z,\hat{\alpha}} &= p + \int_t^s \mu(X_u^{t,x,\hat{\alpha}})P_u^{t,z,\hat{\alpha}} du + \int_t^s \sigma P_u^{t,z,\hat{\alpha}} dW_u \\ I_s^{t,z,\hat{\alpha}} &= i\mathbb{1}_{t\leq s<T_{p(t)+1}} + \sum_{i=p(t)+1}^{N-1} \mathbb{1}_{X_{T_i}^{t,x,\hat{\alpha}}>\Gamma} \mathbb{1}_{T_i\leq s<T_{i+1}} \end{aligned}$$

admits a unique solution, denoted by $\hat{Z}_s^{t,z}$ given an initial condition $Z_t = z$, and the process $\{\hat{\alpha}(s,\hat{Z}_s^{t,z})\ ,\ t \leq s \leq T\}$ lives in $\mathcal{A}_{t,i}(x)$. Then

$$w \quad = \quad v \quad \text{on } [0,T] \times \mathcal{Z}$$

and $\hat{\alpha}$ is an optimal Markovian control.

Proof. In the proof, to simplify the notation, we introduce $K_s^{t,k,\alpha} := (X_s^{t,x,\alpha}, P_s^{t,z,\alpha})$ and $k := (x,p)$ for any $z \in \mathcal{Z}$ and $\alpha \in \mathcal{A}_{t,i}(x)$.
(i) We prove by induction that $w \geq v$ on $[T_j, T_{j+1}]$ for any $j \in \{0,\dots,N-1\}$.
We first consider the case $j = N-1$ and $i = 0$, that means the fisherman can't fish on $[T_{N-1}, T_N]$, thus $v(t,z) = \mathbb{E}[-e^{-\beta(T-t)} f((\Gamma - X_T^{t,x,0})^+, P_T^{t,p,0})]$.

Since w^0 is $C^{1,2}([T_{N-1},T_N) \times \mathbb{R}_+ \times \mathbb{R}_+^*) \cap C^0([T_{N-1},T_N] \times \mathbb{R}_+ \times \mathbb{R}_+^*)$, we have for any $(t,x,p) \in [T_{N-1},T_N) \times \mathbb{R}_+ \times \mathbb{R}_+^*$, $\alpha \in \mathcal{A}_{t,0}(x)$, $s \in [t, T_N)$, and any stopping time τ valued in $[t,T]$, by Itô's formula

$$\begin{aligned} e^{-\beta s\wedge\tau} w_0(s\wedge\tau, K_{s\wedge\tau}^{t,k,\alpha}) &= e^{-\beta t} w_0(t,k) + \int_t^{s\wedge\tau} e^{-\beta u}\big(\partial_t w_0(u, K_u^{t,k,\alpha}) + \mathcal{L}^0 w_0(u, K_u^{t,k,\alpha})\big) du \\ &\quad + \int_t^{s\wedge\tau} e^{-\beta u}\big(\partial_x w_0(u, K_u^{t,k,\alpha})\gamma X_u^{t,x,\alpha} dB_u + \partial_p w_0(u, K_u^{t,k,\alpha})\sigma P_u^{t,z,\alpha} dW_u\big) \end{aligned}$$

We choose $\tau = \tau_n := \inf\{s \geq t : \int_t^s \big(|\partial_x w_0(u, K_u^{t,k,\alpha}) X_u^{t,x,\alpha}|^2 + |\partial_p w_0(u, K_u^{t,k,\alpha}) P_u^{t,z,\alpha}|^2\big) du \geq n\} \wedge T$ and we remark $(\tau_n)_{n\geq 1}$ is an increasing sequence going to T when n goes to ∞. By taking the expectation, we get

$$\mathbb{E}\big[e^{-\beta s\wedge\tau_n} w_0(s\wedge\tau_n, K_{s\wedge\tau_n}^{t,k,\alpha})\big] = e^{-\beta t} w_0(t,k) + \mathbb{E}\Big[\int_t^{s\wedge\tau_n} e^{-\beta u}\big(\partial_t w_0(u, K_u^{t,k,\alpha}) + \mathcal{L}^0 w_0(u, K_u^{t,k,\alpha})\big) du\Big] .$$

Since w_0 satisfies (3.4), we have

$$\mathbb{E}\big[e^{-\beta s\wedge\tau_n} w_0(s\wedge\tau_n, K_{s\wedge\tau_n}^{t,k,\alpha})\big] \leq e^{-\beta t} w_0(t,k) .$$

By the quadratic growth condition on w_0 and the integrability condition on $K^{t,k,\alpha}$ we may apply the dominated convergence theorem and send n to infinity

$$\mathbb{E}\big[e^{-\beta s} w_0(s, K_s^{t,k,\alpha})\big] \leq e^{-\beta t} w_0(t,k) .$$

By sending s to T_N we obtain by the dominated convergence theorem

$$\mathbb{E}\big[e^{-\beta T_N} w_0(T_N^-, K_{T_N^-}^{t,k,\alpha})\big] \leq e^{-\beta t} w_0(t,k) .$$

Which implies

$$e^{-\beta t} v(t,z) = \mathbb{E}\big[-e^{-\beta T_N} f((\Gamma - X_{T_N}^{t,x,0})^+, P_{T_N}^{t,p,0})\big] \leq e^{-\beta t} w_0(t,k) .$$

We now consider the case $j = N-1$ and $i = 1$. Since w^1 is $C^{1,2}([T_{N-1},T_N) \times \mathbb{R}_+ \times \mathbb{R}_+^*) \cap C^0([T_{N-1},T_N] \times \mathbb{R}_+ \times \mathbb{R}_+^*)$, we have for any $(t,x,p) \in [T_{N-1},T_N) \times \mathbb{R}_+ \times \mathbb{R}_+^*$, $\alpha \in \mathcal{A}_{t,1}(x)$, $s \in [t,T_N)$, and any stopping time τ valued in $[t,T]$, by Itô's formula

$$\begin{aligned} e^{-\beta s\wedge\tau} w_1(s\wedge\tau, K_{s\wedge\tau}^{t,k,\alpha}) &= e^{-\beta t} w_1(t,k) + \int_t^{s\wedge\tau} e^{-\beta u}\big(\partial_t w_1(u, K_u^{t,k,\alpha}) + \mathcal{L}^{\alpha_u} w_1(u, K_u^{t,k,\alpha})\big) du \\ &\quad + \int_t^{s\wedge\tau} e^{-\beta u}\big(\partial_x w_1(u, K_u^{t,k,\alpha})\gamma X_u^{t,x,\alpha} dB_u + \partial_p w_1(u, K_u^{t,k,\alpha})\sigma P_u^{t,z,\alpha} dW_u\big) \end{aligned}$$

