SPEED OF PROPAGATION FOR HAMILTON-JACOBI EQUATIONS WITH MULTIPLICATIVE ROUGH TIME DEPENDENCE AND CONVEX HAMILTONIANS

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Abstract. We show that the initial value problem for Hamilton-Jacobi equations with multiplicative rough time dependence, typically stochastic, and convex Hamiltonians satisfies finite speed of propagation. We prove that in general the range of dependence is bounded by a multiple of the length of the "skeleton" of the path, that is a piecewise linear path obtained by connecting the successive extrema of the original one. When the driving path is a Brownian motion, we prove that its skeleton has almost surely finite length. We also discuss the optimality of the estimate.

1. Introduction

We consider the initial value problem for Hamilton-Jacobi equations with multiplicative rough time dependence, that is

$$du = H(Du, x) \cdot d\xi \ \text{ in } \ \mathbb{R}^d \times (0, T] \qquad u(\cdot, 0) = u_0 \ \text{ in } \ \mathbb{R}^d, \tag{1.1}$$

with

$$H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \text{ convex and Lipschitz continuous in the first argument} \tag{1.2}$$

and

$$\xi \in C_0([0, T]), \tag{1.3}$$

where $C_0([0, T])$ denotes the space of continuous paths $\xi : [0, T] \to \mathbb{R}$ such that $\xi(0) = 0$.

When ξ is a C^1 or BV-path, (1.1) is the standard Hamilton-Jacobi equation that is studied using the Crandall-Lions theory of viscosity solutions. For such paths, in place of (1.1) we will often write

$$u_t = H(Du, x)\dot{\xi} \ \text{ in } \ \mathbb{R}^d \times (0, T] \qquad u(\cdot, 0) = u_0 \ \text{ in } \ \mathbb{R}^d \tag{1.4}$$

When ξ is merely continuous, in (1.1) $\cdot$ simply denotes the way the path enters the equation. When ξ is a Brownian motion, then $\cdot d\xi$ stands for the classical Stratonovich differential.

Lions and Souganidis introduced in [13] the notion of stochastic or pathwise viscosity solutions for a general class of equations which contain (1.1) as a special case and studied its well-posedness; for this as well as further properties see Lions-Souganidis [13, 14, 15, 12, 16].

One of the questions raised in [16] was whether (1.1) has a finite speed of propagation, which is one of the important characteristics of the hyperbolic nature of the equations for regular paths.

Roughly speaking, finite speed of propagation means that, if two solutions agree at some time in a ball, then they agree on a forward cone with a time dependent radius.

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A partial result in this direction was shown in Lions and Souganidis [14] (see also Souganidis [16]), while Gassiat showed in [5] that, in general, when H is neither convex nor concave (1.1) does not have the finite speed of propagation property.

In this work, assuming (1.2) and (1.3), we establish finite speed of propagation in the sense formulated precisely next.

Given $T > 0$ and $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ let

$$\begin{aligned}\rho_H(\xi, T) := \sup\Big\{R \geqslant 0 :\ &\text{there exist solutions } \ u^1, u^2 \text{ of (1.1) and } x \in \mathbb{R}^d,\\ &\text{such that } u^1(\cdot,0) = u^2(\cdot,0) \text{ in } B_R(x) \text{ and } u^1(x,T) \neq u^2(x,T)\Big\},\end{aligned} \tag{1.5}$$

where $B_R(x)$ is the ball in $\mathbb{R}^d$ centered at x with radius R.

The classical theory for Hamilton-Jacobi equations (see Lions [9] and Crandall and Lions [2]) yields that, if ξ is a C^1- or, more generally, a BV-path, then

$$\rho_H(\xi, T) \leqslant L\|\xi\|_{TV([0,T])}, \tag{1.6}$$

where

$$\|\xi\|_{TV([0,T])} := \sup_{0=t_0 \leqslant \ldots \leqslant t_n = T} \sum_{i=0}^{n-1} |\xi(t_{i+1}) - \xi(t_i)|$$

is the total variation semi-norm of ξ and L is the Lipschitz constant of H. It is easy to see that (1.6) is sharp when $\dot{\xi} \equiv 1$.

For general rough, that is only continuous, signal ξ it was shown in [14], [16] that, if $H(p, x) = H_1(p) - H_2(p)$, where H_1, H_2 satisfy (1.2) with Lipschitz constant L and $H_1(0) = H_2(0) = 0$, then, for any constant A, if

$$u(0, \cdot) \equiv A \text{ on } B_R(0),$$

then

$$u(t, \cdot) \equiv A \text{ on } B_{R(t)}(0), \quad \text{for } R(t) := R - L(\max_{s\in[0,T]} \xi(s) - \min_{s\in[0,T]} \xi(s)).$$

This does not, however, imply a finite domain of dependence.

In fact, it was shown in [5] that when $H(p) = |p_1| - |p_2|$ equality is attained in (1.6) for all continuous ξ, a fact which implies that there is no finite domain of dependence if $\xi \notin BV([0, T])$. In other words, the counter-example in [5] shows that for non-convex Hamiltonian H, all of the oscillations of ξ, measured in terms of the TV-norm, are relevant for the dynamics of (1.1).

In contrast, in this paper we show that, if H is convex, there is an estimate, which is better than (1.6), and, in particular, implies that the rate of dependence $\rho_H(\xi, T)$ is almost surely finite when ξ is a Brownian path. This new bound relies on a better understanding of which oscillations of the signal ξ are effectively relevant for the dynamics of (1.1).

In this spirit, we prove that, if H is convex, then ξ can be replaced by its skeleton. This is a reduced path $R_{0,T}(\xi)$ which keeps track solely of the oscillations of ξ that are relevant for the dynamics of (1.1) without changing the solution to (1.1). Hence, in the convex case only the oscillations of ξ encoded in $R_{0,T}(\xi)$ are relevant for (1.1). In the one-dimensional setting and for smooth, strictly convex, x-independent Hamiltonians a related result has been obtained independently and by different methods in Hoel, Karlsen, Risebro, and Storrøsten [7].

We also establish that the reduced path of a Brownian motion has almost surely finite variation, a fact which implies that $\rho_H(\xi, T)$ is almost surely finite.

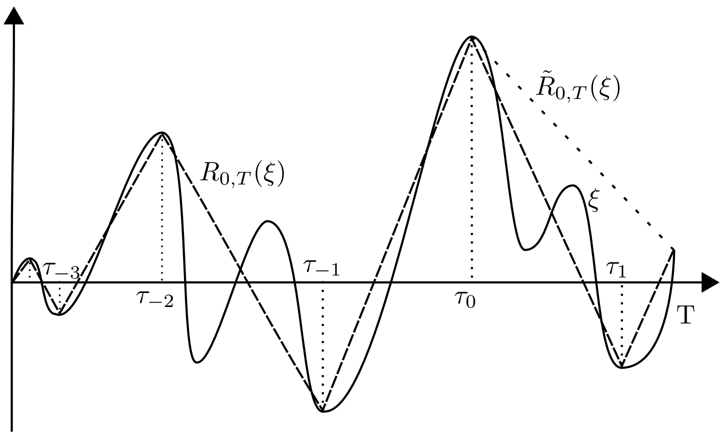

FIGURE 1.1. The (fully) reduced path

Given $\xi \in C_0([0,T])$, the sequence $(\tau_i)_{i\in\mathbb{Z}}$ of successive extrema of ξ is defined by

$$\tau_0 := \sup\left\{t \in [0,T],\ \ \xi(t) = \max_{0\leqslant s\leqslant T} \xi(s) \text{ or } \xi(t) = \min_{0\leqslant s\leqslant T} \xi(s)\right\}, \tag{1.7}$$

and, for all $i \geqslant 0$,

$$\tau_{i+1} = \begin{cases} \arg\max_{[\tau_i,T]} \xi & \text{if} \quad \xi(\tau_i) < 0, \\ \arg\min_{[\tau_i,T]} \xi & \text{if} \quad \xi(\tau_i) > 0, \end{cases} \tag{1.8}$$

and, for all $i \leqslant 0$,

$$\tau_{i-1} = \begin{cases} \arg\max_{[0,\tau_i]} \xi & \text{if} \quad \xi(\tau_i) < 0, \\ \arg\min_{[0,\tau_i]} \xi & \text{if} \quad \xi(\tau_i) > 0. \end{cases} \tag{1.9}$$

The skeleton (resp. full skeleton) or reduced (resp. fully reduced) path $R_{0,T}(\xi)$ (resp. $\tilde{R}_{0,T}(\xi)$) of $\xi \in C_0([0,T])$ is defined as follows (see Figure 1.1).

Definition 1.1. Let $\xi \in C_0([0,T])$.
(i) The reduced path $R_{0,T}(\xi)$ is a piecewise linear function which agrees with ξ on $(\tau_i)_{i\in\mathbb{Z}}$.
(ii) The fully reduced path $\tilde{R}_{0,T}(\xi)$ is a piecewise linear function agreeing with ξ on $(\tau_{-i})_{i\in\mathbb{N}} \cup \{T\}$.
(iii) A path $\xi \in C_0([0,T])$ is reduced (resp. fully reduced) if $\xi = R_{0,T}(\xi)$ (resp. $\xi = \tilde{R}_{0,T}(\xi)$).

