EFFECTIVE OPERATOR FOR ROBIN EIGENVALUES IN DOMAINS WITH CORNERS

MAGDA KHALILE, THOMAS OURMIÈRES-BONAFOS, AND KONSTANTIN PANKRASHKIN

Abstract. We study the eigenvalues of the Laplacian with a strong attractive Robin boundary condition in curvilinear polygons. It was known from previous works that the asymptotics of several first eigenvalues is essentially determined by the corner openings, while only rough estimates were available for the next eigenvalues. Under some geometric assumptions, we go beyond the critical eigenvalue number and give a precise asymptotics of any individual eigenvalue by establishing a link with an effective Schrödinger-type operator on the boundary of the domain with boundary conditions at the corners.

Contents

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1. Introduction

1.1. **Problem setting and previous results.** Given a domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with a suitably regular boundary $\partial\Omega$ and a parameter $\alpha > 0$, we denote by R^Ω_α the Laplacian in $L^2(\Omega)$ with the Robin condition $\partial u/\partial\nu = \alpha u$ at the boundary, where ν is the outer unit normal. The operator is rigorously defined using its quadratic form

$$H^1(\Omega) \ni u \mapsto \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \alpha \int_{\partial\Omega} u^2 \,\mathrm{d}s$$

with $\mathrm{d}s$ being the $(d-1)$-dimensional Hausdorff measure, provided that the form is lower semibounded and closed. The spectral properties of the operator R^Ω_α have attracted a lot of attention during the last years, and a recent review of various results and open problems can be found in the paper [10] by Bucur, Freitas, Kennedy. In the present paper we will be interested in the behavior of the eigenvalues $E_n(R^\Omega_\alpha)$ in the asymptotic regime $\alpha \to +\infty$. Let us recall some available results in this direction.

It seems that the study of the above asymptotic regime was first proposed by Lacey, Ockedon, Sabina [40] when considering a reaction-diffusion system, and Giorgi and Smits [21, 22] obtained a number of estimates with links to the theory of enhanced surface superconductivity. Remark that for bounded Lipschitz domains Ω it follows from the general theory of Sobolev spaces that there exists $C > 0$ with $E_1(R^\Omega_\alpha) \geq -C\alpha^2$ for large α (see Corollary 13 below). Lacey, Ockedon, Sabina in [40] conjectured that under suitable regularity assumptions on Ω the lower bound can be upgraded to an asymptotics

$$E_1(R^\Omega_\alpha) \sim -C_\Omega \alpha^2, \tag{1}$$

with some $C_\Omega > 0$, and they have shown that $C_\Omega = 1$ for C^4 smooth domains. Levitin and Parnovski in [38] have shown the asymptotics (1) for piecewise smooth domains satisfying the interior cone condition, and they have shown that the constant C_Ω is explicitly determined through the spectra of model Robin Laplacians by

$$(-C_\Omega) = \inf_{x\in\partial\Omega} \inf \operatorname{spec}(R^{T_x}_1), \tag{2}$$

where T_x is the tangent cone to Ω at x and spec stands for the spectrum of the operator. Bruneau and Popoff in [9] gave an improved remainder estimate under the slightly stronger assumption that Ω is a so-called corner domain. We also mention the recent preprint [35] by Kovařík and Pankrashkin on non-Lipschitz domains, for which the eigenvalue behavior is completely different.

More precise estimates are available for smooth domains. The lower bound by Lou and Zhu [41] and the upper bound due to Daners and Kennedy [13] imply that if Ω is a bounded C^1 domain, then for each fixed $n \in \mathbb{N}$ one has $E_n(R^\Omega_\alpha) \sim -\alpha^2$. It seems that a more precise asymptotics was first obtained by Pankrashkin in [48]: it was shown that if $\Omega \subset \mathbb{R}^2$ is bounded with a C^3 boundary, then $E_1(R^\Omega_\alpha) = -\alpha^2 - H_*\alpha + \mathcal{O}(\alpha^{\frac{2}{3}})$, where H_* is the maximum of the curvature of the boundary. Exner, Minakov and Parnovski in [15] show that the asymptotics

$$E_n(R^\Omega_\alpha) = -\alpha^2 - H_*\alpha + \mathcal{O}(\alpha^{\frac{2}{3}}) \tag{3}$$

holds for any fixed $n \in \mathbb{N}$, and then Exner and Minakov [14] obtained similar results for non-compact domains. Helffer and Kachmar [26] obtained a complete asymptotic expansion for eigenvalues under the additional assumption that the curvature of the boundary admits a single non-degenerate maximum. Pankrashkin and Popoff in [52] started the study of the multidimensional case: if $\Omega \subset \mathbb{R}^d$ is a C^3 domain, then the asymptotics (3) holds with $H_* := \max H$ and H is defined as the sum of the principal curvatures at the boundary, i.e. $H = (d-1)$ times the mean curvature. An analog of the asymptotics (3) for the first eigenvalue of Robin p-Laplacians was obtained by Kovařík and Pankrashkin in [34]. Among possible applications of the asymptotics (3) one may mention various optimization issues concerning the eigenvalues of

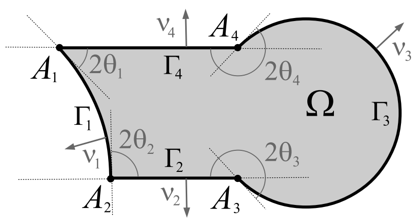

FIGURE 1. An example of a curvilinear polygon Ω with four vertices and sides of constant curvature. The vertices A_1 and A_2 are convex, and the vertices A_3 and A_4 are concave. One has $H_1 < 0$, $H_3 > 0$ and $H_2 = H_4 = 0$.

R^Ω_α. It was conjectured by Bareket [5] that among the domains Ω of fixed volume, for any $\alpha > 0$ the quantity $E_1(R^\Omega_\alpha)$ is maximized by the balls. In this most general form, the conjecture was disproved by Freitas and Krejčiřík [20], but an additional analysis shows that the conjecture may hold in a weaker form under additional restrictions on the geometry of Ω, we refer to the papers by Antunes, Freitas, Krejčiřík [1], Bandle and Wagner [4], Ferone, Trombetti, Nitsch [17], Trani [57]. As noted by Pankrashkin and Popoff in [52], if the ball is the maximizer of $E_1(R^\Omega_\alpha)$ for all $\alpha > 0$ in some class of smooth domains Ω, then it is also the minimizer for the maximum mean curvature H_* in the same class of domains, and this observation leads to some new inequalities for H_*, see e.g. Ferone, Nitsch, Trombetti [18], and it was used to construct a number of counterexamples, for example, the asymptotics (3) was used by Krejčiřík and Lotoreichik [36, 37] in the study of isoperimetric inequalities for Robin laplacians in exterior domains.

In [53] Pankrashkin and Popoff proposed an effective operator to study the eigenvalues of R^Ω_α. Namely, it was shown for C^3 domains Ω, either bounded or with a controllable behavior at infinity, that for any fixed $n \in \mathbb{N}$ one has the asymptotics

$$E_n(R^\Omega_\alpha) = -\alpha^2 + E_n(L_\alpha) + \mathcal{O}(1), \tag{4}$$

where L_α is the Schrödinger operator in $L^2(\partial\Omega)$ acting as $L_\alpha = -\Delta_{\partial\Omega} - \alpha H$ with $\Delta_{\partial\Omega}$ being the Laplace-Beltrami operator on $\partial\Omega$. Kachmar, Keraval, Raymond [31] and Helffer, Kachmar, Raymond [27] have shown that the same effective operator appears in other spectral questions for R^Ω_α, e.g. the Weyl asymptotics and the tunneling effect for R^Ω_α are also controlled by those for L_α at the leading orders. Pankrashkin [50] and Bruneau, Pankrashkin, Popoff [8] used the effective operator in order to study the accumulation of eigenvalues for Robin Laplacians on some non-compact domains.

We also mention some related papers going slightly beyond the initial problem setting. Colorado and García-Melián [12] obtained some results in the same spirit for Laplacians with the boundary condition $\partial u/\partial\nu = \alpha p u$ for variable functions p and $\alpha \to +\infty$. Filinovskii in [19] obtained the estimate $\liminf_{\alpha\to+\infty} \alpha^{-1}\partial E_1(R^\Omega_\alpha)/\partial\alpha \le -1$. Helffer and Pankrashkin [28] studied the exponential splitting between the first two eigenvalues of R^Ω_α in a domain Ω with two congruent corners. Cakoni, Chaulet and Haddar [11] have shown that, in a sense, the only finite accumulation points of the eigenvalues of R^Ω_α for large positive α are the Dirichlet Laplacian eigenvalues of Ω.

1.2. **Main results.** In the present paper, we would like to combine the existing results and techniques in order to study the eigenvalues of R^Ω_α for the case of $\Omega \subset \mathbb{R}^2$ being a *curvilinear polygon* and to better understand the role of corners in the spectral properties. A complete definition of curvilinear polygons will be given later in the text (Subsection 6.1), and for the moment we restrict ourselves to a less formal intuitive definition: one says that a bounded planar domain Ω is a curvilinear polygon if its boundary is smooth except near M points (vertices) $A_1, \dots, A_M$, and if Γ_j and Γ_{j+1} are two smooth pieces of boundary meeting at A_j, then the half-angle θ_j between them (measured inside Ω) is non-degenerate and non-trivial, i.e. $\theta_j \notin \{0, \pi/2, \pi\}$. We say that a vertex A_j is convex if $\theta_j < \pi/2$, otherwise it is called concave. Furthermore, let H_j be the curvature defined on Γ_j, with the convention that $H_j \ge 0$ for convex domain. We refer to Figure 1 for an illustration.

Using the general result (2) one is reduced first to the study of Robin Laplacians in all possible tangent sectors, which have a simple structure in two dimensions. Namely, consider the infinite planar sectors $\mathcal{S}_\theta := \big\{(x_1, x_2) : \big|\arg(x_1+\mathrm{i}x_2)\big| < \theta\big\} \subset \mathbb{R}^2$, see Figure 2, then the tangent sector to Ω at A_j is a rotated copy of $\mathcal{S}_{\theta_j}$, while at all other points the tangent sectors are isometric to $\mathcal{S}_{\frac{\pi}{2}}$, which is just the half-plane. Denote by T_θ the Laplacian in $\mathcal{S}_\theta$ with the normalized Robin boundary condition $\partial u/\partial\nu = u$. Its spectral properties were studied in detail by Khalile and Pankrashkin [33] and are summarized below in Proposition 20. For the current presentation we remark that the essential spectrum is always $[-1, +\infty)$, and, in addition, it has $\kappa(\theta) < \infty$ discrete eigenvalues $\mathcal{E}_1(\theta), \dots \mathcal{E}_{\kappa(\theta)}(\theta)$, while $\kappa(\theta) = 0$ for $\theta \geq \pi/2$ (i.e. there are no discrete eigenvalues at all if the sector is concave), and $\mathcal{E}_1(\theta) = -1/\sin^2\theta$ for $\theta < \pi/2$. Furthermore, one has $\kappa(\theta) = 1$ for $\frac{\pi}{6} \leq \theta < \frac{\pi}{2}$. Hence, with Ω we associate the following objects:

$$
\begin{aligned}
K &:= \kappa(\theta_1) + \dots + \kappa(\theta_M),\\
\mathcal{E} &:= \text{ the disjoint union of } \big\{\mathcal{E}_n(\theta_j),\ n = 1, \dots, \kappa(\theta_j)\big\} \text{ for } j \in \{1, \dots, M\},\\
\mathcal{E}_n &:= \text{ the } n\text{th element of } \mathcal{E} \text{ when numbered in the non-decreasing order.}
\end{aligned}
$$

Khalile in [32] gives an improved version of (2) for curvilinear polygons, namely, for each $n \in \{1, \dots, K\}$ one has $E_n(R^\Omega_\alpha) = \mathcal{E}_n\alpha^2 + \mathcal{O}(\alpha^{\frac{4}{3}})$, while the remainder estimate can be improved for polygons with straight sides, and $E_{K+n}(R^\Omega_\alpha) \sim -\alpha^2$ for each $n \in \mathbb{N}$. (We remark that paper [32] was in turn motivated by the earlier work by Bonnaillie-Noël and Dauge [7] on magnetic Neumann Laplacians in corner domains.) Therefore, the behavior of the first K eigenvalues at the leading order is determined by the corners only, so one might call them *corner-induced.* In the present work we would like to understand in greater detail the asymptotics of the higher eigenvalues $E_{K+n}(R^\Omega_\alpha)$ with a fixed $n \in \mathbb{N}$, which will be referred to as *side-induced.* As the main term $(-\alpha^2)$ in the asymptotics is the same as in the smooth case, one might expect that their behavior should take into account the geometry of the boundary away from the corners, so that a kind of an effective Schrödinger-type operator may appear by analogy with (4). On the other hand, one might expect that the corners should contribute to the effective operator: due to the singularities at the vertices, some boundary conditions might be needed in order to make the effective operator self-adjoint. It seems that the only result obtained in this direction is the one by Pankrashkin [49]: if Ω is the exterior of a convex polygon with side lengths ℓ_j, then for any fixed n one has $E_n(R^\Omega_\alpha) = E_n(\oplus_j D_j) + \mathcal{O}(\alpha^{-\frac{1}{2}})$ as $\alpha \to +\infty$, where D_j is the Dirichlet Laplacian on $(0, \ell_j)$. Remark that this result is in agreement with what precedes: as all the corners are concave, one simply has $K = 0$. We are going to obtain a result in the same spirit for a more general case, in particular, by allowing the presence of convex corners. In order to concentrate on the contribution of the corners, we additionally assume that

the curvature H_j of each boundary piece Γ_j is constant, and we denote $H_* := \max H_j$.

(The general case appears to be much more involved technically, some remarks are given in Section 7 at the end of the paper.) Furthermore, let ℓ_j denote the length of Γ_j. Our analysis will be based on the notion of *non-resonant* convex vertex (it will be seen from the proof that concave vertices are much easier to deal with), which is formulated in terms of a model Robin eigenvalue problem on a truncated sector. Namely, for $\theta \in (0, \pi/2)$ and $r > 0$ let $A^\pm_r$ be the two points lying on the two boundary rays of the sector $\mathcal{S}_\theta$ at the distance $r > 0$ from the origin O, and let B_r be the intersection point of the straight lines passing through $A^\pm_r$ perpendicular to

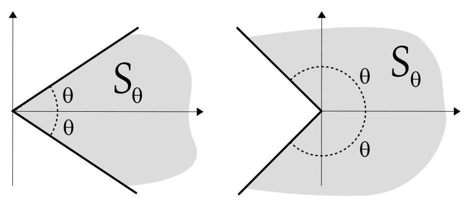

FIGURE 2. The infinite sector $\mathcal{S}_\theta$ for $\theta < \pi/2$ (left) and $\theta > \pi/2$ (right).

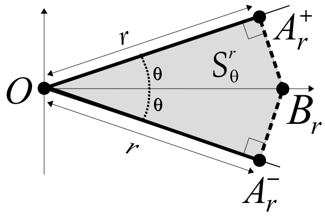

FIGURE 3. The quadrangle $\mathcal{S}^r_\theta$.

the boundary, see Figure 3. Denote by $\mathcal{S}^r_\theta$ the quadrangle $OA_r^+B_rA_r^-$ and by N_θ^r the Laplacian $u \mapsto -\Delta u$ in $\mathcal{S}^r_\theta$ with the Robin boundary condition $\partial u/\partial\nu = u$ at $OA_r^\pm$ and the Neumann boundary condition at $A_r^\pm B_r$. Using rather standard methods one sees that the first $\kappa(\theta)$ eigenvalues of N_θ^r converge to those of T_θ as $r \to +\infty$ (Lemma 27), and the non-resonance condition is a hypothesis on the behavior of the next eigenvalue. We say that a half-angle θ is *non-resonant* if for some $C > 0$ one has $E_{\kappa(\theta)+1}(N_\theta^r) \geq -1 + C/r^2$ for large r. One shows in Proposition 30, using a combination of a separation of variables with a monotonicity argument that all half-angles $\theta \in \big[\frac{\pi}{4}, \frac{\pi}{2}\big)$ are non-resonant. The above condition will play a key role in our analysis, and our main result is as follows:

Theorem 1. *Assume that all corners are concave or non-resonant, then for any fixed $n \in \mathbb{N}$ one has the asymptotics*

$$E_{K+n}(R^\Omega_\alpha) = -\alpha^2 - H_*\alpha - \frac{H_*^2}{2} + E_n\Big(\bigoplus\nolimits_{j:H_j=H_*} D_j\Big) + \mathcal{O}\Big(\frac{\log\alpha}{\sqrt{\alpha}}\Big),$$

where D_j is the Dirichlet Laplacian on $(0,\ell_j)$.

Using the above observation that all obtuse angles are non-resonant we obtain the following important particular case, by putting together all the assumptions:

Corollary 2. *Let $\Omega \subset \mathbb{R}^2$ be a curvilinear polygon with M vertices, half-angles θ_j and sides of length ℓ_j and of constant curvatures H_j. Assume that $\theta_j \geq \pi/4$ for all j, then for any $n \in \mathbb{N}$ there holds, with $H_* := \max H_j$,*

$$E_{K+n}(R^\Omega_\alpha) = -\alpha^2 - H_*\alpha - \frac{H_*^2}{2} + E_n\Big(\bigoplus\nolimits_{j:H_j=H_*} D_j\Big) + \mathcal{O}\Big(\frac{\log\alpha}{\sqrt{\alpha}}\Big),$$

where K is the number of convex vertices and D_j is the Dirichlet Laplacian on $(0,\ell_j)$.

Finally, let us formulate explicitly the case concerning the usual polygons (which corresponds to $H_j \equiv 0$):

Corollary 3. *Let $\Omega \subset \mathbb{R}^2$ be a polygon with M vertices, half-angles θ_j and side lengths ℓ_j. Assume that $\theta_j \geq \pi/4$ for all j, then for any $n \in \mathbb{N}$ there holds*

$$E_{K+n}(R^\Omega_\alpha) = -\alpha^2 + E_n\Big(\bigoplus\nolimits_{j=1}^{M} D_j\Big) + \mathcal{O}\Big(\frac{\log\alpha}{\sqrt{\alpha}}\Big),$$

where K is the number of convex vertices and D_j is the Dirichlet Laplacian on $(0,\ell_j)$.

The text is organized as follows. In Section 2 we recall basic tools from the functional analysis (min-max based eigenvalue estimates, distance between subspaces, Sobolev trace theorems) and study or recall the spectral properties of some model operators (Robin Laplacians on intervals and infinite sectors). We opted for a very detailed self-contained presentation of these preparatory constructions in order to make the reading available to a broader audience. Section 3 is devoted to the study of Robin Laplacians in convex sectors truncated in a special way: we obtain some estimates for the eigenvalues and decay estimate for the eigenfunctions, then introduce the new notion of non-resonant angle and show that it is satisfied by the obtuse angles. In Section 4 we analyze vertex neighborhoods of curvilinear sectors. We introduce neighborhoods of a special form and then show that they can be obtained by applying suitable diffeomorphisms on the truncated sectors from the preceding section. This is then used to give two-sided estimates for the eigenvalues of Robin Laplacians in these neighborhoods. Section 5 presents a spectral analysis of Laplacians in tubular neighborhoods of smooth curves, which is essentially based on a separation of variables written in special terms. In Section 6 we prove the main results. We introduce a special decomposition of curvilinear polygons into vertex neighborhoods of a special shape, side neighborhoods and the interior, and then analyze each part in detail. The upper bound (Proposition 48) appears to be rather elementary, while the proof of the lower bound (Proposition 51) uses all the preceding results: first, we show that the lowest eigenfunctions of the polygon are close, in the sense of a distance between subspaces, to the lowest eigenfunctions of the vertex neighborhoods, and then we analyze their orthogonal complement. The resulting effective operator is obtained by adapting a machinery used initially by Post [54] for the analysis of thin branching domains. In the last Section 7 we discuss possible extensions of the results, in particular, we show that some angles do not satisfy the non-resonance condition as they give a different eigenvalue asymptotics and we explain some links between our study and the spectral analysis of waveguides.

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2. Preliminaries

2.1. **Notation.** For $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ we will use the length $|x| = \sqrt{x_1^2 + x_2^2}$, the scalar product $x \cdot y = x_1 y_1 + x_2 y_2$ and the wedge product $x \wedge y = x_1 y_2 - x_2 y_1$.

In this paper we only deal with real-valued operators, so we prefer to work with real Hilbert spaces in order to have a simpler writing. Let $\mathcal{H}$ be a Hilbert space and $u, v \in \mathcal{H}$, then we denote by $\langle u, v\rangle_{\mathcal{H}}$ the scalar product of u and v. It will be sometimes shortened to $\langle u, v\rangle$ if there is no ambiguity in the choice of the Hilbert space, and the same applies to the associated norm $\|\cdot\|_{\mathcal{H}}$. For a self-adjoint operator A in $\mathcal{H}$ we denote by $\operatorname{spec}(A)$, $\operatorname{spec}_{\mathrm{disc}}(A)$ and $\operatorname{spec}_{\mathrm{ess}}(A)$ the spectrum of A, its discrete spectrum and its essential spectrum, respectively. For $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, by $E_n(A)$ we denote the nth discrete eigenvalue of A (if it exists) when enumerated in the non-decreasing order counting the multiplicities. If the operator A is semibounded from below, then $\mathcal{Q}(A)$ denotes the domain of its sesquilinear form, and the value of the sesqulinear form on two vectors $u, v \in \mathcal{Q}(A)$ will be denoted by $A[u, v]$.

2.2. **Min-max principle and its consequences.** Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and A be a lower semibounded self-adjoint operator in $\mathcal{H}$, with $A \geq -c$ for some $c \in \mathbb{R}$. Recall that $\mathcal{Q}(A)$ equipped with the scalar product

$$\mathcal{Q}(A) \times \mathcal{Q}(A) \ni (u, v) \mapsto := A[u, v] + (c + 1)\langle u, v\rangle_{\mathcal{H}}$$

is a Hilbert space. The following result giving a variational characterization of eigenvalues is a standard tool in the spectral theory of self-adjoint operators, see e.g. [56, Section XIII.1]:

Proposition 4 (Min-max principle). *Let $\Sigma := \inf \operatorname{spec}_{\mathrm{ess}}(A)$ if $\operatorname{spec}_{\mathrm{ess}}(A) \neq \emptyset$, otherwise set $\Sigma := +\infty$. Let $n \in \mathbb{N}$ and D be a dense subspace of $\mathcal{Q}(A)$. Define the nth* Rayleigh quotient *$\Lambda_n(A)$ of A by*

$$\Lambda_n(A) := \inf_{\substack{G \subset D \\ \dim G = n}} \sup_{u \in G \setminus \{0\}} \frac{A[u, u]}{\|u\|^2_{\mathcal{H}}},$$

then one and only one of the following two assertions is true:

- $\Lambda_n(A) < \Sigma$ *and* $E_n(A) = \Lambda_n(A)$.
- $\Lambda_n(A) = \Sigma$ *and* $\Lambda_m(A) = \Lambda_n(A)$ *for all* $m \geq n$.

In the following assertions we recall (with complete proofs) a number of classical inequalities for Rayleigh quotients under various assumptions. Remark that these estimates are directly transformed into estimates for the eigenvalues if the operators in question are with compact resolvents.

Corollary 5 (Inequality for restricted sesquilinear form)**.** *Let A and A' be lower semibounded self-adjoint operators in infinite-dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ respectively. Assume that there exists a linear map $J : \mathcal{Q}(A) \to \mathcal{Q}(A')$ such that $\|Ju\|_{\mathcal{H}'} = \|u\|_{\mathcal{H}}$ and $A'[Ju, Ju] \le A[u, u]$ for all $u \in \mathcal{Q}(A)$, then $\Lambda_n(A') \le \Lambda_n(A)$ for any $n \in \mathbb{N}$.*

Proof. For any finite-dimensional subspace $G \subset \mathcal{Q}(A)$ we have $\dim J(G) = \dim G$. Therefore, using the definitions we have:

$$\Lambda_n(A') = \inf_{\substack{G'\subset\mathcal{Q}(A')\\ \dim G'=n}} \sup_{\substack{u'\in G'\\ u\neq 0}} \frac{A'[u',u']}{\|u'\|^2_{\mathcal{H}'}} \le \inf_{\substack{G\subset\mathcal{Q}(A)\\ \dim G=n}} \sup_{\substack{u'\in J(G)\\ u'\neq 0}} \frac{A'[u',u']}{\|u'\|^2_{\mathcal{H}'}}$$
$$= \inf_{\substack{G\subset\mathcal{Q}(A)\\ \dim G=n}} \sup_{\substack{u\in G\\ u\neq 0}} \frac{A'[Ju,Ju]}{\|Ju\|^2_{\mathcal{H}'}} \le \inf_{\substack{G\subset\mathcal{Q}(A)\\ \dim G=n}} \sup_{\substack{u\in G\\ u\neq 0}} \frac{A[u,u]}{\|u\|^2_{\mathcal{H}}} = \Lambda_n(A). \quad \square$$

Corollary 6 (Inequality for finite-rank perturbations)**.** *Let A and B be lower semibounded self-adjoint operators in an infinite-dimensional Hilbert space $\mathcal{H}$. Assume that there exists $d \in \mathbb{N}$ and a d-dimensional subspace D such that $\mathcal{Q}(A) = \mathcal{Q}(B) \oplus D$ and that $A[u, u] = B[u, u]$ for all $u \in \mathcal{Q}(B)$, then $\Lambda_n(B) \le \Lambda_{n+d}(A)$ for all $n \in \mathbb{N}$.*

Proof. It follows from the assumption that any $(n+d)$-dimensional subspace of $\mathcal{Q}(A)$ contains an n-dimensional subspace of $\mathcal{Q}(B)$, and, moreover, any n-dimensional subspace of $\mathcal{Q}(B)$ is recovered in this way. Therefore,

$$\Lambda_{n+d}(A) = \inf_{\substack{G\subset\mathcal{Q}(A)\\ \dim G=n+d}} \sup_{\substack{u\in G\\ u\neq 0}} \frac{A[u,u]}{\|u\|^2} \ge \inf_{\substack{G\subset\mathcal{Q}(A)\\ \dim G=n+d}} \sup_{\substack{S\subset G\cap\mathcal{Q}(B)\\ \dim S=n}} \sup_{\substack{u\in S\\ u\neq 0}} \frac{A[u,u]}{\|u\|^2}$$
$$\ge \inf_{\substack{S\subset\mathcal{Q}(B)\\ \dim S=n}} \sup_{\substack{u\in S\\ u\neq 0}} \frac{A[u,u]}{\|u\|^2} = \inf_{\substack{S\subset\mathcal{Q}(B)\\ \dim S=n}} \sup_{\substack{u\in S\\ u\neq 0}} \frac{B[u,u]}{\|u\|^2} \equiv \Lambda_n(B). \quad \square$$

Furthermore, the following min-max-based eigenvalue estimate will be of use to compare the eigenvalues of operators acting in different spaces. It was introduced and used by Exner and Post in [16, Lemma 2.1] as well as by Post in [54, Lemma 2.2] in a slightly more general setting, but for the sake of completeness we prefer to give a formulation and a full proof adapted to our needs:

Proposition 7 (An estimate using an identification map)**.** *Let $\mathcal{H}$ and $\mathcal{H}'$ be infinite-dimensional Hilbert spaces, B be a non-negative self-adjoint operator with a compact resolvent in $\mathcal{H}$ and B' be a lower semibounded self-adjoint operator in $\mathcal{H}'$. Pick $n \in \mathbb{N}$ and assume there exists a linear (identification) map $J : \mathcal{Q}(B) \to \mathcal{Q}(B')$ and two constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $\varepsilon_1 < 1/\big(1 + E_n(B)\big)$ and that for any $u \in \mathcal{Q}(B)$ there holds*

$$\|u\|^2_{\mathcal{H}} - \|Ju\|^2_{\mathcal{H}'} \le \varepsilon_1\Big(B[u,u] + \|u\|^2_{\mathcal{H}}\Big), \tag{5}$$
$$B'[Ju,Ju] - B[u,u] \le \varepsilon_2\Big(B[u,u] + \|u\|^2_{\mathcal{H}}\Big), \tag{6}$$

then $\Lambda_n(B') \le E_n(B) + \dfrac{\big(E_n(B)\varepsilon_1 + \varepsilon_2\big)\big(1 + E_n(B)\big)}{1 - \big(1 + E_n(B)\big)\varepsilon_1}.$