We choose $\tau = \tau_n := \inf\{s \geq t : \int_t^s \big(|\partial_x w_1(u, K_u^{t,k,\alpha}) X_u^{t,x,\alpha}|^2 + |\partial_p w_1(u, K_u^{t,k,\alpha}) P_u^{t,z,\alpha}|^2\big) du \geq n\} \wedge T$ and we remark $(\tau_n)_{n\geq 1}$ is an increasing sequence going to T when n goes to infinity. By taking the expectation, we get

$$\mathbb{E}\big[e^{-\beta s\wedge\tau_n} w_1(s\wedge\tau_n, K_{s\wedge\tau_n}^{t,k,\alpha})\big] = e^{-\beta t} w_1(t,k) + \mathbb{E}\Big[\int_t^{s\wedge\tau_n} e^{-\beta u}\big(\partial_t w_1(u, K_u^{t,k,\alpha}) + \mathcal{L}^{\alpha_u} w_1(u, K_u^{t,k,\alpha})\big) du\Big] .$$

By using (3.5) we get

$$\mathbb{E}\big[e^{-\beta s\wedge\tau_n} w_1(s\wedge\tau_n, K^{t,k,\alpha}_{s\wedge\tau_n})\big] \quad\le\quad e^{-\beta t} w_1(t,k) - \mathbb{E}\Big[\int_t^{s\wedge\tau_n} e^{-\beta u}(P^{t,z,\alpha}_u \alpha_u - C(\alpha_u))du\Big] \,.$$

By sending n to infinity we obtain by the dominated convergence theorem

$$\mathbb{E}\big[e^{-\beta s} w_1(s, K^{t,k,\alpha}_{s})\big] \quad\le\quad e^{-\beta t} w_1(t,k) - \mathbb{E}\Big[\int_t^{s} e^{-\beta u}(P^{t,z,\alpha}_u \alpha_u - C(\alpha_u))du\Big] \,.$$

By sending s to T_N^- we obtain by the dominated convergence theorem

$$\mathbb{E}\big[e^{-\beta T_N} w_1(T_N^-, K^{t,k,\alpha}_{T_N^-})\big] \quad\le\quad e^{-\beta t} w_1(t,k) - \mathbb{E}\Big[\int_t^{T_N} e^{-\beta u}(P^{t,z,\alpha}_u \alpha_u - C(\alpha_u))du\Big] \,.$$

Which implies for any $\alpha \in \mathcal{A}_{t,1}(x)$

$$\mathbb{E}\Big[\int_t^{T_N} e^{-\beta u}(P^{t,z,\alpha}_u \alpha_u - C(\alpha_u))du - e^{-\beta T_N} f((\Gamma - X^{t,x,\alpha}_{T_N})^+, P^{t,p,\alpha}_{T_N})\Big] \quad\le\quad e^{-\beta t} w_1(t,k) \,.$$

Thus $v(t,z) \le w(t,z)$ for any $[T_{N-1}, T_N] \times \mathcal{Z}$.

We now suppose the result holds on $[T_j, T_{j+1}]$ for one $j \in \{1, \dots, N-1\}$. We first consider the case $i = 0$. Since w^0 is $C^{1,2}([T_{j-1}, T_j) \times \mathbb{R}_+ \times \mathbb{R}_+^*) \cap C^0([T_{j-1}, T_j] \times \mathbb{R}_+ \times \mathbb{R}_+^*)$, we have for any $(t,x,p) \in [T_{j-1}, T_j) \times \mathbb{R}_+ \times \mathbb{R}_+^*$, $\alpha \in \mathcal{A}_{t,0}(x)$, $s \in [t, T_j)$, and any stopping time τ valued in $[t, T_j]$, by Itô's formula

$$\begin{aligned} e^{-\beta s\wedge\tau} w_0(s\wedge\tau, K^{t,k,\alpha}_{s\wedge\tau}) \quad=\quad & e^{-\beta t} w_0(t,z) + \int_t^{s\wedge\tau} e^{-\beta u}\big(\partial_t w_0(u, K^{t,k,\alpha}_u) + \mathcal{L}^0 w_0(u, K^{t,k,\alpha}_u)\big)du \\ & + \int_t^{s\wedge\tau} e^{-\beta u}\big(\partial_x w_0(u, K^{t,k,\alpha}_u)\gamma X^{t,x,\alpha}_u dB_u + \partial_p w_0(u, K^{t,k,\alpha}_u)\sigma P^{t,z,\alpha}_u dW_u\big) \end{aligned}$$

By using the same technics that previously we get

$$\mathbb{E}\big[e^{-\beta T_j} w_0(T_j^-, K^{t,k,\alpha}_{T_j^-})\big] \quad\le\quad e^{-\beta t} w_0(t,k) \,.$$

By using the condition at time T_j^- for w_0 we get

$$\begin{aligned} e^{-\beta t} w_0(t,k) \quad\ge\quad & \mathbb{E}\big[e^{-\beta T_j}\big(w_0(T_j, K^{t,k,\alpha}_{T_j})\mathbb{1}_{X^{t,x,\alpha}_{T_j} \le \Gamma} + w_1(T_j, K^{t,k,\alpha}_{T_j})\mathbb{1}_{X^{t,x,\alpha}_{T_j} > \Gamma}\big)\big] \\ \ge\quad & \mathbb{E}\big[e^{-\beta T_j} w(T_j, Z^{t,z,\alpha}_{T_j})\big] \\ \ge\quad & \mathbb{E}\big[e^{-\beta T_j} v(T_j, Z^{t,z,\alpha}_{T_j})\big] = e^{-\beta t} v(t,z) \,. \end{aligned}$$