Let u^ξ be the solution to (1.1). We show in Theorem 2.9 in the next section that

$$u^\xi(\cdot,T) = u^{R_{0,T}(\xi)}(\cdot,T), \tag{1.10}$$

which immediately implies the following result.

Theorem 1.2. *Assume* (1.2). *Then, for all* $\xi \in C_0([0,T])$,

$$\rho_H(\xi,T) \leqslant L\ \|R_{0,T}(\xi)\|_{TV([0,T])}. \tag{1.11}$$

The second main result of the paper, which is a probabilistic one and of independent interest, concerns the total variation of the reduced path of a Brownian motion. To state it, we introduce the random variable $\theta : [0,\infty) \to [0,\infty)$ given by

$$\theta(a) := \inf\{t \geqslant 0 : \max_{[0,t]} B - \min_{[0,t]} B = a\}. \tag{1.12}$$

We prove that the length of the reduced path is a random variable with almost Gaussian tails. We also show that if, instead of fixing the time horizon T, we fix the range, that is the maximum minus the minimum of B, then the length has Poissonian tails

Theorem 1.3. *Let B be a Brownian motion and fix $T > 0$. Then, for each $\gamma \in (0,2)$, there exists $C = C(\gamma, T) > 0$ such that, for any $x \geqslant 2$,*

$$\mathbb{P}\left(\|R_{0,T}(B)\|_{TV([0,T])} \geqslant x\right) \leqslant C \exp\left(-Cx^\gamma\right), \tag{1.13}$$

and

$$\lim_{x\to\infty} \frac{\ln \mathbb{P}\left(\|R_{0,\theta(1)}(B)\|_{TV([0,\theta(1)])} \geqslant x\right)}{x \ln(x)} = -1. \tag{1.14}$$

We also study the sharpness of the upper bound. For simplicity we only treat the case $H(p) = |p|$.

Theorem 1.4. *Let $H(p) = |p|$ on $\mathbb{R}^d$ with $d \geqslant 1$. Then, for all $T > 0$ and $\xi \in C_0([0,T])$,*

$$\rho_H(\xi, T) \geqslant \|\tilde{R}_{0,T}(\xi)\|_{TV([0,T])}. \tag{1.15}$$

When $d = 1$, then

$$\rho_H(\xi, T) = \|\tilde{R}_{0,T}(\xi)\|_{TV([0,T])}.$$

The paper is organized as follows. In section 2 we improve upon results of [13, 14, 11] about representation formulae, the control of the oscillations in time and the domain of dependence of the solutions of (1.4) with piecewise linear paths. We then extend these estimates by density to general continuous paths. In order to avoid stating many assumptions on H, we introduce a new condition about solutions of (1.4) which is satisfied by the general class of Hamiltonians for which there is a well-posed theory of pathwise solutions as developed in [12]. All these lead to the proof of Theorem 1.2. In section 3 we discuss the example which shows that the upper bound obtained in Theorem 1.2 is sharp. Section 4 is devoted to the study of "random" properties of the reduced path of the Brownian motion (Theorem 1.3).

2. Reduction to the skeleton path and domain of dependence

Notation and preliminaries. For all $\xi \in C_0([0,T])$ and $u_0 \in BUC(\mathbb{R}^d)$, let S^ξ be the flow of solutions of (1.1). A simple rescaling shows that without loss of generality we may assume that

$$L = 1.$$

In view of (1.2) and the normalization of the Lipschitz constant we have

$$H(p,x) = \sup_{v \in B_1(0)} \{p \cdot v - L(v,x)\},$$

where $L(v,x) = \sup_{p\in\mathbb{R}^d} \{p \cdot v - H(p,x)\}$.

We assume that H satisfies all assumptions needed (see [2]) for $u_t = H(Du,x)\dot{\xi}$ to be well posed when ξ is smooth and we denote by $S_{\pm H}(t) : \mathrm{BUC}(\mathbb{R}^d) \to \mathrm{BUC}(\mathbb{R}^d)$ the solution operator when $\dot{\xi} \equiv \pm 1$, that is, for $u_0 \in \mathrm{BUC}(\mathbb{R}^d)$, $S_{\pm H}(t)u_0$ is the unique solution of

$$u_t = \pm H(Du,x) \text{ in } \mathbb{R}^d \times (0,T] \quad u(\cdot,0) = u_0 \text{ in } \mathbb{R}^d. \tag{2.1}$$

Moreover, for $t \leqslant 0$, $S_H(t) := S_{-H}(-t)$.

Given $S, S' : \mathrm{BUC}(\mathbb{R}^d) \to \mathrm{BUC}(\mathbb{R}^d)$, we say that $S \leqslant S'$ if $Su \leqslant S'u$ for all u in $\mathrm{BUC}(\mathbb{R}^d)$.

In the sequel we write $\xi_{s,t} := \xi_t - \xi_s$ for the increments of ξ over the interval $[s,t]$.

Let $\xi \in C([0,T])$ be a piecewise linear path, that is, for a partition $0 = t_0 \leqslant \ldots \leqslant t_N = T$ of $[0,T]$, and $a_i, b_i \in \mathbb{R}$, $i = 1, \ldots N$,

$$\xi(t) = \sum_{i=0}^{N-1} 1_{[t_i,t_{i+1})}(a_i(t-t_i) + b_i).$$

We then set

$$S_H^{\xi}(0,T) := S_H^{\xi}(t_{N-1},t_N) \circ \cdots \circ S_H^{\xi}(t_0,t_1)$$

and note that

$$S_H^{\xi}(0,T) = S_H(\xi_{t_{N-1},t_N}) \circ \cdots \circ S_H(\xi_{t_0,t_1}).$$

We show later that $\xi \mapsto S_H^{\xi}(0,\cdot)$ is uniformly continuous in sup-norm, which allows to extend $S_H^{\xi}(0,T)$ to all continuous ξ.

Monotonicity properties. The control representation of the solution u of (2.1) (see, for example, Lions [9]) with $\xi_t \equiv t$ and $u_0 \in \mathrm{BUC}(\mathbb{R}^d)$ is

$$u(x,t) = S_H(t)u_0(x) = \sup_{q \in \mathcal{A}} \left\{ u_0(X(t)) - \int_0^t L(q(s),X(s))ds : \; X(0) = x, \, \dot{X}(s) = q(s) \text{ for } s \in [0,t] \right\},$$

and

$$S_{-H}(t)u_0(y) = \inf_{r \in \mathcal{A}} \left\{ u_0(Y(t)) + \int_0^t L(r(s),Y(s))ds : \; Y(0) = y, \, \dot{Y}(s) = -r(s) \text{ for } s \in [0,t] \right\},$$

where $\mathcal{A} = L^{\infty}(\mathbb{R}_+; \overline{B}_1(0))$ is the set of controls.

The next property is a refinement of an observation in [11].

Lemma 2.1. *Fix $t > 0$ and $u_0 \in BUC(\mathbb{R}^d)$. Then*

$$S_H(t) \circ S_H(-t)u_0 \leqslant u_0 \leqslant S_H(-t) \circ S_H(t)u_0.$$

Proof. Since the arguments are identical we only show the proof of the inequality on the left.

We have

$$S_H(t) \circ S_H(-t)u_0(x) = \sup_{q \in \mathcal{A}} \inf_{r \in \mathcal{A}} \Big\{ u_0(Y(t)) + \int_0^t L(r(s),Y(s))ds - \int_0^t L(q(s),X(s))ds :$$
$$Y(0) = X(t), \, \dot{Y}(s) = -r(s), \, X(0) = x, \, \dot{X}(s) = q(s) \text{ for } s \in [0,t] \Big\}.$$

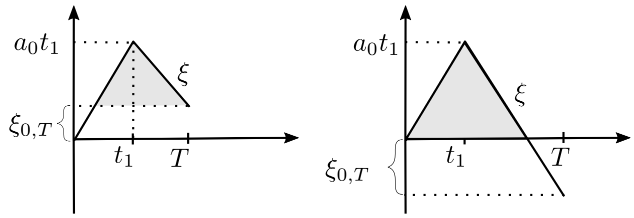

FIGURE 2.1. Reduction

Given $q \in \mathcal{A}$ choose $r(s) = q(t-s)$ in the infimum above. Since $Y(s) = X(t-s)$, it follows that

$$S_H(t) \circ S_H(-t)u_0(x) \leqslant \sup_{q \in \mathcal{A}} \Big\{ u_0(X(0)) + \int_0^t L(q(t-s), X(t-s))ds - \int_0^t L(q(s), X(s))ds : \\ X(0) = x, \dot{X}(s) = q(s) \text{ for } s \in [0,t] \Big\} \\ = u_0(x).$$

□

The next result is an easy consequence of Lemma 2.1 and the definition of S_H^ξ for piecewise linear paths.