Proof. We denote $E_n := E_n(B)$ and let F_n be the subspace of $\mathcal{H}$ spanned by the n first eigenfunctions of B. For any $u \in F_n$ with $u \neq 0$ we have $B[u,u]/\|u\|^2 \leq E_n$ due to the min-max principle, and it follows from (5) that

$$\|Ju\|^2 \geq \|u\|^2 - \varepsilon_1\big(B[u,u] + \|u\|^2\big) \geq \big(1 - \varepsilon_1(E_n+1)\big)\|u\|^2,$$

and due to the assumption on ε_1 we have $Ju \neq 0$, and $\dim J(F_n) = \dim F_n = n$. Using now (6) and the positivity of B we estimate

$$\begin{aligned}
\frac{B'[Ju,Ju]}{\|Ju\|^2} &\leq \frac{B[u,u] + \varepsilon_2\big(B[u,u] + \|u\|^2\big)}{\|Ju\|^2} \leq \frac{B[u,u] + \varepsilon_2\big(B[u,u] + \|u\|^2\big)}{\|u\|^2 - \varepsilon_1\big(B[u,u] + \|u\|^2\big)}\\
&= \frac{B[u,u]}{\|u\|^2}\frac{1}{1 - \varepsilon_1\Big(\dfrac{B[u,u]}{\|u\|^2} + 1\Big)} + \varepsilon_2\frac{\dfrac{B[u,u]}{\|u\|^2} + 1}{1 - \varepsilon_1\Big(\dfrac{B[u,u]}{\|u\|^2} + 1\Big)}\\
&= \frac{B[u,u]}{\|u\|^2} + \frac{B[u,u]}{\|u\|^2}\frac{\varepsilon_1\Big(\dfrac{B[u,u]}{\|u\|^2} + 1\Big)}{1 - \varepsilon_1\Big(\dfrac{B[u,u]}{\|u\|^2} + 1\Big)} + \varepsilon_2\frac{\dfrac{B[u,u]}{\|u\|^2} + 1}{1 - \varepsilon_1\Big(\dfrac{B[u,u]}{\|u\|^2} + 1\Big)}\\
&\leq E_n + E_n\frac{\varepsilon_1(E_n+1)}{1 - \varepsilon_1(E_n+1)} + \frac{\varepsilon_2(E_n+1)}{1 - \varepsilon_1(E_n+1)}\\
&= E_n + \frac{(E_n\varepsilon_1 + \varepsilon_2)(E_n+1)}{1 - \varepsilon_1(E_n+1)}.
\end{aligned}$$

Noting that

$$\Lambda_n(B') \leq \sup_{u' \in J(F_n),\, u' \neq 0} \frac{B'[u',u']}{\|u'\|^2} \equiv \sup_{u \in F_n,\, u \neq 0} \frac{B'(Ju,Ju)}{\|Ju\|^2},$$

and using the preceding estimate one arrives at the conclusion. □

2.3. **Distance between closed subspaces.** We will use the well-known notion of a distance between two closed subspaces:

Definition 8 (Distance between subspaces)**.** Let E and F be closed subspaces of a Hilbert space $\mathcal{H}$ and denote by P_E and P_F the orthogonal projectors in $\mathcal{H}$ on E and F respectively. The *distance* $d(E,F)$ *between* E *and* F is defined by

$$d(E,F) := \sup_{x \in E,\, x \neq 0} \frac{\|x - P_F x\|}{\|x\|} \equiv \|P_E - P_F P_E\| \equiv \|P_E - P_E P_F\|.$$

One easily sees that the distance is not symmetric, i.e. $d(E,F) \neq d(F,E)$ in general, but the triangular inequality is satisfied, whose proof we include for completeness:

Lemma 9 (Triangular inequality)**.** *If E, F and G are closed subspaces of a Hilbert space $\mathcal{H}$, then $d(E,G) \leq d(E,F) + d(F,G)$.*

Proof. Using the notation used in Definition 8 we represent

$$\begin{aligned}
P_E - P_G P_E &= (P_E - P_F P_E) + (P_F - P_G P_F)P_E - P_G(P_E - P_F P_E)\\
&= (1 - P_G)(P_E - P_F P_E) + (P_F - P_G P_F)P_E.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(E,G) \equiv \|P_E - P_G P_E\| &\leq \|1 - P_G\| \cdot \|P_E - P_F P_E\| + \|P_F - P_G P_F\| \cdot \|P_E\|\\
&= \|P_E - P_F P_E\| + \|P_F - P_G P_F\| = d(E,F) + d(F,G).
\end{aligned}$$ □

Furthermore, we will need the following result due to Helffer and Sjöstrand [29, Proposition 2.5] allowing to estimate the distance between two subspaces in a special case. We provide its short proof to have a self-contained presentation:

Proposition 10 (An estimate for the distance between subspaces)**.** *Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and $I \subset \mathbb{R}$ be a compact interval. For some $n \in \mathbb{N}$ let $\mu_1, ..., \mu_n \in I$ and $\psi_1, ..., \psi_n \in \mathcal{D}(A)$ be linearly independent vectors, then we denote*

$$\varepsilon := \max_{j\in\{1,...,n\}} \big\|(A-\mu_j)\psi_j\big\|, \quad \eta := \tfrac{1}{2}\operatorname{dist}\big(I, (\operatorname{spec} A)\backslash I\big),$$
$$\lambda := \text{ the smallest eigenvalue of the Gram matrix } \big(\langle\psi_j, \psi_k\rangle\big)_{j,k\in\{1,...,n\}}.$$

If $\eta > 0$, then the distance $d(E, F)$ between the subspaces

$$E := \operatorname{span}\{\psi_1, ..., \psi_n\}, \quad F := \text{the spectral subspace associated with } A \text{ and } I$$

satisfies $d(E, F) \le \dfrac{\varepsilon}{\eta}\sqrt{\dfrac{n}{\lambda}}$.

Proof. Let $I =: (a, b)$. For $R > 0$, we denote by Υ_R the boundary of the rectangle $(a-\eta, b+\eta) + \mathrm{i}(-R, R) \subset \mathbb{C}$ oriented in the anti-clockwise direction. By assumption, none of μ_j and no point of $\operatorname{spec} A$ belong to Υ_R, while I lies in the interior of Υ_R. The orthogonal projector P_F on F is the spectral projector of A on I, hence,

$$P_F = \frac{1}{2\pi\mathrm{i}}\int_{\Upsilon_R}(A-z)^{-1}\,\mathrm{d}z.$$

Denote $r_j := (A-\mu_j)\psi_j$, then for $\lambda \in \Upsilon_R$ one has $(A-z)\psi_j = (\mu_j - z)\psi_j + r_j$,

$$(A-z)^{-1}\psi_j = \frac{1}{\mu_j - z}\,\psi_j - \frac{1}{\mu_j - z}\,(A-z)^{-1}r_j,$$

and the substitution into the above formula for P_F gives

$$P_F\psi_j = \frac{1}{2\pi\mathrm{i}}\int_{\Upsilon_R}\frac{\psi_j}{\mu_j - z}\,\mathrm{d}z - \frac{1}{2\pi\mathrm{i}}\int_{\Upsilon_R}\frac{(A-z)^{-1}r_j}{\mu_j - z}\,\mathrm{d}z \equiv \psi_j - \frac{1}{2\pi\mathrm{i}}\int_{\Upsilon_R}\frac{(A-z)^{-1}r_j}{\mu_j - z}\,\mathrm{d}z.$$

Now we would like to pass to the limit $R \to +\infty$. For z lying on the horizontal parts $L_R^\pm := (a-\eta, b+\eta) \pm \mathrm{i}R$ of Υ_R one has

$$\Big\|\frac{(A-z)^{-1}r_j}{\mu_j - z}\Big\| \le \frac{\|r_j\|}{|\mu_j - z|\,\operatorname{dist}(z, \operatorname{spec} A)} \le \frac{\varepsilon}{R^2}, \qquad \frac{1}{2\pi\mathrm{i}}\int_{L_R^\pm}\frac{(A-z)^{-1}r_j}{\mu_j - z}\,\mathrm{d}z \to 0,$$

and in the formula for $P_F\psi_j$ one keeps the integration on the vertical parts of Υ_R only,

$$P_F\psi_j = \psi_j - \frac{1}{2\pi\mathrm{i}}\bigg(\int_{b+\eta-\mathrm{i}\infty}^{b+\eta+\mathrm{i}\infty} - \int_{a-\eta-\mathrm{i}\infty}^{a-\eta+\mathrm{i}\infty}\bigg)\frac{(A-z)^{-1}r_j}{\mu_j - z}\,\mathrm{d}z. \tag{7}$$

For $\gamma \in \{a-\eta, b+\eta\}$ and $z = \gamma + \mathrm{i}t$ with $t \in \mathbb{R}$ one has

$$\Big\|\frac{(A-z)^{-1}r_j}{\mu_j - z}\Big\| \le \frac{\|r_j\|}{|\mu_j - z|\,\operatorname{dist}(z, \operatorname{spec} A)} \le \frac{\varepsilon}{\sqrt{\eta^2+t^2}\,\sqrt{\eta^2+t^2}} \equiv \frac{\varepsilon}{\eta^2+t^2},$$

and the substitution into (7) gives

$$\|P_F\psi_j - \psi_j\| \le \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varepsilon}{\eta^2+t^2}\,\mathrm{d}t = \frac{\varepsilon}{\eta}.$$

Now let $\psi \in E$ be arbitrary, then $\psi = \sum_{j=1}^n \xi_j \psi_j$ with some constants ξ_j, while $\|\psi\|^2 = \langle \psi, \psi \rangle \geq \lambda(|\xi_1|^2 + \cdots + |\xi_n|^2)$, and, using the Cauchy-Schwarz inequality,

$$\|P_F\psi - \psi\| \leq \sum_{j=1}^n |\xi_j| \cdot \|P_F\psi_j - \psi_j\| \leq \frac{\varepsilon}{\eta} \sum_{j=1}^n |\xi_j| \leq \frac{\varepsilon}{\eta} \sqrt{\sum_{j=1}^n |\xi_j|^2} \sqrt{\sum_{j=1}^n 1} \leq \frac{\varepsilon}{\eta} \sqrt{\frac{n}{\lambda}} \|\psi\|. \qquad \square$$

2.4. **Sobolev spaces and Laplacians with mixed boundary conditions.** We will need a special form of some inequalities in Sobolev spaces. The following result is well known, see e.g. the book by Grisvard [24, Theorem 1.5.1.10]:

Lemma 11 (Sobolev trace inequality). *Let $U \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary ∂U, then there exists a constant $c > 0$ such that*

$$\int_{\partial U} u^2 \, \mathrm{d}s \leq c\Big(\varepsilon \int_U |\nabla u|^2 \, \mathrm{d}x + \frac{1}{\varepsilon} \int_U u^2 \, \mathrm{d}x\Big) \text{ for all } u \in H^1(U) \text{ and } \varepsilon \in (0, 1].$$

In what follows we will deal with numerous Laplacians with various combinations of boundary conditions. In order to simplify the writing, we introduce the following definition:

Definition 12 (Laplacians with mixed boundary conditions). Let $U \subset \mathbb{R}^d$ be an open set and Γ_D, Γ_N, Γ_R be disjoint subsets of ∂U such that $\overline{\Gamma_D \cup \Gamma_N \cup \Gamma_R} = \partial U$. In addition, let $\alpha \in \mathbb{R}$, then by the Laplacian in Ω with the Dirichlet boundary condition at Γ_D, the Neumann boundary condition at Γ_N and the α-Robin boundary condition at Γ_R we mean the self-adjoint operator A in $L^2(U)$ with

$$A[u, u] = \int_U |\nabla u|^2 \, \mathrm{d}x - \alpha \int_{\Gamma_R} |u|^2 \, \mathrm{d}s, \quad \mathcal{Q}(A) = \Big\{u \in H^1(U) : \, u = 0 \text{ at } \Gamma_D\Big\},$$

where $\mathrm{d}s$ is the $(d-1)$-dimensional Hausdorff measure on ∂U, provided that the above expression defines a closed semibounded from below sesquilinear form (which is the case for the bounded Lipschitz domains U due to Lemma 11). Informally, the operator A acts then as $u \mapsto -\Delta u$ on suitably regular functions u in U satisfying $u = 0$ at Γ_D, $\partial_\nu u = 0$ at Γ_N, $\partial_\nu u = \alpha u$ at Γ_R, where ∂_ν stands for the outer normal derivative.

Corollary 13 (Generic lower bound for Robin Laplacians). *Let $U \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. For $\alpha > 0$, let R^U_α be the Laplacian in U with the α-Robin boundary condition at the whole boundary, then there exists $C > 0$ such that $R^U_\alpha \geq -C\alpha^2$ for α sufficiently large.*

Proof. Using Lemma 11 for $\varepsilon := 1/(c\alpha)$ one arrives at the result with $C = c^2$. $\square$

We will need a variant of the above inequalities for scaled domains:

Corollary 14 (Sobolev inequality on scaled domains). *Let $U \subset \mathbb{R}^d$ be a bounded Lipschitz domain, then there exists $c > 0$ such that*

$$\int_{\partial(tU)} f^2 \, \mathrm{d}s \leq c\Big(t\varepsilon \int_{tU} |\nabla f|^2 \, \mathrm{d}x + \frac{1}{t\varepsilon} \int_{tU} f^2 \, \mathrm{d}x\Big) \quad \text{for all } t > 0, \ f \in H^1(tU), \ \varepsilon \in (0, 1].$$

Proof. For $f \in L^2(tU)$ denote by $f_t \in L^2(U)$ the function given by $f_t(x) = f(tx)$, then $f \in H^1(tU)$ if and only if $f_t \in H^1(U)$. Using the Sobolev inequality (Lemma 11) for the fixed domain U we see that there is a constant $c > 0$ such that

$$\int_{\partial U} f_t^2 \, \mathrm{d}s \leq c\Big(\varepsilon \int_U |\nabla f_t|^2 \, \mathrm{d}x + \frac{1}{\varepsilon} \int_U f_t^2 \, \mathrm{d}x\Big) \text{ for all } f \in H^1(tU), \ \varepsilon \in (0, 1]. \tag{8}$$

and using the change of variables $x = y/t$ one easily obtains

$$\int_{\partial U} f_t^2 \, \mathrm{d}s = \int_{\partial U} f(tx)^2 \, \mathrm{d}s = t^{1-d} \int_{\partial(tU)} f(y)^2 \, \mathrm{d}s,$$

$$\int_U |\nabla f_t|^2\,\mathrm{d}x = \int_{\partial U} t^2 \big|(\nabla f)(tx)\big|^2\,\mathrm{d}x = t^{2-d}\int_{tU} |\nabla f(y)|^2\,\mathrm{d}y,$$
$$\int_U f_t^2\,\mathrm{d}x = \int_U f(tx)^2\,\mathrm{d}x = t^{-d}\int_U f(y)^2\,\mathrm{d}y.$$

The substitution of these three equalities into (8) gives the result. □

Corollary 15 (Robin Laplacians on scaled domains)**.** *Let $U \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and $t > 0$. For $\alpha > 0$, let R_α^{tU} be the Laplacian in tU with the α-Robin boundary condition at the whole boundary, then there exists $C > 0$ such that $R_\alpha^{tU} \geq -C\alpha^2$ for αt sufficiently large.*

Proof. We continue using the notation of Corollary 14, then

$$\begin{aligned} R_\alpha^{tU}[f,f] &= \int_{tU} |\nabla f|^2\,\mathrm{d}x - \alpha\int_{\partial(tU)} f^2\,\mathrm{d}s \\ &\geq (1 - c\alpha t\varepsilon)\int_{tU} |\nabla f|^2\,\mathrm{d}x - \frac{c\alpha}{t\varepsilon}\int_U f^2\,\mathrm{d}x, \quad f \in H^1(tU),\ \varepsilon \in (0,1]. \end{aligned}$$

Hence, taking $\varepsilon := 1/(c\alpha t)$ we arrive at $R_\alpha^{tU} \geq -c^2\alpha^2$. □

Remark 16 (Dirichlet-Neumann bracketing)**.** A number of estimates for Laplacians with mixed boundary conditions can be obtained using the min-max principle, which will be our main tool in this work. Let us mention explicitly the most typical situations, which will be used on a permanent basis.

(a) Monotonicity: Let $U \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz boundary and Γ_D, Γ_N, Γ_R be disjoint subsets of ∂U such that $\overline{\Gamma_D \cup \Gamma_N \cup \Gamma_R} = \partial U$, and let L be the Laplacian in U with the α-Robin boundary conditions at Γ_R with some $\alpha > 0$, the Neumann boundary condition at Γ_N and the Dirichlet boundary condition at Γ_D. Furthermore, let Γ'_D, Γ'_N, Γ'_R be another decomposition of ∂U and L' be the associated Laplacian with the respective boundary condition. If $\Gamma'_R \subset \Gamma_R$ and $\Gamma'_D \supset \Gamma_D$, then for all $n \in \mathbb{N}$ one has $E_n(L) \leq E_n(L')$. Indeed, we are in the situation of Corollary 5 with J defined as the identity: one has $\mathcal{Q}(L') \subset \mathcal{Q}(L)$ due to $\Gamma'_D \supset \Gamma_D$, and for $u \in \mathcal{Q}(L')$ one has

$$L[u,u] = L'[u,u] - \alpha\int_{\Gamma_R \setminus \Gamma'_R} u^2\,\mathrm{d}s \leq L'[u,u].$$

(b) Dirichlet bracketing: Let $U \subset \mathbb{R}^d$ be an open set and Γ_D, Γ_N, Γ_R be disjoint subsets of ∂U such that $\overline{\Gamma_D \cup \Gamma_N \cup \Gamma_R} = \partial U$, and let L be the Laplacian in U with the α-Robin boundary conditions at Γ_R with some $\alpha > 0$, the Neumann boundary condition at Γ_N and the Dirichlet boundary condition at Γ_D. Furthermore, let $U' \subset U$ be a bounded Lipschitz domain. Set $\Gamma'_R := \Gamma_R \cap \partial U'$ and $\Gamma'_N := \Gamma_N \cap \partial U'$ and let L' be the Laplacian in U' with the α-Robin boundary condition on Γ'_R, the Neumann boundary condition on Γ'_N and the Dirichlet boundary condition at the rest of the boundary, then for all $n \in \mathbb{N}$ one has $\Lambda_n(L) \leq E_n(L')$. Indeed, this case is covered by Corollary 5, it is sufficient to define $J : \mathcal{Q}(L') \to \mathcal{Q}(L)$ as the extension by zero, then the assumption are satisfied for $A := L'$ and $A' := L$.

(c) Neumann bracketing: Let $U \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz boundary and Γ_D, Γ_N, Γ_R be disjoint subsets of ∂U such that $\overline{\Gamma_D \cup \Gamma_N \cup \Gamma_R} = \partial U$, and let L be the Laplacian in U with the α-Robin boundary conditions at Γ_R with some $\alpha > 0$, the Neumann boundary condition at Γ_N and the Dirichlet boundary condition at Γ_D. Furthermore, let $\Sigma \subset U$ be a Lipschitz curve such that $U \setminus \Sigma$ is the disjoint union of two open sets U_1 and U_2, both with Lipschitz boundaries. Denote by L_j the Laplacians in U_j with the α-Robin boundary conditions at $\Gamma_R \cap \partial U_j$, the Dirichlet boundary conditions at $\Gamma_D \cap \partial U_j$ and the Neumann boundary conditions at the rest of the boundaries, then for all $n \in \mathbb{N}$ one has $E_n(L) \geq E_n(L_1 \oplus L_2)$. Indeed, we are again in the situation of Corollary 5: the sesquilinear form of L is the restriction of the sesquilinear form of $L_1 \oplus L_2$, the functions matching at both sides of Σ, so the assumptions are satisfied by the identity map $J : \mathcal{Q}(L) \to \mathcal{Q}(L_1 \oplus L_2)$.

2.5. **One-dimensional model operators.** Let us discuss some spectral properties of Laplacians on finite intervals with a combination of boundary conditions. Some estimates already appeared in earlier papers on Robin laplacians, we prefer to provide full details, as the computations are very elementary.

Proposition 17 (Robin-Dirichlet Laplacians on a finite interval)**.** *For $\delta > 0$ and $\alpha > 0$, let L_D be the Laplacian on $(0, \delta)$ with the α-Robin boundary condition at 0 and the Dirichlet boundary condition at δ, then for $\alpha\delta \to +\infty$ there holds $E_1(L_D) = -\alpha^2\big(1 + \mathcal{O}(e^{-\delta\alpha})\big)$ and $E_2(L_D) \geq 0$.*

Proof. Looking for negative eigenvalues $E = -k^2$ with $k > 0$, then using the boundary condition at δ we see that the associated eigenfunction f is of the form $f(t) = \sinh k(\delta - t)$. Substituting into the boundary condition at 0 one obtains $0 = f'(0) + \alpha f(0) = -k \cosh k\delta + \alpha \sinh k\delta$, which then rewrites as

$$F(k\delta) = \alpha\delta, \quad F(x) := x \coth x. \tag{9}$$

One has $F(0) = 1$, $F(+\infty) = +\infty$, and

$$F'(x) = \coth x - \frac{x}{\sinh^2 x} = \frac{\sinh x \, \cosh x - x}{\sinh^2 x} = \frac{\sinh(2x) - 2x}{2 \sinh^2 x} > 0 \text{ for } x > 0,$$

i.e. $F : (0, +\infty) \to (0, +\infty)$ is strictly increasing and bijective, and for $\alpha\delta$ the above equation (9) admits a unique solution, which then satisfies $k\delta \to +\infty$. Remark that this already shows that the second eigenvalue is non-negative. To obtain an asymptotics for k we rewrite (9) as $k = \alpha \tanh(k\delta)$. Due to $k\delta \to +\infty$ we have $\frac{1}{2} \leq \tanh(k\delta) \leq 1$ implying $\alpha/2 \leq k \leq \alpha$. Then using the equation again we have $\alpha \tanh\left(\frac{1}{2}\,\alpha\delta\right) \leq k \leq \alpha$, while one has the elementary asymptotics $\tanh\left(\frac{1}{2}\,\alpha\delta\right) = 1 + \mathcal{O}(e^{-\alpha\delta})$. As $E_1(L_D) = -k^2$, we arrive at the result. ☐

Proposition 18 (Robin-Neumann Laplacians on a finite interval)**.** *For $\delta > 0$, $\alpha > 0$ and $\beta \geq 0$, let L_N denote the Laplacian on $(0, \delta)$ with the α-Robin boundary condition at 0 and the β-Robin boundary condition at δ, then for $\alpha\delta \to +\infty$ and $\beta\delta \to 0^+$ one has $E_1(L_N) = -\alpha^2\big(1+\mathcal{O}(e^{-\alpha\delta})\big)$ and $E_2(L_N) \geq 1/\delta^2$.*

Proof. We start by estimating the second eigenvalue. Let B_β be the Laplacian on $(0, \delta)$ with the Dirichlet boundary condition at 0 and the β-Robin boundary condition at δ, then the sesquilinear form of B_β is a restriction of the sesquilinear form of L_N, and $\mathcal{Q}(L_N) = H^1(0, \delta)$ only differs from $\mathcal{Q}(B_\beta) = \big\{f \in H^1(0, \delta) : f(0) = 0\big\}$ by a one-dimensional subspace. It follows by the min-max principle (Corollary 6) that $E_2(L_N) \geq E_1(B_\beta)$. Now, let us obtain a lower bound for B_β. For $\beta = 0$ one obtains simply the Laplacian on $(0, \delta)$ with the Dirichlet boundary condition at 0 and the Neumann boundary condition at δ, and $E_1(B_0) = \pi^2/(4\delta^2)$. Using Corollary 14 we see that for some $c > 0$ there holds

$$f(\delta)^2 \leq c\Big(\delta \int_0^\delta (f')^2 \, \mathrm{d}t + \frac{1}{\delta} \int_0^\delta f^2 \, \mathrm{d}t\Big) \text{ for all } f \in H^1(0, \delta).$$

It follows that for $f \in \mathcal{Q}(B_\beta) \equiv \mathcal{Q}(B_0)$ one has

$$B_\beta[f, f] = \int_0^\delta (f')^2 \, \mathrm{d}t - \beta f(\delta)^2 \geq (1 - c\beta\delta) \int_0^\delta (f')^2 \, \mathrm{d}t - \frac{c\beta}{\delta} \int_0^\delta f^2 \, \mathrm{d}t, \quad ,$$

and using the min-max principle implies that for $\beta\delta \to 0^+$ one has

$$E_1(B_\delta) \geq (1 - c\beta\delta) E_1(B_0) - \frac{c\beta\delta}{\delta^2} = \frac{(1 - c\beta\delta)\pi^2 - 4c\beta\delta}{4\delta^2} \geq \frac{1}{\delta^2}$$

Now let us study the first eigenvalue. A value $E = -k^2$ with $k > 0$ is an eigenvalue iff one can find $(C_1, C_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that the function $f : x \mapsto C_1 e^{kx} + C_2 e^{-kx}$ belongs to the operator domain. The boundary conditions give

$$\begin{aligned} 0 &= f'(0) + \alpha f(0) = (\alpha + k)C_1 + (\alpha - k)C_2, \\ 0 &= f'(\delta) - \beta f(\delta) = (k - \beta)e^{k\delta}C_1 - (k + \beta)e^{-k\delta}C_2, \end{aligned}$$

and one has a non-zero solution iff the determinant of the system vanishes, i.e. iff k satisfies the equation $(k + \alpha)(k + \beta)e^{-k\delta} = (k - \alpha)(k - \beta)e^{k\delta}$, which we rewrite as

$$g(k) = h(k), \quad g(k) := \frac{k + \alpha}{k - \alpha}, \quad h(k) := \frac{k - \beta}{k + \beta} e^{2k\delta}. \tag{10}$$

Both functions g and h are continuous, and g is strictly decreasing on $(\alpha, +\infty)$ with $g(\alpha^+) = +\infty$ and $g(+\infty)$=1. On the other hand, the function h is strictly increasing in $(\alpha, +\infty)$ being the product of two strictly increasing positive functions (we assume without loss of generality that $\alpha > \beta$), and $h(\alpha^+) = e^{2\alpha\delta}(\alpha - \beta)/(\alpha + \beta) < +\infty$ and $h(+\infty) = +\infty$. Therefore, there exists a unique solution k of (10) with $k \in (\alpha, +\infty)$. To obtain the required estimate we use again the monotonicity of h on $(\alpha, +\infty)$:

$$\frac{k + \alpha}{k - \alpha} = g(k) = h(k) > h(\alpha^+) = \frac{\alpha - \beta}{\alpha + \beta}\, e^{2\alpha\delta} = \frac{\alpha\delta - \beta\delta}{\alpha\delta + \beta\delta}\, e^{2\alpha\delta}.$$

Using the assumptions $\alpha\delta \to +\infty$ and $\beta\delta \to 0^+$ we minorate the last term very roughly by $e^{\alpha\delta}$, which gives

$$\frac{k + \alpha}{k - \alpha} \ge e^{\alpha\delta}, \text{ and then } k \le \alpha\, \frac{1 + e^{-\alpha\delta}}{1 - e^{-\alpha\delta}} = \alpha\big(1 + \mathcal{O}(e^{-\alpha\delta})\big).$$

By combining with $k > \alpha$ we arrive at the sought estimate for $E_1(L_N) = -k^2$. □

2.6. **Robin Laplacians in infinite sectors.** Now, let us recall some basic facts on Robin laplacians in infinite sectors, which will be a starting point for the subsequent analysis of curvilinear polygons.

Definition 19 (Infinite sector $\mathcal{S}_\theta$). Let $\theta \in (0, \pi)$, then by $\mathcal{S}_\theta$ we denote the following infinite sector of opening angle 2θ:

$$\mathcal{S}_\theta = \big\{(x_1, x_2) \in \mathbb{R}^2 : -\theta < \arg(x_1 + \mathrm{i}x_2) < \theta\big\}, \quad 0 < \theta < \pi,$$

see Figure 2 in the introduction. Remark that for $\theta = \pi/2$ one obtains simply the half-plane $\mathbb{R}_+ \times \mathbb{R}$.

The following proposition summarizes the basic properties of the associated Robin Laplacians proved in the paper [33] by Khalile and Pankrashkin:

Proposition 20 (Robin Laplacians $T_{\theta,\alpha}$ in infinite sectors). *For $\theta \in (0, \pi)$ and $\alpha > 0$, let $T_{\theta,\alpha}$ be the Laplacian on $\mathcal{S}_\theta$ with the α-Robin boundary condition at the whole boundary, then:*

- *the operator $T_{\theta,\alpha}$ is well-defined, lower semibounded and is unitarily equivalent to $\alpha^2 T_{\theta,1}$ for all $\theta \in (0, \pi)$ and $\alpha > 0$,*
- $\mathrm{spec}_{\mathrm{ess}}(T_{\theta,\alpha}) = [-\alpha^2, +\infty)$ *for all $\theta \in (0, \pi)$ and $\alpha > 0$,*
- *the discrete spectrum of $T_{\theta,\alpha}$ is non-empty if and only if $\theta < \frac{\pi}{2}$, in particular,*

  $$E_1(T_{\theta,\alpha}) = -\alpha^2/\sin^2\theta \text{ for } \theta \in \big(0, \tfrac{\pi}{2}\big),$$

  and if one denotes

  $$\kappa(\theta) := \text{the number of discrete eigenvalues of } T_{\theta,\alpha},$$

  which is independent of α, then
  - *$\kappa(\theta) < +\infty$ and $\theta \mapsto \kappa(\theta)$ is non-increasing,*
  - *for all $\frac{\pi}{6} \le \theta < \frac{\pi}{2}$ one has $\kappa(\theta) = 1$,*

- *there exist $b>0$ and $B>0$ such that if $n\in\big\{1,\dots,\kappa(\theta)\big\}$ and $\psi_{n,\alpha}$ is an eigenfunction of $T_{\theta,\alpha}$ for the nth eigenvalue, then for any $\alpha>0$ one has the Agmon-type decay estimate*

$$\iint_{\mathcal{S}_\theta} e^{b\alpha|x|}\Big(\frac{1}{\alpha^2}\big|\nabla\psi_{n,\alpha}(x)\big|^2+\psi_{n,\alpha}(x)^2\Big)\,dx\le B\|\psi_{n,\alpha}\|^2_{L^2(\mathcal{S}_\theta)}. \tag{11}$$

Remark that the above properties of $T_{\theta,\alpha}$ are also of relevance for Steklov-type eigenvalue problems in domains with corners, see e.g. the papers by Ivrii [30] and Levitin, Parnovski, Polterovich, Sher [39]. For a subsequent use we give a special name to the eigenvalues of the above operator with $\alpha=1$:

Definition 21 (Eigenvalues $\mathcal{E}_j(\theta)$)**.** For $\theta\in\big(0,\frac{\pi}{2}\big)$ and $n\in\{1,\dots,\kappa(\theta)\big\}$ we denote

$$\mathcal{E}_n(\theta):=E_n(T_{\theta,1}).$$

Remark that due to Proposition 20 one has $\mathcal{E}_1(\theta)=-1/\sin^2\theta$ and

$$\mathcal{E}_n(\theta)<-1,\quad E_n(T_{\theta,\alpha})=\mathcal{E}_n(\theta)\,\alpha^2<-\alpha^2,\quad \theta\in\Big(0,\frac{\pi}{2}\Big),\ n\in\big\{1,\dots,\kappa(\theta)\big\},\ \alpha>0.$$

3. Analysis in truncated convex sectors

3.1. **Robin Laplacians in truncated convex sectors.** Recall that the infinite sectors $\mathcal{S}_\theta$ are defined above in Definition 19. Let us introduce their truncated versions.