We now consider the case $i = 1$. Since w^1 is $C^{1,2}([T_{j-1}, T_j) \times \mathbb{R}_+ \times \mathbb{R}_+^*) \cap C^0([T_{j-1}, T_j] \times \mathbb{R}_+ \times \mathbb{R}_+^*)$, we have for any $(t,x,p) \in [T_{j-1}, T_j) \times \mathbb{R}_+ \times \mathbb{R}_+^*$, $\alpha \in \mathcal{A}_{t,1}(x)$, $s \in [t, T_j)$, and any stopping time τ valued in $[t, T_j]$, by Itô's formula

$$\begin{aligned} e^{-\beta s\wedge\tau} w_1(s\wedge\tau, K^{t,k,\alpha}_{s\wedge\tau}) \quad=\quad & e^{-\beta t} w_1(t,k) + \int_t^{s\wedge\tau} e^{-\beta u}\big(\partial_t w_1(u, K^{t,k,\alpha}_u) + \mathcal{L}^{\alpha_u} w_1(u, K^{t,k,\alpha}_u)\big)du \\ & + \int_t^{s\wedge\tau} e^{-\beta u}\big(\partial_x w_1(u, K^{t,k,\alpha}_u)\gamma X^{t,x,\alpha}_u dB_u + \partial_p w_1(u, K^{t,k,\alpha}_u)\sigma P^{t,z,\alpha}_u dW_u\big) \,. \end{aligned}$$

By using the previous arguments we obtain

$$\mathbb{E}\big[e^{-\beta T_j} w_1(T_j^-, K^{t,k,\alpha}_{T_j^-})\big] \;\; \leq \;\; e^{-\beta t} w_1(t,k) - \mathbb{E}\Big[\int_t^{T_j} e^{-\beta u}(P_u^{t,z,\alpha}\alpha_u - C(\alpha_u))du\Big] \, .$$

By using the condition at time T_j^- for w_1 we get

$$\begin{aligned} e^{-\beta t} w_1(t,k) \;\; &\geq \;\; \mathbb{E}\big[e^{-\beta T_j}\big(w_0(T_j, K^{t,k,\alpha}_{T_j})\mathbb{1}_{X^{t,x,\alpha}_{T_j}\leq \Gamma} + w_1(T_j, K^{t,k,\alpha}_{T_j})\mathbb{1}_{X^{t,x,\alpha}_{T_j}>\Gamma}\big)\big] \\ &\quad + \mathbb{E}\Big[\int_t^{T_j} e^{-\beta u}(P_u^{t,z,\alpha}\alpha_u - C(\alpha_u))du\Big] \\ &\geq \;\; \mathbb{E}\big[e^{-\beta T_j} w(T_j, Z^{t,z,\alpha}_{T_j})\big] + \mathbb{E}\Big[\int_t^{T_j} e^{-\beta u}(P_u^{t,z,\alpha}\alpha_u - C(\alpha_u))du\Big] \\ &\geq \;\; \mathbb{E}\big[e^{-\beta T_j} v(T_j, Z^{t,z,\alpha}_{T_j})\big] + \mathbb{E}\Big[\int_t^{T_j} e^{-\beta u}(P_u^{t,z,\alpha}\alpha_u - C(\alpha_u))du\Big] \, . \end{aligned}$$

Then for any $\bar\alpha \in \mathcal{A}_{T_j, I^{t,i}_{T_j}}(X^{t,x,\alpha}_{T_j})$ we get

$$\begin{aligned} e^{-\beta t} w_1(t,k) \;\; &\geq \;\; \mathbb{E}\Big[\int_{T_j}^{T} e^{-\beta s}(P_s^{T_j, Z^{t,z,\alpha}_{T_j}, \bar\alpha}\bar\alpha_s - C(\bar\alpha_s))ds - e^{-\beta T} f((\Gamma - X_T^{T_j, X^{t,x,\alpha}_{T_j}, \bar\alpha})^+, P_T^{T_j, P^{t,p,\alpha}_{T_j}, \bar\alpha})\Big] \\ &\quad + \mathbb{E}\Big[\int_t^{T_j} e^{-\beta u}(P_u^{t,z,\alpha}\alpha_u - C(\alpha_u))du\Big] \, , \end{aligned}$$

which implies for any $\alpha \in \mathcal{A}_{t,i}(x)$ we get

$$w_1(t,k) \;\; \geq \;\; \mathbb{E}\Big[\int_t^{T} e^{-\beta(u-t)}(P_u^{t,z,\alpha}\alpha_u - C(\alpha_u))du - e^{-\beta(T-t)} f((\Gamma - X_T^{t,x,\alpha})^+, P_T^{t,p,\alpha})\Big] \, .$$

Thus $w_1(t,x,p) \geq v(t,z)$.

(ii) We prove by induction that $w = v$ on $[T_j, T_{j+1}]$ for any $j \in \{0, \dots, N-1\}$.
We first consider the case $j = N-1$ and $i = 0$. We apply Itô's formula to $e^{-\beta u} w(u, \hat Z_u^{t,z})$ between $t \in [T_{N-1}, T_N)$ and $s \in [t,T)$ (after a localization for removing the stochastic integral term in the expectation)

$$\mathbb{E}\big[e^{-\beta T_N} w(T_N^-, \hat Z^{t,z}_{T_N^-})\big] \;\; = \;\; e^{-\beta t} w(t,z) + \mathbb{E}\Big[\int_t^{T_N} e^{-\beta u}\big(\partial_t w(u, \hat Z_u^{t,z}) + \mathcal{L}^0 w(u, \hat Z_u^{t,z})\big)du\Big] \, .$$

Thus we get

$$w(t,z) \;\; = \;\; \mathbb{E}\big[- e^{-\beta(T_N - t)} f((\Gamma - X_T^{t,x,0})^+, P_T^{t,p,0})\big] \;\; = \;\; v(t,z) \, .$$

We now consider the case $j = N-1$ and $i = 1$. We apply Itô's formula to $e^{-\beta u} w(u, \hat Z_u^{t,z})$ between $t \in [T_{N-1}, T_N)$ and T_N (after a localization for removing the stochastic integral term in the expectation)

$$\mathbb{E}\big[e^{-\beta(T_N - t)} w(T_N^-, \hat Z^{t,z}_{T_N^-})\big] \;\; = \;\; w(t,z) + \mathbb{E}\Big[\int_t^{T_N} e^{-\beta(u-t)}\big(\partial_t w(u, \hat Z_u^{t,z}) + \mathcal{L}^{\hat\alpha(u, \hat Z_u^{t,z})} w(u, \hat Z_u^{t,z})\big)du\Big] \, .$$