Lemma 2.2. *Let* $\xi_t = 1_{t \in [0,t_1]}(a_0 t) + 1_{t \in [t_1,T]}(a_1(t-t_1) + a_0 t_1)$. *If* $a_0 \geqslant 0$ *and* $a_1 \leqslant 0$ *(resp.* $a_0 \leqslant 0$ *and* $a_1 \geqslant 0$*), then*

$$S_H^\xi(0,T) \geqslant S_H(\xi_{0,T}) \qquad (\textit{resp. } S_H^\xi(0,T) \leqslant S_H(\xi_{0,T}).)$$

Proof. Since the claim is immediate if $a_0 = 0$ or $a_1 = 0$, we assume next that $a_0 > 0$ and $a_1 < 0$ (see Figure 2.1).

If $\xi_{0,T} \leqslant 0$, then

$$S_H(a_1(T-t_1)) = S_{-H}(-a_1(T-t_1)) = S_{-H}(-a_1(T-t_1) - a_0 t_1) \circ S_{-H}(a_0 t_1) \\ = S_{-H}(-\xi_{0,T}) \circ S_{-H}(a_0 t_1) = S_H(\xi_{0,T}) \circ S_H(-a_0 t_1),$$

and, hence, in view of Lemma 2.1,

$$S_H^\xi(0,T) = S_H(\xi_{0,T}) \circ S_H(-a_0 t_1) \circ S_H(a_0 t_1) \geqslant S_H(\xi_{0,T}).$$

If $\xi_{0,T} \geqslant 0$ (see Figure 2.1), then, again using Lemma 2.1, we find

$$S_H^\xi(0,T) = S_H(a_1(T-t_1)) \circ S_H(-a_1(T-t_1) + a_0 t_1 + a_1(T-t_1)) \\ = S_H(a_1(T-t_1)) \circ S_H(-a_1(T-t_1)) \circ S_H(a_0 t_1 + a_1(T-t_1)) \leqslant S_H(\xi_{0,T}).$$

For the second inequality we note that $S_{-H}^{-\xi}(0,T) = S_H^\xi(0,T)$, $S_{-H}(-t) = S_H(t)$. It then follows from the the first part that

$$S_H^\xi(0,T) = S_{-H}^{-\xi}(0,T) \geqslant S_{-H}(-\xi_{0,T}) = S_H(\xi_{0,T}).$$

□

The next observation provides the first indication of the possible reduction encountered when using the max or min of a given path. For the statement, given piecewise linear path ξ, we set

$$\tau_{max} := \sup\{t \in [0,T] :\ \xi_t = \max_{s\in[0,T]} \xi_s\} \ \text{ and } \tau_{min} := \sup\{t \in [0,T] :\ \xi_t = \min_{s\in[0,T]} \xi_s\}.$$

Lemma 2.3. *Fix a piecewise linear path ξ. Then*

$$S^{\xi}_H(\tau_{max}, T) \circ S_H(\xi_{0,\tau_{max}}) \leqslant S^{\xi}_H(0,T) \leqslant S_H(\xi_{\tau_{min},T}) \circ S^{\xi}_H(0,\tau_{min}).$$

Proof. Since the proofs of both inequalities are similar, we only show the details for the first.

Note that without loss of generality we may assume that $\mathrm{sgn}(\xi_{t_{i-1},t_i}) = -\mathrm{sgn}(\xi_{t_i,t_{i+1}})$ for all $[t_{i-1}, t_{i+1}] \subseteq [0, \tau_{max}]$.

It follows that, if $\xi_{|[0,\tau_{max}]}$ is linear, then $S^{\xi}_H(0,\tau_{max}) = S_H(\xi_{0,\tau_{max}})$.

If not, since $\xi_{0,\tau_{max}} \geqslant 0$, there is an index j such that $\xi_{t_{j-1},t_{j+1}} \geqslant 0$ and $\xi_{t_{j-1},t_j} \leqslant 0$. It then follows from Lemma 2.2 that

$$S^{\xi}_H(0,\tau_{max}) \leqslant S^{\tilde{\xi}}_H(0,\tau_{max}),$$

where $\tilde{\xi}$ is piecewise linear and coincides with ξ for all $t \in \{t_i :\ i \neq j\}$.

A simple iteration yields $S^{\xi}_H(0,\tau_{max}) \leqslant S_H(\xi_{0,\tau_{max}})$, and, since $S^{\xi}_H(0,T) = S^{\xi}_H(\tau_{max},T) \circ S^{\xi}_H(0,\tau_{max})$, this concludes the proof.

□

We combine the conclusions of the previous lemmata to establish the following monotonicity result.

Corollary 2.4. *Let ξ, ζ be piecewise linear, $\xi(0) = \zeta(0)$, $\xi(T) = \zeta(T)$ and $\xi \leqslant \zeta$ on $[0,T]$. Then*

$$S^{\xi}_H(0,T) \leqslant S^{\zeta}_H(0,T). \tag{2.2}$$

Proof. We assume that ξ and ζ are piecewise linear on each interval $[t_i, t_{i+1}]$ of a joint subdivision $0 = t_0 \leqslant \ldots \leqslant t_N = T$ of $[0,T]$.

If $N = 2$, we show that, for all $\gamma \geqslant 0$ and all $a, b \in \mathbb{R}$,

$$S_H(a+\gamma) \circ S_H(b-\gamma) \leqslant S_H(a) \circ S_H(b). \tag{2.3}$$

If $a \geqslant 0$, this follows from the fact that, in view of Lemma 2.2,

$$S_H(\gamma) \circ S_H(b-\gamma) \leqslant S_H(b).$$

If $a + \gamma \leqslant 0$, then again Lemma 2.2 yields

$$S_H(a) \circ S_H(b) = S_H(a+\gamma) \circ S_H(-\gamma) \circ S_H(b) \geqslant S_H(a+\gamma) \circ S_H(b-\gamma).$$

Finally, if $a \leqslant 0 \leqslant a + \gamma$ we have

$$S_H(a) \circ S_H(b) \geqslant S_H(a+b) \geqslant S_H(a+\gamma) \circ S_H(b-\gamma).$$

The proof for $N > 2$ follows by induction on N. Let ρ be piecewise linear on the same partition and coincide with ζ on t_0, t_1, and with ξ on $t_2, \ldots, t_N$. The induction hypothesis then yields

$$S^{\xi}_H(0,t_2) \leqslant S^{\rho}_H(0,t_2) \ \text{ and } \ S^{\rho}_H(t_1,T) \leqslant S^{\zeta}_H(t_1,T)$$

from which we deduce

$$S^{\xi}_H(0,T) \leqslant S^{\rho}_H(0,T) \leqslant S^{\zeta}_H(0,T).$$

□

A uniform modulus of continuity. To extend the information obtained about the possible cancellations and oscillations from piecewise linear to arbitrary continuous paths, we need a well-posed theory for the pathwise viscosity solutions. Such a theory has been developed by the last two authors in [11] and [12]. The former reference imposes conditions on the joint dependence of the Hamiltonians in (p, x) but does not require convexity. A special (resp. a more general) class of convex or concave Hamiltonians, which do not require such conditions, is studied in Friz, Gassiat, Lions and Souganidis [3] (resp. Lions and Souganidis [15]). An alternative, although less intrinsic, approach is to show that the solution operator has a unique extension from piecewise linear paths to arbitrary continuous ones.

To avoid stating additional conditions and since finding the optimal assumptions on the joint dependence on (p, x) of the Hamiltonians is not the main focus of this paper, we bypass this issue here. Instead, we formulate a general assumption that allows to have a unique extension of the solution operator to all continuous paths, which is enough to analyze the domain of dependence. We only remark that this assumption is satisfied by the Hamiltonians considered in [12] as well as some other ones that can be analyzed by the same methods.

For $t \in (0, T)$, the minimal action, also known as the fundamental solution, associated with Hamiltonians satisfying (1.2) is given by

$$\mathcal{L}(x, y, t) := \inf\{\int_0^t L(\dot\gamma(s), \gamma(s))ds : \quad \gamma \in C^{0,1}([0, T]) \text{ such that } \gamma(0) = x, \gamma(t) = y\};$$

when we need to emphasize the dependence of $\mathcal{L}$, we write $\mathcal{L}^H$.

We recall (see, for example, [9]) that, for all $t, s \geqslant 0$ and $x, y, z \in \mathbb{R}^d$,

$$\mathcal{L}(x, z, t + s) \leqslant \mathcal{L}(x, y, t) + \mathcal{L}(y, z, s). \tag{2.4}$$

Moreover, for any $u_0 \in \mathrm{BUC}(\mathbb{R}^d)$, $t \geqslant 0$ and $x \in \mathbb{R}^d$,

$$u(x, t) = S_H(t)u_0(x) = \sup_{y\in\mathbb{R}^d} [u_0(y) - \mathcal{L}(x, y, t)]\,. \tag{2.5}$$

Finally, since $-S_{-H}(t)u_0 = S_{\breve{H}}(-u_0)$ with $\breve{H}(p, x) = H(-p, x)$, we also have, for any $u_0 \in \mathrm{BUC}(\mathbb{R}^d)$, $t \geqslant 0$ and $x \in \mathbb{R}^d$,

$$S_{-H}(t)u_0 = \inf_{y\in\mathbb{R}^d} \left[u_0(y) + \mathcal{L}^{\breve{H}}(x, y, t)\right]. \tag{2.6}$$

We assume that, for all $r > 0$,

$$\limsup_{\delta\to0} \inf_{r\leqslant|x-y|} \mathcal{L}(x, y, \delta) = +\infty, \tag{2.7}$$

and

$$\lim_{\delta\to0} \lim_{r\to0} \sup_{|x-y|\leqslant r} \mathcal{L}(x, y, \delta) = 0. \tag{2.8}$$

Note that (2.8) is some sort of controllability assumption, while (2.7) follows from a uniform in x upper bound on H.