Definition 22 (Truncated convex sector $\mathcal{S}^r_\theta$)**.** Let $\theta\in\big(0,\frac{\pi}{2}\big)$ and $r>0$. Consider the points $A^\pm_r=r(\cos\theta,\pm\sin\theta)\in\partial\mathcal{S}_\theta$ and $B_r=r\big(1/\cos\theta,0\big)\in\mathcal{S}_\theta$, and denote by $\mathcal{S}^r_\theta$ the interior of the quadrangle $OA^+_rB_rA^-_r$ (remark that the sides $B_rA^\pm_r$ are orthogonal to $\partial\mathcal{S}_\theta$ at $A^\pm_r$, see Fig. 4). We will distinguish between two parts of the boundary of $\mathcal{S}^r_\theta$, namely, we set

$$\partial_*\mathcal{S}^r_\theta:=\partial\mathcal{S}^r_\theta\cap\partial\mathcal{S}_\theta:=\text{polygonal chain } A^+_rOA^-_r,$$
$$\partial_{\text{ext}}\mathcal{S}^r_\theta:=\partial\mathcal{S}^r_\theta\setminus\partial_*\mathcal{S}^r_\theta:=\text{polygonal chain } A^+_rB_rA^-_r.$$

In what follows we will need some properties of three operators associated with $\mathcal{S}^r_\theta$, namely, for $\theta\in\big(0,\frac{\pi}{2}\big)$, $\alpha>0$ and $r>0$ we introduce:

$$\begin{aligned}
D^r_{\theta,\alpha}:=&\text{ the Laplacian in } \mathcal{S}^r_\theta \text{ with the } \alpha\text{-Robin boundary condition at } \partial_*\mathcal{S}^r_\theta\\
&\text{ and the \textit{Dirichlet} boundary condition at } \partial_{\text{ext}}\mathcal{S}^r_\theta,\\
N^r_{\theta,\alpha}:=&\text{ the Laplacian in } \mathcal{S}^r_\theta \text{ with the } \alpha\text{-Robin boundary condition at } \partial_*\mathcal{S}^r_\theta\\
&\text{ and the \textit{Neumann} boundary condition at } \partial_{\text{ext}}\mathcal{S}^r_\theta,\\
R^r_{\theta,\alpha}:=&\text{ the Laplacian in } \mathcal{S}^r_\theta \text{ with the } \alpha\text{-Robin boundary condition \textit{at the}}\\
&\text{\textit{ whole boundary.}}
\end{aligned}\tag{12}$$

We remark that $N^r_{\theta,\alpha}$ will play a key role in the subsequent considerations (in particular, see Subsection 3.2), while the other two operators will be used mostly for auxiliary constructions. The following properties of the three operators are easily established by a standard routine computation:

Lemma 23 (Scaling and truncated sectors)**.** *For $t,r>0$ denote by Ξ_t the unitary operators (dilations) $\Xi_t:L^2(\mathcal{S}^{tr}_\theta)\to L^2(\mathcal{S}^r_\theta)$, $(\Xi_tu)(x)=t\,u(tx)$. Let $X^r_{\theta,\alpha}$ be any of the three operators $D^r_{\theta,\alpha}$, $N^r_{\theta,\alpha}$, $R^r_{\theta,\alpha}$, then $X^r_{\theta,t\alpha}\Xi_t=t^2\Xi_tX^{tr}_{\theta,\alpha}$, which then gives the eigenvalue identities*

$$E_n(X^r_{\theta,\alpha})=\alpha^2E_n(X^{\alpha r}_{\theta,1}) \text{ for all } n\in\mathbb{N}. \tag{13}$$

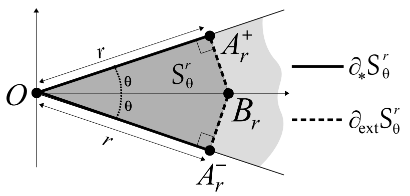


Figure 4. The truncated sector $\mathcal{S}^r_\theta$ is shaded (see Definition 22). The part of the boundary $\partial_*\mathcal{S}^r_\theta$ is indicated by the thick solid line, and the part of the boundary $\partial_{\text{ext}}\mathcal{S}^r_\theta$ is shown as the thick dashed line.

Let us show that in a suitable asymptotic regime the lowest eigenvalues of the Robin-Dirichlet Laplacians $D^r_{\theta,\alpha}$ are close to the Robin eigenvalues of the associated infinite sectors:

Lemma 24 (First eigenvalues of the Robin-Dirichlet Laplacian $D^r_{\theta,\alpha}$)**.** *For some $c > 0$ one has*

$$E_n(D^r_{\theta,\alpha}) = E_n(T_{\theta,\alpha}) + \mathcal{O}(\alpha^2 e^{-c\alpha r}) \equiv \alpha^2\big(\mathcal{E}_n(\theta) + \mathcal{O}(e^{-c\alpha r})\big) \text{ for } n \in \big\{1,\dots,\kappa(\theta)\big\}$$

and $E_{\kappa(\theta)+1}(D^r_{\theta,\alpha}) \geq -\alpha^2$ as $\alpha r \to +\infty$.

Proof. The result is quite standard and is based on the fact that the Robin eigenfunctions of the infinite sectors satisfy an Agmon-type estimate at infinity, but we provide a proof for the sake of completeness. In view of the above scaling (13) it is sufficient to study the case $\alpha = 1$ and $r \to +\infty$. Recall that

$$D^r_{\theta,1}[u,u] = \iint_{\mathcal{S}^r_\theta} |\nabla u|^2 \,\mathrm{d}x - \int_{\partial_* \mathcal{S}^r_\theta} u^2 \,\mathrm{d}s, \quad \mathcal{Q}(D^r_{\theta,1}) = \big\{H^1(\mathcal{S}^r_\theta) : u = 0 \text{ on } \partial_{\mathrm{ext}}\mathcal{S}^r_\theta\big\},$$

$$T_{\theta,1}[u,u] = \iint_{\mathcal{S}_\theta} |\nabla u|^2 \,dx - \int_{\partial\mathcal{S}_\theta} u^2 \,\mathrm{d}s, \quad \mathcal{Q}(T_{\theta,1}) = H^1(\mathcal{S}_\theta).$$

The Dirichlet bracketing argument (see Remark 16) implies $E_n(D^r_{\theta,1}) \geq \Lambda_n(T_{\theta,1})$ for any $r > 0$ and $n \in \mathbb{N}$. For $n \in \big\{1,\dots,\kappa(\theta)\big\}$ one has $\Lambda_n(T_{\theta,1}) = E_n(T_{\theta,1}) \equiv \mathcal{E}_n(\theta)$, while $\Lambda_{\kappa(\theta)+1}(T_{\theta,1}) = \inf \mathrm{spec}_{\mathrm{ess}}\, T_{\theta,1} = -1$. This proves the required lower bounds.

To prove the upper bound, let us pick $n \in \big\{1,\dots,\kappa(\theta)\big\}$ and let ψ_j, $j = 1,\dots,n$, be eigenfunctions of the operator $T_{\theta,1}$ in the infinite sector corresponding to the n first eigenvalues and chosen to form an orthonormal family, i.e.

$$\langle \psi_j, \psi_k\rangle_{L^2(\mathcal{S}_\theta)} = \delta_{j,k}, \quad T_{\theta,1}[\psi_j,\psi_k] = \mathcal{E}_j(\theta)\,\delta_{j,k}, \quad j,k = 1,\dots,n.$$

Let $\chi_0, \chi_1 : \mathbb{R} \to [0,1]$ be smooth functions such that $\chi_0 = 1$ in $\big(-\infty, \frac{1}{2}\big]$, $\chi_0 = 0$ in $[1,\infty)$ and $\chi_0^2 + \chi_1^2 = 1$. We define $\chi^r_j : \mathbb{R}^2 \to \mathbb{R}$ by $\chi^r_j(x) = \chi_j\big(|x|/r\big)$, $j = 0,1$, and $\psi^r_j : \mathcal{S}_\theta \to \mathbb{R}$ by $\psi^r_j := \chi^r_0 \psi_j$, $j = 1,\dots,n$, and keep the same symbols for the restrictions of these functions to $\mathcal{S}^r_\theta$. Remark that the functions ψ^r_j belong to $H^1(\mathcal{S}^r_\theta)$ and vanish at $\partial_{\mathrm{ext}}\mathcal{S}^r_\theta$, i.e. they belong to $\mathcal{Q}(D^r_{\theta,1})$ and can be used to estimate the Rayleigh quotients. Let us now use the Agmon-type estimate (11) with suitable $b > 0$ and $B > 0$ for the eigenfunctions ψ_j. Denote

$$C^r_{j,k} := \iint_{\mathcal{S}_\theta} (\chi^r_1)^2 \psi_j \psi_k \,dx,$$

then $|C^r_{j,k}| \leq \frac{1}{2}\,(C^r_{j,j} + C^r_{k,k})$ and

$$C^r_{j,j} = \iint_{\mathcal{S}_\theta} (\chi^r_1)^2 \psi_j^2 \,dx \leq \iint_{\mathcal{S}_\theta : |x| > r/2} \psi_j^2 \,dx \leq e^{-\frac{br}{2}} \iint_{\mathcal{S}_\theta : |x| > r/2} e^{b|x|} \psi_j^2 \,dx \leq B e^{-\frac{br}{2}}.$$

Therefore, for large r one has $C^r_{j,k} = \mathcal{O}(e^{-cr})$ with $c := \frac{1}{2}\,b$ and

$$\langle \psi^r_j, \psi^r_k\rangle_{L^2(\mathcal{S}^r_\theta)} = \langle \psi_j, \psi_k\rangle_{L^2(\mathcal{S}_\theta)} - C^r_{j,k} = \delta_{j,k} + \mathcal{O}(e^{-cr}).$$

In particular, for large r the functions ψ^r_j are linearly independent. Using similar estimates we obtain

$$\iint_{\mathcal{S}_\theta} \nabla(\chi^r_1 \psi_j)^2 \,dx = \mathcal{O}(e^{-cr}), \quad \iint_{\mathcal{S}^r_\theta} \nabla\psi^r_j \cdot \nabla\psi^r_k \,dx = \iint_{\mathcal{S}_\theta} \nabla\psi_j \cdot \nabla\psi_k \,dx + \mathcal{O}(e^{-cr}).$$

To estimate the quantities

$$G^r_{j,k} := \int_{\partial\mathcal{S}_\theta} (\chi^r_1)^2 \psi_j\, \psi_k \,\mathrm{d}s$$

we remark again that $|G^r_{j,k}| \le \frac{1}{2}\,(G^r_{j,j} + G^r_{k,k})$, and using $\chi^r_1 \psi_j$ as a test function in the inequality $T_{\theta,1} \ge -(\sin\theta)^{-2}$ for the Robin Laplacian in the sector we obtain

$$G^r_{j,j} = \int_{\partial \mathcal{S}_\theta} (\chi^r_1)^2 \psi_j^2 \,\mathrm{d}s \le \iint_{\mathcal{S}_\theta} \nabla(\chi^r_1 \psi_j)^2\,dx + \frac{1}{\sin^2\theta} \iint_{\mathcal{S}_\theta} \chi^r_1 \psi_j^2\,dx = \mathcal{O}(e^{-cr}),$$

which implies $G^r_{j,k} = \mathcal{O}(e^{-cr})$ and

$$\int_{\partial_* S^r_\theta} \psi^r_j \psi^r_k \,\mathrm{d}s = \int_{\partial \mathcal{S}_\theta} \psi_j\,\psi_k\,\mathrm{d}s - G^r_{j,k} = \int_{\partial \mathcal{S}_\theta} \psi_j\,\psi_k\,\mathrm{d}s + \mathcal{O}(e^{-cr}).$$

Denote $L_r := \mathrm{span}(\psi^r_1, \dots, \psi^r_n)$, which is an n-dimensional subspace of $\mathcal{Q}(D^r_{\theta,1})$ for large r. For any function ψ of the form

$$\psi = \xi_1 \psi^r_1 + \dots + \xi_n \psi^r_n \in L_r, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$$

one has, due to the preceding estimates, $\|\psi\|^2_{L^2(\mathcal{S}^r_\theta)} = |\xi|^2\big(1 + \mathcal{O}(e^{-cr})\big)$ and

$$\begin{aligned} D^r_{\theta,1}[\psi,\psi] &= \sum_{j,k=1}^{n} \Big(T_{\theta,1}[\psi_j,\psi_k] \,+ \mathcal{O}(e^{-cr})\Big)\xi_j \xi_k \\ &= \sum_{j,k=1}^{n} \Big(\mathcal{E}_j(\theta)\,\delta_{j,k} + \mathcal{O}(e^{-cr})\Big)\xi_j\xi_k \le \big(\mathcal{E}_n(\theta) + \mathcal{O}(e^{-cr})\big)|\xi|^2, \end{aligned}$$

and an application of the min-max principle gives

$$E_n(D^r_{\theta,1}) \le \sup_{\psi \in L_r,\,\psi\neq 0} \frac{D^r_{\theta,1}[\psi,\psi]}{\|\psi\|^2_{L^2(\mathcal{S}^r_\theta)}} \le \mathcal{E}_n(\theta) + \mathcal{O}(e^{-cr}). \qquad \square$$

In order to obtain an analogous result on the behavior of the first $\kappa(\theta)$ eigenvalues of $N^r_{\theta,\alpha}$ we need a preliminary estimate.

Definition 25 (Polygons $\mathcal{P}^{r,\rho}_\theta$)**.** For $\theta \in \big(0, \frac{\pi}{2}\big)$ and $0 < \rho < r$ we consider the polygons

$$\mathcal{P}^{r,\rho}_\theta := \mathcal{S}^r_\theta \setminus \overline{\mathcal{S}^\rho_\theta} \equiv \text{the hexagon } B_\rho A^+_\rho A^+_r B_r A^-_r A^-_\rho,$$

where one uses the same notation as in the definition of $\mathcal{S}^r_\theta$, see Figures 4 and 5(a). We again split the boundary of $\mathcal{P}^{r,\rho}_\theta$ into two parts by setting

$$\partial_* \mathcal{P}^{r,\rho}_\theta := \partial \mathcal{P}^{r,\rho}_\theta \cap \partial \mathcal{S}_\theta := \text{the union of the segments } [A^\pm_\rho, A^\pm_r], \quad \partial_{\mathrm{ext}} \mathcal{P}^{r,\rho}_\theta := \partial \mathcal{P}^{r,\rho}_\theta \setminus \partial_* \mathcal{P}^{r,\rho}_\theta.$$

Lemma 26 (Robin-Neumann Laplacian $P^{r,\rho}_{\theta,\alpha}$ in $\mathcal{P}^{r,\rho}_\theta$)**.** *Let $P^{r,\rho}_{\theta,\alpha}$ denote the Laplacian in $\mathcal{P}^{r,\rho}_\theta$ with the α-Robin boundary condition at $\partial_* \mathcal{P}^{r,\rho}_\theta$ and the Neumann boundary condition at $\partial_{\mathrm{ext}} \mathcal{P}^{r,\rho}_\theta$, then $E_1(P^{r,\rho}_{\theta,\alpha}) \ge -\alpha^2\big(1 + \mathcal{O}(e^{-\alpha\rho\tan\theta})\big)$ as $\alpha\rho$ tends to $+\infty$.*

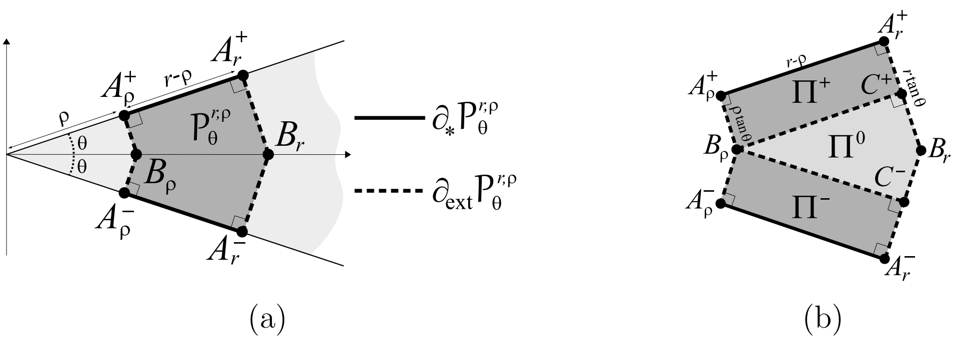


FIGURE 5. (a) Polygon $\mathcal{P}^{r,\rho}_\theta$, see Definition 25. (b) Decomposition of $\mathcal{P}^{r,\rho}_\theta$ for the proof of Lemma 26. The solid/dashed lines correspond to Robin/Neumann boundary conditions.

Proof. Denote for shortness $P := P^{r,\rho}_{\theta,\alpha}$. Let us decompose the polygon $\mathcal{P}^{r,\rho}_{\theta}$ as shown in Figure 5(b). Namely, let $C^{\pm}$ be orthogonal projection of B_ρ on the segment $[A^{\pm}_r, B_r]$ and let U be the domain obtained from $\mathcal{P}^{r,\rho}_{\theta}$ by taking out the segments $[B_\rho, C^{\pm}]$, then U is the disjoint union of two rectangles $\Pi^{\pm} := B_\rho A^{\pm}_\rho A^{\pm}_r C^{\pm}$ and the quadrangle $\Pi^0 := B_\rho C^+ B_r C^-$. Let Λ be the Laplacian in U with the α-Robin boundary condition on $\partial_*\mathcal{P}^{r,\rho}_{\theta} \subset \partial U$ and the Neumann boundary condition at the remaining boundary. Due to the Neumann bracketing argument (see Remark 16) one has $E_1(P) \geq E_1(\Lambda)$. Hence, it is sufficient to show the sought lower bound for $E_1(\Lambda)$.

The operator Λ is the direct sum $\Lambda_0 \oplus \Lambda_+ \oplus \Lambda_-$ with Λ_j acting in $L^2(\Pi^j)$. Namely, Λ_0 is just the Neumann Laplacian in Π^0, and, therefore, $E_1(\Lambda_0) = 0$. Furthermore, $\Lambda^{\pm}$ are Laplacians in the rectangles $\Pi^{\pm}$ with the α-Robin boundary conditions on the sides $A^{\pm}_\rho A^{\pm}_r$ and the Neumann boundary condition at the remaining boundary. Therefore, they admit a separation of variables and are both unitary equivalent to $L_N \otimes 1 + 1 \otimes T$, where the operator L_N is the Laplacian on $(0, \rho\tan\theta)$ with the α-Robin boundary condition at 0 and the Neumann boundary condition at $\rho\tan\theta$ and T is the Neumann Laplacian on $(0, r-\rho)$. Therefore, $E_1(\Lambda^{\pm}) = E_1(L_N) + E_1(T) = E_1(L_N)$. The operator L_N is covered by Proposition 18 (with $\beta = 0$), and $E_1(L_N) = -\alpha^2\big(1 + \mathcal{O}(e^{-\alpha\rho\tan\theta})\big) < 0$ for $\alpha\rho \to +\infty$. Therefore, for $\alpha\rho \to +\infty$ one has

$$
\begin{aligned}
E_1(P) \geq E_1(\Lambda) = E_1(\Lambda_0 \oplus \Lambda_+ \oplus \Lambda_-) &= \min_{j\in\{0,+,-\}} E_1(\Lambda^j) \\
&= E_1(\Lambda^+) = E_1(L_N) = -\alpha^2\big(1 + \mathcal{O}(e^{-\alpha\rho\tan\theta})\big). \quad \square
\end{aligned}
$$

Lemma 27 (First eigenvalues of the Robin-Neumann Laplacian $N^r_{\theta,\alpha}$)**.** *For $\alpha r \to +\infty$ there holds*

$$
\begin{aligned}
E_n(N^r_{\theta,\alpha}) &= \Big[\mathcal{E}_n(\theta) + \mathcal{O}\Big(\frac{1}{(\alpha r)^2}\Big)\Big]\,\alpha^2, \quad n \in \big\{1, \dots, \kappa(\theta)\big\}, \\
E_{\kappa(\theta)+1}(N^r_{\theta,\alpha}) &\geq -\alpha^2 + o(\alpha^2).
\end{aligned}
$$

Proof. The standard monotonicity argument (see Remark 16) shows that an upper bound for the eigenvalues of $N^r_{\theta,\alpha}$ are the eigenvalues of $D^r_{\theta,\alpha}$. Hence, the upper bound for $E_n(N^r_{\theta,\alpha})$ follows from the upper bound for the eigenvalues of $D^r_{\theta,\alpha}$ obtained in Lemma 24 above.

Let us pass to the proof of the lower bound. Let $\chi_0, \chi_1 : \mathbb{R} \to [0,1]$ be smooth functions such that $\chi_0 = 1$ in $\big(-\infty, \frac{1}{2}\big]$, $\chi_0 = 0$ in $[1, \infty)$ and $\chi_0^2 + \chi_1^2 = 1$. We define $\chi^r_j : \mathbb{R}^2 \to \mathbb{R}$ by $\chi^r_j(x) = \chi_j\big(|x|/r\big)$, $j = 0, 1$. Recall that

$$
N^r_{\theta,\alpha}[u,u] = \iint_{\mathcal{S}^r_\theta} |\nabla u|^2 \,\mathrm{d}x - \alpha \int_{\partial_*\mathcal{S}^r_\theta} u^2 \,\mathrm{d}s, \quad \mathcal{Q}(N^r_{\theta,\alpha}) = H^1(\mathcal{S}^r_\theta),
$$

and by direct computation for any $u \in \mathcal{Q}(N^r_{\theta,\alpha})$ one has

$$
\begin{aligned}
N^r_{\theta,\alpha}[u,u] &= N^r_{\theta,\alpha}[\chi^r_0 u, \chi^r_0 u] + N^r_{\theta,\alpha}[\chi^r_1 u, \chi^r_1 u] - \iint_{\mathcal{S}^r_\theta} \big(|\nabla\chi^r_0|^2 + |\nabla\chi^r_1|^2\big) u^2 \,dx \\
&\geq N^r_{\theta,\alpha}[\chi^r_0 u, \chi^r_0 u] + N^r_{\theta,\alpha}[\chi^r_1 u, \chi^r_1 u] - \frac{a}{r^2}\,\|u\|^2_{L^2(\mathcal{S}^r_\theta)}, \quad a := \|\chi_0'\|^2_\infty + \|\chi_1'\|^2_\infty. \quad (14)
\end{aligned}
$$

One has $\chi^r_0 u \in H^1(\mathcal{S}^r_\theta)$ and $\chi^r_0 u = 0$ at $\partial_{\rm ext}\mathcal{S}^r_\theta$. At the same time, the function $\chi^r_1 u$ vanishes inside the disk $|x| \leq \frac{1}{2}r$ and, hence, is supported in the quadrangle $\mathcal{P}^{r,\rho}_\theta$ with $\rho := \frac{1}{2} r\cos\theta$ and belongs to $H^1(\mathcal{P}^{r,\rho}_\theta)$. Therefore, one has $\chi^r_0 u \in \mathcal{Q}(D^r_{\theta,\alpha})$ and $\chi^r_1 u \in \mathcal{Q}(P^{r,\rho}_{\theta,\alpha})$, and the inequality (14) rewrites as

$$
N^r_{\theta,\alpha}[u,u] \geq D^r_{\theta,\alpha}[\chi^r_0 u, \chi^r_0 u] + P^{r,\rho}_{\theta,\alpha}[\chi^r_1 u, \chi^r_1 u] - (a/r^2)\|u\|^2_{L^2(\mathcal{S}^r_\theta)},
$$

and we recall that $\|u\|^2_{L^2(\mathcal{S}^r_\theta)} = \|\chi^r_0 u\|^2_{L^2(\mathcal{S}^r_\theta)} + \|\chi^r_0 u\|^2_{L^2(\mathcal{P}^{r,\rho}_\theta)}$. In other words, if one defines $J : \mathcal{Q}(N^r_{\theta,\alpha}) \to \mathcal{Q}(D^r_{\theta,\alpha} \oplus P^{r,\rho}_{\theta,\alpha})$ by $Ju = (\chi^r_0 u, \chi^r_1 u)$, then

$$\|Ju\| = \|u\|, \quad (N^r_{\theta,\alpha} + a/r^2)[u,u] \geq (D^r_{\theta,\alpha} \oplus P^{r,\rho}_{\theta,\alpha})[Ju, Ju],$$

and the min-max principle (Corollary 5) gives the eigenvalue inequalities

$$E_n(N^r_{\theta,\alpha}) \geq E_n(D^r_{\theta,\alpha} \oplus P^{r,\rho}_{\theta,\alpha}) - a/r^2, \quad r > 0,\ n \in \mathbb{N}. \tag{15}$$

Now let us pick $n \in \{1, \dots, \kappa(\theta)\}$ and consider the regime $\alpha r \to +\infty$. Then one also has $\alpha\rho \to +\infty$, and the estimate of Lemma 26 for the first eigenvalue of $P^{r,\rho}_{\theta,\alpha}$ gives $E_1(P^{r,\rho}_{\theta,\alpha}) \geq -\alpha^2 + o(\alpha^2)$. On the other hand, the estimate of Lemma 24 for the first eigenvalues of $D^r_{\theta,\alpha}$ shows that $E_n(D^R_{\theta,\alpha}) = \alpha^2\big(\mathcal{E}_n(\theta) + \mathcal{O}(e^{-c\alpha r})\big)$ with some $c > 0$, which is smaller than $-\alpha^2 + o(\alpha^2)$ due to the inequality $\mathcal{E}_n(\theta) < -1$. Hence,

$$E_n(D^r_{\theta,\alpha} \oplus P^{r,\rho}_{\theta,\alpha}) = E_n(D^r_{\theta,\alpha}) = \alpha^2\big(\mathcal{E}_n(\theta) + \mathcal{O}(e^{-c\alpha r})\big).$$

Substituting this last estimate into the inequality (15) one arrives to

$$E_n(N^r_{\theta,\alpha}) \geq \Big[\mathcal{E}_n(\theta) + \mathcal{O}(e^{-c\alpha r}) - \frac{a}{(\alpha r)^2}\Big]\,\alpha^2 = \Big[\mathcal{E}_n(\theta) + \mathcal{O}\Big(\frac{1}{(\alpha r)^2}\Big)\Big]\,\alpha^2. \qquad \square$$

Furthermore, using (15) for $n = \kappa(\theta) + 1$ we have

$$\begin{aligned} E_{\kappa(\theta)+1}(N^r_{\theta,\alpha}) &\geq \min\Big\{E_{\kappa(\theta)+1}(D^r_{\theta,\alpha}), E_1(P^{r,\rho}_{\theta,\alpha})\Big\} - \frac{a}{r^2} \\ &= \min\Big\{E_{\kappa(\theta)+1}(D^r_{\theta,\alpha}), E_1(P^{r,\rho}_{\theta,\alpha})\Big\} + o(\alpha^2). \end{aligned}$$

As already mentioned, we have $E_1(P^{r,\rho}_{\theta,\alpha}) \geq -\alpha^2 + o(\alpha)^2$, while $E_{\kappa(\theta)+1}(D^r_{\theta,\alpha}) \geq -\alpha^2$ by Lemma 24, which concludes the proof.

Finally, let us give a rough estimate for the first eigenvalue of $R^r_{\theta,\alpha}$ (it will be improved later).

Lemma 28 (Robin Laplacian $R^r_{\theta,\alpha}$ in a truncated sector)**.** *For some $c > 0$ there holds $R^r_{\theta,\alpha} \geq -c\alpha^2$ as $\alpha r \to +\infty$.*

Proof. Using the scaling as in Lemma 23 we obtain $E_1(R^r_{\theta,\alpha}) = r^{-2} R^1_{\theta,\alpha r}$, and due to Corollary 13 there exists $c > 0$ such that $R^1_{\theta,\alpha r} \geq -c(\alpha r)^2$ for $\alpha r \to +\infty$. $\square$

3.2. **Non-resonant sectors.** Recall that the Robin-Neumann Laplacians $N^r_{\theta,\alpha}$ in the truncated convex sectors $\mathcal{S}^r_\theta$ are defined in (12), and that due to the asymptotics of Lemma 27 their first $\kappa(\theta)$ eigenvalues are, in a sense, close to the first $\kappa(\theta)$ eigenvalues of the Robin Laplacian $T_{\theta,\alpha}$ in the associated infinite sectors $\mathcal{S}_\theta$ in the regime $\alpha r \to +\infty$. For the subsequent study we will use the notion of a non-resonant angle, which involves a hypothesis on the behavior of the next eigenvalue of $N^r_{\theta,\alpha}$ in the same asymptotic regime. Namely, we will use the following definition:

Definition 29 (Non-resonant half-angle)**.** We say that a half-angle $\theta \in \big(0, \frac{\pi}{2}\big)$ is *non-resonant* if there exists a constant $C > 0$ such that

$$E_{\kappa(\theta)+1}(N^r_{\theta,\alpha}) \geq -\alpha^2 + C/r^2 \text{ as } \alpha > 0 \text{ is fixed and } r \text{ is large.}$$

Remark that due to the scaling rule (13) the property does not depend on α and can be equivalently reformulated as

$$E_{\kappa(\theta)+1}(N^r_{\theta,\alpha}) \geq -\alpha^2 + C/r^2 \text{ as } \alpha r \text{ is large.}$$

Our first aim is to show that a large range of half-angles satisfies the non-resonance property:

Proposition 30 (Straight and obtuse angles are non-resonant)**.** *All half-angles θ with $\frac{\pi}{4} \leq \theta < \frac{\pi}{2}$ are non-resonant.*

Proof. Without loss of generality we set $\alpha = 1$ and remove the dependence on α from the notation and write N^r_θ instead of $N^r_{\theta,1}$.