Which implies

$$\begin{aligned}
w(t,z) &= \mathbb{E}\Big[\int_t^{T_N} e^{-\beta(u-t)}(P_u^{t,z,\hat\alpha}\hat\alpha(u,\hat Z_u^{t,z}) - C(\hat\alpha(u,\hat Z_u^{t,z})))du - e^{-\beta(T-t)}f((\Gamma - X_T^{t,x,\hat\alpha})^+, P_T^{t,p,\hat\alpha})\Big] \\
&= J(t,z,\hat\alpha)\ .
\end{aligned}$$

Thus $w(t,z) = J(t,z,\hat\alpha) = v(t,z)$ on $[T_{N-1},T_N]\times\mathbb{R}_+\times\mathbb{R}_+^*$ with $i=1$.
We now suppose the result holds on $[T_j,T_{j+1}]$ for one $j\in\{1,\dots,N-1\}$. We first consider the case $i=0$. Since w is $C^{1,2}([T_{j-1},T_j)\times\mathbb{R}_+\times\mathbb{R}_+^*)\cap C^0([T_{j-1},T_j]\times\mathbb{R}_+\times\mathbb{R}_+^*)$, we have for any $(t,x,p)\in[T_{j-1},T_j)\times\mathbb{R}_+\times\mathbb{R}_+^*$ by using the previous technics

$$\mathbb{E}\big[e^{-\beta(T_j-t)}w(T_j^-,\hat Z_{T_j^-}^{t,z})\big] \quad = \quad w(t,z)\ .$$

By using the condition at time T_j^- for w we get

$$\begin{aligned}
e^{-\beta t}w(t,z) &= \mathbb{E}\big[e^{-\beta T_j}\big(w(T_j,X_{T_j}^{t,x,\hat\alpha},P_{T_j}^{t,z,\hat\alpha},0)\mathbb{1}_{X_{T_j}^{t,x}\le\Gamma} + w(T_j,X_{T_j}^{t,x,\hat\alpha},P_{T_j}^{t,z,\hat\alpha},1)\mathbb{1}_{X_{T_j}^{t,x}>\Gamma}\big)\big] \\
&= \mathbb{E}\big[e^{-\beta T_j}w(T_j,\hat Z_{T_j}^{t,z})\big] \\
&= \mathbb{E}\big[e^{-\beta T_j}v(T_j,\hat Z_{T_j}^{t,z})\big] = e^{-\beta t}v(t,z)
\end{aligned}$$

We now consider the case $i=1$. Since w is $C^{1,2}([T_{j-1},T_j)\times\mathbb{R}_+\times\mathbb{R}_+^*)\cap C^0([T_{j-1},T_j]\times\mathbb{R}_+\times\mathbb{R}_+^*)$, we have for any $(t,x,p)\in[T_{j-1},T_j)\times\mathbb{R}_+\times\mathbb{R}_+^*$, $\alpha\in\mathcal{A}_{t,1}(x)$, by Itô's formula

$$\begin{aligned}
\mathbb{E}[e^{-\beta T_j}w(T_j,\hat Z_{T_j}^{t,z})] &= e^{-\beta t}w(t,z) + \mathbb{E}\Big[\int_t^{T_j} e^{-\beta u}\big(\partial_t w(u,\hat Z_u^{t,z}) + \mathcal{L}^{\hat\alpha(u,\hat Z_u^{t,z})}w(u,\hat Z_u^{t,z})\big)du\Big] \\
&= e^{-\beta t}w(t,z) - \mathbb{E}\Big[\int_t^{T_j} e^{-\beta u}(P_u^{t,z,\hat\alpha}\hat\alpha(u,\hat Z_u^{t,z}) - C(\hat\alpha(u,\hat Z_u^{t,z})))du\Big]
\end{aligned}$$

By using the condition at time T_j^- for w we get

$$\begin{aligned}
e^{-\beta t}w(t,z) &= \mathbb{E}[\int_{T_j}^T e^{-\beta u}(P_u^{T_j,\hat Z_{T_j}^{t,z},\hat\alpha}\hat\alpha(u,\hat Z_u^{T_j,\hat Z_{T_j}^{t,z}}) - C(\hat\alpha(u,\hat Z_u^{T_j,\hat Z_{T_j}^{t,z}})))du] \\
&\quad -\mathbb{E}[e^{-\beta T}f((\Gamma - X_T^{T_j,X_{T_j}^{t,x,\hat\alpha},\hat\alpha})^+, P_T^{T_j,P_{T_j}^{t,p,\hat\alpha},\hat\alpha})] \\
&\quad +\mathbb{E}\Big[\int_t^{T_j} e^{-\beta u}(P_u^{t,z,\hat\alpha}\hat\alpha(u,\hat Z_u^{t,z}) - C(\hat\alpha(u,\hat Z_u^{t,z})))du\Big] \\
&= \mathbb{E}[\int_t^T e^{-\beta u}(P_u^{t,z,\hat\alpha}\hat\alpha(u,\hat Z_u^{t,z}) - C(\hat\alpha(u,\hat Z_u^{t,z})))du - e^{-\beta T}f((\Gamma - X_T^{t,x,\hat\alpha})^+, P_T^{t,p,\hat\alpha})]
\end{aligned}$$

□

4 Numerical Results

4.1 The discrete problem

In this section we introduce the numerical tools that we use to solve the HJB equations linked to v_0 and v_1 and associated to the stochastic control problem. We use an implicit finite difference scheme mixed with an iterative procedure which leads to the resolution of a Controlled Markov Chain problem. This class of problems is intensely studied by Kushner and Dupuis [7]. The convergence of the solution of the numerical scheme towards the solution of the HJB equation, when the time-space step goes to zero, can be shown using the standard local consistency argument i.e. the first and the second moments of the approximating Markov chain converge to those of the continuous process (X, P). We refer to [4, 5, 6] for numerical schemes involving a Controlled Markov Chain control problem.