Proposition 2.5. *If* (2.7) *and* (2.8) *hold, then, for each* $u_0 \in BUC(\mathbb{R}^d)$ *and* $T \geqslant 0$*, the family*

$$\left\{S^{\xi}_H(0, T)u_0 : \quad \xi \text{ piecewise linear}\right\}$$

has a uniform modulus of continuity.

The claim above is a consequence of the following estimate.

Proposition 2.6. *Let* $u = S_H^{\xi}(0,t)u_0$ *with* ξ *piecewise linear and* $u_0 \in BUC(\mathbb{R}^d)$*. Then, for all* $t \geqslant 0$ *and all* $x, y \in \mathbb{R}^d$,

$$u(x,t) - u(y,t) \leqslant \inf_{\delta>0} \left(\mathcal{L}(y,x,\delta) + \sup_{x',y' \in \mathbb{R}^d} \left[u_0(x') - u_0(y') - \mathcal{L}(y',x',\delta) \right] \right). \tag{2.9}$$

Proof. By induction it is enough to prove the estimate for $u = S_H(t)u_0$ and $u = S_H(-t)u_0$.

We begin with the former and we fix x, y and $x_1 \in \mathbb{R}^d$. Assuming in what follows the inf in the definition of $\mathcal{L}$ is attained, otherwise we work with approximate minimizers, we choose γ to be a minimizer for $\mathcal{L}(y, x_1, t+\delta)$ and set $\tilde{y}_1 = \gamma(t)$. It follows from (2.4) that

$$\begin{aligned}\mathcal{L}(y,x,\delta) + \mathcal{L}(x,x_1,t) \geqslant \mathcal{L}(y,x_1,t+\delta) &= \int_0^t L(\dot{\gamma}(s),\gamma(s))ds + \int_t^{t+\delta} L(\dot{\gamma}(s),\gamma(s))ds \\ &\geqslant \mathcal{L}(y,\tilde{y}_1,t) + \mathcal{L}(\tilde{y}_1,x_1,\delta).\end{aligned}$$

Hence

$$\begin{aligned}&(u_0(x_1) - \mathcal{L}(x,x_1,t)) - \sup_{y_1 \in \mathbb{R}^d} \{u_0(y_1) - \mathcal{L}(y,y_1,t)\} - \mathcal{L}(y,x,\delta) \\ &\leqslant u_0(x_1) - u_0(\tilde{y}_1) - \mathcal{L}(x,x_1,t) + \mathcal{L}(y,\tilde{y}_1,t) - \mathcal{L}(y,x,\delta) \\ &\leqslant u_0(x_1) - u_0(\tilde{y}_1) - \mathcal{L}(\tilde{y}_1,x_1,\delta) \leqslant \sup_{x',y' \in \mathbb{R}^d} \left\{u_0(x') - u_0(y') - \mathcal{L}(y',x',\delta)\right\}.\end{aligned}$$

It follows that

$$u(x,t) - u(y,t) - \mathcal{L}(y,x,\delta) \leqslant \sup_{x',y'} \left\{u_0(x') - u_0(y') - \mathcal{L}(y',x',\delta)\right\}$$

and we conclude by taking the inf over δ.

In view of (2.6), a similar argument gives the estimate for $u = S_{-H}(t)u_0$. □

Proof of Proposition 2.5. Fix u_0 and let

$$\eta(\delta) := \sup_{x',y' \in \mathbb{R}^d} \left(u_0(x') - u_0(y') - \mathcal{L}(y',x',\delta)\right)$$

and

$$\nu(x,y) := \inf_{\delta>0} \left(\mathcal{L}(y,x,\delta) + \eta(\delta)\right), \quad \omega(r) := \sup_{|x-y| \leqslant r} \max[\nu(x,y), \nu(y,x)].$$

If follows from Proposition 2.6 that, if $v = S_H^{\xi} u_0$, then $|v(x) - v(y)| \leqslant \omega(|x-y|)$. On the other hand, in view of (2.7), $\lim_{\delta \to 0} \eta(\delta) = 0$, and, hence, using (2.8) we conclude that $\lim_{r \to 0} \omega(r) = 0$. □

Extension and reduction. The extension result is stated next. In this subsection, we always assume that either H is independent of x or that (2.7) and (2.8) hold. In what follows we write $\|\cdot\|_{\infty,\mathcal{O}}$ for the L^{∞}-norm over $\mathcal{O}$.

Corollary 2.7. *The map $\xi \mapsto S_H(\xi)$ is uniformly continuous in the sup-norm in the sense that, if $(\xi^n)_{n\in\mathbb{N}}$ is a sequence of piecewise-linear functions on $[0,T]$ with $\lim_{n,m\to\infty}\|\xi^n-\xi^m\|_{\infty,[0,T]}=0$, then, for all $u\in BUC(\mathbb{R}^d)$,*

$$\lim_{n,m\to\infty}\|S_H^{\xi^n}(0,T)u-S_H^{\xi^m}(0,T)u\|_{\infty,\mathbb{R}^d}=0. \tag{2.10}$$

It follows that we can extend $\xi\mapsto S_H(\xi)$ to all continuous paths. Indeed $\xi^n\to\xi$ in sup-norm as $n\to\infty$, then

$$S_H^{\xi}(0,T)u:=\lim_{n\to\infty}S_H^{\xi^n}(0,T)u. \tag{2.11}$$

Proof of Corollary 2.7 Fix $\delta>0$ and let ξ,ζ be piecewise linear such that $\|\xi-\zeta\|_\infty\leqslant\delta$ on $[0,T]$. We extend ξ,ζ to all of $\mathbb{R}$ as constants on $(-\infty,0)$ and $(T,+\infty)$ and choose $\eta\in[-1,1]$ such that $\xi(T)=\zeta(T)+\eta\delta$.
Let $\xi^{\pm\delta}$ be defined by

$$\xi^{\pm\delta}:=\begin{cases}\xi\pm\delta & \text{on } [0,T],\\ \xi & \text{on } (-\infty,-\delta)\cup(T+\delta,+\infty),\end{cases}$$

and

$$\dot{\xi}^{\pm\delta}=\pm1 \ \text{ on } (-\delta,0) \ \text{ and } \ \dot{\xi}^{\pm\delta}=\mp1-\eta \text{ on } (T,T+\delta).$$

It follows that $\xi^{\pm}(-\delta)=\zeta(-\delta)$, $\xi^{\pm}(T+\delta)=\zeta(T+\delta)$, $\xi^{-\delta}\leqslant\zeta\leqslant\xi^{\delta}$, and, since $S_H^{\xi^{\pm\delta}}(-\delta,T+\delta)=S_H(\mp\delta-\eta\delta)\circ S_H^{\xi}(0,T)\circ S_H(\pm\delta)$, Corollary 2.4 yields

$$S_H(\delta(1-\eta))\circ S_H^{\xi}(0,T)\circ S_H(-\delta)\leqslant S_H^{\zeta}(0,T)\leqslant S_H(-\delta(1+\eta))\circ S_H^{\xi}(0,T)\circ S_H(\delta)$$

which implies that

$$S_H^{\xi}(0,T)-\mathcal{S}_H(-\delta(1+\eta))\circ S_H^{\xi}(0,T)\circ S_H(\delta)\leqslant S_H^{\xi}(0,T)-S_H^{\zeta}(0,T),$$

and

$$S_H^{\xi}(0,T)-S_H^{\zeta}(0,T)\leqslant S_H^{\xi}(0,T)-S_H(\delta(1-\eta))\circ S_H^{\xi}(0,T)\circ S_H(-\delta).$$

We now need to check that both sides of the above inequality go to 0 as $\delta\to0$. This follows if

$$\lim_{\delta\to0}\|S_H(\delta)\circ S_H^{\xi}(0,T)u-S_H(-\delta)\circ S_H^{\xi}(0,T)u\|_{\infty,\mathbb{R}^d}=0$$

independently of ξ, which is a consequence of Proposition 2.5.

□

The next conclusion is an immediate consequence of Lemma 2.3 and Corollary 2.7.