Consider first the case $\theta = \pi/4$. As $\kappa(\pi/4) = 1$ (see Proposition 20), we need to prove that there exists a constant $C > 0$ satisfying

$$E_2(N^r_{\frac{\pi}{4}}) \geq -1 + C/r^2 \text{ as } r \text{ is large.} \tag{16}$$

Remark that $\mathcal{S}^r_{\frac{\pi}{4}}$ is simply a square of side length r, and $N^r_{\frac{\pi}{4}}$ is the Laplacian with 1-Robin boundary condition on two neighboring sides and the Neumann boundary condition on the other two sides. Using the one-dimensional Laplacians L_N on $(0, r)$ with the 1-Robin boundary condition at 0 and the Neumann boundary condition at r one can separate variables, which shows that $N^r_{\frac{\pi}{4}}$ is unitarily equivalent to $L_N \otimes 1 + 1 \otimes L_N$, and then, $E_2(N^r_{\frac{\pi}{4}}) = E_1(L_N) + E_2(L_N)$, i.e. by Proposition 18 one has

$$E_2(N^r_{\frac{\pi}{4}}) \geq -1 + 1/r^2 + \mathcal{O}(e^{-r}) \text{ as } r \to +\infty,$$

which gives the sought inequality (16). Hence, the claim holds for $\theta = \pi/4$.

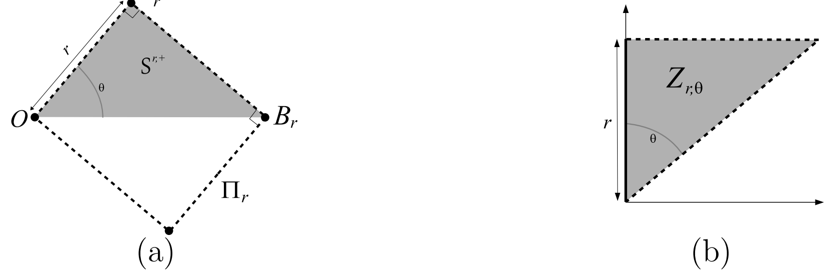
Now let us consider an arbitrary $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$. Recall first that we still have $\kappa(\theta) = 1$ (Proposition 20), hence, we need to show that there exists $C > 0$ such that

$$E_2(N^r_\theta) \geq -1 + C/r^2 \text{ as } r \text{ is large.} \tag{17}$$

Using the symmetry with respect to the axis Ox_1 one easily sees that N^r_θ is unitarily equivalent to $T^{r,D}_\theta \oplus T^{r,N}_\theta$, where $T^{r,D/N}_\theta$ stand for the Laplacians in

$$\mathcal{S}^{r,+}_\theta := \mathcal{S}^r_\theta \cap \big\{(x_1, x_2) : x_2 > 0\big\} = \text{triangle } OA^+_r B_r$$

with the 1-Robin boundary condition at OA^+_r, the Neumann boundary condition at $A^+_r B_r$ and the Dirichlet/Neumann boundary condition at OB_r (we refer to Figures 4 and 6(a) for an illustration). Let us study first the Dirichlet part $T^{r,D}_\theta$. Let Π_r be the rectangle constructed on the vectors OA^+_r and $A^+_r B_r$, see Figure 6(a), then $S^{r,+}_\theta \subset \Pi_r$. Using the standard Dirichlet bracketing (Remark 16) we obtain $E_n(T^{r,D}_\theta) \geq E_n(Q_r)$ for any $n \in \mathbb{N}$, where Q_r is the Laplacian in Π_r with the 1-Robin boundary condition at OA^+_r, the Neumann boundary condition at $A^+_r B_r$ and the Dirichlet boundary condition at the remaining part of the boundary. Remark that $|A^+_r B_r| = r \tan\theta$, and the operator Q_r admits then a separation of variables and is unitarily equivalent to $L_D \otimes 1 + 1 \otimes D_r$, where D_r is the Laplacian on $(0, r)$ with the Dirichlet boundary condition at 0 and the Neumann boundary condition at r, and L_D is the one-dimensional Laplacian on the interval $(0, r\tan\theta)$ with the 1-Robin boundary condition at 0 and the Dirichlet boundary condition on the other end. Therefore, $E_1(T^{r,D}_\theta) = E_1(L_D) + E_1(D_r) = E_1(L_D) + \pi^2/(4r^2)$. Due to Proposition 17 we have $E_1(L_D) = -1 + \mathcal{O}(e^{-r\tan\theta})$, therefore, $E_1(T^{r,D}_\theta) \geq$



(a) (b)

Figure 6. Constructions for the proof of Proposition 30. (a) The completion of the triangle $\mathcal{S}^{r,+}_\theta$ (shaded) to a rectangle Π_r (surrounded by the dashed line). (b) The triangle $Z_{r,\theta}$ is a rotated copy of $\mathcal{S}^{r,+}_\theta$. The solid/dashed lines correspond to Robin/Neumann boundary conditions.

$-1 + C_D/r^2$ for large r with any fixed $C_D \in (0, \pi^2/4)$. Therefore, the sought estimate (17) becomes equivalent to the existence of $C_N > 0$ for which there holds

$$E_2(T_\theta^{r,N}) \geq -1 + C_N/r^2 \text{ as } r \to +\infty, \tag{18}$$

which we already know to hold for $\theta = \frac{\pi}{4}$.

In order to study $T_\theta^{r,N}$ we apply a rotation bringing the triangle $\mathcal{S}_\theta^{r,+}$ onto the triangle

$$Z_{r,\theta} := \big\{(x_1, x_2) : \quad 0 < x_1 < r \tan\theta, \quad x_1 \operatorname{cotan}\theta < x_2 < r\big\},$$

so that $T_\theta^{r,N}$ becomes unitary equivalent to the Laplacian $Q_{r,\theta}$ in $L^2(Z_{r,\theta})$ with the 1-Robin boundary condition along the axis Ox_2 and the Neumann boundary condition at the remaining boundary, and $E_n(T_\theta^{r,N}) = E_n(Q_{r,\theta})$ for any $n \in \mathbb{N}$, and one easily sees that

$$Q_{r,\theta}[u,u] = \iint_{Z_{r,\theta}} |\nabla u|^2 \,\mathrm{d}x - \int_0^r u(0,x_2)^2 \,\mathrm{d}x_2, \quad \mathcal{Q}(Q_{r,\theta}) = H^1(Z_{r,\theta}),$$

see Figure 6(b) for an illustration. Using the unitary transform

$$V : L^2(Z_{r\tan\theta,\frac{\pi}{4}}) \to L^2(Z_{r,\theta}), \quad (Vu)(x_1,x_2) = \sqrt{\tan\theta}\, u(x_1, x_2\tan\theta),$$

which satisfies $V\big(H^1(Z_{r\tan\theta,\frac{\pi}{4}})\big) = H^1(Z_{r,\theta})$, we obtain, with $u_j := \partial u/\partial x_j$,

$$\begin{aligned}
Q_{R,\theta}[Vu,Vu] &= \tan\theta \iint_{Z_{R,\theta}} \Big(u_1(x_1,x_2\tan\theta)^2 + \tan^2\theta\, u_2(x_1,x_2\tan\theta)^2\Big)\, dx \\
&\qquad - \tan\theta \int_0^r u(0,x_2\tan\theta)^2 \,\mathrm{d}x_2 \\
&= \iint_{Z_{r\tan\theta,\frac{\pi}{4}}} \Big(u_1(x_1,x_2)^2 + \tan^2\theta\, u_2(x_1,x_2)^2\Big)\, dx - \alpha \int_0^{r\tan\theta} u(0,x_2)^2 \,\mathrm{d}x_2 \\
&= Q_{r\tan\theta,\frac{\pi}{4}}[u,u] + (\tan^2\theta - 1) \iint_{Z_{r\tan\theta,\frac{\pi}{4}}} u_2^2 \, dx.
\end{aligned}$$

For $\theta \in \big[\frac{\pi}{4}, \frac{\pi}{2}\big)$ we have $\tan\theta \geq 1$, hence, $Q_{r,\theta}[Vu,Vu] \geq Q_{r\tan\theta,\frac{\pi}{4}}[u,u]$ for all functions $u \in H^1(Z_{r,\theta})$, and by the min-max principle we have

$$E_n(T_\theta^{r,N}) = E_n(Q_{r,\theta}) \geq E_n(Q_{r\tan\theta,\frac{\pi}{4}}) = E_n(T_{\frac{\pi}{4}}^{r\tan\theta,N}).$$

It was already shown in the first part of the proof that for some $C > 0$ we have $E_2(T_{\frac{\pi}{4}}^{r\tan\theta,N}) \geq -1 + C/(r\tan\theta)^2$ for large r, so the substitution into the preceding inequality gives the sought estimate (18) with $C_N = C \operatorname{cotan}^2\theta$. □

We will see later (Subsection 7.1) by an indirect argument that there are angles θ which do not satisfy the non-resonance property (which can then be referred as *resonant* ones).

4. Analysis in truncated curvilinear convex sectors

4.1. **Geometry of curvilinear sectors.** Let us introduce a geometric setting which will be used throughout the whole section.

Let $\Gamma_\pm$ be two C^3 curves meeting at a point at an angle $2\theta \in (0,\pi)$. In this section we would like to construct some neighborhoods and cut-off functions near the intersection point. More precisely, let $s_* > 0$ and $\gamma_\pm : [-s_*, s_*] \to \mathbb{R}^2$ be the arc length parametrizations of $\Gamma_\pm$, i.e. both $\gamma_\pm$ are injective C^3 functions with $|\gamma_\pm'| = 1$ and $\Gamma_\pm = \gamma_\pm\big([-s_*, s_*]\big)$. By applying suitable rotations and translations we assume without loss of generality that

$$\gamma_\pm(0) = (0,0), \quad \gamma_\pm'(0) = (\cos\theta, \pm\sin\theta), \quad \theta \in \Big(0, \frac{\pi}{2}\Big). \tag{19}$$

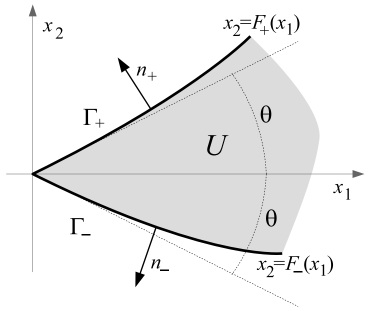


FIGURE 7. The curves $\Gamma_\pm$ and the curvilinear sector U. The thin dotted lines correspond to the tangents to $\Gamma_\pm$ at the origin.

In view of the above assumptions, near the point $(0,0)$ the curves $\Gamma_\pm$ are the graphs of C^3 functions $F_\pm$ with $\pm F_+(t) > \pm F_-(t)$ for $\pm t > 0$, and we will be interested in some constructions in the curvilinear sector

$$U := \Big\{(x_1, x_2) : \ 0 < x_1 < b, \ F_-(x_1) < x_2 < F_+(x_1)\Big\}, \quad b > 0,$$

see Figure 7. For subsequent use we also introduce unit normal vectors $n_\pm(s)$ to $\Gamma_\pm$ at $\gamma_\pm(s)$ which depend smoothly on s and point to the outside of U for small s. In particular, one has then $n_\pm(0) = (-\sin\theta, \pm\cos\theta)$. As $n_\pm$ are unit vectors, one has $n'_\pm(s) = k_\pm(s)\gamma'_\pm(s)$, where $k_\pm$ are C^1 functions (which coincide up to the sign with the algebraic curvatures on $\Gamma_\pm$), and $\gamma'_\pm(s) \wedge n_\pm(s) \equiv \pm 1$.

Lemma 31 (Construction of a curvilinear angle bisector)**.** *There exist $t_1 > 0$ and a C^2 smooth function $Y : (-t_1, t_1) \to \mathbb{R}^2$ such that for $t \in (0, t_1)$ the point $Y(t)$ is the unique point of U which is at the distance t from both Γ_+ and Γ_-, and the points $A_\pm(t) \in \Gamma_\pm$ satisfying $\big|A_\pm(t) - Y(t)\big| = t$ are uniquely defined. Furthermore, $A_\pm(t) := \gamma_\pm\big(\lambda_\pm(t)\big)$, where $\lambda_\pm$ are C^2 functions defined near 0, and*

$$\lambda_\pm(0) = 0, \quad \lambda'_\pm(0) = \operatorname{cotan}\theta, \quad Y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad Y'(0) = \frac{1}{\sin\theta}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The resulting curve

$$\Sigma := \Big\{\big(t, Y(t)\big) : t \in (-t_1, t_1)\Big\}, \tag{20}$$

can be viewed as the curvilinear angle bisector due to its geometric property: each point of Σ is at equal distances from the curved sides $\Gamma_\pm$.

Proof. For $t_0 > 0$ and $s_0 \in (0, s_*)$ consider the maps (see Figure 8)

$$\Phi_\pm : (-s_0, s_0) \times (-t_0, t_0) \to \mathbb{R}^2, \quad \Phi_\pm(s,t) = \gamma_\pm(s) - t n_\pm(s).$$

It is a well known result from the differential geometry that $\Phi_\pm$ are injective for $t_0 > 0$ small enough, and that $\operatorname{dist}\big(\Phi_\pm(s,t), \Gamma_\pm\big) = |t|$ and that they are C^2-diffeomorphisms from $(-s_0, s_0) \times (-t_0, t_0)$ to its images under $\Phi_\pm$. One has

$$\frac{\partial \Phi_\pm}{\partial s}(s,t) = \gamma'_\pm(s) - t n'_\pm(s) = \big(1 - t k_\pm(s)\big)\,\gamma'_\pm(s).$$

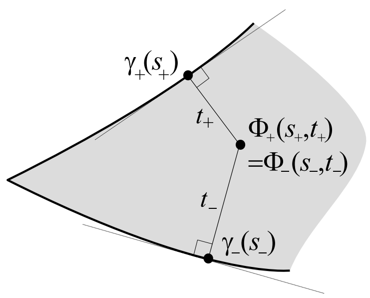


FIGURE 8. The maps $\Phi_\pm$.

Define $G : (-s_0, s_0) \times (-s_0, s_0) \times (-t_0, t_0) \to \mathbb{R}^2$ by $G(s_+, s_-, t) := \Phi_+(s_+, t) - \Phi_-(s_-, t)$, then $G(0,0,0) = \gamma_+(0) - \gamma_-(0) = (0,0)$ and $\partial G/\partial s_\pm(s_+, s_-, t) = \pm\big(1 - tk_\pm(s_\pm)\big)\gamma'_\pm(s_\pm)$, and the two vectors $\partial G/\partial s_\pm(0,0,0) = \pm\gamma'_\pm(0)$ are linearly independent. Hence, it follows by the implicit function theorem that there exist $t_1 > 0$ and $s_1 > 0$ and C^2 functions $\lambda_\pm : (-t_1, t_1) \to (-s_1, s_1)$ with $\lambda_\pm(0) = 0$ such that for $(s_+, s_-, t) \in (-s_1, s_1) \times (-s_1, s_1) \times (-t_1, t_1)$ one has the equivalence: $G(s_+, s_-, t) = 0$ if and only if $s_\pm = \lambda_\pm(t)$. If one defines a C^2 function $Y : (-t_1, t_1) \to \mathbb{R}^2$ by $Y(t) := \Phi_\pm\big(\lambda_\pm(t), t\big)$, then for any $t \in (0, t_1)$ the point $Y(t)$ is the unique point of U satisfying $\operatorname{dist}\big(Y(t), \Gamma_\pm\big) = t$, and the points $A_\pm(t)$ of $\Gamma_\pm$ which are the closest to $Y(t)$ are $A_\pm(t) = \gamma_\pm\big(\lambda_\pm(t)\big)$. One differentiates the equality $G\big(\lambda_+(t), \lambda_-(t), t\big) = 0$ with respect to t to arrive at

$$\lambda'_+(t)\Big[1 - tk_+\big(\lambda_+(t)\big)\Big]\gamma'_+\big(\lambda_+(t)\big) - \lambda'_-(t)\Big[1 - tk_-\big(\lambda_-(t)\big)\Big]\gamma'_-\big(\lambda_-(t)\big) - \Big[n_+\big(\lambda_+(t)\big) - n_-\big(\lambda_-(t)\big)\Big] = 0.$$

For $t = 0$ one has $\lambda'_+(0)\gamma'_+(0) - \lambda'_-(0)\gamma'_-(0) = n_+(0) - n_-(0)$, i.e.

$$\begin{pmatrix} \cos\theta & -\cos\theta \\ \sin\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \lambda'_+(0) \\ \lambda'_-(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 2\cos\theta \end{pmatrix},$$

which gives

$$\begin{pmatrix} \lambda'_+(0) \\ \lambda'_-(0) \end{pmatrix} = \frac{1}{2\sin\theta\cos\theta} \begin{pmatrix} \sin\theta & \cos\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 2\cos\theta \end{pmatrix} = \begin{pmatrix} \operatorname{cotan}\theta \\ \operatorname{cotan}\theta \end{pmatrix}.$$

Then

$$Y'(t) = \frac{d}{dt}\,\Phi_+\big(\lambda_+(t), t\big) = \lambda'_+(t)\Big[1 - tk_+\big(\lambda_+(t)\big)\Big]\gamma'_+\big(\lambda_+(t)\big) - n_+\big(\lambda_+(t)\big),$$
$$Y'(0) = \lambda'_+(0)\gamma'_+(0) - n_+(0) = \operatorname{cotan}\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} - \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \frac{1}{\sin\theta}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \qquad \square$$

Using the objects defined in Lemma 31 we introduce the following sets V_t:

Definition 32 (Sets V_t and their boundaries)**.** For $t \in (0, t_1)$ denote by V_t the interior of the curvilinear quadrangle bounded by the pieces of $\Gamma_\pm$ enclosed between the points $(0,0)$ and $A_\pm(t)$ and by the straight line segments connecting $Y(t)$ to $A_\pm(t)$. We refer to Figure 9 for an illustration. One will distinguish between two parts of its boundary, i.e. one denotes

$$\partial_* V_t := \partial V_t \cap (\Gamma_+ \cup \Gamma_-), \text{ and } \partial_{\text{ext}} V_t := \partial V_t \setminus \partial_* V_t.$$

Then, we would like to "straighten" V_t in a controlable way in order to obtain a truncated curvilinear sector $\mathcal{S}^r_\theta$ (see Definition 22).

Lemma 33 (Straightening a curvilinear quadrangle)**.** *There is a bi-Lipschitz map Φ between two neighborhoods of the origin with $\Phi'(x) = I_2 + \mathcal{O}\big(|x|\big)$ for $x \to 0$ and a C^2 smooth function r defined near 0 with $r(0) = 0$ and $r'(0) = \operatorname{cotan}\theta$ such that $\Phi(\mathcal{S}^{r(t)}_\theta) = V_t$, $\Phi(\partial_*\mathcal{S}^{r(t)}_\theta) = \partial_* V_t$ and $\Phi(\partial_{\text{ext}}\mathcal{S}^{r(t)}_\theta) = \partial_{\text{ext}} V_t$ for all sufficiently small $t > 0$.*

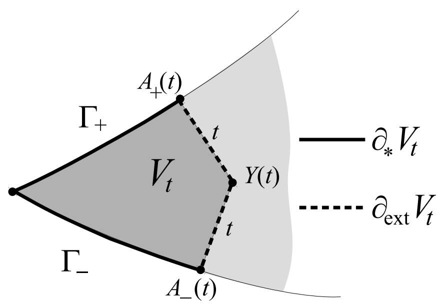


FIGURE 9. Construction of the domain V_t

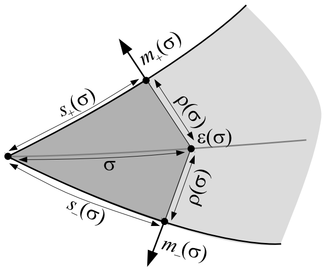


FIGURE 10. Parametrization with the arc-length. The unit vectors $m_\pm$ are orthogonal to the boundary. The arrows indicate the length of the corresponding arcs. For small σ one has $\rho(\sigma) = \sigma \sin\theta + \mathcal{O}(\sigma^2)$ and $s_\pm(\sigma) = \sigma\cos\theta + \mathcal{O}(\sigma^2)$.

Proof. Without loss of generality we may assume that $t_1 > 0$ is sufficiently small such that $Y'(t) \neq 0$ for $t \in [-t_1, t_1]$. Let us introduce an arc-length parametrization of the curvilinear angle bisector Σ introduced in (20): consider the function σ with $\sigma(0) = 0$ and $\sigma' = |Y'|$, i.e. $\sigma(t)$ is the length of $Y\big([0,t]\big)$. One has $\sigma'(0) = \big|Y'(0)\big| = 1/\sin\theta$ and $\sigma' = |Y'| > 0$ on $[-t_1, t_1]$. Hence, $\sigma : [-t_1, t_1] \to [-\sigma_-, \sigma_+]$ is a C^2 diffeomorphism for some $\sigma_\pm > 0$. Denote by $\rho : [-\sigma_-, \sigma_+] \to [-t_1, t_1]$ its inverse, which is then also C^2 and satisfies $\rho(0) = 0$ and $\rho'(0) = 1/\sigma'(0) = \sin\theta$. Finally, let us pick a small $\delta > 0$ and define $\varepsilon := Y \circ \rho : (-\delta, \delta) \to \mathbb{R}^2$, then one has $|\varepsilon'| = 1$, $\varepsilon'(0) = (1,0)^T$, and $Y\big([0,t]\big) = \varepsilon\big([0, \sigma(t)]\big)$ for small $t > 0$, i.e. ε is an arc-length parametrization of Σ near the origin. By construction, the point $\varepsilon(\sigma)$ is then the unique point of U with $\operatorname{dist}\big(\varepsilon(\sigma), \Gamma_\pm\big) = \rho(\sigma)$, and for small σ one has $\rho(\sigma) = \sigma\sin\theta + \mathcal{O}(\sigma^2)$. Furthermore, if one sets $s_\pm(\sigma) := \lambda_\pm\big(\rho(\sigma)\big)$, then $s_\pm(\cdot)$ are C^2 functions with $s'_\pm(0) = \lambda'_\pm(0)\rho'(0) = \cos\theta$, and the points $B_\pm(\sigma) := \gamma_\pm\big(s_\pm(\sigma)\big)$ of $\Gamma_\pm$ are the closest to $\varepsilon(\sigma)$. We also reparametrize the normal vectors to $\Gamma_\pm$ by setting $m_\pm(\sigma) := n_\pm\big(s_\pm(\sigma)\big)$, then one has $B_\pm(\sigma) = \varepsilon(\sigma) + \rho(\sigma) m_\pm(\sigma)$. The above constructions are illustrated in Figure 10.

For the C^2 maps $\Psi_\pm : (\sigma, \tau) \mapsto \varepsilon(\sigma) + \tau m_\pm(\sigma)$ one has $\Psi_\pm(0,0) = (0,0)$ and

$$\Psi'_\pm(0,0) = \begin{pmatrix} \dfrac{\partial\Psi_\pm}{\partial\sigma}(0,0) & \dfrac{\partial\Psi_\pm}{\partial\tau}(0,0) \end{pmatrix} = \begin{pmatrix} \varepsilon'(0) & m_\pm(0) \end{pmatrix} = \begin{pmatrix} 1 & -\sin\theta \\ 0 & \pm\cos\theta \end{pmatrix},$$

i.e. the Jacobian matrix $\Psi'_\pm(0,0)$ is invertible. Therefore, the maps $\Psi_\pm$ are diffeomorphisms between suitable neighborhoods of the origin. Furthermore, if for $t > 0$ one introduces the curvilinear triangles $\Lambda_t := \big\{(\sigma,\tau) : \quad 0 < \sigma < \sigma(t), \quad 0 < \tau < \rho(\sigma)\big\}$, then the image $\Psi_\pm(\overline{\Lambda_t})$ is exactly the closure of the upper/lower $V_t^\pm$ part of V_t, i.e. of the part of V_t lying above/below Σ, and $\Psi_+(\cdot, 0) = \Psi_-(\cdot, 0)$. We now use this observation to construct a map Φ with the sought properties. Namely, in addition to the above curvilinear triangles Λ_t let us consider its "straightened" version $L_t = \big\{(\sigma,\tau) : \quad 0 < \sigma < \sigma(t), \quad 0 < \tau < \sigma\sin\theta\big\}$. obtained by replacing ρ through its linear approximation at 0. The map $H : (\sigma,\tau) \mapsto \big(\sigma, \rho(\sigma)\tau/(\sigma\sin\theta)\big)$ satisfies then $H'(0,0) = I_2$, hence, it is a diffeomorphism between suitable neighborhoods of the origin, and for sufficiently small $t > 0$ it is bijective from $\overline{L_t}$ to $\overline{\Lambda_t}$.

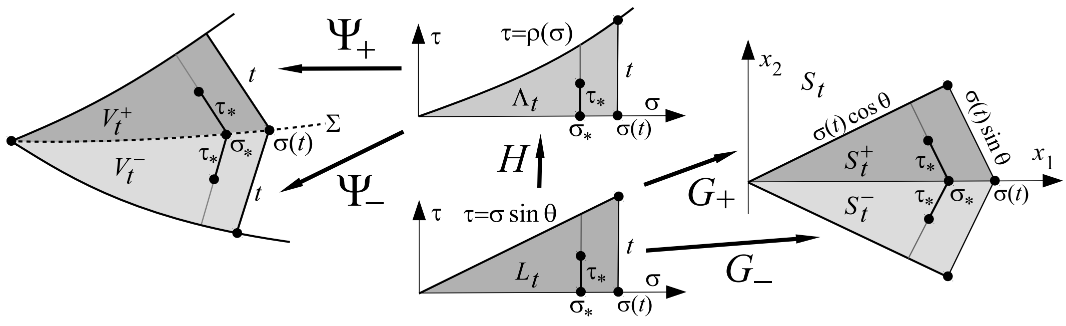


FIGURE 11. The maps $\Psi_\pm$, H and $G_\pm$ in the proof of Lemma 33.

Now let us consider the truncated sector $S_t := \mathcal{S}_\theta^{\sigma(t)\cos\theta}$ and their upper/lower parts $S_t^\pm := S_t \cap \{(x_1, x_2) : \pm x_2 > 0\}$. One easily sees that the maps

$$G_\pm : \mathbb{R}^2 \to \mathbb{R}^2, \quad (\sigma, \tau) \mapsto \sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tau \begin{pmatrix} -\sin\theta \\ \pm\cos\theta \end{pmatrix}$$

are diffeomorphisms, and $\overline{S_t^\pm} = G_\pm(\overline{L}_t)$ for small $t > 0$, and the inverses are given by

$$G_\pm^{-1}(x_1, x_2) = \begin{pmatrix} 1 & \pm\tan\theta \\ 0 & \pm\dfrac{1}{\cos\theta} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We refer to Figure 11 for a graphical representation of the above maps.