We begin by localizing the problem on the bounded domain $[0,T]\times[0,x_{max}]\times[p_{min},p_{max}]$, where x_{max}, p_{min} and p_{max} are nonnegative constants. Then we assume the following Neumann boundary conditions on the localized boundary

$$\begin{aligned}\frac{\partial v}{\partial x}(t,0,p) &= \frac{\partial v}{\partial x}(t,x_{max},p) \;=\; 0\;,\\ \frac{\partial v}{\partial p}(t,x,p_{min}) &= \frac{\partial v}{\partial p}(t,x,p_{max}) \;=\; 0\;.\end{aligned}$$

Let δ, h and k be the discretization steps along the directions t, x and p respectively. For (t,x,p) in the time-space grid

$$\mathcal{G}_{\delta,h,k} := \{t_i=(i-1)\delta,\ i=1,\dots,n_t\}\times\{x_j=(j-1)h,\ j=1,\dots,n_x\}\times\{p_l=p_{min}+(l-1)k,\ l=1,\dots,n_p\},$$

where $n_t = T/\delta+1$, $n_x = x_{max}/h+1$ and $n_p = (p_{max}-p_{min})/k+1$.
We consider approximations of the following form

$$\begin{aligned}\frac{\partial v}{\partial t}(t,x,p) &\sim \frac{v(t+\delta,x,p)-v(t,x,p)}{\delta}\;,\\ \frac{\partial v}{\partial x}(t,x,p) &\sim \pm\frac{v(t,x\pm h,p)-v(t,x,p)}{h}\;,\\ \frac{\partial v}{\partial p}(t,x,p) &\sim \pm\frac{v(t,x,p+k)-v(t,x,p)}{k}\;,\\ \frac{\partial^2 v}{\partial x^2}(t,x,p) &\sim \frac{v(t,x+h,p)+v(t,x-h,p)-2v(t,x,p)}{h^2}\;,\\ \frac{\partial^2 v}{\partial p^2}(t,x,p) &\sim \frac{v(t,x,p+k)+v(t,x,p-k)-2v(t,x,p)}{k^2}\;,\\ \frac{\partial^2 v}{\partial x\partial y}(t,x,p) &\sim \frac{2v(t,x,p)+v(t,x+h,p+k)+v(t,x-h,p-k)}{2hk}\\ &\quad -\frac{v(t,x+h,p)+v(t,x,p+k)+v(t,x-h,p)+v(t,x,p-k)}{2hk}\;.\end{aligned}$$

Let's introduce the following quantities

$$\begin{aligned}\eta_x(x,a) &:= \eta x(\lambda-x)-a\;,\\ Q^{\delta,h,k}(x,p,a) &:= 1+\frac{|\eta_x(x,a)|\delta}{h}+\frac{\mu(x)p\delta}{k}+\frac{\gamma^2x^2\delta}{h^2}+\frac{\sigma^2p^2\delta}{k^2}-\frac{\rho\sigma\gamma xp\delta}{hk}\;,\\ \Delta t^{\delta,h,k}(x,p,a) &:= \frac{\delta}{Q^{\delta,h,k}(x,p,a)}\;.\end{aligned}$$

In the below table we define the Markov chain states and the associated transition probabilities that we obtain when we apply the finite difference approach.

Markov Chain State	Transition Probability
$z_1 = (t, x+h, p)$	$\pi_1(x,p,a) = \left(\frac{(\eta_x(x,a))^+\delta}{h} + \frac{\gamma^2 x^2 \delta}{2h^2} - \frac{\rho\sigma\gamma x p\delta}{2hk}\right)/Q^{\delta,h,k}(x,p,a)$
$z_2 = (t, x-h, p)$	$\pi_2(x,p,a) = \left(\frac{(\eta_x(x,a))^-\delta}{h} + \frac{\gamma^2 x^2 \delta}{2h^2} - \frac{\rho\sigma\gamma x p\delta}{2hk}\right)/Q^{\delta,h,k}(x,p,a)$
$z_3 = (t, x, p+k)$	$\pi_3(x,p,a) = \left(\frac{\mu(x)p\delta}{k} + \frac{\sigma^2 p^2 \delta}{2k^2} - \frac{\rho\sigma\gamma x p\delta}{2hk}\right)/Q^{\delta,h,k}(x,p,a)$
$z_4 = (t, x, p-k)$	$\pi_4(x,p,a) = \left(\frac{\sigma^2 p^2 \delta}{2k^2} - \frac{\rho\sigma\gamma x p\delta}{2hk}\right)/Q^{\delta,h,k}(x,p,a)$
$z_5 = (t, x+h, p+k)$	$\pi_5(x,p,a) = \left(\frac{\rho\sigma\gamma x p\delta}{2hk}\right)/Q^{\delta,h,k}(x,p,a)$
$z_6 = (t, x-h, p-k)$	$\pi_6(x,p,a) = \left(\frac{\rho\sigma\gamma x p\delta}{2hk}\right)/Q^{\delta,h,k}(x,p,a)$
$z_7 = (t+\delta, x, p)$	$\pi_7(x,p,a) = 1/Q^{\delta,h,k}(x,p,a)$

Table 1 : *The approximating Markov Chain*

Thus, using the above notations and discretizing the space of controls as follows

$$\{0, \dots, \bar{a}\} \quad := \quad \{a = (m-1)\bar{a}/(n_a - 1), \ m = 1, \dots, n_a\}$$

where $n_a \in \mathbb{N}^*$, we approximate the HJB equations associated to the functions v_0 and v_1 by the following iterative scheme

$$\begin{aligned} v_0^{\delta,n+1}(t,x,p) &= \left\{ \frac{\sum_{i=1}^7 \pi_i(x,p,0) v_0^{\delta,n}(z_i)}{1 + \beta \Delta t^{\delta,h,k}(x,p,0)} \right\}, \quad t \in [0,T] - \{T_j\}_{1 \le j \le N} \\ v_0^{\delta,n+1}(T_j - \delta, x, p) &= v_0^{\delta,n}(T_j,x,p)\mathbb{1}_{x \le \Gamma} + v_1^{\delta,n}(T_j,x,p)\mathbb{1}_{x > \Gamma}, \quad j \in \{1,..,N-1\} \\ v_0^{\delta,n+1}(T_N - \delta, x, p) &= -f((\Gamma - x)^+, p) \\ v_0^{\delta,0} &\equiv 0 \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} v_1^{\delta,n+1}(t,x,p) &= \max_{\{0,\dots,\bar{a}\}} \left\{ \frac{\sum_{i=1}^7 \pi_i(x,p,a) v_1^{\delta,n}(z_i) + (pa - C(a))\Delta t^{\delta,h,k}(x,p,a)}{1 + \beta \Delta t^{\delta,h,k}(x,p,a)} \right\}, \ t \in [0,T] - \{T_j\}_{1 \le j \le N} \\ v_1^{\delta,n+1}(T_j - \delta, x, p) &= v_0^{\delta,n}(T_j,x,p)\mathbb{1}_{x \le \Gamma} + v_1^{\delta,n}(T_j,x,p)\mathbb{1}_{x > \Gamma}, \quad j \in \{1,..,N-1\} \\ v_1^{\delta,n+1}(T_N - \delta, x, p) &= -f((\Gamma - x)^+, p) \\ v_1^{\delta,0} &\equiv 0 \,. \end{aligned} \tag{4.7}$$

For all $(x,p) \in [0, x_{max}] \times [p_{min}, p_{max}]$, the above iterative scheme combined with the boundary conditions is explicit and fully implementable on the enlarged grid

$$\begin{aligned} \mathcal{G}^+_{\delta,h,k} \quad := \quad & \{t_i = (i-1)\delta, \ i = 1, \dots, n_t\} \times \{x_j = (j-1)h, \ j = 0, \dots, n_x + 1\} \\ & \times \{p_l = p_{min} + (l-1)k, \ l = 0, \dots, n_p + 1\} \end{aligned}$$

for a given stopping criterion ε (i.e. the iterative scheme is stopped when the relative error is less than ε).