Corollary 2.8. *Let ξ be a continuous path such that $\xi_T=\max_{[0,T]}\xi$ and $\xi_0=\min_{[0,T]}\xi$. Then,*

$$S_H^{\xi}(0,T)=S_H(\xi_{0,T}).$$

Similarly, if $\xi_T=\min_{[0,T]}\xi$ and $\xi_0=\max_{[0,T]}\xi$, then

$$S_H^{\xi}(0,T)=S_{-H}(-\xi_{0,T}).$$

It follows that we can have a general representation for the solution to (1.1) as a (countable) composition of the flows $S_H(t),S_H(-t)$.

Theorem 2.9. *Let ξ be a continuous path. Then*

$$S^{\xi}(0,T)=S^{R_{0,T}(\xi)}(0,T).$$

Proof. We apply Corollary 2.8 inductively to the successive extrema as defined in (1.7), (1.8), (1.9). It only remains to show that this procedure converges for $i \to \pm\infty$. This follows from the continuity of ξ in combination with Corollary 2.7.

□

3. The optimality of the domain of dependence

We consider the initial value problem

$$du = |Du| \cdot d\xi \text{ in } \mathbb{R}^d \times (0,T] \quad u(\cdot,0) = u_0(\cdot) \text{ in } \mathbb{R}^d \tag{3.1}$$

and prove Theorem 1.4.

We remark that, in view of the geometric properties of (3.1), it is enough to consider the evolution of the level set

$$P^+(t) = \{x \in \mathbb{R} : \ u(x,t) \geqslant 0\}.$$

Indeed, (3.1) is a level-set PDE, that is, if u is a solution, then also $\Phi(u)$ is a solution. At this point the choice of the Stratonovich differential in (3.1) is important (see Souganidis [1], [16] and Lions, Souganidis [11]). It follows that $P^+(t)$ depends only on $P^+(0)$ and not on the particular form of u_0. In fact, in the case of (3.1) this can be read off the explicit solution formula, for all $\delta > 0$,

$$S_{|\cdot|}(\delta)u(x) = \sup_{|x-y| \leqslant \delta} u(y), \qquad S_{|\cdot|}(-\delta)u(x) = \inf_{|x-y| \leqslant \delta} u(y).$$

In particular, in $d = 1$ and with the convention that $[c,d] = \varnothing$ if $c > d$, it follows that, for all $\delta \in \mathbb{R}$,

$$\mathcal{S}_{|\cdot|}(\delta)([a,b]) = [a-\delta, b+\delta]. \tag{3.2}$$

We notice that, informally, for general initial conditions, P^+ expands with speed $|d\xi|$ when $d\xi > 0$, and contracts with speed $|d\xi|$ when $d\xi < 0$.

The key behind the construction of the lower bound is the observation, already made in [11], that there is some irreversibility in the dynamics. For example, once a hole is filled, that is two connected components of P^+ are joined by an increase in ξ, it cannot be recreated later when ξ decreases. Symmetrically, if a component of P^+ is destroyed by a decrease in ξ, it does not re-appear later. This intuition leads to the lower bound for $\rho_H(\xi,T)$ derived below.

In what follows, to simplify the notation we omit the dependence of the solution operator and the speed of propagation on H, that is, we simply write S, S^ξ and $\rho(\xi,T)$. We fix $d = 1$ and establish first the lower bound in Theorem 1.4, and then look at the upper bound. Note that considering initial conditions depending only on the first coordinate implies that the lower bound also holds for $d \geqslant 2$.

Lower bound for the speed of propagation. The result is stated next.

Proposition 3.1. *Let ξ be a continuous path. Then*

$$\rho_H(\xi,T) \geqslant \|\tilde{R}^\xi(0,T)\|_{TV([0,T])}. \tag{3.3}$$

Proof. Without loss of generality we assume that ξ is a reduced path. Moreover, since the claim stays the same if we replace ξ by $-\xi$, we further assume that $\xi(\tau_0) = \max_{0\leqslant s\leqslant T} \xi(s)$.

We first consider the case where $N := \max\{n \leqslant 0 : \ \tau_n = 0\}$ is finite. Since ξ is constant if $N = 0$, we further assume $N \leqslant -1$ and fix a sequence x_i, $N-1 \leqslant i \leqslant 1$ such that $x_1 = 0$ and, for all $N < k \leqslant 0$,

$$2\left|\xi_{0,\tau_{k-2}}\right| < x_{k+1} - x_k < 2\left|\xi_{0,\tau_k}\right|. \tag{3.4}$$

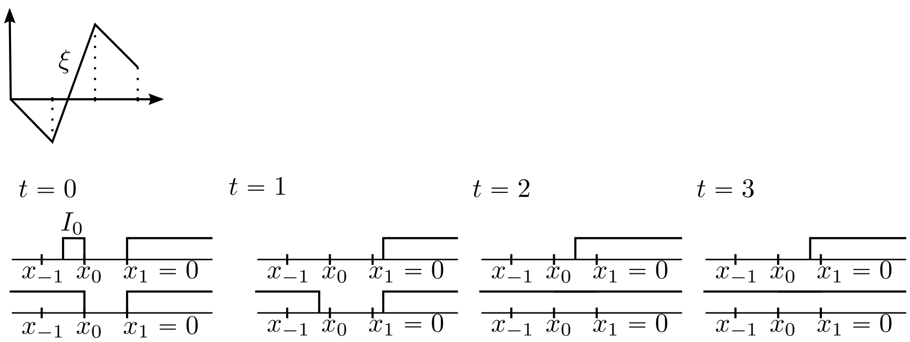

FIGURE 3.1. Lower bound

Set

$$I_k = \begin{cases} [x_{2k-1}, x_{2k}] & \text{if } 2k-1 > N \\ \varnothing & \text{otherwise,} \end{cases} \tag{3.5}$$

and

$$P^1 = \bigcup_{k\leqslant 0} I_k \cup [0,+\infty), \qquad P^2 = (-\infty, x_N] \cup P^1. \tag{3.6}$$

Since ξ is a reduced path and due to (3.2), the evolution of P^1, P^2 can be easily described by induction on k as follows.

The component I_k evolves individually, that is it does not intersect any other connected components, before τ_{2k}. This follows from the fact that $|x_{2k-1} - x_{2k-2}|$ and $|x_{2k} - x_{2k+1}|$ are smaller than $2\xi_{0,\tau_{2k}}$.

Since $-2\xi_{0,\tau_{2k-3}} < |x_{2k} - x_{2k-1}| < -2\xi_{0,\tau_{2k-1}}$, the component I_k of P^1 disappears at time τ_{2k-1} but not at any of the earlier τ_i's.

Finally, given that $x_{2k-1} - x_{2k-2} < 2\xi_{0,\tau_{2k-2}}$, the component I_k of P^2 has joined the components I_j with $j < k$ by the time τ_{2k-2}.

It follows that

$$\mathcal{S}(\xi, \tau_0)(P^1) = [-\xi_{0,\tau_0}, +\infty) \ \text{ and } \ \mathcal{S}^\xi(0,T)(P^1) = [-\xi_{0,T}, +\infty)$$

and

$$\mathcal{S}^\xi(0,T)(P^2) = \mathbb{R}.$$

Since P^1 and P^2 only differ for $x \leqslant x_N$, this implies

$$\rho_H(\xi, T) \geqslant (-\xi_{0,T} - x_N)_+ .$$

Choosing the $(x_k - x_{k-1})$ as large as possible in (3.4) we obtain

$$\rho(\xi, T) \geqslant -\xi_{0,T} + 2\sum_{k\leqslant 0} |\xi_{0,\tau_k}| = |\xi_{\tau_0,T}| + \sum_{k\leqslant 0} |\xi_{0,\tau_k}| + \sum_{k\leqslant -1} |\xi_{0,\tau_k}| .$$

Finally, using that, for $k \leqslant 0$, $\left|\xi_{\tau_{k-1},\tau_k}\right| = |\xi_{0,\tau_k}| + \left|\xi_{0,\tau_{k-1}}\right|$, we obtain

$$\rho(\xi, T) \geqslant |\xi_{\tau_0,T}| + \sum_{k\leqslant 0} |\xi_{\tau_{k-1},\tau_k}| = \left\|\tilde{R}_{0,T}(\xi)\right\|_{TV}$$

which concludes the proof in the case where $\tau_N = 0$ for some N.

Next we treat the general case. We fix $N \leqslant -1$ arbitrary, and as before we define x_k, I_k and P^1, P^2 satisfying (3.4), (3.5) and (3.6). Now since (3.4) implies that

$$x_{2k} - x_{2k-1} > - \inf_{[0,\tau_N]} \xi, \;\; x_{2k+1} - x_{2k} > \sup_{[0,\tau_N]} \xi,$$

the I_k's do not interact before time τ_N.