Now let use define Φ by $\Phi(x_1, x_2) = \Psi_\pm \circ H \circ G_\pm^{-1}(x_1, x_2)$ for $\pm x_2 > 0$, which then extends by continuity to $x_2 = 0$ due to

$$\Psi_\pm \circ H \circ G_\pm^{-1}(x_1, 0) = \Psi_\pm \circ H(x_1, 0) = \Psi_\pm(x_1, 0) = \varepsilon(x_1).$$

By construction, the map Φ is C^2 on $\{\pm x_2 \geq 0\}$ and continuous along $x_2 = 0$, hence it is Lipschitz. Furthermore, by construction it defines bijections $S_t^\pm \to V_t^\pm$, $S_t \to V_t$ as well as $\partial_* S_t \to \partial_* V_t$ and $\partial_{\mathrm{ext}} S_t \to \partial_{\mathrm{ext}} V_t$. To estimate the Jacobian matrix Φ' we compute

$$\begin{aligned}(\Psi_\pm \circ H \circ G_\pm^{-1})'(0,0) &= \Psi_\pm'(0,0) H'(0,0) (G_\pm^{-1})'(0,0) \\ &= \begin{pmatrix} 1 & -\sin\theta \\ 0 & \pm\cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \pm\tan\theta \\ 0 & \pm\dfrac{1}{\cos\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

As $\Phi_\pm$ are C^1, it follows that $\Phi_\pm' = I_2 + \mathcal{O}(t)$ in V_t, which shows the requested property for Φ'. As $\Phi_\pm^{-1}$ are C^1 near the origin and Φ^{-1} is continuous by construction, it follows that Φ^{-1} is Lipschitz, therefore, the map Φ is bi-Lipschitz. Hence, we obtain the claim with $r(t) = \sigma(t)\cos\theta$, and $r'(0) = \sigma'(0)\cos\theta = \operatorname{cotan}\theta$. □

For later references we mention explicitly the following corollary, which is quite obvious from the geometric point of view:

Corollary 34 (Norm estimate in V_t)**.** *There exist $0 < a < b$ such that for all sufficiently small $t > 0$ there holds $|x| < bt$ for $x \in V_t$, and $|x| > at$ for $x \in V_s \setminus \overline{V_t}$ and $s > t$.*

Proof. Let us use a map Φ and a function r as in Lemma 33. Remark first that

$$|y| < \frac{r}{\cos\theta} \text{ for } y \in \mathcal{S}_\theta^r \text{ and } r > 0, \qquad |y| > r \text{ for } y \in \mathcal{S}_\theta^R \setminus \overline{\mathcal{S}_\theta^r} \text{ and } R > r > 0. \tag{21}$$

As $v \in V_t$ iff $v = \Phi(y)$ with $y \in \mathcal{S}_\theta^{r(t)}$ and $r(t) = \mathcal{O}(t)$, by applying the Taylor expansion of Φ near the origin one obtains $\frac{1}{2}|y| \leq |v| \leq 2|y|$. Using the estimates (21) one arrives at the result. □

We complete this subsection by providing a construction of cut-off functions with some special properties:

Lemma 35 (Cut-off functions in a curvilinear sector)**.** *Let $0 < a < b$, then there exist $\delta_0 > 0$, $\eta > 0$, $K > 0$ and C^2 functions $\varphi_\delta : \overline{V_\eta} \to \mathbb{R}$ with $\delta \in (0, \delta_0)$ such that:*

(a) *$0 \leq \varphi_\delta \leq 1$, and for all $\beta \in \mathbb{N}^2$ with $1 \leq |\beta| \leq 2$ there holds $\|\partial^\beta \varphi_\delta\|_\infty \leq K\delta^{-|\beta|}$,*
(b) *$\varphi_\delta = 1$ in $V_{a\delta}$,*
(c) *$\varphi_\delta = 0$ in $V_\eta \setminus \overline{V_{b\delta}}$,*
(d) *the normal derivative of φ_δ at $\Gamma_\pm$ is zero.*

Proof. For small $t_0 > 0$ and $s_0 > 0$ consider the maps

$$\Phi_\pm : (-s_0, s_0) \times (-t_0, t_0) \to \mathbb{R}^2, \quad \Phi_\pm(s,t) = \gamma_\pm(s) - t n_\pm(s).$$

It is a well known result of differential geometry that $\Phi_\pm$ are injective for $t_0 > 0$ small enough, with $\operatorname{dist}\big(\Phi_\pm(\cdot,t),\Gamma_\pm\big) = |t|$ for $|t| < t_0$, and that they are C^2-diffeomorphisms from $(-s_0, s_0) \times (-t_0, t_0)$ to its images under $\Phi_\pm$. Remark that one has

$$\frac{\partial \Phi_\pm}{\partial s}(s,t) = \gamma_\pm'(s) - t n_\pm'(s) = \big(1 - t k_\pm(s)\big)\, \gamma_\pm'(s), \quad \frac{\partial \Phi_\pm}{\partial t}(s,t) = -n_\pm(s),$$

i.e. if one writes $(\tau_1^\pm, \tau_2^\pm) := \gamma_\pm'$ and $(n_1^\pm, n_2^\pm) := n_\pm$, then

$$\Phi_\pm'(s,t) = \begin{pmatrix} \big(1 - t k_\pm(s)\big)\tau_1^\pm(s) & -n_1^\pm(s) \\ \big(1 - t k_\pm(s)\big)\tau_2^\pm(s) & -n_2^\pm(s) \end{pmatrix}.$$

By choosing $\eta > 0$ sufficiently small one can then invert the maps $(s,t) \mapsto \Phi_\pm(s,t)$ near the origin in order to obtain C^2 functions $s_\pm$ and $t_\pm$ on V_η, and the inverse function theorem shows that

$$\nabla s_\pm(x) = \pm \frac{1}{1 - t_\pm(x) K_\pm(x)} \Big(N_2^\pm(x), -N_1^\pm(x) \Big), \quad K_\pm := k_\pm \circ s_\pm, \quad N_j^\pm := n_j^\pm \circ s_\pm. \tag{22}$$

In particular, $s_\pm(0,0) = 0$ and $\nabla s_\pm(0,0) = (\cos\theta, \pm \sin\theta)$, therefore,

$$s_\pm(x_1,x_2) = (\cos\theta, \pm\sin\theta) \cdot (x_1,x_2) + \mathcal{O}(x_1^2 + x_2^2) \text{ for } (x_1,x_2) \to (0,0).$$

We further remark that for small s one has obviously $s_\pm\big(\gamma_\pm(s)\big) = s$, while

$$s_\pm\big(\gamma_\mp(s)\big) = (\cos\theta, \pm\sin\theta) \cdot \gamma_\mp'(0)s + \mathcal{O}(s^2) \equiv \cos(2\theta)\, s + \mathcal{O}(s^2) \text{ for } s \to 0. \tag{23}$$

Let us pick some $c \in (a \operatorname{cotan}\theta, b \operatorname{cotan}\theta)$ and then a sufficiently small $\varepsilon > 0$ satisfying

$$[c - \varepsilon, c + \varepsilon] \subset (a \operatorname{cotan}\theta, b \operatorname{cotan}\theta), \qquad \cos(2\theta)(c + \varepsilon) < c - \varepsilon. \tag{24}$$

We remark that the second condition follows from the first one for $\theta \geq \frac{\pi}{4}$. Let $\psi : \mathbb{R} \to [0,1]$ be a C^∞ function with $\psi(s) = 1$ for $s < c - \varepsilon$ and $\psi(s) = 0$ for $s > c + \varepsilon$. For small $\delta > 0$ we define then $\varphi_\delta : \overline{V_\eta} \to \mathbb{R}$ by $\varphi_\delta(x) = \psi\big(s_+(x)/\delta\big)\psi\big(s_-(x)/\delta\big)$. Note that the property (a) is automatically satisfied due to the the C^2 smoothness of the functions $s_\pm$.

In order to see the properties (b) and (c) we first remark that due to Lemma 31 the definition of the domain V_t for small t can be reformulated as $V_t := \big\{x \in V_\eta :\ s_\pm(x) < \lambda_\pm(t)\big\}$, and for small δ and a fixed $A > 0$ one has $\lambda_\pm(A\delta) = A\delta \operatorname{cotan}\theta + \mathcal{O}(\delta^2)$. In particular, for $x \in V_{a\delta}$ one has $s_\pm(x) \leq a\delta \operatorname{cotan}\theta + \mathcal{O}(\delta^2) < (c - \varepsilon)\delta$ as δ is small, which shows that $\varphi_\delta(x) = 1$ and proves the claim (b). Furthermore, for $x \notin V_{b\beta}$ one of the following two inequalities holds: $s_\pm(x) > \lambda_\pm(b\delta)$. As $\lambda_\pm(b\delta) = b\delta \operatorname{cotan}\theta + \mathcal{O}(\delta^2) > (c + \varepsilon)\delta$, it follows that at least one of the terms $s_\pm(x)/\delta$ is greater than $c + \varepsilon$. As ψ vanishes in $(c + \varepsilon, +\infty)$, it follows that $\varphi_\delta(x) = 0$. This proves the claim (c).

Let us finally show the property (d). For a better readability we give the computation of the normal derivative on Γ_+ only, the case of Γ_- is considered in a completely similar way. For $x = \gamma_+(s) \in \Gamma_+$ with $s > 0$ one has

$$\begin{aligned} \frac{\partial \varphi_\delta}{\partial n_+}(x) = n_+(s) \cdot (\nabla\varphi_\delta)\big(\gamma_+(s)\big) = \frac{1}{\delta}\, n_+(s) \cdot \Big[(\nabla s_+)\big(\gamma_+(s)\big) \psi'\Big(\frac{s_+\big(\gamma_+(s)\big)}{\delta}\Big) \psi\Big(\frac{s_-\big(\gamma_+(s)\big)}{\delta}\Big) \\ + (\nabla s_-)\big(\gamma_+(s)\big) \psi'\Big(\frac{s_-\big(\gamma_+(s)\big)}{\delta}\Big) \psi\Big(\frac{s_+\big(\gamma_+(s)\big)}{\delta}\Big) \Big]. \end{aligned}$$

By (22) one has $(\nabla s_+)\big(\gamma_+(s)\big) = \big(n_2^+(s), -n_1^+(s)\big)$, which gives $n_+(s) \cdot (\nabla s_+)\big(\gamma_+(s)\big) = 0$, and the preceding expression simplifies to

$$\frac{\partial \varphi_\delta}{\partial n_+}\big(\gamma_+(s)\big) = \Big[\frac{1}{\delta}\, n_+(s) \cdot (\nabla s_-)\big(\gamma_+(s)\big)\Big] \psi'\Big(\frac{s_-\big(\gamma_+(s)\big)}{\delta}\Big) \psi\Big(\frac{s}{\delta}\Big).$$

Let us show that the product of the last two terms is zero for small δ, i.e. that $\psi'\big(\xi(s)\big)\psi(s/\delta) = 0$ for $\xi(s) := s_-\big(\gamma_+(s)\big)/\delta$. First, by construction of ψ the second factor vanishes for $s \ge (c+\varepsilon)\delta$. Therefore, one needs to show that $\psi'\big(\xi(s)\big) = 0$ for all $0 < s \le (c+\varepsilon)\delta$ as δ is sufficiently small. Using the Taylor expansion (23) for small δ we have $\xi(s) = \cos(2\theta)s/\delta + \mathcal{O}(\delta)$. If $\theta \ge \frac{\pi}{4}$, then $\cos(2\theta) \le 0$, and $\xi(s) \le \mathcal{O}(\delta) < c - \varepsilon$. If $\theta < \frac{\pi}{4}$, then $\cos(2\theta) > 0$, and due to the choice of ε made in (24) one obtains $\xi(s) \le \cos(2\theta)(c+\varepsilon) + \mathcal{O}(\delta) < c - \varepsilon$. Therefore, in both cases one has $\xi(s) < c - \varepsilon$ for all $0 < s < (c+\varepsilon)\delta$ as δ is sufficiently small. As ψ was chosen constant on $(-\infty, c-\varepsilon)$, we have $\psi'\big(\xi(s)\big) = 0$. □

4.2. **Robin Laplacians in truncated curvilinear sectors.** We will need some properties of Robin-Dirichlet and Robin-Neumann Laplacians in the above truncated sectors V_δ. Namely, for $\alpha > 0$ and $\delta > 0$ sufficiently small, consider

$$\begin{aligned} D^\delta_{\Gamma,\alpha} :=& \text{ the Laplacian in } V_\delta \text{ with the } \alpha\text{-Robin boundary condition at } \partial_* V_\delta, \\ & \text{ and the \textit{Dirichlet} boundary condition at } \partial_{\text{ext}} V_\delta, \\ N^\delta_{\Gamma,\alpha} :=& \text{ the Laplacian in } V_\delta \text{ with the } \alpha\text{-Robin boundary condition at } \partial_* V_\delta \\ & \text{ and the \textit{Neumann} boundary condition at } \partial_{\text{ext}} V_\delta, \\ R^\delta_{\Gamma,\alpha} :=& \text{ the Laplacian in } V_\delta \text{ with the } \alpha\text{-Robin boundary condition \textit{at the whole boundary.}} \end{aligned} \tag{25}$$

Furthermore, for $0 < \rho < \delta$ we consider the curvilinear hexagons $\mathcal{P}^{\delta,\rho}_\Gamma$ and their partial boundaries given by

$$\mathcal{P}^{\delta,\rho}_\Gamma := V_\delta \setminus \overline{V_\rho}, \quad \partial_* \mathcal{P}^{\delta,\rho}_\Gamma := \partial \mathcal{P}^{\delta,\rho}_\Gamma \cap \partial_* V_\delta, \quad \partial_{\text{ext}} \mathcal{P}^{\delta,\rho}_\Gamma := \partial \mathcal{P}^{\delta,\rho}_\Gamma \setminus \partial_* \mathcal{P}^{\delta,\rho}_\Gamma,$$

and the associated operators

$$\begin{aligned} P^{\delta,\rho}_{\Gamma,\alpha} :=& \text{ the Laplacian in } \mathcal{P}^{\delta,\rho}_\Gamma \text{ with the } \alpha\text{-Robin boundary condition} \\ & \text{ at } \partial_* \mathcal{P}^{\delta,\rho}_{\Gamma,\alpha} \text{ and the Neumann boundary condition at } \partial_{\text{ext}} \mathcal{P}^{\delta,\rho}_\Gamma, \end{aligned}$$

see Figure 12. These hexagons and the associated operators are auxiliary objects and will not be used outside this section. The four above operators are curvilinear analogs of the operators $(D/N/R)^r_{\theta,\alpha}$ and $P^{r,\rho}_{\theta,\alpha}$ introduced and studied in Section 3. This observation can be used to estimate the eigenvalues.

Lemma 36 (Comparison of eigenvalues for small curvilinear and straight polygons)**.** *There exist $a > 0$, $a_0 > 0$, $\delta_0 > 0$ and a C^2 function $r(0)$ with $r = 0$ and $r'(0) = \operatorname{cotan}\theta$ such that for all $\delta \in (0, \delta_0)$, $\rho \in (0, \delta)$, $\alpha > 0$, $n \in \mathbb{N}$ there holds*

$$\begin{aligned} (1 - a_0\delta) E_n(N^{r(\delta)}_{\theta,\alpha(1+a\delta)}) &\le E_n(N^\delta_{\Gamma,\alpha}) \le (1 + a_0\delta) E_n(N^{r(\delta)}_{\theta,\alpha(1-a\delta)}), \\ (1 - a_0\delta) E_n(D^{r(\delta)}_{\theta,\alpha(1+a\delta)}) &\le E_n(D^\delta_{\Gamma,\alpha}) \le (1 + a_0\delta) E_n(D^{r(\delta)}_{\theta,\alpha(1-a\delta)}), \\ (1 - a_0\delta) E_n(R^{r(\delta)}_{\theta,\alpha(1+a\delta)}) &\le E_n(R^\delta_{\Gamma,\alpha}) \le (1 + a_0\delta) E_n(R^{r(\delta)}_{\theta,\alpha(1-a\delta)}), \\ (1 - a_0\delta) E_n(P^{r(\delta),r(\rho)}_{\theta,\alpha(1+a\delta)}) &\le E_n(P^{\delta,\rho}_{\Gamma,\alpha}) \le (1 + a_0\delta) E_n(P^{r(\delta),r(\rho)}_{\theta,\alpha(1-a\delta)}). \end{aligned}$$

Proof. By Lemma 33 there exists $\delta_0 > 0$ and a bi-Lipschitz map Φ between two neighborhoods of the origin with $\Phi'(x) = I_2 + \mathcal{O}\big(|x|\big)$ and a C^2 smooth function r defined near 0 with $r(0) = 0$

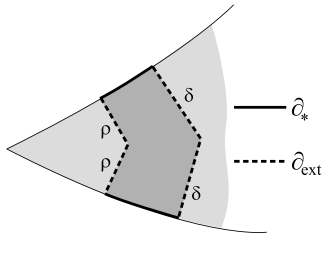


FIGURE 12. The curvilinear polygon $\mathcal{P}^{\delta,\rho}_{\Gamma,\alpha}$ is shaded. The solid/dashed lines correspond to the Robin/Neumann boundary conditions.

and $r'(0)=\operatorname{cotan}\theta$ such that $\Phi(\mathcal{S}_\theta^{r(\delta)})=V_\delta$, with $\Phi(\partial_*\mathcal{S}_\theta^{r(\delta)})=\partial_* V_\delta$ and $\Phi(\partial_{\text{ext}}\mathcal{S}_\theta^{r(\delta)})=\partial_{\text{ext}}V_\delta$, for $\delta\in(0,\delta_0)$. Remark that this implies automatically the equalities

$$\mathcal{P}_\Gamma^{\delta,\rho}=\Phi(\mathcal{P}_\theta^{r(\delta),r(\rho)}),\quad \partial_*\mathcal{P}_\Gamma^{\delta,\rho}=\Phi(\partial_*\mathcal{P}_\theta^{r(\delta),r(\rho)}),\quad \partial_{\text{ext}}\mathcal{P}_\Gamma^{\delta,\rho}=\Phi(\partial_{\text{ext}}\mathcal{P}_\theta^{r(\delta),r(\rho)}).$$

The analysis of all the four operators is done in an almost identical way, so we consider the lower bound of $D^\delta_{\Gamma,\alpha}$ only.

The map $\mathcal{U}_\delta: L^2(V_\delta)\to L^2(\mathcal{S}_\theta^{r(\delta)})$, $u:=\mathcal{U}_\delta v:=v\circ\Phi$, is bijective, furthermore, it is also bijective between $H^1(V_\delta)$ and $H^1(\mathcal{S}_\theta^{r(\delta)})$, and $v=0$ on $\partial_{\text{ext}}V_\delta$ iff $u=0$ on $\partial_{\text{ext}}\mathcal{S}_\theta^{r(\delta)}$. Furthermore, using the change of variables one has

$$\iint_{V_\delta} v^2\,dx=\iint_{\mathcal{S}_\theta^{r(\delta)}} u^2\det\Phi'\,dx.$$

One can find $b>0$ such that $1-b\delta\le\det\Phi'\le 1+b\delta$ in $\mathcal{S}_\theta^{r(\delta)}$, and then

$$(1-b\delta)\iint_{\mathcal{S}_\theta^{r(\delta)}}u^2\,dx\le\iint_{V_\delta}v^2\,dx\le(1+b\delta)\iint_{\mathcal{S}_\theta^{r(\delta)}}u^2\,dx.$$

Furthermore, the standard change of variables shows that

$$\iint_{V_\delta}|\nabla v|^2\,dx=\iint_{\mathcal{S}_\theta^{r(\delta)}}\sum\nolimits_{j,k}G^{j,k}\partial_j u\,\partial_k u\det\Phi'\,dx,$$

where the matrix $(G^{j,k})$ is the inverse of $(\partial_j\Phi\cdot\partial_k\Phi)=\big(\delta_{j,k}+\mathcal{O}(\delta)\big)$. Therefore, with some $b_0>0$ for any u one has

$$(1-b_0\delta)|\nabla u|^2\le\sum\nolimits_{j,k}G^{j,k}\partial_j u\,\partial_k u\le(1+b_0\delta)|\nabla u|^2,$$

so by adjusting the above value of $b>0$ we have

$$(1-b\delta)\iint_{\mathcal{S}_\theta^{r(\delta)}}|\nabla u|^2\,dx\le\iint_{V_\delta}|\nabla v|^2\,dx\le(1+b\delta)\iint_{\mathcal{S}_\theta^{r(\delta)}}|\nabla u|^2\,dx.$$

Finally, if $I_\delta\ni t\to\gamma(t)$ is an arc-length parametrization of $\partial_*\mathcal{S}_\theta^{r(\delta)}$, i.e. $|\gamma'|=1$, then $I_\delta\ni t\to\Phi\big(\gamma(t)\big)$ is a parametrization of ∂_*V_δ, and

$$\int_{\partial_*V_\delta}v^2\,\mathrm{d}s=\int_{I_\delta}u(t)^2\big|\Phi'\big(\gamma(t)\big)\gamma'(t)\big|\,\mathrm{d}t.$$

Due to $\Phi'(x)=I_2+\mathcal{O}\big(|x|\big)$ we have, with an adjusted value of $b>0$,

$$1-b\delta\le\big|\Phi'\big(\gamma(t)\big)\gamma'(t)\big|\le1+b\delta,$$

$$(1-b\delta)\int_{\partial_*\mathcal{S}_\theta^{r(\delta)}}u^2\,\mathrm{d}s\le\int_{\partial_*V_\delta}v^2\,\mathrm{d}s\le(1+b\delta)\int_{\partial_*\mathcal{S}_\theta^{r(\delta)}}u^2\,\mathrm{d}s.$$

Recall that

$$D^\delta_{\Gamma,\alpha}[v,v]=\iint_{V_\delta}|\nabla v|^2\,\mathrm{d}x-\alpha\int_{\partial_*V_\delta}v^2\,\mathrm{d}s,\quad v\in\mathcal{Q}(D^\delta_{\Gamma,\alpha}).$$

Therefore, for $v\ne0$ one can combine the preceding estimates to arrive at

$$\frac{D^\delta_{\Gamma,\alpha}[v,v]}{\|v\|^2_{L^2(V_\delta)}}\ge\frac{1-b\delta}{1+b\delta}\frac{\displaystyle\iint_{\mathcal{S}_\theta^{r(\delta)}}|\nabla u|^2\,\mathrm{d}x}{\displaystyle\iint_{\mathcal{S}_\theta^{r(\delta)}}u^2\,\mathrm{d}x}-\alpha\frac{1+b\delta}{1-b\delta}\frac{\displaystyle\int_{\partial_*\mathcal{S}_\theta^{r(\delta)}}u^2\,\mathrm{d}s}{\displaystyle\iint_{\mathcal{S}_\theta^{r(\delta)}}u^2\,\mathrm{d}x}.$$

Let us choose $a_0>0$ and $a>0$ to have, for all sufficiently small $\delta>0$,

$$\frac{1-b\delta}{1+b\delta}\ge1-a_0\delta,\quad\frac{1}{1-a_0\delta}\cdot\frac{1+b\delta}{1-b\delta}<1+a\delta,$$

then the preceding inequality implies

$$\frac{D^{\delta}_{\Gamma,\alpha}[v,v]}{\|v\|^2_{L^2(V_\delta)}} \ge (1-a_0\delta)\frac{\displaystyle\iint_{\mathcal{S}_\theta^{r(\delta)}} |\nabla u|^2\,\mathrm{d}x - \alpha(1+a\delta)\int_{\partial_*\mathcal{S}_\theta^{r(\delta)}} u^2\,\mathrm{d}s}{\displaystyle\iint_{\mathcal{S}_\theta^{r(\delta)}} u^2\,\mathrm{d}x}.$$

This can be rewritten as

$$\frac{D^{\delta}_{\Gamma,\alpha}[v,v]}{\|u\|^2_{L^2(V_\delta)}} \ge (1-a_0\delta)\frac{D^{r(\delta)}_{\theta,\alpha(1+a\delta)}[\mathcal{U}_\delta v,\mathcal{U}_\delta v]}{\|\mathcal{U}_\delta v\|^2_{\mathcal{S}_\theta^{r(\delta)}}}.$$

As noted above, the map $\mathcal{U}_\delta : \mathcal{Q}(D^{\delta}_{\Gamma,\alpha}) \to \mathcal{Q}(D^{r(\delta)}_{\theta,\alpha(1+a\delta)})$ is bijective, and the preceding estimate is uniform in u. Therefore, the min-max principle gives

$$E_n(D^{\delta}_{\Gamma,\alpha}) \ge (1-a_0\delta)E_n(D^{r(\delta)}_{\theta,\alpha(1+a\delta)}), \quad n\in\mathbb{N}. \qquad \square$$

Corollary 37 (First eigenvalues of Robin-Dirichlet/Neumann Laplacians in truncated curvilinear convex sectors)**.** *There is $b>0$ such that for $\delta\to 0^+$ and $\alpha\delta\to+\infty$ one has*

$$E_n(D^{\delta}_{\Gamma,\alpha}) = \mathcal{E}_n(\theta)\,\alpha^2 + \mathcal{O}(\alpha^2\delta + \alpha^2 e^{-b\alpha\delta}), \qquad n\in\big\{1,\dots,\kappa(\theta)\big\},$$
$$E_n(N^{\delta}_{\Gamma,\alpha}) = \mathcal{E}_n(\theta)\,\alpha^2 + \mathcal{O}(\alpha^2\delta + 1/\delta^2), \qquad n\in\big\{1,\dots,\kappa(\theta)\big\},$$

and $E_{\kappa(\theta)+1}(D^{\delta}_{\Gamma,\alpha}) \ge E_{\kappa(\theta)+1}(N^{\delta}_{\Gamma,\alpha}) \ge -\alpha^2 + o(\alpha^2)$.

Proof. With the help of Lemma 36 for $n\in\big\{1,\dots,\kappa(\theta)\big\}$ one has

$$\big(1+\mathcal{O}(\delta)\big)E_n(D^{r(\delta)}_{\theta,\alpha(1+a\delta)}) \le E_n(D^{\delta}_{\Gamma,\alpha}) \le \big(1+\mathcal{O}(\delta)\big)E_n(D^{r(\delta)}_{\theta,\alpha(1-a\delta)}),$$
$$\big(1+\mathcal{O}(\delta)\big)E_n(N^{r(\delta)}_{\theta,\alpha(1+a\delta)}) \le E_n(N^{\delta}_{\Gamma,\alpha}) \le \big(1+\mathcal{O}(\delta)\big)E_n(N^{r(\delta)}_{\theta,\alpha(1-a\delta)}).$$

As $r(\delta)\alpha\to+\infty$ for $\alpha\delta\to+\infty$, we can apply Lemmas 24 to $D^{r(\delta)}_{\theta,\alpha(1\pm a\delta)}$ and Lemma 27 to $N^{r(\delta)}_{\theta,\alpha(1\pm a\delta)}$ to get, with some $c>0$,

$$E_n(D^{\delta}_{\Gamma,\alpha}) = \big(1+\mathcal{O}(\delta)\big)\Big(\mathcal{E}_n(\theta)+\mathcal{O}(e^{-c\alpha\delta})\Big)\alpha^2 = \Big(\mathcal{E}_n(\theta)+\mathcal{O}(\delta+e^{-c\alpha\delta})\Big)\alpha^2,$$
$$E_n(N^{\delta}_{\Gamma,\alpha}) = \big(1+\mathcal{O}(\delta)\big)\Big(\mathcal{E}_n(\theta)+\mathcal{O}(\frac{1}{(\alpha\delta)^2})\Big)\alpha^2 = \mathcal{E}_n(\theta)\alpha^2 + \mathcal{O}(\alpha^2\delta+\frac{1}{\delta^2})\Big),$$

and

$$E_{\kappa(\theta)+1}(N^{\delta}_{\Gamma,\alpha}) \ge (1-a_0\delta)E_n(N^{r(\delta)}_{\theta,\alpha(1+a\delta)}) \ge (1-a_0\delta)\big(-1+o(1)\big)\alpha^2(1+a\delta)^2 = \alpha^2+o(\alpha^2).$$

The remaining inequality $E_{\kappa(\theta)+1}(D^{\delta}_{\Gamma,\alpha}) \ge E_{\kappa(\theta)+1}(N^{\delta}_{\Gamma,\alpha})$ is valid due to the standard monotonicity argument (see Remark 16). $\square$

Corollary 38 (Robin Laplacians in truncated curvilinear convex sectors)**.** *There exists $c>0$ such that $E_1(R^{\delta}_{\Gamma,\alpha}) \ge -c\alpha^2$ for $\delta\to0^+$ and $\alpha\delta\to+\infty$.*

Proof. One applies Lemma 36 in order to reduce the problem to the study of $E_1(R^{r(\delta)}_{\theta,\alpha(1+a\delta)})$, which is then estimated using Lemma 28. $\square$

Corollary 39 (Lower bound for $P^{\delta,\rho}_{\Gamma,\alpha}$)**.** *For $\delta\to0^+$, $\alpha\delta\to+\infty$ and $\rho<\delta$ there holds $E_1(P^{\delta,\rho}_{\Gamma,\alpha}) \ge -\alpha^2+o(\alpha^2)$.*

Proof. One uses first Lemma 36 in order to reduce the problem to the study of $E_1(P^{r(\delta),r(\rho)}_{\theta,\alpha(1+a\delta)})$, which is then estimated using Lemma 26. $\square$

4.3. **Eigenfunctions of the Robin-Neumann Laplacians.** We will need an Agmon-type decay estimate for the first $\kappa(\theta)$ eigenfunctions of $N^{\delta}_{\Gamma,\alpha}$, which is established in the following lemma:

Lemma 40 (Agmon estimate for the first eigenfunctions of $N^{\delta}_{\Gamma,\alpha}$)**.** *There exist $c>0$ and $C>0$ such that if $n\in\big\{1,\dots,\kappa(\theta)\big\}$ and $\psi^{\delta,n}_{\Gamma,\alpha}$ is an eigenfunction of $N^{\delta}_{\Gamma,\alpha}$ for the nth eigenvalue, then for $\delta\to 0^+$ and $\alpha\delta\to+\infty$ there holds*

$$\iint_{V_\delta} e^{c\alpha|x|}\Big(\frac{1}{\alpha^2}|\nabla\psi^{\delta,n}_{\Gamma,\alpha}|^2+|\psi^{\delta,n}_{\Gamma,\alpha}|^2\Big)\,dx\le C\,\big\|\psi^{\delta,n}_{\Gamma,\alpha}\big\|^2_{L^2(V_\delta)}.$$

Proof. During the proof we denote for shortness

$$N:=N^{\delta}_{\Gamma,\alpha},\quad \psi:=\psi^{\delta,n}_{\Gamma,\alpha},\quad \mathcal{E}:=\mathcal{E}_{\kappa(\theta)}<-1.$$

For $b>0$ to be chosen later let us consider the function $\phi: V_\delta\ni x\mapsto b|x|\in\mathbb{R}$, then $|\nabla\phi|=b$, and a standard computation gives

$$\begin{aligned}
N\big[e^{\alpha\phi}\psi,e^{\alpha\phi}\psi\big]&\equiv\iint_{V_\delta}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,\mathrm{d}x-\alpha\int_{\partial_*V_\delta}e^{2\alpha\phi}\psi^2\,\mathrm{d}s\\
&=\iint_{V_\delta}e^{2\alpha\phi}\Big((-\Delta\psi)\psi+\alpha^2|\nabla\phi|^2\psi^2\Big)\,\mathrm{d}x\\
&=\iint_{V_\delta}e^{2\alpha\phi}\big(E_n(N)+b^2\alpha^2\big)\psi^2\,\mathrm{d}x.
\end{aligned}$$