Remark 4.1. *It's clear that the first and the second moments of the Markov chain defined in Table 1 converge to those of the continuous process* (X, P) *as the time and space steps go to zero. Hence, the convergence of our scheme may be obtained using the same analysis developed in [7].*

4.2 Numerical Interpretations

The numerical computation are done using the following set of data:

– Dynamics values:

 ▶ $\eta = 0.7$, $\lambda = 0.5$, $\gamma = 0.2$, $\mu = 0.1$, $\sigma = 0.1$, $\rho = 0.01$.

 ▶ $T = 1$, $\beta = 0.1$.

 ▶ Drift function: $\mu(x) = \mu + \xi_1 e^{-\xi_2 x}$, $\xi_1 = 0.5$, $\xi_2 = 0.2$.

 ▶ Penalty function : $f(x, p) = \kappa p x$, $\kappa = 5$.

 ▶ Cost function : $C(x) = x^2$.

 ▶ Regulation parameters : $N = 10$ (number of cheks) , $\Gamma = 0.2308$.

– Grid values:

 ▶ Localisation: $x_{max} = 1$, $p_{min} = 0.1$, $p_{max} = 1.1$, $\bar{a} = 0.5$.
 ▶ Discretization: $n_x = 40$, $n_p = 40$, $n_t = 100$ and $n_a = 10$.
 ▶ Stopping criterion : $\varepsilon = 0.01$.

Figure 1: The shape of the value functions v_1 and v_0 for a fixed time t.
We plot the shape of the value functions v_1 and v_0 sliced in the plane (x, p) for a fixed date t. We can see that, as expected, $v_1 \geq v_0$ because if we can fish we will have a greater payoff. The two functions are nondecreasing w.r.t. x which is natural due to the fact that the bigger is the fish population, the more we can fish and the less we are penalized at time T. On the other hand, the two functions are nondecreasing in p when $x > \Gamma$, because the higher the price, the richer we get when we harvest and sell. Conversely, when $x < \Gamma$, the value functions v_1 and v_0 are non-increasing in p which is due to the penalty function f that is nondecreasing w.r.t. p (i.e. the higher is the price the more we are penalized by the regulator).

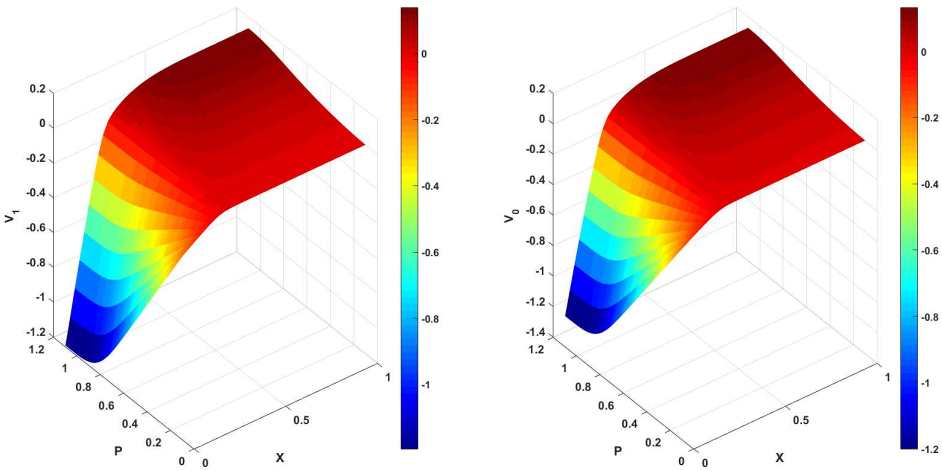

Figure 1: *The shape of the value functions v_1 and v_0 for a fixed time t*

Figure 2: The shape of the value functions v_1 and v_0 for a fixed price p.
We plot the shape of the value functions v_1 and v_0 sliced in the plane (t, x) for a fixed price p. As expected, v_1 and v_0 are decreasing w.r.t. t if x is large (we are not penalized since $x > \Gamma$ and we fish less when t is close to T) and increasing if x is small (we are penalized less as the process X increases w.r.t. t) .

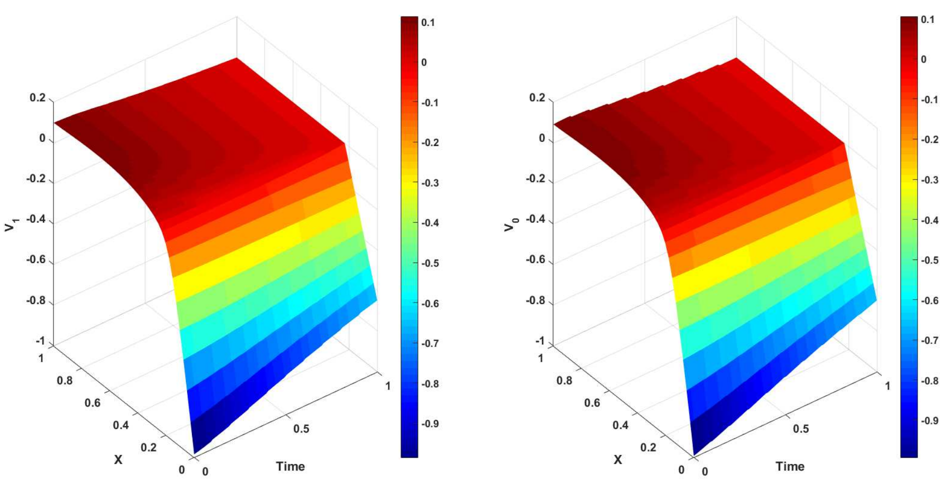

Figure 2: *The shape of the value functions v_1 and v_0 for a fixed price p*

Figure 3: The shape of the optimal control for a fixed price p (different regions).
We plot the shape of the optimal fish strategy $\alpha^*(t, x, p)$ sliced in the plane (x, t) for a fixed