Let $\tilde{x}_k = x_k + (-1)^k \xi_{0,\tau_N}$ and

$$I_k = \begin{cases} [\tilde{x}_{2k-1}, \tilde{x}_{2k}] & \text{if } 2k-1 > N \\ \varnothing & \text{otherwise.} \end{cases}$$

Then

$$\mathcal{S}^\xi(0,\tau_N)(P^1) = \cup_{k\leqslant 0} \tilde{I}_k \cup [-\xi_{0,\tau_N}, +\infty), \quad \mathcal{S}^\xi(\tau_N, T)(P^2) = (-\infty, \tilde{x}_N] \cup_{k \leqslant 0} \tilde{I}_k.$$

Note that

$$\tilde{x}_{k+1} - \tilde{x}_k = x_{k+1} - x_k + (-1)^k 2\xi_{0,\tau_N}$$

is bounded from above by

$$2 \left| \; \xi_{0,\tau_k} \right| + (-1)^k 2\xi_{0,\tau_N} = 2 \left| \; \xi_{\tau_N,\tau_k} \right|$$

and, when $k \geqslant N+2$, from below by $2 \left| \; \xi_{\tau_N,\tau_{k-2}} \right|$.

Hence, the evolution on $[\tau_N, T]$ is then given as in the case $\tau_N = 0$, and we obtain again

$$\mathcal{S}^\xi(0,T)(P^1) = [-\xi_{0,T}, +\infty) \;\text{ and }\; \mathcal{S}^\xi(0,T)(P^2) = \mathbb{R}.$$

Taking again t$x_k - x_{k-1}$ as large as possible yields

$$\rho(\xi, T) \geqslant \left\| \tilde{R}_{0,T}(\xi) \right\|_{TV([\tau_N,T])},$$

and letting $N \to -\infty$ finishes the proof. □

Optimality in one space dimension. We assume $d = 1$ and consider x-independent Hamiltonians. In this case the representation obtained in Section 2 is even simpler, since only the fully reduced path is needed.

Proposition 3.2. *Let $H : \mathbb{R} \to \mathbb{R}$ be continuous and convex. Then*

$$u(\cdot, T) = S^\xi(0,T) = S^{\tilde{R}_{0,T}(\xi)}(0,T), \tag{3.7}$$

and

$$\rho_H(\xi, T) \leqslant \|H'\|_\infty \|\tilde{R}_{0,T}(\xi)\|_{TV([0,T])}. \tag{3.8}$$

Proof. The claim is shown for H smooth and strictly convex in [7] using a regularization argument. The result extends to convex H by approximation, since (1.1) is stable under the passage to the limit in H by standard viscosity theory; see [11]. □

When $d \geqslant 2$, (3.7) is not true in general. Indeed, this can be easily seen by the counter-example depicted in Figure 3.2 and Figure 3.3, which correspond to the continuous, piece-wise linear path ξ with

$$\dot{\xi}(t) = \begin{cases} 4 & \text{for } (0,1) \\ -2 & \text{on } (1,2) \\ 1 & \text{on } (2,3) \end{cases} \quad \text{and } \dot{\tilde{R}}_{0,3}(\xi) = \begin{cases} 4 & \text{on } (0,1) \\ -\frac{1}{2} & \text{on } (1,3); \end{cases}$$

it is easy to observe that in this case $S^\xi_{|\cdot|}(0,3) \neq S^{\tilde{R}_{0,3}(\xi)}_{|\cdot|}(0,3)$.

FIGURE 3.2. Evolution of $S^{\xi}_{|\cdot|}(0,t)$ at $t=0$, $t=1$, $t=2$, $t=3$

FIGURE 3.3. Evolution of $S^{\tilde{R}(\xi)}_{|\cdot|}(0,t)$ at $t=0$, $t=1$, $t=3$

Note however that the claims in Proposition 3.2 hold in arbitrary dimension for $H(p) = \frac{1}{2}|p|^2$ (cf. Gassiat and Gess [4]) and more generally for a class of uniformly convex H (cf. Lions and Souganidis [10]).

Since (3.7) is not valid in general in dimension $d \geqslant 2$, the speed of propagation may depend on the total variation of the full reduced path $R_{0,T}(\xi)$. Note that we do not know, even for $H = |\cdot|$, if one always has equality in (1.11) in that case. The following proposition gives an example of a situation where $\rho_H(\xi,T) = \|R_{0,T}(\xi)\|_{TV([0,T])} > \|\tilde{R}_{0,T}(\xi)\|_{TV([0,T])}$.

Proposition 3.3. *Let $\delta_1 > \delta_2 > \delta_3 > 0$ and ξ continuous on $[0,3]$ with*

$$\dot{\xi} = \begin{cases} \delta_1 & \text{on } (0,1), \\ -\delta_2 & \text{on } (1,2), \\ +\delta_3 & \text{on } (2,3). \end{cases}$$

Then

$$\rho_{|\cdot|}(\xi,3) = \delta_1 + \delta_2 + \delta_3 = \|\xi\|_{TV([0,3])}.$$

Proof. Fix $0 < L < 2\delta_2$ and $\eta > 0$, and consider initial conditions (see Figure 3.5 and Figure 3.4)

$$P_1 = [0,L] \times \{-\delta_1, +\delta_1\}, \quad P_2 = P_1 \cup (\{-\delta_1\} \times \mathbb{R}).$$

Then $(-\eta,0), (L+\eta,0) \notin \mathcal{S}^{\xi}_{|\cdot|}(0,1)P_1$, and, hence,

$$(B_{\delta_2}(-\eta,0) \cup B_{\delta_2}(L+\eta,0)) \cap \mathcal{S}^{\xi}_{|\cdot|}(0,2)P_1 = \varnothing.$$

Since $L < 2\delta_2$, for η small enough, the interior of $B_{\delta_2}(-\eta,0) \cap B_{\delta_2}(L+\eta,0)$ is non-empty. It follows that, for all $\eta_1 \in (0,\eta)$ small enough,

$$B_{\delta_3}(L-\delta_2+\delta_3-\eta_1,0) \subseteq B_{\delta_2}(-\eta,0) \cup B_{\delta_2}(L+\eta,0),$$

FIGURE 3.4. Evolution of $S^{\xi}_{|\cdot|}(0,\cdot)P_1$

FIGURE 3.5. Evolution of $S^{\xi}_{|\cdot|}(0,\cdot)P_2$

and, consequently,

$$(L-\delta_2+\delta_3-\eta_1,0)\notin \mathcal{S}^{\xi}_{|\cdot|}(0,3)P_1.$$

We next note that $[-2\delta_1,L]\times[-2\delta_1,2\delta_1]\subseteq \mathcal{S}^{\xi}_{|\cdot|}(0,1)P_2$ and thus

$$[-2\delta_1+\delta_2,L-\delta_2]\times[-2\delta_1+\delta_2,2\delta_1-\delta_2]\subseteq \mathcal{S}^{\xi}_{|\cdot|}(0,2)P_2.$$

It follows that

$$B_{\delta_3}(L-\delta_2+\delta_3-\eta,0)\cap \mathcal{S}^{\xi}_{|\cdot|}(0,2)P_2\neq\varnothing,$$

and, hence, for each $\eta>0$,

$$(L-\delta_2+\delta_3-\eta,0)\in \mathcal{S}^{\xi}_{|\cdot|}(0,3)P_2.$$

In conclusion, for all $\eta>0$ small enough,

$$(L-\delta_2+\delta_3-\eta,0)\in \mathcal{S}^{\xi}_{|\cdot|}(0,3)(P^2)\backslash \mathcal{S}^{\xi}_{|\cdot|}(0,3)(P^1)$$

so that $\rho_{|\cdot|}(\xi,3)\geqslant \delta_1+L-\delta_2+\delta_3-\eta$. Letting $L\to 2\delta_2$ and $\eta\to 0$ finishes the proof. □

4. THE BROWNIAN CASE

We begin with some preliminary discussion and a few results that are needed for the proof of Theorem 1.3.

The key observation in the proof of Theorem 1.3 is that the length of $R_{0,T}(B)$ on $[0,\tau_0]$, where τ_0 is given by (1.7), is the same as that of a left-continuous path obtained by removing all excursions of B between its minimum and maximum.

We fix an arbitrary continuous path $B : [0, +\infty) \to \mathbb{R}$ with $B(0) = 0$. Let $M(t) := \sup_{s \leqslant t} B(s)$, $m(t) := \inf_{s \leqslant t} B(s)$, $R(t) := M(t) - m(t)$, recall that for $a \geqslant 0$

$$\theta(a) = \inf \{t \geqslant 0, \ \ R(t) = a\},$$

and, $r \geqslant 0$, define

$$S(r) := B(\theta(r)).$$

Then S is a left continuous process with right limits, the dynamics of which are simple to describe (see the proof of Lemma 4.1 below): Away from the jumps, S has a drift given by $sign(S(r))$, and the jumps are given by $\Delta S(r) := S(r^+) - S(r) = -sign(S(r))r$. In particular,

$$L(r) := \|S\|_{TV([0,r])} = r + \sum_{0 \leqslant s < r} s \mathbf{1}_{\Delta S(s) \neq 0}. \tag{4.1}$$

The following lemma relates L and $R_{0,T}(B)$.