Due to Corollary 37 one has $E_n(N^{\delta}_{\Gamma,\alpha})=\big(\mathcal{E}_n(\theta)+o(1)\big)\alpha^2$ for $\alpha\delta\to+\infty$. Therefore, if we pick an arbitrary $\varepsilon>0$, then $E_n(N^{\delta}_{\Gamma,\alpha})\le(\mathcal{E}+\varepsilon)\alpha^2$, and

$$N\big[e^{\alpha\phi}\psi,e^{\alpha\phi}\psi\big]\le(\mathcal{E}+b^2+\varepsilon)\alpha^2\iint_{V_\delta}e^{2\alpha\phi}\psi^2dx. \tag{26}$$

On the other hand, let us pick $\eta\in(0,1)$ whose exact value will be chosen later, and set $\rho:=L/\alpha$ with a value $L>0$ to be chosen later as well, then

$$\begin{aligned}
N\big[e^{\alpha\phi}\psi,e^{\alpha\phi}\psi\big]&\equiv\iint_{V_\delta}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,\mathrm{d}x-\alpha\int_{\partial_*V_\delta}e^{2\alpha\phi}\psi^2\,\mathrm{d}s\\
&=\eta\iint_{V_\delta}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,\mathrm{d}x+(1-\eta)\bigg[\iint_{V_\rho}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,\mathrm{d}x-\frac{\alpha}{1-\eta}\int_{\partial_*V_\rho}e^{2\alpha\phi}\psi^2\,\mathrm{d}s\\
&\quad+\iint_{V_\delta\setminus\overline{V_\rho}}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,\mathrm{d}x-\frac{\alpha}{1-\eta}\int_{\partial_*V_\delta\setminus\partial_*V_\rho}e^{2\alpha\phi}\psi^2\,\mathrm{d}s\bigg]\\
&=\eta\iint_{V_\delta}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,\mathrm{d}x\\
&\quad+(1-\eta)\Big(N^{\rho}_{\Gamma,\frac{\alpha}{1-\eta}}\big[e^{\alpha\phi}\psi,e^{\alpha\phi}\psi\big]+P^{\delta,\rho}_{\Gamma,\frac{\alpha}{1-\eta}}\big[e^{\alpha\phi}\psi,e^{\alpha\phi}\psi\big]\Big)\\
&\ge\eta\iint_{V_\delta}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,\mathrm{d}x+(1-\eta)E_1\big(N^{\rho}_{\Gamma,\frac{\alpha}{1-\eta}}\big)\iint_{V_\rho}e^{2\alpha\phi}\psi^2\,\mathrm{d}x\\
&\quad+(1-\eta)E_1\big(P^{\delta,\rho}_{\Gamma,\frac{\alpha}{1-\eta}}\big)\iint_{\mathcal{P}^{\delta,\rho}_{\Gamma}}e^{2\alpha\phi}\psi^2\,\mathrm{d}x.
\end{aligned}$$

By applying Corollary 37 for $N^{\rho}_{\Gamma,\frac{\alpha}{1-\eta}}$ and Corollary 39 to $P^{\delta,\rho}_{\Gamma,\frac{\alpha}{1-\eta}}$ we see that the constant L in the definition of ρ can be chosen sufficiently large to have, for large α,

$$E_1\big(N^{\rho}_{\Gamma,\frac{\alpha}{1-\eta}}\big)\ge\big(\mathcal{E}_1(\theta)-\varepsilon\big)\frac{\alpha^2}{(1-\eta)^2},\quad E_1\big(P^{\delta,\rho}_{\Gamma,\frac{\alpha}{1-\eta}}\big)\ge-\frac{(1+\varepsilon)\alpha^2}{(1-\eta)^2},$$

and the substitution into the preceding inequality gives

$$N\big[e^{\alpha\phi}\psi,e^{\alpha\phi}\psi\big]\ge\eta\iint_{V_\delta}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,dx+\frac{\mathcal{E}_1(\theta)-\varepsilon}{1-\eta}\,\alpha^2\iint_{V_\rho}e^{2\alpha\phi}\psi^2dx$$
$$-\frac{1+\varepsilon}{1-\eta}\,\alpha^2\iint_{\mathcal{P}_\Gamma^{\delta,\rho}}e^{2\alpha\phi}\psi^2dx.$$

Using $\mathcal{P}_\Gamma^{\delta,\rho}=V_\delta\setminus\overline{V_\rho}$ and substituting this inequality into (26) one arrives at

$$\eta\iint_{V_\delta}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,dx+\frac{\mathcal{E}_1(\theta)-\varepsilon}{1-\eta}\,\alpha^2\iint_{V_\rho}e^{2\alpha\phi}\psi^2dx-\frac{1+\varepsilon}{1-\eta}\,\alpha^2\iint_{\mathcal{P}_\Gamma^{\delta,\rho}}e^{2\alpha\phi}\psi^2dx$$
$$\le(\mathcal{E}+b^2+\varepsilon)\alpha^2\iint_{\mathcal{P}_\Gamma^{\delta,\rho}}e^{2\alpha\phi}\psi^2dx+(\mathcal{E}+b^2+\varepsilon)\alpha^2\iint_{V_\rho}e^{2\alpha\phi}\psi^2dx,$$

which we rewrite as

$$\eta\iint_{V_\delta}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,dx+\Big(-\mathcal{E}-b^2-\varepsilon-\frac{1+\varepsilon}{1-\eta}\Big)\alpha^2\iint_{V_\delta\setminus\overline{V_\rho}}e^{2\alpha\phi}\psi^2dx$$
$$\le\Big(\mathcal{E}+b^2+\varepsilon-\frac{\mathcal{E}_1(\theta)-\varepsilon}{1-\eta}\Big)\alpha^2\iint_{V_\rho}e^{2\alpha\phi}\psi^2dx.$$

We have

$$a_0:=-\mathcal{E}-b^2-\varepsilon-\frac{1+\varepsilon}{1-\eta}==\frac{-\mathcal{E}-1+\big(\eta b^2-b^2+\eta\mathcal{E}-2\varepsilon+\varepsilon\eta\big)}{1-\eta},$$

and due to $\mathcal{E}<-1$ one can choose $\varepsilon>0$, $\eta>0$ and $b>0$ sufficiently small to have $a_0>0$. Therefore, with the notation

$$b_0:=\mathcal{E}+b^2+\varepsilon-\frac{\mathcal{E}_1(\theta)-\varepsilon}{1-\eta}=\frac{\mathcal{E}-\mathcal{E}_1(\theta)-\eta\mathcal{E}+2\varepsilon-\varepsilon\eta}{1-\eta}+b^2>b^2>0$$

we arrive at

$$\eta\iint_{V_\delta}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,dx+a_0\alpha^2\iint_{V_\delta\setminus\overline{V_\rho}}e^{2\alpha\phi}\psi^2dx\le b_0\alpha^2\iint_{V_\rho}e^{2\alpha\phi}\psi^2dx.$$

Due to Corollary 34, with some $a>0$ one has $|x|\le a\rho$ for all $x\in V_\rho$. Therefore, in V_ρ one has $\alpha\phi\le\alpha baL/\alpha=baL$, and the preceding inequalitiy takes the form

$$\eta\iint_{V_\delta}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,dx+a_0\alpha^2\iint_{V_\delta\setminus\overline{V_\rho}}e^{2\alpha\phi}\psi^2dx\le A\alpha^2\iint_{V_\rho}\psi^2dx,\quad A:=b_0e^{2baL},$$

and then

$$\iint_{V_\delta}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,dx+2b^2\alpha^2\iint_{V_\delta}\big|e^{2\alpha\phi}\psi\big|^2\,dx$$
$$=\frac{1}{\eta}\,\eta\iint_{V_\delta}\big|\nabla(e^{\alpha\phi}\psi)\big|^2\,dx+\frac{2b^2}{a_0}\,a_0\alpha^2\iint_{V_\delta\setminus\overline{V_\rho}}e^{2\alpha\phi}\psi^2dx+2b^2\alpha^2\iint_{V_\rho}\psi^2dx$$
$$\le\Big(\frac{1}{\eta}\,A+\frac{2b^2}{a_0}\,A+2b^2\Big)\alpha^2\iint_{V_\rho}\psi^2dx=:A_0\alpha^2\iint_{V_\rho}\psi^2dx\le A_0\alpha^2\|\psi\|^2_{L^2(V_\delta)}.\quad(27)$$

Finally, using the elementary inequality $xy\le\frac14x^2+y^2$ for $x,y\in\mathbb{R}$ we estimate

$$\big|\nabla(e^{\alpha\phi}\psi)\big|^2=|e^{\alpha\phi}\nabla\psi|^2+\alpha^2|\psi e^{\alpha\phi}\nabla\phi|^2+2\alpha(e^{\alpha\phi}\nabla\psi)\cdot(\psi e^{\alpha\phi}\nabla\phi)$$
$$\ge|e^{\alpha\phi}\nabla\psi|^2+b^2\alpha^2|e^{\alpha\phi}\psi|^2-2\big|e^{\alpha\phi}\nabla\psi\big|\,\big|b\alpha e^{\alpha\phi}\psi\big|$$
$$\ge|e^{\alpha\phi}\nabla\psi|^2+b^2\alpha^2|e^{\alpha\phi}\psi|^2-2\left(\frac14\big|e^{\alpha\phi}\nabla\psi\big|^2+\big|b\alpha e^{\alpha\phi}\psi\big|^2\right)$$

$$\geq \frac{1}{2}|e^{\alpha\phi}\nabla\psi|^2 - b^2\alpha^2|e^{\alpha\phi}\psi|^2.$$

The substitution into (27) gives

$$\iint_{V_\delta} e^{2b\alpha|x|}\Big(\frac{1}{2}|\nabla\psi|^2 + b^2\alpha^2\psi^2\Big)\,dx \leq A_0\alpha^2\|\psi\|^2_{L^2(V_\delta)},$$

and one arrives to the claim by taking $c := 2b$ and $C := A_0(2+1/b^2)$. □

4.4. **Estimates for non-resonant convex sectors.** The above results are valid for all possible values of the half-angle θ. Let us consider in greater detail the case when θ is non-resonant. The estimates of this subsection will be of crucial importance for the subsequent analysis.

Corollary 41 (Eigenvalues of non-resonant truncated convex sectors)**.** *Assume that the half-angle θ is non-resonant, then for any $A\in\mathbb{R}$ there exists $c>0$ such that for $\alpha\delta\to+\infty$ and $\delta\to 0^+$ with $\alpha^2\delta^3\to 0^+$ there holds*

$$E_{\kappa(\theta)+1}(D^\delta_{\Gamma,\alpha}) \geq E_{\kappa(\theta)+1}(N^\delta_{\Gamma,\alpha}) \geq -\alpha^2 + A\alpha + c/\delta^2.$$

Proof. Using first Lemma 36 we see that there exist $a>0$, $a_0>0$ and a C^2 function r defined near zero with $r=0$ and $r'(0)=\operatorname{cotan}\theta$ such that for $\delta\to 0^+$ and $\alpha\delta\to+\infty$ there holds $E_n(N^\delta_{\Gamma,\alpha}) \geq (1-a_0\delta)E_n(N^{r(\delta)}_{\theta,\alpha(1+a\delta)})$. Recall that due to the definition of a non-resonance half-angle (Definition 29) with some $C>0$ we have $E_{\kappa(\theta)+1}(N^r_{\theta,\alpha}) \geq -\alpha^2 + C/r^2$ as αr is large. In the asymptotic regime under consideration we have $\alpha(1+a\delta)r(\delta)\sim\alpha\delta\operatorname{cotan}\theta\to+\infty$, hence,

$$E_n(N^{r(\delta)}_{\theta,\alpha(1+a\delta)}) \geq -\alpha^2(1+a\delta)^2 + \frac{C}{r(\delta)^2} \geq -\alpha^2 - 3a\alpha^2\delta + \frac{C_0}{\delta^2}, \quad C_0 := \frac{C\tan^2\theta}{2},$$

$$\begin{aligned}(1-a_0\delta)E_n(N^{r(\delta)}_{\theta,\alpha(1+a\delta)}) &\geq (1-a_0\delta)\Big(-\alpha^2 - 3a\alpha^2\delta + \frac{C_0}{\delta^2}\Big)\\ &\geq -\alpha^2 - 3a\alpha^2\delta + \frac{C_0}{\delta^2} - \frac{a_0C_0}{\delta} \geq -\alpha^2 - 3a\alpha^2\delta + \frac{C_0}{2\delta^2}\\ &= -\alpha^2 + A\alpha + \frac{1}{\delta^2}\Big(\frac{1}{2}\,C_0 - 3a\alpha^2\delta^3 - A\alpha\delta^2\Big).\end{aligned}$$

For $\alpha^2\delta^3\to 0^+$ one has $\alpha\delta^2 = \alpha^2\delta^3/(\alpha\delta)\to 0^+$, and for any fixed $c\in(0,C_0/2)$ there holds

$$E_{\kappa(\theta)+1}(N^\delta_{\Gamma,\alpha}) \geq -\alpha^2 + A\alpha + c/\delta^2.$$

The inequality $E_{\kappa(\theta)+1}(D^\delta_{\Gamma,\alpha}) \geq E_{\kappa(\theta)+1}(N^\delta_{\Gamma,\alpha})$ follows from the min-max principle by the standard monotonicity argument (see Remark 16). □

The following result puts the preceding considerations into a special form to be used later in the proof of the main results:

Corollary 42 (Trace and norm estimates in a non-resonant truncated convex sector)**.** *Assume that θ is non-resonant and denote by $\mathcal{L}^\delta_{\Gamma,\alpha}$ the subspace spanned by the eigenfunctions corresponding to the first $\kappa(\theta)$ eigenvalues of $N^\delta_{\Gamma,\alpha}$. Then for any $A\in\mathbb{R}$ there exists $b>0$ such that for $\delta\to 0^+$ and $\alpha\delta\to+\infty$ with $\alpha^2\delta^3\to 0^+$ there holds*

$$\|v\|^2_{L^2(V_\delta)} \leq b\,\delta^2\Big(N^\delta_{\Gamma,\alpha}[v,v] + (\alpha^2 - A\alpha)\|v\|^2_{L^2(V_\delta)}\Big), \quad v\in H^1(V_\delta)\cap\big(\mathcal{L}^\delta_{\Gamma,\alpha}\big)^\perp, \tag{28}$$

$$\int_{\partial_{\text{ext}}V_\delta} v^2\,\mathrm{d}s \leq b\alpha\delta^2\Big(N^\delta_{\Gamma,\alpha}[v,v] + (\alpha^2 - A\alpha)\|v\|^2_{L^2(V_\delta)}\Big), \quad v\in H^1(V_\delta)\cap\big(\mathcal{L}^\delta_{\Gamma,\alpha}\big)^\perp. \tag{29}$$

Proof. The norm estimate (28) directly follows from the preceding Corollary 41 with the help of the spectral theorem. To obtain (29) consider first the operator $R^\delta_{\Gamma,\alpha}$ defined as the Laplacian in V_δ with the α-Robin boundary condition at the whole boundary, then it was shown in Corollary 38 that with some $c_0 > 0$ for $\delta \to 0^+$ and $\alpha\delta \to +\infty$ one has $R^\delta_{\Gamma,\alpha} \ge -c_0\alpha^2$. Using the fact that

$$\mathcal{Q}(N^\delta_{\Gamma,\alpha}) = \mathcal{Q}(R^\delta_{\Gamma,\alpha}) = H^1(V_\delta), \quad R^\delta_{\Gamma,\alpha}[v,v] = N^\delta_{\Gamma,\alpha}[v,v] - \alpha \int_{\partial_{\mathrm{ext}} V_\delta} v^2 \,\mathrm{d}s$$

for large α we can rewrite the preceding inequality for $R^\delta_{\Gamma,\alpha}$ as

$$\int_{\partial_{\mathrm{ext}} V_\delta} v^2 \,\mathrm{d}s \le \frac{1}{\alpha} N^\delta_{\Gamma,\alpha}[v,v] + c_0\alpha \|v\|^2_{L^2(V_\delta)} \text{ for all } v \in H^1(V_\delta).$$

Assume in addition that $v \perp \mathcal{L}^\delta_{\Gamma,\alpha}$, then one can find an upper bound of the second term on the right-hand side with the help of (28), and

$$\begin{aligned}
\int_{\partial_{\mathrm{ext}} V_\delta} v^2 \,\mathrm{d}s &\le \frac{1}{\alpha} N^\delta_{\Gamma,\alpha}[v,v] + c_0 b\, \alpha\delta^2 \Big(N^\delta_{\Gamma,\alpha}[v,v] + (\alpha^2 - A\alpha)\|v\|^2_{L^2(V_\delta)} \Big) \\
&= \Big(\frac{1}{\alpha} + c_0 b\, \alpha\delta^2 \Big) N^\delta_{\Gamma,\alpha}[v,v] + c_0 b\, \alpha\delta^2 (\alpha^2 - A\alpha)\|v\|^2_{L^2(V_\delta)} \\
&\le \Big(\frac{1}{\alpha} + c_0 b\, \alpha\delta^2 \Big) \Big(N^\delta_{\Gamma,\alpha}[v,v] + (\alpha^2 - A\alpha)\|v\|^2_{L^2(V_\delta)} \Big),
\end{aligned}$$

where on the last step we used the fact $\alpha^2 - A\alpha > 0$ for large α. Now it remains to note that $1/\alpha = o(\alpha\delta^2)$ because $\alpha\delta \to +\infty$ as $\alpha \to +\infty$. □

5. Analysis in thin neighborhoods of smooth open arcs

5.1. **Geometric setting and change of variables.** Let $\ell > 0$ and $\gamma : [0,\ell] \to \mathbb{R}^2$ be an injective C^3 function with $|\gamma'| = 1$, then ℓ is exactly the length of the smooth open arc $\Gamma := \gamma\big([0,\ell]\big)$. Let $\nu : [0,\ell] \to \mathbb{R}^2$ be the unit normal on Γ such that $\nu(s)$ is orthogonal to the tangent vector $\tau(s) := \gamma'(s)$ to Γ at the point $\gamma(s)$ and that $\nu(s)$ is obtained from $\tau(s)$ by the rotation by $\frac{\pi}{2}$ in the clockwise direction, then one always has $\nu(s) \wedge \tau(s) = 1$. The *curvature* $H(s)$ of Γ at the point $\gamma(s)$ is defined by $\nu'(s) = H(s)\tau(s)$. Under the assumptions made, the function $H : [0,\ell] \to \mathbb{R}$ is at least C^1. Now let $\delta_0 > 0$ and pick two C^1 functions

$$\lambda_\pm : [0,\delta_0) \to [0,\infty) \text{ with } \lambda_\pm(0) = 0 \text{ and } \lambda'_\pm(0) \ge 0,$$

and for $\delta \in (0,\delta_0)$ introduce the following objects:

$$\begin{aligned}
&\text{the maps } \Phi : (0,\ell) \times (0,\delta) \ni (s,t) \mapsto \gamma(s) - t\nu(s) \in \mathbb{R}^2, \\
&\text{the intervals } I_\delta := \big(\lambda_-(\delta), \ell - \lambda_+(\delta)\big), \\
&\text{the rectangles } \Pi_\delta := I_\delta \times (0,\delta), \quad \text{the images } W_\delta := \Phi(\Pi_\delta).
\end{aligned}$$

It is a classical result of the differential geometry that if $\delta_0 > 0$ is chosen sufficiently small, then for all $\delta \in (0,\delta_0)$ the maps Φ are diffeomorphisms between Π_δ and their images W_δ, with $\operatorname{dist}\big(\Phi(s,t),\Gamma\big) = t$, i.e. the sets W_δ are a kind of one-sided neighborhoods of the reduced arcs $\Gamma_\delta := \gamma(I_\delta)$. We decompose the boundary ∂W_δ into three parts:

$$\partial_* W_\delta := \Phi\big(I_\delta \times \{0\}\big) \equiv \Gamma_\delta, \quad \partial_{\mathrm{out}} W_\delta := \Phi\big(I_\delta \times \{\delta\}\big), \quad \partial_{\mathrm{ext}} W_\delta := \partial W_\delta \setminus \big(\partial_* W_\delta \cup \partial_{\mathrm{out}} W_\delta\big),$$

see Figure 13. We are going to give some estimates for the eigenvalues of Laplacians on W_δ with the α-Robin boundary condition at $\partial_* W_\delta$ and Dirichlet or Neumann boundary conditions at the remaining boundary.

The computations of the following lemma are standard, but we prefer to give full details in order to have a self-contained presentation.

Lemma 43 (Weighted change of variables in W_δ)**.** *Consider the unitary transform*

$$G_\delta : L^2(W_\delta) \to L^2(\Pi_\delta), \quad (G_\delta u)(s,t) = \big(1 - tH(s)\big)^{\frac{1}{2}} u\big(\Phi(s,t)\big),$$

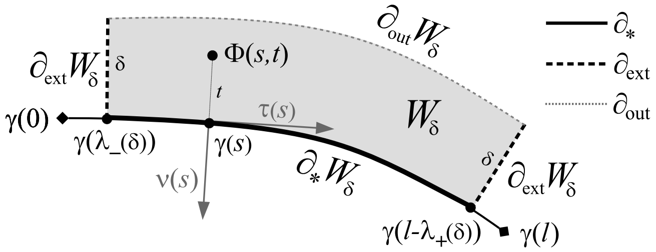


FIGURE 13. The curvilinear rectangle W_δ is shaded. The boundary part $\partial_* W_\delta \equiv \Gamma_\delta$ is shown by the thick solid line, the part $\partial_{\text{ext}} W_\delta$ is shown by the thick dashed line, and $\partial_{\text{out}} W_\delta$ is marked by the thin dashed line.

then $u \in H^1(W_\delta)$ if and only if $g := G_\delta u \in H^1(\Pi_\delta)$, and there exists $b > 0$ such that for sufficiently small δ, for all functions u and g as above and for all $\alpha > 0$ one has the two-sided estimate

$$B_-[g,g] \le \iint_{W_\delta} |\nabla u|^2 dx - \alpha \int_{\partial_* W_\delta} u^2 \,\mathrm{d}s \le B_+[g,g],$$

$$\textit{where } B_\pm[g,g] := \int_{I_\delta}\int_0^\delta \Big[\Big(1 \pm b\delta\Big)\Big(\frac{\partial g}{\partial s}\Big)^2 + \Big(\frac{\partial g}{\partial t}\Big)^2 - \Big(\frac{H^2}{4} \mp b\delta\Big) g^2\Big] \,\mathrm{d}t\,\mathrm{d}s - \int_{I_\delta}\Big(\alpha + \frac{H}{2}\Big) g(s,0)^2 \,\mathrm{d}s \pm b\int_{I_\delta} g(s,\delta)^2 \,\mathrm{d}s$$

Proof. The equivalence $u \in H^1$ iff $g \in H^1$ is obvious in view of the smoothness of H and Φ, so let us concentrate on the bounds for the integrals. For $u \in H^1(W_\delta)$ consider first $v := u \circ \Phi$, then the standard change of variables gives

$$\begin{aligned} A &:= \iint_{W_\delta} |\nabla u|^2 \,\mathrm{d}x - \int_{\partial_* W_\delta} u^2 \,\mathrm{d}s \\ &= \int_{I_\delta}\int_0^\delta \Big[\frac{1}{1-tH}\Big(\frac{\partial v}{\partial s}\Big)^2 + (1-tH)\Big(\frac{\partial v}{\partial t}\Big)^2\Big] \,\mathrm{d}t\,\mathrm{d}s - \alpha \int_{I_\delta} v(s,0)^2 \,\mathrm{d}s. \end{aligned}$$

As $v = (1-tH)^{-\frac12} g$, a direct computation gives

$$\begin{aligned} A &= \int_{I_\delta}\int_0^\delta \Big[\frac{1}{(1-tH)^2}\Big(\frac{\partial g}{\partial s} + \frac{tH'}{2(1-tH)}\,g\Big)^2 + \Big(\frac{\partial g}{\partial t} + \frac{H}{2(1-tH)}\,g\Big)^2\Big] \,\mathrm{d}t\,\mathrm{d}s - \alpha \int_{I_\delta} g(s,0)^2 \,\mathrm{d}s \\ &= \int_{I_\delta}\int_0^\delta \Big[\frac{1}{(1-tH)^2}\Big(\frac{\partial g}{\partial s}\Big)^2 + \frac{tH'}{(1-tH)^3}\,g\frac{\partial g}{\partial s} + \frac{(tH')^2}{4(1-tH)^4} g^2 \\ &\quad + \Big(\frac{\partial g}{\partial t}\Big)^2 + \frac{H}{1-tH} g\frac{\partial g}{\partial t} + \frac{H^2}{4(1-tH)^2}\,g^2\Big] \,\mathrm{d}t\,\mathrm{d}s - \alpha \int_{I_\delta} g(s,0)^2 \,\mathrm{d}s. \end{aligned}$$

Using the integration by parts we obtain

$$\begin{aligned} \int_0^\delta \frac{H}{1-tH} g\frac{\partial g}{\partial t} \,\mathrm{d}t &= \frac12 \int_0^\delta \frac{H}{1-tH}\frac{\partial (g^2)}{\partial t} \,\mathrm{d}t \\ &= \frac{H}{1-\delta H} g(\cdot,\delta)^2 - \frac{H}{2}\,g(\cdot,0)^2 - \int_0^\delta \frac{H^2}{2(1-tH)^2}\,g^2 \,\mathrm{d}t, \end{aligned}$$

and then

$$
\begin{aligned}
A = \int_{I_\delta}\int_0^\delta \Big[&\frac{1}{(1-tH)^2}\Big(\frac{\partial g}{\partial s}\Big)^2 + \frac{tH'}{(1-tH)^3}\, g\frac{\partial g}{\partial s} + \Big(\frac{\partial g}{\partial t}\Big)^2\\
&+ \Big(\frac{(tH')^2}{4(1-tH)^4} - \frac{H^2}{4(1-tH)^2}\Big)\, g^2\Big]\,\mathrm{d}t\,\mathrm{d}s - \int_{I_\delta}\Big(\alpha+\frac{H}{2}\Big) g(s,0)^2\,\mathrm{d}s\\
&+ \int_{I_\delta}\frac{H}{1-\delta H} g(s,\delta)^2\,\mathrm{d}s.
\end{aligned}
\tag{30}
$$

For a suitable $a>0$, a sufficiently small δ and $t\in(0,\delta)$ there holds, uniformly in s,

$$
\Big| g\frac{\partial g}{\partial s}\Big| \le \frac{1}{2}\Big(\frac{\partial g}{\partial s}\Big)^2 + \frac{1}{2}\,g^2, \quad |tH'|\le a\delta, \quad \Big|\frac{1}{(1-tH)^j}-1\Big| \le a\delta \text{ with } j\in\{1,2,3,4\},
$$

so putting all these estimates into (30) one arrives at the result. □

5.2. **Bounds for the case of a constant curvature.** Everywhere in the present section we assume that the curvature H is constant and obtain some bound for the eigenvalues of various Laplacians in W_δ. Remark that the computations can alternatively be done using Bessel functions (as the curve γ is then either a circle arc or a straight line), but we prefer to do it starting from the general principles in order to avoid considering various special cases.

Lemma 44. *Assume that H is constant and denote by D^W the Laplacian in W_δ with the α-Robin boundary condition at $\partial_* W_\delta$ and the Dirichlet boundary condition at the remaining part of the boundary, then there exists $b>0$ such that for $\delta\to 0^+$ and $\alpha\delta\to+\infty$ one has*

$$
E_n(D^W) \le -\alpha^2 - \alpha H - \frac{H^2}{2} + (1+b\delta)E_n(D_\delta) + b(\delta+\alpha^2 e^{-\alpha\delta}), \quad n\in\mathbb{N},
$$

where D_δ is the Dirichlet Laplacian on I_δ.