price p. We can see two main regions : a Fish region (with different harvesting rates) and a No-fish region (dark blue). The interpretations of these results are as follows: when we are far from maturity T, it's not optimal to fish when X is under λ because under this quantity the fish quantity increases naturally so we want to let this happen to reach the maturity with X over Γ and hence, to avoid the penalization. As we are closer to T, it is optimal to fish when $X < \lambda$ with different rates a which permits to attain T without being penalized (i.e. $X_T > \Gamma$) and optimizing the profit generated by selling the harvested fish. The rates are greater as the fish population grows and this is due to the cost function C (the cost of harvesting).

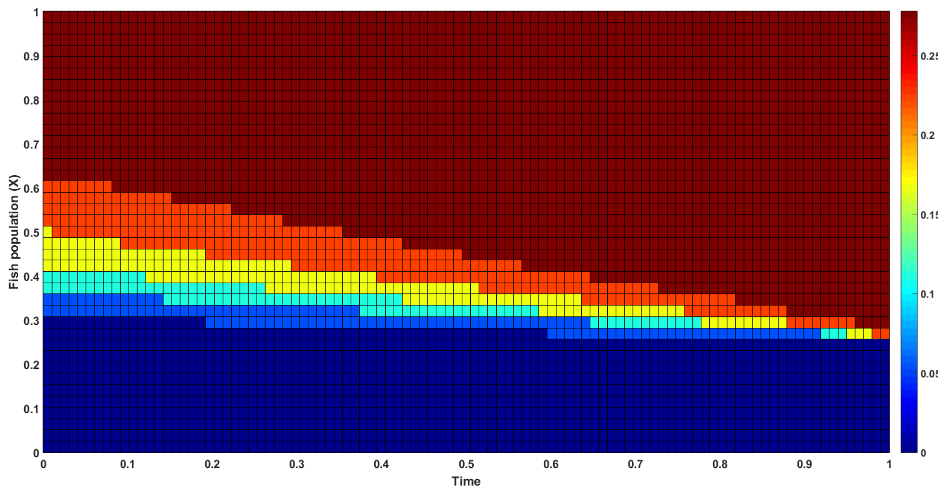

Figure 3: *The shape of the optimal control for a fixed price p*

Figure 4: The optimal control v.s. naive controls.
After computing the optimal control and the value function via the iterative procedure, we simulate the correlated Brownian motions B and W on the horizon $[0, T]$ and we adjust the dynamics of X and P according to the optimal control computed previously. Figure 4 represents a mean over three thousand simulated paths of X and P controlled by the optimal strategy α^* and two other naive strategies α^1 and α^2. The first naive strategy consists in fishing the maximum $\bar{a}$ at all time till T (in red) and the second one consists in waiting until a certain time t_0 which is chosen by the fisherman (in Figure 4 we take $t_0 = 0.5$) then fishing the maximum $\bar{a}$ till time T (in green). In Figure 4, the starting point is $X_0 = 0.7$ and $P_0 = 0.5$ and the P&Ls (i.e. $\sum_{i=1}^{n_t} e^{\beta(T-t_i)}(P_{t_i}\alpha_{t_i} - \alpha_{t_i}^2)\delta - f((\Gamma - X_T^\alpha)^+, P_T^\alpha))$ of the three strategies are respectively (with 95% confidence level bounds) : P&L$(\alpha^*) =$ $0.0873(\pm 0.0002)$, P&L$(\alpha^1) = -0.0182(\pm 0.0024)$ and P&L$(\alpha^2) = 0.0315(\pm 0.0005)$.
We can see that our computed strategy is better than the two others. Indeed, with our control, the fisherman begins to fish continuously with a rate a which is smaller than the maximum $\bar{a}$, that allows him to attain time T with a fish population above Γ avoiding by this the penalization that occurs if $X_T < \Gamma$. On the one hand, with the strategy α^1 at time T we fish more but we are penalized because the fish population is under Γ at time T. On

the other hand, using the strategy α^2, we are not penalized because the fish population at time T is above Γ but we fish less longer in time (we start fishing at $t_0 = 0.5$), hence we obtain less revenue.