Lemma 4.1. *For all $T \geqslant 0$,*

$$\|R_{0,T}(B)\|_{TV([0,\tau_0])} = L(R(T)).$$

Proof. Recall (1.7), (1.8) and (1.9) and note that

$$\|R_{0,T}(B)\|_{TV([0,\tau_0])} = \sum_{i \leqslant 0} \|R_{0,T}(B)\|_{TV([\tau_{i-1},\tau_i])} = \sum_{i \leqslant 0} |B(\tau_{i-1}) - B(\tau_i)|.$$

Fix $i \leqslant 0$ and assume that $B(\tau_{i-1}) = m(\tau_{i-1}) < 0$. If $r \in (R(\tau_{i-1}), R(\tau_i)]$, then monotonicity of θ and the definition of τ_i give $\theta(r) \leqslant \theta(R(\tau_i)) = \tau_i$, and, hence, $m(\theta(r)) = m(\tau_{i-1})$.
Moreover, for $r \in (R(\tau_{i-1}), R(\tau_i)]$, we have $B(\theta(r)) = M(\theta(r))$ and thus

$$r = R(\theta(r)) = M(\theta(r)) - m(\tau_{i-1}) = B(\theta(r)) - B(\tau_{i-1}).$$

In conclusion, for all $r \in (R(\tau_{i-1}), R(\tau_i)]$,

$$S(r) = B(\theta(r)) = B(\tau_{i-1}) + r = S(R(\tau_{i-1})) + r,$$

that is S has a jump of size $R(\tau_{i-1})$ at $r = R(\tau_{i-1})$ and is affine with slope 1 on $(R(\tau_{i-1}), R(\tau_i)]$.
If $B(\tau_{i-1}) = M(\tau_{i-1}) > 0$, the same reasoning shows that S has a jump of size $-R(\tau_{i-1})$ at $r = R(\tau_{i-1})$ and is affine with slope -1 on $(R(\tau_{i-1}), R(\tau_i)]$.
Finally we get

$$\begin{aligned} L(R(\tau_0)) = \|S\|_{TV([0,R(\tau_0)])} &= \sum_{i \leqslant 0} \|S\|_{TV([R(\tau_{i-1}),R(\tau_i)])} \\ &= \sum_{i \leqslant 0} |R(\tau_{i-1})| + |R(\tau_i) - R(\tau_{i-1})| = \sum_{i \leqslant 0} R(\tau_i) = \sum_{i \leqslant 0} |B(\tau_i) - B(\tau_{i-1})|. \end{aligned}$$

In view of the definition of τ_0 and R we have $R(\tau_0) = R(T)$ and thus $L(R(T)) = L(R(\tau_0))$, which finishes the proof. □

Next we assume that B is a linear Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we describe some of the properties of the time change S that we will use below. The results have been obtained in Imhof [8, Theorem p. 352] and Vallois [17, Theorem 1].

The first result is that

$$\{s > 0, \ \Delta S(s) \neq 0\} \ \text{ is a Poisson point process on } (0, +\infty) \text{ with intensity measure } \frac{ds}{s}. \tag{4.2}$$

For fixed $r > 0$ set $\sigma_0(r) := r$ and, for all $k \in \mathbb{N}$, define the successive jump times by

$$\sigma_{k+1}(r) = \sup\{s < \sigma_k(r) : \Delta S(s) \neq 0\},$$

It follows from (4.2) that

$$(\sigma_{n+1}(r)/\sigma_n(r))_{n\geqslant 0} \overset{(d)}{=} (U_n)_{n\geqslant 0}, \tag{4.3}$$

where $(U_n)_{n\geqslant 0}$ is a sequence of i.i.d. random variables each with uniform distribution on $[0,1]$, and $\overset{(d)}{=}$ denotes equality in the sense of distributions.

Moreover, if

$$\theta_n(r) := \inf\{t \geqslant 0, R(t) = \sigma_n(r)\},$$

then

$$(|B((\theta_{n+1}(r)+t) \wedge \theta_n(r)) - B(\theta_{n+1}(r))| : t \geqslant 0)_{n\geqslant 0} \overset{(d)}{=} (X^n(t \wedge \zeta_n(r)); t \geqslant 0)_{n\geqslant 0} \tag{4.4}$$

where the processes X^n are i.i.d. 3-dimensional Bessel processes, independent from S, starting from 0 and

$$\zeta_n(r) := \inf\{t > 0 : \ X^n(t) = \sigma_n(r)\}.$$

At this point we recall that, in view of the scaling properties of the Brownian motion, it is enough to prove Theorem 1.3 with $T = 1$, that is to study the reduced path $R_{0,1}(B)$.

To prove (1.13) we need the following two lemmata.

Lemma 4.2. *There exists $C > 0$ such that, for all $r, x \geqslant 0$,*

$$\mathbb{P}\left(R(1) \geqslant r | L(r) - r \geqslant x\right) \leqslant C \exp\left(-Cx^2\right). \tag{4.5}$$

Proof. Note that $R(1) \geqslant r$ is equivalent to $\theta(r) \leqslant 1$.

It follows from (4.3) and (4.4) that there exist i.i.d. uniformly distributed in $[0,1]$ random variables U_n such that

$$L(r) = r + \sum_{n>1} \sigma_n(r) = r\left(1 + \sum_{n=0}^{\infty} U_1 \cdots U_n\right).$$

Moreover, in view of the scaling properties of the Bessel processes,

$$\theta(r) = \sum_{n\geqslant 0} \left(\theta_n(r) - \theta_{n+1}(r)\right)) \overset{(d)}{=} \sum_{n\geqslant 0} \zeta_n(r)) \overset{(d)}{=} \sum_{n\geqslant 0} (\sigma_n(r))^2 \tilde{\zeta}_n,$$

where

$$\tilde{\zeta}_n = \inf\left\{t > 0 : \tilde{X}^n(t) = 1\right\}.$$

Since the i.i.d. Bessel random variables $\tilde{X}^n$ are independent from the random variables U_n, it follows that

$$(\theta(r), L(r)) \overset{(d)}{=} \left(r^2 \sum_{n=0}^{\infty} (U_1 \dots U_n)^2 \tilde{\zeta}_n, r\left(1 + \sum_{n=0}^{\infty} U_1 \cdots U_n\right)\right).$$

Using once again scaling and the fact that Bessel processes have the Gaussian tails, we also find that, for some $C_0 > 0$ and all $n \in \mathbb{N}$,

$$\mathbb{P}\left(\tilde{\zeta}_n^{-1/2} \geqslant y\right) = \mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} \tilde{X}^n_t \geqslant y\right) \leqslant \exp(-C_0 y^2). \tag{4.6}$$

Then, Jensen's inequality yields that, for any nonnegative sequence α_n,

$$\left(\sum_{n\geqslant 0} \alpha_n^2 \tilde{\zeta}_n\right) \geqslant \left(\sum_{n\geqslant 0} \alpha_n\right)^3 \left(\sum_{n\geqslant 0} \alpha_n^{1/2} \tilde{\zeta}_n^{-1/2}\right)^{-2}.$$

and thus

$$\mathbb{P}\left(\sum_{n=0}^{\infty} \alpha_n^2 \tilde{\zeta}_n \leqslant s\right) \leqslant \mathbb{P}\left(\sum_{n\geqslant 0} \alpha_n^{1/2} \tilde{\zeta}_n^{-1/2} \geqslant s^{-1/2} \left(\sum_{n\geqslant 0} \alpha_n\right)^{3/2}\right).$$

On the other hand, (4.6), the independence of the $\tilde{\zeta}_n$ and a straightforward Chernoff bound yield, for all $y \geqslant 0$ and some $C > 0$,

$$\mathbb{P}\left(\sum_{n\geqslant 0} \alpha_n^{1/2} \tilde{\zeta}_n^{-1/2} \geqslant y\right) \leqslant \exp\left(-C \frac{y^2}{\sum_{n\geqslant 0} \alpha_n}\right)$$

for some $C > 0$, so that

$$\mathbb{P}\left(\sum_{n=0}^{\infty} \alpha_n^2 \tilde{\zeta}_n \leqslant s\right) \leqslant \exp\left(-Cs^{-1} \left(\sum_{n\geqslant 0} \alpha_n\right)^2\right).$$

Applying the inequality above with $\alpha_n = U_1 \dots U_n$ and $s = r^{-2}$, we get

$$\mathbb{P}\left(\theta(r) \leqslant 1 | L(r) - r \geqslant x\right) \leqslant \exp\left(-Cr^2 (x/r)^2\right) \leqslant \exp(-Cx^2).$$

□

Lemma 4.3. *There exists $C > 0$ such that, for all $x \geqslant 2r \geqslant 0$ and $\delta \in (0, 1]$,*

$$\mathbb{P}\left(L(r(1+\delta)) - L(r) - r\delta \geqslant x\right) \leqslant C \exp\left(-\frac{x}{2r} |\ln(\delta)|\right). \tag{4.7}$$

Proof. Note that

$$\begin{aligned} L(r(1+\delta)) - L(r) - r\delta &= \sum_{r\leqslant s< r(1+\delta)} s 1_{\Delta S(s)\neq 0} \\ &\leqslant \ 2r \# \left\{s \in [r, r(1+\delta)),\ \ \Delta S(s) \neq 0\right\}. \end{aligned}$$

In view of (4.2), $\#\left\{s \in [r, r(1+\delta)),\ \ \Delta S(s) \neq 0\right\}$ is a Poisson random variable with parameter

$$\int_r^{r(1+\delta)} \frac{du}{u} = \ln(1+\delta) \leqslant \delta.$$

In addition, any Poisson random variable P with parameter δ satisfies the classical inequality

$$\begin{aligned} \mathbb{P}(P \geqslant y) &\leqslant \inf_{\gamma} \left\{\mathbb{E}[e^{\gamma P}] e^{-\gamma y}\right\} = \inf_{\gamma} \exp(-\gamma y + \delta(e^{\gamma} - 1)) = \exp(-y \ln\left(\frac{y}{\delta}\right) + y - \delta) \\ &\leqslant C \exp(-y |\ln(\delta)|). \end{aligned}$$

It follows that

$$\mathbb{P}\left(L(r(1+\delta)) - L(r) - r\delta \geqslant x\right) \leqslant C \exp\left(-\frac{x}{2r} |\ln(\delta)|\right).$$

□

We are now ready for the proof of Theorem 1.3.