Proof. Due to Lemma 43, for some $b_0>0$ one has $E_n(D^W)\le E_n(B_+)$, where B_+ is the self-adjoint operator in $H^1(\Pi_\delta)$ with

$$
\begin{aligned}
\mathcal{Q}(B_+) &= \big\{ g\in H^1\big(I_\delta\times(0,\delta)\big) :\ g(\cdot,\delta)=0,\ g(\iota,\cdot)=0 \text{ for each } \iota\in\partial I_\delta\big\},\\
B_+[g,g] &:= \int_{I_\delta}\int_0^\delta \Big[(1+b_0\delta)\Big(\frac{\partial g}{\partial s}\Big)^2 + \Big(\frac{\partial g}{\partial t}\Big)^2\\
&\qquad - \Big(\frac{H^2}{4}-b_0\delta\Big) g^2\Big]\,\mathrm{d}t\,\mathrm{d}s - \int_{I_\delta}\Big(\alpha+\frac{H}{2}\Big) g(s,0)^2\,\mathrm{d}s.
\end{aligned}
$$

One easily sees that for a constant H the operator B_+ admits a separation of variables and is unitary equivalent to $C_+ := (1+b_0\delta)D_\delta\otimes 1 + 1\otimes L_D - (H^2/4 - b_0\delta)$, where L_D is the Laplacian on $(0,\delta)$ with the $(\alpha+H/2)$-Robin boundary condition at 0 and the Dirichlet boundary condition at δ, so using Proposition 17 with some $b_1>0$ we have $E_1(L_D) = -(\alpha+H/2)^2 + b_1\alpha^2 e^{-\alpha\delta}$ and $E_2(L_D)\ge 0$. Therefore, for each fixed $n\in\mathbb{N}$ due to $E_n(D_\delta)=\mathcal{O}(1)$ we have

$$
\begin{aligned}
E_n(D^W) \le E_n(B_+) = E_n(C_+) &= E_1(L_D) + (1+b_0\delta)E_n(D_\delta) - \Big(\frac{H^2}{4}-b_0\delta\Big),\\
&\le -\alpha^2 - \alpha H - \frac{H^2}{2} + (1+b_0\delta)E_n(D_\delta) + b_0\delta + b_1\alpha^2 e^{\alpha\delta},
\end{aligned}
$$

and arrive at the result by taking $b:=\max\{b_0,b_1\}$. □

Lemma 45. *Assume that H is constant and denote by N^W the Laplacian in W_δ with the α-Robin boundary condition on $\partial_* W_\delta$ and the Neumann boundary condition at the remaining* *boundary. There are functions $\psi \in L^2(0,\delta)$ with $\|\psi\|^2_{L^2(0,\delta)} = 1$ such that if one defines the change of variables*

$$G_\delta : L^2(W_\delta) \to L^2(\Pi_\delta), \quad (G_\delta u)(s,t) = \big(1 - tH(s)\big)^{\frac{1}{2}} u\big(\Phi(s,t)\big),$$

and the map

$$P : L^2(W_\delta) \to L^2(I_\delta), \quad (Pu)(s) := \int_0^\delta \psi(t)(G_\delta u)(s,t)\,\mathrm{d}t, \tag{31}$$

then one has for $\delta \to 0^+$ and $\alpha\delta \to +\infty$ and with some $b > 0$

$$\begin{aligned} N^W[u,u] \geq -\Big(\alpha^2 + \alpha H + \frac{H^2}{2}\Big)\|Pu\|^2_{L^2(I_\delta)} + \big(1 - b\delta\big)\big\|(Pu)'\big\|^2_{L^2(I_\delta)} \\ - b\big(\delta + \alpha^2 e^{-\alpha\delta}\big)\|Pu\|^2_{L^2(I_\delta)} \text{ for all } u \in H^1(W_\delta). \end{aligned} \tag{32}$$

In particular,

$$N^W \geq -\Big(\alpha^2 + \alpha H + \frac{H^2}{2}\Big) + \mathcal{O}(\delta + \alpha^2 e^{-c\alpha\delta}). \tag{33}$$

Proof. Denote $g := G_\delta u \in L^2(\Pi_\delta)$, then due to the standard change of variables (Lemma 43) one can find $b_0 > 0$ and $\beta > 0$ to have, for all $u \in H^1(W_\delta)$,

$$\begin{aligned} N^W[u,u] \geq B_-[g,g] := \int_{I_\delta}\int_0^\delta \Big[\big(1 - b_0\delta\big)g_s^2 + g_t^2 \\ - \Big(\frac{H^2}{4} + b_0\delta\Big)g^2\Big]\,\mathrm{d}t\,\mathrm{d}s - \int_{I_\delta}\Big(\alpha + \frac{H}{2}\Big)g(s,0)^2\,\mathrm{d}s - \beta\int_{I_\delta} g(s,\delta)^2\,\mathrm{d}s, \end{aligned}$$

where we have set $g_s := \partial g/\partial s$ and $g_t := \partial g/\partial t$. Denote by L_N the one-dimensional Laplacian in $(0,\delta)$ with the $(\alpha + H/2)$-Robin boundary condition at 0 and the β-Robin boundary condition at δ, and let ψ be its eigenfunction for the first eigenvalue, normalized by $\|\psi\|_{L^2(0,\delta)} = 1$. With this choice of ψ, define the map P as in (31). For shortness we denote $f := Pu$ and define $z \in L^2(\Pi_\delta)$ by $z(s,t) := g(s,t) - f(s)\psi(t)$, then, with $z_s := \partial z/\partial s$, we have the identities

$$\begin{aligned} \int_0^\delta \psi(t)z(\cdot,t)\,\mathrm{d}t = 0, \quad \int_0^\delta \psi(t)z_s(\cdot,t)\,\mathrm{d}t = 0, \\ \|u\|^2_{L^2(W_\delta)} = \|g\|^2_{L^2(\Pi_\delta)} = \|f\|^2_{L^2(I_\delta)} + \|z\|^2_{L^2(\Pi_\delta)}, \end{aligned} \tag{34}$$

and due to the spectral theorem for the operator L_N there holds

$$\begin{aligned} \int_{I_\delta}\int_0^\delta g_t^2\,\mathrm{d}t\,\mathrm{d}s - \int_{I_\delta}\Big(\alpha + \frac{H}{2}\Big)g(s,0)^2\,\mathrm{d}s - \beta\int_{I_\delta} g(s,\delta)^2\,\mathrm{d}s \\ \geq \int_{I_\delta}\int_0^\delta \Big(E_1(L_N)f(s)^2\psi(t)^2 + E_2(L_N)z(s,t)^2\Big)\,\mathrm{d}t\,\mathrm{d}s \\ = E_1(L_N)\|f\|^2_{L^2(I_\delta)} + E_2(L_N)\|z\|^2_{L^2(\Pi_\delta)}, \end{aligned}$$

an using the second equality in (34) we also have

$$\int_{I_\delta}\int_0^\delta g_s^2\,\mathrm{d}t\,\mathrm{d}s = \|f'\|^2_{L^2(I_\delta)} + \|z_s\|^2_{L^2(\Pi_\delta)} \geq \|f'\|^2_{L^2(I_\delta)}.$$

Therefore,

$$\begin{aligned} B_-[g,g] \geq (1 - b_0\delta)\|f'\|^2_{L^2(I_\delta)} + \Big(E_1(L_N) - \frac{H^2}{4} - b_0\delta\Big)\|f\|^2_{L^2(I_\delta)} \\ + \Big(E_2(L_N) - \frac{H^2}{4} - b_0\delta\Big)\|z\|^2_{L^2(\Pi_\delta)}. \end{aligned}$$

Using Proposition 18 in order to estimate the eigenvalues of L_N one has then, with a suitable $a_0 > 0$,

$$\begin{aligned} E_1(L_N) - \frac{H^2}{4} - b_0\delta &= -\Big(\alpha + \frac{H}{2}\Big)^2 - a_0\alpha^2 e^{-\alpha\delta} - \frac{H^2}{4} - b_0\delta \\ &\geq -\alpha^2 - \alpha H - \frac{H^2}{2} - a_1\big(\delta + \alpha^2 e^{-\alpha\delta}\big), \quad a_1 := \max\{a_0, b_0\}, \\ E_2(L_N) - \frac{H^2}{4} - b_0\delta &\geq \frac{1}{\delta^2} - \frac{H^2}{4} - b_0\delta \geq 0, \end{aligned}$$

and then

$$B_-[g,g] \geq (1 - b_0\delta)\|f'\|^2_{L^2(I_\delta)} - \Big(\alpha^2 - \alpha H - \frac{H^2}{2}\Big)\|f\|^2_{L^2(I_\delta)} - a_1\big(\delta + \alpha^2 e^{-\alpha\delta}\big)\|f\|^2_{L^2(I_\delta)}.$$

Hence, one arrives at the sought inequality (32) by taking $b := \max\{b_0, a_1\}$. To prove the lower bound (33) is it sufficient to use $\|f\|^2_{L^2(I_\delta)} \leq \|u\|_{L^2(W_\delta)}$. ☐

6. Robin Laplacians in curvilinear polygons

6.1. **Decomposition of curvilinear polygons.** Let us describe more precisely the class of domains Ω we are going to deal with as well as its decomposition into pieces of special shape.

A bounded domain $\Omega \subset \mathbb{R}^2$ will be called a *curvilinear polygon* with $M \geq 1$ vertices if there exist $A_1, \dots, A_M \in \mathbb{R}^2$ and $\ell_1, \dots, \ell_M > 0$ such that:

- there are injective C^3 maps $\gamma_j : \mathbb{R} \to \mathbb{R}^2$ with $|\gamma_j'| = 1$ such that

$$\gamma_j(0) = A_j, \quad \gamma_j(\ell_j) = A_{j+1}, \quad j \in \{1, \dots, M\},$$

  where we identify $A_0 \equiv A_M$ and $A_{M+1} \equiv A_1$, and the same numbering convention applies to the finite arcs $\Gamma_j := \gamma_j\big((0,\ell_j)\big)$ which we assume mutually disjoint and such that $\Gamma := \partial\Omega = \bigcup_{j=1}^M \overline{\Gamma_j}$.
- The orientation of each γ_j is assumed to be chosen in such a way that if $\nu_j(t)$ is the *outer* unit normal to $\partial\Omega$ at a point $\gamma_j(t)$, then $\nu_j(s) \wedge \gamma_j'(s) = 1$, i.e. $\nu_j(s)$ is obtained by rotating the tangent vector $\gamma_j'(t)$ by $\frac{\pi}{2}$ in the clockwise direction, and the curvature $H_j(t)$ of Γ_j at the point $\gamma_j(s)$ is defined by $\nu_j'(s) = H_j(t)\,\gamma_j'(s)$.
- By $\theta_j \in [0, \pi]$ we denote the half-angle of the boundary at a vertex A_j, i.e. the number $\theta_j \in [0,\pi]$ is characterized by the conditions

$$\cos(2\theta_j) = \gamma_{j-1}'(\ell_{j-1}) \cdot \big(-\gamma_j'(0)\big), \quad \sin(2\theta_j) = \gamma_j'(0) \wedge \big(-\gamma_{j-1}'(\ell_{j-1})\big).$$

  Our assumption is that there are neither zero angles nor artificial vertices, i.e.

$$\theta_j \notin \Big\{0, \frac{\pi}{2}, \pi\Big\}, \quad j = 1, \dots, M.$$

The above points $A_j \in \partial\Omega$ will be called the *vertices* of Ω. Furthermore, one says that A_j is a *convex* vertex if $\theta_j < \frac{\pi}{2}$ and is a *concave* one otherwise, and we denote

$$\mathcal{J}_{\text{cvx}} := \{j : A_j \text{ is convex}\}.$$

We refer to Figure 1 in the introduction for an illustration, and in that case one has $\mathcal{J}_{\text{cvx}} = \{1, 2\}$.

Let us now proceed with a special decomposition of Ω. For small $\delta > 0$, denote

$$\Omega_\delta = \big\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\big\}, \quad \Omega_\delta^c := \Omega \setminus \overline{\Omega_\delta}.$$

We further decompose Ω_δ near each vertex as follows:

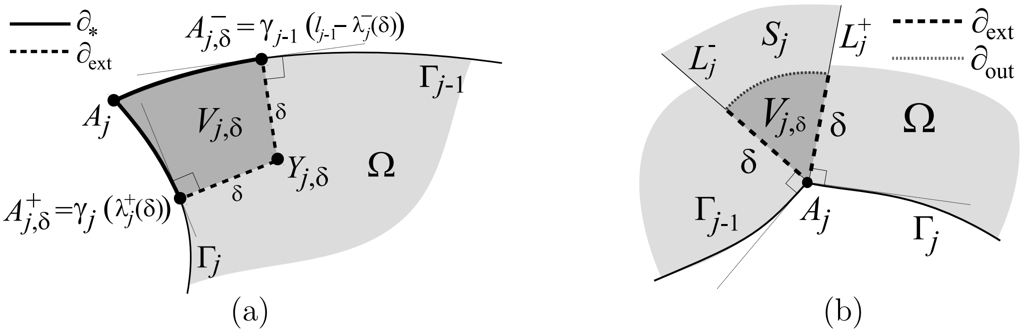


FIGURE 14. The construction of the neighborhoods $V_{j,\delta}$: (a) convex vertex, (b) concave vertex. The partial boundary $\partial_* V_{j,\delta}$ is shown with the thick solid line, the part $\partial_{\text{ext}} V_{j,\delta}$ is indicated with the thick dashed line, and the part $\partial_{\text{out}} V_{j,\delta}$ with the gray dotted line.

- Let A_j be a convex vertex. The following constructions are consequences of Lemma 31 and are illustrated in Figure 14(a). For sufficiently small δ there exists a unique point $Y_{j,\delta} \in \Omega$ such that $\operatorname{dist}(Y_{j,\delta}, \Gamma_{j-1}) = \operatorname{dist}(Y_{j,\delta}, \Gamma_j) = \delta$, and there are uniquely defined numbers $\lambda_j^\pm(\delta) > 0$ such that the points
$$A^-_{j,\delta} := \gamma_{j-1}\big(\ell_j - \lambda_j^-(\delta)\big), \quad A^+_{j,\delta} := \gamma_j\big(\lambda_j^+(\delta)\big)$$
satisfy $|Y_{j,\delta} - A^-_{j,\delta}| = |Y_{j,\delta} - A^+_{j,\delta}| = \delta$. The quantities $\lambda_j^\pm(\delta) > 0$ satisfy
$$\lambda_j^\pm(\delta) = \delta \operatorname{cotan} \theta_j + \mathcal{O}(\delta^2) \quad \text{for } \delta \to 0^+. \tag{35}$$
We denote by $V_{j,\delta}$ the curvilinear quadrangle whose boundary consists of the arcs $\gamma_{j-1}\big([\ell_j - \lambda_j^-(\delta), \ell_j]\big)$, $\gamma_j\big([0, \lambda_j^+(\delta)]\big)$ and the segments $A^\pm_{j,\delta} Y_{j,\delta}$, and we decompose its boundary into the following parts:
$$\partial_* V_{j,\delta} := \partial V_{j,\delta} \cap \partial\Omega, \quad \partial_{\text{ext}} V_{j,\delta} := \partial V_{j,\delta} \setminus \partial_* V_{j,\delta}, \quad \partial_{\text{out}} V_{j,\delta} := \emptyset.$$
- Let A_j be a concave vertex. The constructions are illustrated in Figure 14(b). Let L_j^- be the half-line emanating from A_j, orthogonal to Γ_{j-1} at A_j and directed inside Ω. By L_j^+ we denote the half-line emanating from A_j, orthogonal to Γ_j at A_j and directed inside Ω. Denote by S_j the infinite sector bounded by L_j^- and L_j^+ which lies inside Ω near A_j. Then we set
$$V_{j,\delta} := S_j \cap B(A_j, \delta), \quad \lambda_j^\pm(\delta) := 0,$$
and decompose its boundary as follows:
$$\partial_* V_{j,\delta} := \emptyset, \quad \partial_{\text{out}} V_{j,\delta} := \partial V_{j,\delta} \cap \partial\Omega^c_\delta, \quad \partial_{\text{ext}} V_{j,\delta} := \partial V_{j,\delta} \setminus \partial_{\text{out}} V_{j,\delta}.$$

The "length deficiency" $\lambda_j^+(\delta) + \lambda_j^-(\delta)$ is exactly the length of $\partial_* V_{j,\delta}$ for both convex and concave vertices.

The set $\mathcal{W}_\delta := \Omega_\delta \setminus \overline{\bigcup_{j=1}^M V_{j,\delta}}$ is then the union of M disjoint curvilinear rectangles, namely, if one denotes
$$I_{j,\delta} := \Big(\lambda_j^+(\delta), \ell_j - \lambda_{j+1}^-(\delta)\Big), \quad \Pi_{j,\delta} := I_{j,\delta} \times (0, \delta),$$
$$W_{j,\delta} := \Phi_j(\Pi_{j,\delta}), \quad \Phi_j(s,t) := \gamma_j(s) - t\nu_j(s),$$
then $\mathcal{W}_\delta = \bigcup_{j=1}^M W_{j,\delta}$. Each domain $W_{j,\delta}$ belongs to the class discussed in Subsection 5.1, and we decompose its boundary according to the same rules, i.e.
$$\partial_* W_{j,\delta} := \partial W_{j,\delta} \cap \partial\Omega, \quad \partial_{\text{out}} W_{j,\delta} := \partial W_{j,\delta} \cap \partial\Omega^c_\delta, \quad \partial_{\text{ext}} W_{j,\delta} := \partial W_{j,\delta} \setminus \Big(\partial_* W_{j,\delta} \cup \partial_{\text{out}} W_{j,\delta}\Big).$$

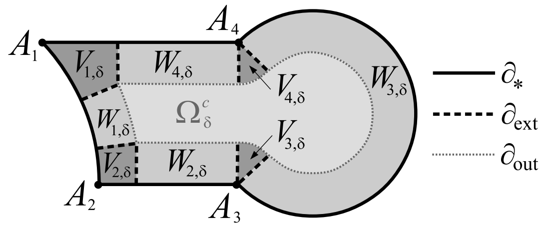


FIGURE 15. Decomposition of a curvilinear polygon.

The resulting decomposition of Ω is illustrated in Figure 15. Remark that by construction we have the equality

$$\bigcup_{j=1}^{M} \partial_{\text{ext}} V_{j,\delta} = \bigcup_{j=1}^{M} \partial_{\text{ext}} W_{j,\delta}, \tag{36}$$

which will be of importance later.

In this section we will be interested in the eigenvalues of the operator R^Ω_α defined as

$R^\Omega_\alpha :=$ the Laplacian in Ω with the α-Robin boundary condition at the whole boundary

in the asymptotic regime $\alpha \to +\infty$. Some additional assumptions on Ω will appear in the next subsections.

6.2. **Corner-induced eigenvalues.** Now let us proceed with the spectral estimates for R^Ω_α. In the rest of the text we assume that

$$\text{all curvatures } H_j \text{ are constant} \tag{37}$$

(i.e. each curve Γ_j is either a line segment or a circle arc), and we explicitly mention that H_j can be different for different j. With each $j \in \{1, \dots, M\}$ we associate the corresponding number $\kappa(\theta_j)$ of discrete eigenvalues of the Robin Laplacians in the infinite sector of aperture $2\theta_j$ (see Section 2.6) and set

$$K := \kappa(\theta_1) + \dots + \kappa(\theta_M) \equiv \sum\nolimits_{j \in \mathcal{J}_{\text{cvx}}} \kappa(\theta_j),$$

$$\mathcal{E} := \text{ the disjoint union of } \big\{\mathcal{E}_n(\theta_j),\ n = 1, \dots, \kappa(\theta_j)\big\} \text{ for } j \in \mathcal{J}_{\text{cvx}},$$

$$\mathcal{E}_n := \text{ the } n\text{th element of } \mathcal{E} \text{ when numbered in the non-decreasing order,}$$

(see Subsection 2.6 for a detailed notation). For what follows we introduce several operators:

$N^V_j :=$ the Laplacian in $V_{j,\delta}$ with the α-Robin boundary condition at $\partial_* V_{j,\delta}$ and the Neumann boundary condition at the rest of the boundary,

$D^V_j :=$ the Laplacian in $V_{j,\delta}$ with the α-Robin boundary condition at $\partial_* V_{j,\delta}$ and the Dirichlet boundary condition at the rest of the boundary.

We remark that for concave vertices A_j, the respective operators $(N/D)^V_j$ are just the Neumann/Dirichlet Laplacians in $V_{j,\delta}$ due to $\partial_* V_{j,\delta} = \emptyset$. Furthermore, denote

$N^W_j :=$ the Laplacian in $W_{j,\delta}$ with the α-Robin boundary condition at $\partial_* W_{j,\delta}$ and the Neumann boundary condition at the rest of the boundary,

$D^W_j :=$ the Laplacian in $W_{j,\delta}$ with the α-Robin boundary condition at $\partial_* W_{j,\delta}$ and the Dirichlet boundary condition at the rest of the boundary,

Finally, introduce

$N_0 :=$ the Neumann Laplacian in Ω^c_δ.

We will use several times the following estimate for the above operators $(N/D)^V_j$ acting in the vertex neighborhoods $V_{j,\delta}$.

Lemma 46 (Eigenvalues of the operators in the vertex neighborhoods)**.** *For $\delta \to 0^+$ and $\alpha\delta \to 0^+$ one has, with some $c > 0$,*

$$E_n\Big(\bigoplus_{j=1}^{M} N_j^V\Big) = \mathcal{E}_n\,\alpha^2 + \mathcal{O}(\alpha^2\delta + 1/\delta^2) \qquad \text{for } n = 1,\dots,K,$$

$$E_n\Big(\bigoplus_{j=1}^{M} D_j^V\Big) = \mathcal{E}_n\,\alpha^2 + \mathcal{O}(\alpha^2\delta + \alpha^2 e^{-c\alpha\delta}) \qquad \text{for } n = 1,\dots,K,$$

$$E_{K+1}\Big(\bigoplus_{j=1}^{M} N_j^V\Big) \ge -\alpha^2 + o(\alpha^2).$$

Proof. If $j \in \mathcal{J}_{\text{cvx}}$, then by applying Corollary 37 to N_j^V we obtain

$$E_n(N_j^V) = \mathcal{E}_n(\theta_j)\,\alpha^2 + \mathcal{O}(\alpha^2\delta + 1/\delta^2) \text{ for } n = 1,\dots,\kappa(\theta_j), \qquad E_{\kappa(\theta_j)+1}(N_j^V) \ge -\alpha^2 + o(\alpha^2).$$

For $j \notin \mathcal{J}_{\text{cvx}}$ we simply have $N_j^V \ge 0$ as in this case we just deal with the Neumann Laplacian in $V_{j,\delta}$. It follows that for each $n = 1,\dots,K$ one has

$$E_n\bigg(\bigoplus_{j=1}^{M} N_j^V\bigg) = E_n\Big(\bigoplus_{j\in\mathcal{J}_{\text{cvx}}} N_j^V\Big) = \mathcal{E}_n\,\alpha^2 + \mathcal{O}(\alpha^2\delta + 1/\delta^2),$$

$$E_{K+1}\Big(\bigoplus_{j=1}^{M} N_j^V\Big) = \min\Big\{\min_{j\in\mathcal{J}_{\text{cvx}}} E_{\kappa(\theta_j)+1}(N_j^V), \min_{j\notin\mathcal{J}_{\text{cvx}}} E_1(N_j^V)\Big\} \ge -\alpha^2 + o(\alpha^2).$$

Similarly, in Corollary 37 we have proved that, with some $c > 0$,

$$E_n(D_j^V) = \mathcal{E}_n(\theta_j)\,\alpha^2 + \mathcal{O}(\alpha^2\delta + \alpha^2 e^{-c\alpha\delta}), \quad j \in \mathcal{J}_{\text{cvx}}, \quad n = 1,\dots,\kappa(\theta_j),$$

and for $j \notin \mathcal{J}_{\text{cvx}}$ the Dirichlet Laplacians D_j^V in $V_{j,\delta}$ are clearly positive. Therefore,

$$E_n\Big(\bigoplus_{j=1}^{M} D_j^V\Big) = E_n\Big(\bigoplus_{j\in\mathcal{J}_{\text{cvx}}} D_j^V\Big) = \mathcal{E}_n\,\alpha^2 + \mathcal{O}(\alpha^2\delta + \alpha^2 e^{-c\alpha\delta}), \quad n = 1,\dots,K. \qquad \square$$

The following result was already obtained earlier in [32] even in a more general setting, but it follows easily from our construction, so we include its proof for completeness.

Proposition 47 (First K eigenvalues of Robin Laplacians in Ω)**.** *As $\alpha \to +\infty$, one has for each $n = 1,\dots,K$,*

$$E_n(R_\alpha^\Omega) = \mathcal{E}_n\,\alpha^2 + \mathcal{O}(\alpha^{\frac{4}{3}}),$$

while $E_{K+1}(R_\alpha^\Omega) \ge -\alpha^2 + o(\alpha^2)$.

Proof. It follows by the the Dirichlet-Neumann bracketing (see Remark 16) that for any $n \in \mathbb{N}$ we have

$$E_n\bigg(N_0 \oplus \Big(\bigoplus_{j=1}^{M} N_j^V\Big) \oplus \Big(\bigoplus_{j=1}^{M} N_j^W\Big)\bigg) \le E_n(R_\alpha^\Omega) \le E_n\Big(\bigoplus_{j=1}^{M} D_j^V\Big). \tag{38}$$

Now let δ depend on α in such a way that $\delta \to 0^+$ and $\alpha\delta \to +\infty$ for $\alpha \to +\infty$. We have trivially $N_0 \ge 0$, and it is shown in Lemma 45 that $N_j^W \ge -\alpha^2 + o(\alpha^2)$ for all j. Therefore, using the preceding Lemma 46 we have

$$E_n\bigg(N_0 \oplus \Big(\bigoplus_{j=1}^{M} N_j^V\Big) \oplus \Big(\bigoplus_{j=1}^{M} N_j^W\Big)\bigg) = E_n\Big(\bigoplus_{j=1}^{M} N_j^V\Big), \quad n = 1,\dots K.$$

Using this equality in (38) and then applying Lemma 46 we obtain

$$E_n(R_\alpha^\Omega) = \mathcal{E}_n\,\alpha^2 + \mathcal{O}\big(1/\delta^2 + \alpha^2\delta + \alpha^2 e^{-c\alpha\delta}\big), \quad n = 1,\dots,K,$$

and it is sufficient to set $\delta := \alpha^{-\frac{2}{3}}$ to get the desired estimate. Furthermore,

$$E_{K+1}\bigg(N_0 \oplus \Big(\bigoplus_{j=1}^{M} N_j^V\Big) \oplus \Big(\bigoplus_{j=1}^{M} N_j^W\Big)\bigg)$$
$$\geq \min\Big\{E_1(N_0), E_{K+1}\Big(\bigoplus_{j=1}^{M} N_j^V\Big), E_1\Big(\bigoplus_{j=1}^{M} N_j^W\Big)\Big\} \geq -\alpha^2 + o(\alpha^2),$$

which implies $E_{K+1}(R_\alpha^\Omega) \geq -\alpha^2 + o(\alpha^2)$ due to (38). □

In the following two sections we are going to prove the main results of the paper concerning the behavior of the eigenvalues $E_{K+n}(R_\alpha^\Omega)$ with any fixed $n \in \mathbb{N}$. In addition to (37) we will assume that

$$\text{all convex vertices are non-resonant.} \tag{39}$$

We denote:

$$D_j := \text{ the Dirichlet Laplacian on } (0, \ell_j), \quad D_{j,\delta} := \text{ the Dirichlet Laplacian on } I_{j,\delta},$$
$$H_* := \max_{j=1,\dots,M} H_j, \quad \mathcal{J}_* := \Big\{j \in \{1, \dots, M\} : H_j = H_*\Big\}.$$

By definition $I_{j,\delta} := \big(\lambda_j^+(\delta), \ell_j - \lambda_{j+1}^-(\delta)\big)$ and due to (35) for each fixed $n \in \mathbb{N}$ one has $E_n(D_{j,\delta}) = E_n(D_j) + \mathcal{O}(\delta)$.