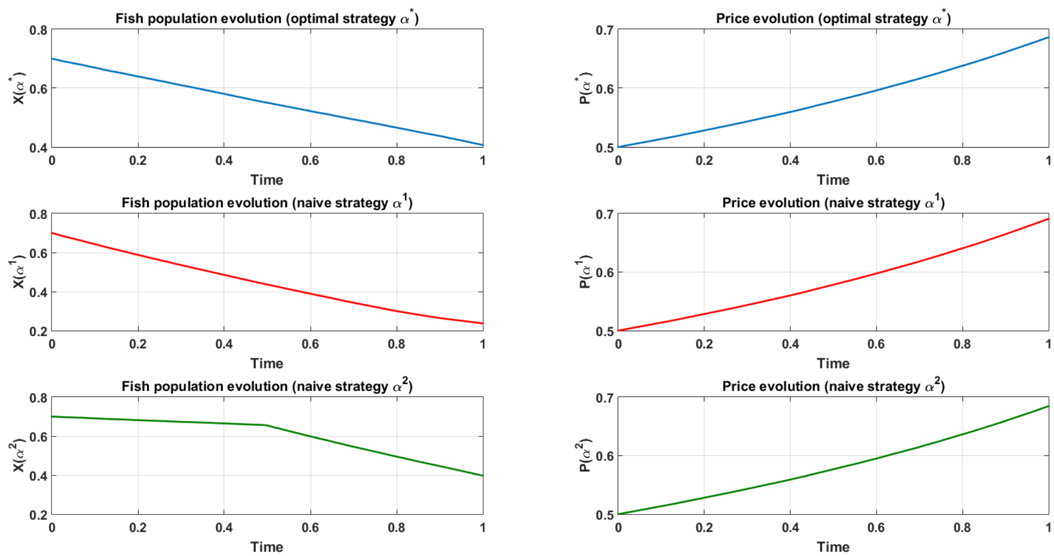

Figure 4: *The optimal control α^* v.s. naive controls α^1 and α^2.*

Figure 5: The optimal control v.s. naive controls (different start).
As in Figure 4, Figure 5 represents a mean over three thousand simulated paths of X and P for the optimal control and the same two other naive strategies. But this time we choose to start with $X_0 = 0.3$ and $P_0 = 0.5$. The P&Ls of the three strategies are respectively (with 95% confidence level bounds) : P&L(α^*) = 0.0302($\pm$0.0008), P&L(α^1) = $-$0.1032($\pm$0.0027) and P&L(α^2) = $-$0.0331($\pm$0.0022). We can see that our strategy is still better than the two others. On the one hand, starting at time $t = 0$ from a position under the threshold λ, the fish population tends to increase (mean-reverting effect), hence, as we can see in Figure 5, our optimal strategy is to wait until X reachs a certain level over Γ before starting to harvest (around time $t = 0.2$). Doing this allows the fisherman to avoid the risk of being under the penalization barrier Γ at time T. On the other hand, using the first naive strategy α^1, the fish population is quickly under Γ and at time $t = 0.2$, the regulator does not allow the fisherman to harvest anymore. Moreover, the fisherman is penalized as the fish population does not surpass Γ at time T. On the contrary, if we wait up to time $t_0 = 0.75$ before starting to fish with the maximum rate $\bar{a}$ (the naive strategy α^2), the fish population will grow (given that $X_t < \lambda$) till time $t_0 = 0.75$ and then decrease (because we start fishing at this date) but will be above Γ at time T. Hence, we are not penalized but still our optimal strategy α^* outperforms α^2.

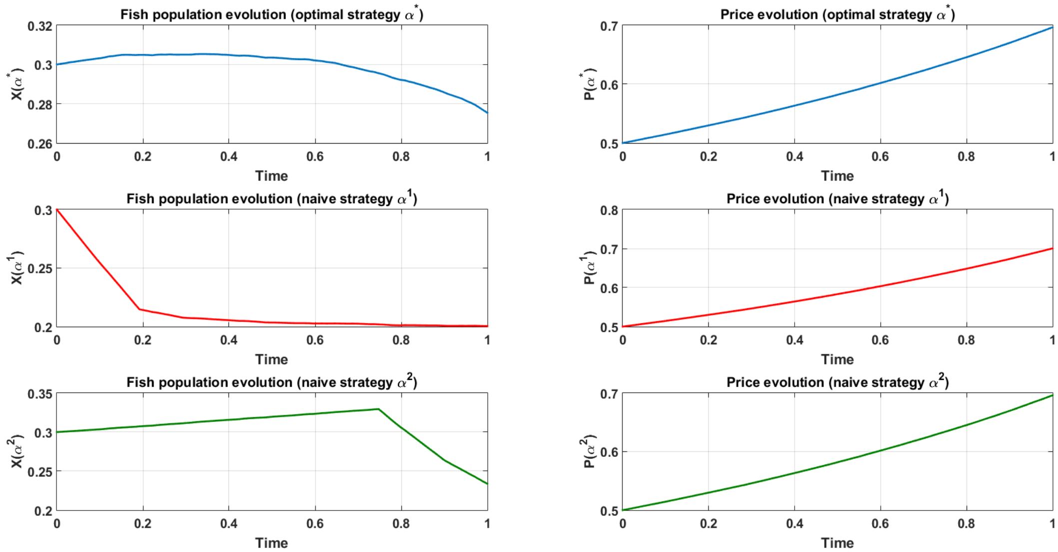

Figure 5: *The optimal control α^* v.s. naive controls α^1 and α^2 ($X_0 = 0.3$ and $P_0 = 0.5$).*

Table 2: P&L for different penalty constant κ.
In Table 2, we choose to compute our optimal control and the corresponding simulations for $\Gamma = 0.4872 > \lambda = 0.3$. With this configuration, we know that if the fish population drops under Γ, it's likely to stay under Γ. Hence, the fisherman has to let at all time the fish population X over Γ to avoid the penalization at time T. Except for Γ and λ, we use the same values of the parameters defined in the beginning of this numerical part and we represent in Table 2 the value of the P&L and the fish population X at time T starting with $X_0 = 0.7$ and $P_0 = 0.5$. These quantities were computed using a mean over three thousand trajectories under the optimal control α^* for different values of the penalty constant κ (with 95% confidence level bounds). We can see that the P&L is a decreasing function w.r.t. κ which is natural because the less you are penalized, the more you take risks and the richer you are. Although, we can remark that for $\kappa = 1$ the fish population at time T is under Γ because the penalization is not sever enough, hence the fisherman prefers to be penalized and fish a little more which makes him richer. Therefore, we assume that for our set of data, to create a fair balance between the biological requirements and the maximization of the profit induced by fishing, a suitable choice for the penalty constant is $\kappa = 2$. This amount of fines insures the double objective of the sustainable fishing: the fish population does not fall below a certain threshold that guarantees its natural renewal, and fisheries make profits to prevent them from going bankrupt.

	$\kappa = 1$	$\kappa = 2$	$\kappa = 3$
P&L	0.0265($\pm$0.0013)	0.0102($\pm$0.0018)	−0.0012($\pm$0.0023)
X_T	0.4855($\pm$0.0021)	0.5066($\pm$0.0020)	0.5182($\pm$0.0022)
	$\kappa = 4$	$\kappa = 5$	$\kappa = 6$
P&L	−0.0111($\pm$0.0029)	−0.0215($\pm$0.0034)	−0.0251($\pm$0.0038)
X_T	0.5232($\pm$0.0022)	0.5238($\pm$0.0023)	0.5308($\pm$0.0023)

Table 2 : *P&L for different penalty constant κ*

5 Conclusion

In this paper, we have investigated the problem of sustainable fishing. We built a model where fishing is continuous depending on the quantity of fish available in the sea. Fisheries try to maximize their profit under the constraint of fines when the quota is exceeded. We have also introduced the fact that the selling price of fish depends on the quantity (stock) of fishes remaining in the sea.

We have shown some interesting results. Firstly, the strategy issued from the resolution of the expected profit maximization is still better than naive strategies. Secondly, we delimit a level of fines which insures the double objective of the sustainable fishing: the fish population stays above a certain threshold that guarantees its natural renewal, so that fisheries make profits. These results allow us to better understand the behavior of the fisheries according to the amount of the fines, and how to define a rule of price for the fines to guarantee a sustainable fishing.

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