Proof of Theorem 1.3. We begin with (1.14) and note that on the interval $[0, \theta(1)]$ we have $\tau_0 = \theta(1)$.

It then follows from the definition of $\theta(1)$ that $R(\tau_0) = R(\theta(1)) = 1$ and thus, using Lemma 4.1, we find

$$\left\|R_{0,\theta(1)}(B)\right\|_{TV([0,\theta(1)])} = L(R(\theta(1))) = L(1).$$

We further note that, in view of (4.3), we have that

$$L(1) \overset{(d)}{=} 1 + U_0 + U_0U_1 + U_0U_1U_2 + \dots,$$

where the random variables U_i are i.i.d. uniformly distributed on $[0, 1]$.

It now follows from [6, Theorem 3.1] that $L(1)$ has Poissonian tails, that is,

$$\lim_{x\to\infty} \frac{\ln \mathbb{P}(L(1) \geqslant x)}{x \ln x} \to -1.$$

We present now the proof of (1.13). Throughout the argument below, C will denote a constant whose value may change from line to line.

We first note that

$$\|R_{0,1}(B)\|_{TV([0,1])} \leqslant \|R_{0,1}(B)\|_{TV([0,\tau_0])} + \|R_{0,1}(B)\|_{TV([\tau_{-1},1])}.$$

Moreover, the symmetry of Brownian motion under time reversal gives

$$\|R_{0,1}(B)\|_{TV([\tau_{-1},1])} \overset{(d)}{=} \|R_{0,1}(B)\|_{TV([0,\tau_0])}.$$

It follows that it suffices to bound the tail probabilities of $\|R_{0,1}(B)\|_{TV([0,\tau_0]))}$ and, hence, in view of Lemma 4.1, the tail of $L(R(1))$.

Fix $\gamma \in (0, 2)$ and let $\alpha < \gamma$ be such that $\gamma = \frac{2(1+\alpha)}{3}$. For a fixed $x > 0$, let $r_0 := \frac{x}{4}$ and, for $k \in \mathbb{N}$, $r_{k+1} := r_k(1 + e^{-x^\alpha})^{-1}$. It is immediate that there exists an N such that $N \leqslant Ce^{Cx^\alpha}$ and $r_N \leqslant x^{-1}$.

Moreover,

$$\mathbb{P}\left(L(R(1)) \geqslant x\right) \leqslant \sum_{k=0}^{N-1} \mathbb{P}\left(r_{k+1} \leqslant R(1) \leqslant r_k; L(r_k) \geqslant x\right) + \mathbb{P}\left(R(1) \geqslant \frac{x}{4}\right) + \mathbb{P}\left(L(r_N) \geqslant x\right).$$

The second term on the right hand side is bounded by $\exp(-Cx^2)$ since $R(1)$ has Gaussian tails. Moreover, Brownian scaling implies that $L(\tau t) \overset{(d)}{=} \tau L(t)$ for all $\tau > 0$ and thus

$$\mathbb{P}(L(r_N) \geqslant x) \leqslant \mathbb{P}\left(L(\frac{1}{x}) \geqslant x\right) \leqslant \mathbb{P}(L(1) \geqslant x^2) \leqslant \exp(-Cx^2).$$

In addition, we note that, for $k \in \{0, \dots, N-1\}$,

$$\begin{aligned}&\mathbb{P}\left(r_{k+1} \leqslant R(1) \leqslant r_k; L(r_k) \geqslant x\right)\\ &\leqslant \mathbb{P}\left(r_{k+1} \leqslant R(1); L(r_{k+1}) \geqslant \frac{x}{2}\right) + \mathbb{P}\left(r_{k+1} \leqslant R(1); L(r_k) - L(r_{k+1}) \geqslant \frac{x}{2}\right).\end{aligned}$$

Lemma 4.2 implies that

$$\mathbb{P}\left(r_{k+1} \leqslant R(1); L(r_{k+1}) \geqslant \frac{x}{2}\right) \leqslant \mathbb{P}\left(r_{k+1} \leqslant R(1); L(r_{k+1}) - r_{k+1} \geqslant \frac{x}{4}\right) \leqslant C\exp(-Cx^2).$$

Then, the Cauchy-Schwarz inequality and Lemma 4.3 give

$$\begin{aligned} P\left(r_{k+1} \leqslant R(1); L(r_k) - L(r_{k+1}) \geqslant \frac{x}{2}\right) &\leqslant P\left(r_{k+1} \leqslant R(1)\right)^{1/2} \mathbb{P}\left(L(r_k) - L(r_{k+1}) \geqslant \frac{x}{2}\right)^{1/2} \\ &\leqslant C\exp(-Cr_{k+1}^2)\exp\left(-C\frac{x^{1+\alpha}}{r_{k+1}}\right) \\ &\leqslant C\exp\left(-Cx^{\frac{2(1+\alpha)}{3}}\right). \end{aligned}$$

It follows that

$$\mathbb{P}\left(L(R(1)) \geqslant x\right) \leqslant Ce^{Cx^{\alpha}}e^{-Cx^{\gamma}} \leqslant Ce^{-Cx^{\gamma}}.$$

□

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References

[1] M. Bardi, M. G. Crandall, L. C. Evans, H. M. Soner, and P. E. Souganidis. *Viscosity solutions and applications*, volume 1660 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin; Centro Internazionale Matematico Estivo (C.I.M.E.), Florence, 1997. Lectures given at the 2nd C.I.M.E. Session held in Montecatini Terme, June 12–20, 1995, Edited by I. Capuzzo Dolcetta and P. L. Lions, Fondazione CIME/CIME Foundation Subseries.

[2] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[3] Peter K Friz, Paul Gassiat, Pierre-Louis Lions, and Panagiotis E Souganidis. Eikonal equations and pathwise solutions to fully non-linear spdes. *Stochastics and Partial Differential Equations: Analysis and Computations*, 5(2):256–277, 2017.

[4] P. Gassiat and B. Gess. Regularization by noise for stochastic Hamilton-Jacobi equations. *arXiv:1609.07074*, September 2016.

[5] Paul Gassiat. A stochastic Hamilton-Jacobi equation with infinite speed of propagation. *C. R. Math. Acad. Sci. Paris*, 355(3):296–298, 2017.

[6] Charles M. Goldie and Rudolf Grübel. Perpetuities with thin tails. *Adv. in Appl. Probab.*, 28(2):463–480, 1996.

[7] H. Hoel, K. H. Karlsen, N. H. Risebro, and E. B. Storrøsten. Path-dependent convex conservation laws. *arXiv:1711.01841*, November 2017.

[8] J.-P. Imhof. A construction of the Brownian path from BES^3 pieces. *Stochastic Process. Appl.*, 43(2):345–353, 1992.

[9] Pierre-Louis Lions. *Generalized solutions of Hamilton-Jacobi equations*, volume 69 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.

[10] Pierre-Louis Lions and Panagiotis E. Souganidis. *In preparation.*

[11] Pierre-Louis Lions and Panagiotis E. Souganidis. Stochastic viscosity solutions. *Book, in preparation.*

[12] Pierre-Louis Lions and Panagiotis E. Souganidis. Stochastic viscosity solutions of spatially dependent hamilton-jacobi equations with multiple paths. *In preparation.*

[13] Pierre-Louis Lions and Panagiotis E. Souganidis. Fully nonlinear stochastic partial differential equations. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(9):1085–1092, 1998.

[14] Pierre-Louis Lions and Panagiotis E. Souganidis. Fully nonlinear stochastic partial differential equations: non-smooth equations and applications. *C. R. Acad. Sci. Paris Sér. I Math.*, 327(8):735–741, 1998.

[15] Pierre-Louis Lions and Panagiotis E. Souganidis. Stochastic viscosity solutions. *Lectures in College de France*, 2009.

[16] Panagiotis E. Souganidis. Fully nonlinear first- and second-order stochastic partial differential equations. *CIME lecture notes*, pages 1–37, 2016.

[17] P. Vallois. Decomposing the Brownian path via the range process. *Stochastic Process. Appl.*, 55(2):211–226, 1995.

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