In the subsequent constructions we choose $\delta > 0$ depending on α in such a way that

$$\delta \to 0^+, \quad \alpha\delta \to +\infty, \quad \alpha^2\delta^3 \to 0^+ \qquad \text{for } \alpha \to +\infty. \tag{40}$$

6.3. **Upper bound for side-induced eigenvalues.** The upper bound for eigenvalues $E_{K+n}(R_\alpha^\Omega)$ easily follows from the preceding estimates:

Proposition 48 (Upper bound for $E_{K+n}(R_\alpha^\Omega)$)**.** *Under the assumptions* (37) *and* (39), *for any fixed $n \in \mathbb{N}$ there holds*

$$E_{K+n}(R_\alpha^\Omega) \leq -\alpha^2 - H_*\alpha - \frac{H_*^2}{2} + E_n\Big(\bigoplus_{j\in\mathcal{J}_*} D_j\Big) + \mathcal{O}\Big(\frac{\log\alpha}{\alpha}\Big) \quad \text{as } \alpha \to +\infty.$$

Proof. The Dirichlet bracketing argument (see Remark 16) shows that for any $m \in \mathbb{N}$ we have

$$E_m(R_\alpha^\Omega) \leq E_m\bigg(\Big(\bigoplus_{j=1}^{M} D_j^V\Big) \oplus \Big(\bigoplus_{j=1}^{M} D_j^W\Big)\bigg). \tag{41}$$

Let us study each of the operators in the direct sums by assuming that δ satisfies (40). Recall first that for some $\varepsilon > 0$ one has, due to Lemma 46,

$$E_K\Big(\bigoplus_{j=1}^{M} D_j^V\Big) \leq -(1+\varepsilon)\alpha^2. \tag{42}$$

Furthermore, applying Corollary 41 on non-resonant truncated sectors to each operator D_j^V with $j \in \mathcal{J}_{\text{cvx}}$, one obtains, with a suitable constant $c_0 > 0$, the lower bound $E_{\kappa(\theta_j)+1}(D_j^V) \geq -\alpha^2 + c_0/\delta^2$, while for $j \notin \mathcal{J}_{\text{cvx}}$ one has $D_j^V \geq 0$, as D_j^V is then simply the Dirichlet Laplacian in $V_{j,\delta}$. Therefore,

$$E_{K+1}\Big(\bigoplus_{j=1}^{M} D_j^V\Big) \geq -\alpha^2 + c_0/\delta^2. \tag{43}$$

Each operator D_j^W can be studied using Lemma 44, hence, there exist $b > 0$ and $c > 0$ such that

$$E_n(D_j^W) \le -\alpha^2 - \alpha H_j - H_j^2/2 + (1 + b\delta)E_n(D_{j,\delta}) + b(\delta + \alpha^2 e^{-\alpha\delta}), \quad n \in \mathbb{N}.$$

For each fixed $n \in \mathbb{N}$ one has $E_n(D_{j,\delta}) = E_n(D_j) + \mathcal{O}(\delta)$, and the last estimate rewrites as

$$E_n(D_j^W) \le -\alpha^2 - \alpha H_j - H_j^2/2 + E_n(D_j) + \mathcal{O}(\delta + \alpha^2 e^{-\alpha\delta}). \tag{44}$$

Set $\delta := (3\log\alpha)/\alpha$, then the initial assumptions (40) are satisfied, one has $\alpha^2 e^{-\alpha\delta} = 1/\alpha = o(\delta)$, the remainder in (44) simplifies to $\mathcal{O}(\delta)$, and, for each fixed n,

$$E_n\Big(\bigoplus_{j=1}^M D_j^W\Big) \le E_n\bigg(\bigoplus_{j=1}^M \Big(-\alpha^2 - \alpha H_j - \frac{H_j^2}{2} + D_j\Big)\bigg) + \mathcal{O}(\delta) = -\alpha^2 + \mathcal{O}(\alpha). \tag{45}$$

In addition, by Lemma 45 we still have $E_1(D_j^W) \ge E_1(N_j^W) \ge -\alpha^2 + o(\alpha^2)$. This last estimate and (42) imply that for any fixed $n \in \mathbb{N}$ we have

$$E_K\Big(\bigoplus_{j=1}^M D_j^V\Big) \le -(1+\varepsilon)\alpha^2 \le -\alpha^2 + o(\alpha^2) \le E_n\Big(\bigoplus_{j=1}^M D_j^W\Big),$$

and due to (43) and (45) we also get

$$E_n\Big(\bigoplus_{j=1}^M D_j^W\Big) = -\alpha^2 + \mathcal{O}(\alpha) \le -\alpha^2 + \frac{c_0}{\delta^2} \le E_{K+1}\Big(\bigoplus_{j=1}^M D_j^V\Big).$$

Therefore,

$$\begin{aligned}
E_{K+n}\bigg(\Big(\bigoplus_{j=1}^M D_j^V\Big) \oplus \Big(\bigoplus_{j=1}^M D_j^W\Big)\bigg) &= E_n\Big(\bigoplus_{j=1}^M D_j^W\Big)\\
&\le E_n\bigg(\bigoplus_{j=1}^M \Big(-\alpha^2 - \alpha H_j - \frac{H_j^2}{2} + D_j\Big)\bigg) + \mathcal{O}\Big(\frac{\log\alpha}{\alpha}\Big)\\
&= E_n\bigg(\bigoplus_{j\in\mathcal{J}_*} \Big(-\alpha^2 - \alpha H_j - \frac{H_j^2}{2} + D_j\Big)\bigg) + \mathcal{O}\Big(\frac{\log\alpha}{\alpha}\Big)\\
&= -\alpha^2 - \alpha H_* - \frac{H_*^2}{2} + E_n\Big(\bigoplus_{j\in\mathcal{J}_*} D_j\Big) + \mathcal{O}\Big(\frac{\log\alpha}{\alpha}\Big),
\end{aligned}$$

and it remains to substitute this last inequality into the initial upper bound (41). □

6.4. **Lower bound for side-induced eigenvalues.** The lower bound for the eigenvalues $E_{K+n}(R_\alpha^\Omega)$ will be obtained with the help of the Proposition 7 constructing a suitable identification map, which needs some preparations. We introduce some additional objects:

$L :=$ the subspace of $L^2(\Omega)$ spanned by the first K eigenfunctions of R_α^Ω,

$L_j :=$ the subspace of $L^2(V_{j,\delta})$ spanned by the first $\kappa(\theta_j)$ eigenfunctions of N_j^V, with $j \in \mathcal{J}_{\mathrm{cvx}}$,

$\sigma_j : L^2(\Omega) \to L^2(V_{j,\delta})$ the operator of restriction, $(\sigma_j u)(x) = u(x)$ for $x \in V_{j,\delta}$,

then the adjoint operators $\sigma_j^* : L^2(V_{j,\delta}) \to L^2(\Omega)$ are the operators of extension by zero.

Recall that the distance $d(E, F)$ between closed subspaces E and F was discussed in Subsection 2.3.

Lemma 49 (Distance between eigensubspaces)**.** *Let $j \in \mathcal{J}_{\mathrm{cvx}}$, then in the limit $\delta \to 0^+$ and $\alpha\delta \to +\infty$ there holds $d(\sigma_j^* L_j, L) = \mathcal{O}(e^{-c\alpha\delta})$ with some fixed $c > 0$.*

Proof. During the proof we denote $\Lambda_j := \sigma_j^* L_j \subset L^2(\Omega)$, and for $v \in L^2(V_{j,\delta})$ we denote $v_* := \sigma_j^* v \in L^2(\Omega)$. Let $0 < a < b < 1$. Due to Lemma 35 one can find smooth cut-off functions $\varphi_\delta \in C^2(\overline{\Omega})$ with the following properties:

- $0 \le \varphi_\delta \le 1$, and for all $\beta \in \mathbb{N}^2$ with $1 \le |\beta| \le 2$ there holds $\|\partial^\beta \varphi_\delta\|_\infty \le C\delta^{-|\beta|}$,
- $\varphi_\delta = 1$ in $V_{j,a\delta}$, and $\varphi_\delta = 0$ in $\Omega \setminus \overline{V_{j,b\delta}}$,
- the normal derivative of φ_δ at $\partial\Omega$ is zero,

where $C > 0$ is some fixed constant. Recall that by Corollary 34 one can find $a_0 > 0$ such that $|x - A_j| > a_0\delta$ for $x \in V_{j,\delta} \setminus \overline{V_{j,a\delta}}$.

Consider the subspace $\varphi_\delta\Lambda_j := \big\{\varphi_\delta v_* : v_* \in \Lambda_j\big\} \subset L^2(\Omega)$, then according to the triangular inequality for the distance (Lemma 9) we have

$$d(\Lambda_j, L) \le d(\Lambda_j, \varphi_\delta\Lambda_j) + d(\varphi_\delta\Lambda_j, L). \tag{46}$$

The first term on the right-hand side can be easily estimated by applying directly the definition of the distance. Namely, due to the Agmon-type estimate for the first $\kappa(\theta_j)$ eigenfunctions of N_j^V (Lemma 40), with some $b > 0$ and $B > 0$ there holds

$$\iint_{V_{j,\delta}} e^{b\alpha|x-A_j|}\Big(\frac{1}{\alpha^2}|\nabla v|^2 + v^2\Big)dx \le B\|v\|^2_{L^2(V_{j,\delta})}, \quad v \in L_j.$$

Writing

$$\int_{V_{j,\delta}\setminus\overline{V_{j,a\delta}}} \Big(\frac{1}{\alpha^2}|\nabla v|^2 + v^2\Big)\,\mathrm{d}x = \int_{V_{j,\delta}\setminus\overline{V_{j,a\delta}}} e^{-b\alpha|x-A_j|}\cdot e^{b\alpha|x-A_j|}\Big(\frac{1}{\alpha^2}|\nabla v|^2 + v_*^2\Big)\,\mathrm{d}x$$

we obtain the following upper bound

$$\begin{aligned}\int_{V_{j,\delta}\setminus\overline{V_{j,a\delta}}} \Big(\frac{1}{\alpha^2}|\nabla v|^2 + v^2\Big)\,\mathrm{d}x &\le e^{-b\alpha a_0\delta}\int_{V_{j,\delta}\setminus\overline{V_{j,a\delta}}} e^{b\alpha|x-A_j|}\Big(\frac{1}{\alpha^2}|\nabla v|^2 + v_*^2\Big)\,\mathrm{d}x\\ &\le e^{-b\alpha a_0\delta}\int_{V_{j,\delta}} e^{b\alpha|x-A_j|}\Big(\frac{1}{\alpha^2}|\nabla v|^2 + v_*^2\Big)\,\mathrm{d}x.\end{aligned}$$

This finally gives

$$\int_{V_{j,\delta}\setminus\overline{V_{j,a\delta}}} \Big(\frac{1}{\alpha^2}|\nabla v|^2 + v^2\Big)\,\mathrm{d}x \le Be^{-2c\alpha\delta}\|v\|^2_{L^2(V_{j,\delta})}, \quad c := ba_0/2. \tag{47}$$

Therefore, for any $v_* \in \Lambda_j$ we have

$$\begin{aligned}\big\|v_* - \varphi_\delta v_*\big\|^2_{L^2(\Omega)} &= \iint_\Omega (1-\varphi_\delta)^2 v_*^2\,dx \le \iint_{\Omega\setminus\overline{V_{j,a\delta}}} v_*^2\,dx\\ &\equiv \iint_{V_{j,\delta}\setminus\overline{V_{j,a\delta}}} v^2\,dx \le Be^{-2c\alpha\delta}\|v\|^2_{L^2(V_{j,\delta})} \equiv Be^{-2c\alpha\delta}\|v_*\|^2_{L^2(\Omega)}.\end{aligned}$$

Denote by P_j the orthogonal projector on $\varphi_\delta\Lambda_j$ in $L^2(\Omega)$, then for any $u \in L^2(\Omega)$ we have by definition $\|u - P_j u\| = \inf_{\phi\in\varphi_\delta\Lambda_j}\|u - \phi\|$. Therefore, for any non-zero $v_* \in \Lambda_j$ we have

$$\frac{\|v_* - P_j v_*\|}{\|v_*\|} \le \frac{\|v_* - \varphi_\delta v_*\|}{\|v_*\|} \equiv \frac{\big\|(1-\varphi_\delta)v_*\big\|}{\|v_*\|} \le \sqrt{B}\,e^{-c\alpha\delta},$$

$$d(\Lambda_j, \varphi_\delta\Lambda_j) = \sup_{v_*\in\Lambda_j,\, v_*\ne 0} \frac{\|v_* - P_j v_*\|}{\|v_*\|} \le \sqrt{B}\,e^{-c\alpha\delta}. \tag{48}$$

Now we need an estimate for the second term on the right-hand side of (46), which will be obtained with the help of Proposition 10. Namely, let v^n with $n \in \big\{1, \dots, \kappa(\theta_j)\big\}$ be eigenfunctions of N_j^V for the eigenvalues $E_n := E_n(N_j^V)$ forming an orthonormal basis of L_j, then, in particular,

$$-\Delta v^n = E_n v^n \text{ in } V_{j,\delta}, \quad \frac{\partial v^n}{\partial \nu} = \alpha v^n \text{ at } \partial_* V_{j,\delta} \subset \partial\Omega,$$

where ν the outer unit normal. Consider the functions $\psi_n := \varphi_\delta v_*^n$, then using the above properties of φ_δ we have

$$\Delta\psi_n = \big((\Delta\varphi_\delta)v^n + 2\nabla\varphi_\delta\cdot\nabla v^n + \varphi_\delta\Delta v^n\big)_* \in L^2(\Omega),$$
$$\frac{\partial\psi_n}{\partial\nu} = \frac{\partial\varphi_\delta}{\partial\nu}v^n + \varphi_\delta\frac{\partial v^n}{\partial\nu} = \varphi_\delta\frac{\partial v^n}{\partial\nu} = \alpha\varphi_\delta v^n = \alpha\psi_n \text{ on } \partial\Omega,$$

which shows that ψ_n belong to the domain of R_α^Ω. To estimate the norms of $(R_\alpha^\Omega - E_n)\psi_n$ we represent first

$$(R_\alpha^\Omega - E_n)\psi_n = (-\Delta - E_n)\psi_n = \big(-(\Delta\varphi_\delta)v^n - 2\nabla\varphi_\delta\cdot\nabla v^n\big)_*$$

and note that the supports of $\nabla\varphi_\delta$ and $\Delta\varphi_\delta$ are contained in $V_{j,b\delta}\setminus\overline{V_{j,a\delta}}$. Therefore, with the help of (47) we can estimate

$$\iint_\Omega \big|(\Delta\varphi_\delta)v_*^n\big|^2\,\mathrm{d}x \le \frac{C^2}{\delta^4}\iint_{V_{j,b\delta}\setminus\overline{V_{j,a\delta}}}(v^n)^2\,\mathrm{d}x \le \frac{BC^2}{\delta^4}e^{-2c\alpha\delta}\|v^n\|^2_{L^2(V_{j,\delta})} \equiv \frac{BC^2}{\delta^4}e^{-2c\alpha\delta},$$
$$\iint_\Omega \big|\nabla\varphi_\delta\cdot\nabla v_*^n\big|^2\,\mathrm{d}x \le \iint_\Omega |\nabla\varphi_\delta|^2|\nabla v_*^n|^2\,\mathrm{d}x \le \frac{C^2}{\delta^2}\iint_{V_{j,b\delta}\setminus\overline{V_{j,a\delta}}}(\nabla v^n)^2\,\mathrm{d}x$$
$$\le \frac{BC^2\alpha^2}{\delta^2}e^{-2c\alpha\delta}\|v^n\|^2_{L^2(V_{j,\delta})} \equiv \frac{BC^2\alpha^2}{\delta^2}e^{-2c\alpha\delta},$$

and by noting that $1/\delta^2 = o(\alpha/\delta)$ we have $\big\|(R_\alpha^\Omega - E_n)\psi_n\big\|_{L^2(\Omega)} = \mathcal{O}\big((\alpha/\delta)\,e^{-c\alpha\delta}\big)$.

Let us estimate the Gram matrix G of (ψ_n). We have, using Cauchy-Schwarz and (47),

$$\Big|\langle\psi_k,\psi_n\rangle_{L^2(\Omega)} - \langle v_*^k, v_*^n\rangle_{L^2(\Omega)}\Big| = \Big|\iint_\Omega(\varphi_\delta^2-1)v_*^k v_*^n\,\mathrm{d}x\Big| \le \iint_{V_{j,\delta}\setminus\overline{V_{j,a\delta}}}|v^k\,v^n|\,\mathrm{d}x$$
$$\le \frac{1}{2}\Big(\iint_{V_{j,\delta}\setminus\overline{V_{j,a\delta}}}(v^k)^2\,\mathrm{d}x + \iint_{V_{j,\delta}\setminus\overline{V_{j,a\delta}}}(v^n)^2\,\mathrm{d}x\Big) \le Be^{-2c\alpha\delta},$$

Therefore, we have $\langle\psi_k,\psi_n\rangle_{L^2(\Omega)} = \delta_{k,n} + \mathcal{O}(e^{-2c\alpha\delta})$, and the lowest eigenvalue λ of G is estimated as $\lambda = 1 + \mathcal{O}(e^{-2c\alpha\delta})$.

Finally let $h := (-\mathcal{E}_K - 1)/2$, then the interval $I := \big((\mathcal{E}_1 - h)\alpha^2, (\mathcal{E}_K + h)\alpha^2\big)$ contains all the above eigenvalues E_n due to Lemma 46, and it also contains the first K eigenvalues of R_α^Ω and satisfies $\operatorname{dist}\big(I, \operatorname{spec}(R_\alpha^\Omega)\setminus I\big) \ge \frac{1}{4}h\alpha^2$ due to Proposition 47. Therefore, we are exactly in the situation of Proposition 10 with the parameters

$$E = \varphi_\delta\Lambda_j, \quad F = L, \quad \varepsilon = \mathcal{O}\Big(\frac{\alpha}{\delta}\,e^{-c\alpha\delta}\Big), \quad \eta \ge \frac{1}{8}h\alpha^2, \quad \lambda = 1 + \mathcal{O}(e^{-2c\alpha\delta}),$$

which gives $d(\varphi_\delta\Lambda_j, L) = \mathcal{O}\big(e^{-c\alpha\delta}/(\alpha\delta)\big)$. By combining this last inequality with (48) in the initial triangular inequality (46) one arrives at the conclusion. □

Lemma 50 (Norm and trace estimate in vertex neighborhoods)**.** *For any $A \in \mathbb{R}$ one can find constants $b > 0$ and $c > 0$ such that for α and δ as in* (40) *there holds, for any $j = 1,\dots,M$,*

$$\|\sigma_j u\|^2_{L^2(V_{j,\delta})} \le b\delta^2\Big(N_j^V[\sigma_j u,\sigma_j u] + (\alpha^2 - A\alpha)\alpha^2\|\sigma_j u\|^2_{L^2(V_{j,\delta})}\Big)$$
$$+ b\alpha^2\delta^2 e^{-c\alpha\delta}\|u\|^2_{L^2(\Omega)}, \tag{49}$$
$$\int_{\partial_{\mathrm{ext}}V_{j,\delta}}(\sigma_j u)^2\,\mathrm{d}s \le b\alpha\delta^2\Big(N_j^V[\sigma_j u,\sigma_j u] + (\alpha^2 - A\alpha)\alpha^2\|\sigma_j u\|^2_{L^2(V_{j,\delta})}\Big)$$

$$+ b\alpha^3\delta^2 e^{-c\alpha\delta}\|u\|^2_{L^2(\Omega)} \tag{50}$$

as $u \in H^1(\Omega)$ *with* $u \perp L$.

Proof. Remark that the sought inequalities look quite similar to those in Corollary 42. The novelty is that we do not assume $\sigma_j u \perp L_j$ (in this case the result would follow directly) but just $u \perp L$. The main technical ingredient of the proof below is to show that the orthogonal projection of $\sigma_j u$ onto L_j is sufficiently small and absorbed by the last summands in the above inequalities. This will be achieved using the distance estimate of Lemma 49.

Assume first that $j \in \mathcal{J}_{\mathrm{cvx}}$. Let P be the orthogonal projector on L in $L^2(\Omega)$ and P_j be the orthogonal projector on L_j in $L^2(V_{j,\delta})$. Denote

$$u^V := \sigma_j u, \quad v_0 := P_j u^V, \quad v := (1-P_j)u^V,$$

which belong to $L^2(V_{j,\delta})$. Due to $u \perp L$ we have $u = (1-P)u$, hence,

$$\|v_0\|_{L^2(V_{j,\delta})} = \|\sigma_j^* v_0\|_{L^2(\Omega)} = \big\|\sigma_j^* P_j \sigma_j (1-P)u\big\|_{L^2(\Omega)} \le \big\|\sigma_j^* P_j \sigma_j (1-P)\big\| \, \|u\|_{L^2(\Omega)}.$$

The operator $\Pi_j := \sigma_j^* P_j \sigma_j$ is exactly the orthogonal projector on $\sigma_j^* L_j$ in $L^2(\Omega)$. Therefore, by Lemma 49 one has, with some $c > 0$,

$$\big\|\sigma_j^* P_j \sigma_j (1-P)\big\| = \|\Pi_j - \Pi_j P\| = d(\sigma_j^* L_j, L) = \mathcal{O}(e^{-c\alpha\delta}).$$

and with some $b > 0$ one has then

$$\|v_0\|_{L^2(V_{j,\delta})} \le b e^{-c\alpha\delta}\|u\|_{L^2(\Omega)}. \tag{51}$$

As P_j is a spectral projector for N_j^V, one has $N_j^V[u^V, u^V] = N_j^V[v_0, v_0] + N_j^V[v, v]$, and due to the spectral theorem we have the inequalities

$$E_1(N_j^V)\|v_0\|^2_{L^2(V_{j,\delta})} \le N_j^V[v_0, v_0] \le E_{\kappa(\theta_j)}(N_j^V)\|v_0\|^2_{L^2(V_{j,\delta})}.$$

In view of Lemma 46 we have $E_n(N_j^V) = \mathcal{O}(\alpha^2)$ for $n = 1, \dots, \kappa(\theta_j)$, which in combination with the norm estimate (51) for v_0 gives

$$\Big|N_j^V[v_0, v_0]\Big| \le a_0\alpha^2 e^{-2c\alpha\delta}\|u\|^2_{L^2(\Omega)}, \quad N_j^V[v, v] \le N_j^V[u^V, u^V] + a_0\alpha^2 e^{-2c\alpha\delta}\|u\|^2_{L^2(\Omega)}. \tag{52}$$

As $v \perp L_j$, one can apply the trace and norm estimate for non-resonant truncated sectors (Corollary 42). Using first the norm estimate one has, with some $c_1 > 0$,

$$\|v\|^2_{L^2(V_{j,\delta})} \le c_1\delta^2\Big(N_j^V[v, v] + (\alpha^2 - A\alpha)\|v\|^2_{L^2(V_{j,\delta})}\Big),$$

and by using (51), (52) and the trivial inequality $\|v\|^2_{L^2(V_{j,\delta})} \le \|u^V\|^2_{L^2(V_{j,\delta})}$ we have

$$\begin{aligned}
\|u^V\|^2_{L^2(V_{j,\delta})} &= \|v\|^2_{L^2(V_{j,\delta})} + \|v_0\|^2_{L^2(V_{j,\delta})} \\
&\le c_1\delta^2\Big(N_j^V[v, v] + (\alpha^2 - A\alpha)\|v\|^2_{L^2(V_{j,\delta})}\Big) + b^2 e^{-2c\alpha\delta}\|u\|^2_{L^2(\Omega)} \\
&\le c_1\delta^2\Big(N_j^V[u^V, u^V] + (\alpha^2 - A\alpha)\|u^V\|^2_{L^2(V_{j,\delta})}\Big) + (a_0 c_1\alpha^2\delta^2 e^{-2c\alpha\delta} + b^2 e^{-2c\alpha\delta})\|u\|^2_{L^2(\Omega)} \\
&\le c_1\delta^2\Big(N_j^V[u^V, u^V] + (\alpha^2 - A\alpha)\|u^V\|^2_{L^2(V_{j,\delta})}\Big) + b_0\alpha^2\delta^2 e^{-2c\alpha\delta}\|u\|^2_{L^2(\Omega)}
\end{aligned}$$

with a sufficiently large $b_0 > 0$, which proves (49). Furthermore, using first the trace estimate of Corollary 42 and then (52) we have, with some $c_2 > 0$,

$$\begin{aligned}
\int_{\partial_{\mathrm{ext}} V_{j,\delta}} v^2 \,\mathrm{d}s &\le c_1\alpha\delta^2\Big(N_j^V[v, v] + (\alpha^2 - A\alpha)\|v\|^2_{L^2(V_{j,\delta})}\Big) \\
&\le c_1\alpha\delta^2\Big(N_j^V[u^V, u^V] + (\alpha^2 - A\alpha)\|u^V\|^2_{L^2(V_{j,\delta})}\Big) \\
&\quad + c_2\alpha^3\delta^2 e^{-2c\alpha\delta}\|u\|^2_{L^2(\Omega)},
\end{aligned} \tag{53}$$

with $c_2 := c_1 a_0$. Furthermore, the operator R_j^V defined as the Laplacian in $V_{j,\delta}$ with the α-Robin boundary condition at the whole boundary satisfy $E_1(R_j^V) \geq -c_0\alpha^2$ with some $c_0 > 0$ (see Lemma 38), i.e.

$$R_j^V[f,f] \equiv N_j^V[f,f] - \alpha \int_{\partial_{\mathrm{ext}} V_{j,\delta}} f^2 \,\mathrm{d}s \geq -c_0\alpha^2 \|f\|^2_{L^2(V_{j,\delta})} \text{ for all } f \in H^1(V_{j,\delta}),$$

hence, one has

$$\int_{\partial_{\mathrm{ext}} V_{j,\delta}} f^2 \,\mathrm{d}s \leq \frac{1}{\alpha}\Big(N_j^V[f,f] + c_0\alpha^2 \|f\|^2_{L^2(V_{j,\delta})}\Big) \text{ for all } f \in H^1(V_{j,\delta}).$$

Using this inequality for $f := v_0$ and making use of the estimates (51) and (52) for the both terms on the right-hand side we arrive at

$$\int_{\partial_{\mathrm{ext}} V_{j,\delta}} v_0^2 \,\mathrm{d}s \leq c_3 \alpha e^{-2c\alpha\delta} \|u\|^2_{L^2(\Omega)} \tag{54}$$

with some $c_3 > 0$. Finally,

$$\int_{\partial_{\mathrm{ext}} V_{j,\delta}} (u^V)^2 \,\mathrm{d}s \equiv \int_{\partial_{\mathrm{ext}} V_{j,\delta}} (v+v_0)^2 \,\mathrm{d}s \leq 2\int_{\partial_{\mathrm{ext}} V_{j,\delta}} v^2 \,\mathrm{d}s + 2\int_{\partial_{\mathrm{ext}} V_{j,\delta}} v_0^2 \,\mathrm{d}s,$$

and by estimating the two terms on the right-hand side by (53) and (54) respectively one arrives at (50).

Now assume that $j \notin \mathfrak{J}_{\mathrm{cvx}}$, then $N_j^V \geq 0$ is just the Neumann Laplacian in $V_{j,\delta}$. In particular, for large α one has the obvious estimate

$$\|\sigma_j u\|^2_{L^2(V_{j,\delta})} \leq \frac{2}{\alpha^2}\Big(N_j^V[\sigma_j u, \sigma_j u] + (\alpha^2 - A\alpha)\|\sigma_j u\|^2_{L^2(V_{j,\delta})}\Big)$$

implying (49) due to $1/\alpha^2 = o(\delta^2)$. To obtain (50) consider the Laplacian R_j^V in $V_{j,\delta}$ with the α-Robin boundary condition at the whole boundary. As $V_{j,\delta}$ is isometric to the scaled domains δU with a fixed U being the sector of opening $2\theta_j$ and radius 1, one has $R_j^V \geq -c_4\alpha^2$ with some $c_4 > 0$ (see Corollary 15), and

$$\begin{aligned}\int_{\partial V_{j,\delta}} f^2 \,\mathrm{d}s &\leq \frac{1}{\alpha}\Big(\int_{V_{j,\delta}} |\nabla f|^2 \,\mathrm{d}x + c_4\alpha^2 \int_{V_{j,\delta}} f^2 \,\mathrm{d}x\Big)\\ &\equiv \frac{1}{\alpha}\Big(N_j^V[f,f] + c_4\alpha^2 \|f\|^2_{L^2(V_{j,\delta})}\Big) \text{ for all } f \in H^1(V_{j,\delta}).\end{aligned}$$

As both terms on the right-hand side are non-negative, one can choose $c_5 \geq 0$ sufficiently large to have

$$\int_{\partial V_{j,\delta}} f^2 \,\mathrm{d}s \leq \frac{c_5}{\alpha}\Big(N_j^V[f,f] + (\alpha^2 - A\alpha)\|f\|^2_{L^2(V_{j,\delta})}\Big) \text{ for all } f \in H^1(V_{j,\delta}).$$

Therefore, by taking $f := \sigma_j u$ one has

$$\int_{\partial_{\mathrm{ext}} V_{j,\delta}} (\sigma_j u)^2 \,\mathrm{d}s \leq \int_{\partial V_{j,\delta}} (\sigma_j u)^2 \,\mathrm{d}s \leq \frac{c_5}{\alpha}\Big(N_j^V[\sigma_j u, \sigma_j u] + (\alpha^2 - A\alpha)\|\sigma_j u\|^2_{L^2(V_{j,\delta})}\Big)$$

and due to $1/\alpha = o(\alpha\delta^2)$ we arrive at (50). Remark that the last summands in (49) and (50) appear for convex vertices only. □

Proposition 51 (Lower bound for $E_{K+n}(R_\alpha^\Omega)$)**.** *Under the assumptions* (37) *and* (39)*, for any fixed $n \in \mathbb{N}$ one has, as $\alpha \to +\infty$,*

$$E_{K+n}(R_\alpha^\Omega) \geq -\alpha^2 - H_*\alpha - \frac{H_*^2}{2} + E_n\Big(\bigoplus_{j \in \mathfrak{J}_*} D_j\Big) + \mathcal{O}\Big(\frac{\log \alpha}{\sqrt{\alpha}}\Big).$$

Proof. During the proof we choose δ satisfying (40). The estimate will be obtained with the help of an identification operator (see Proposition 7). Consider the Hilbert spaces

$$\mathcal{H} := \text{ the orthogonal complement of } L \text{ in } L^2(\Omega), \quad \mathcal{H}' := \bigoplus_{j\in\mathcal{J}_*} L^2(I_{j,\delta}).$$

During the proof for $u \in \mathcal{H}$ we denote

$$\begin{aligned} v_j &:= \text{ the restriction of } u \text{ to } V_{j,\delta}, & \|v_j\| &:= \|v_j\|_{L^2(V_{j,\delta})},\\ w_j &:= \text{ the restriction of } u \text{ to } W_{j,\delta}, & \|w_j\| &:= \|w_j\|_{L^2(W_{j,\delta})},\\ u_c &:= \text{ the restriction of } u \text{ to } \Omega^c_\delta, & \|u_c\| &:= \|u_c\|_{L^2(\Omega^c_\delta)}, \end{aligned}$$

and remark that due to the constructions and the equality (36) we have

$$\sum_{j=1}^{M} \int_{\partial_{\text{ext}} V_{j,\delta}} v_j^2 \,\mathrm{d}s = \sum_{j=1}^{M} \int_{\partial_{\text{ext}} W_{j,\delta}} w_j^2 \,\mathrm{d}s. \tag{55}$$

Applying Lemma 50 with $A := -H_j$ we obtain, with some $b > 0$ and $c > 0$, the inequalities

$$\begin{aligned} \|v_j\|^2 &\le b\delta^2\Big(N_j^V[v_j,v_j] + (\alpha^2 + H_j\alpha)\|v_j\|^2\Big) + b\alpha^2\delta^2 e^{-c\alpha\delta}\|u\|^2,\\ \int_{\partial_{\text{ext}} V_{j,\delta}} v_j^2\,\mathrm{d}s &\le b\alpha\delta^2\Big(N_j^V[v_j,v_j] + (\alpha^2 + H_j\alpha)\|v_j\|^2\Big) + b\alpha^3\delta^2 e^{-c\alpha\delta}\|u\|^2. \end{aligned}$$

Furthermore, by applying Lemma 45 (separation of variables) to each $W_{j,\delta}$ we conclude that there are functions $\psi_j \in L^2(0,\delta)$ with $\|\psi_j\|^2_{L^2(0,\delta)} = 1$ such that if one defines

$$P_j : \mathcal{H} \to L^2(I_{j,\delta}), \quad (P_j u)(s) := \int_0^\delta \psi_j(t)\sqrt{1 - H_j t}\, w_j\big(\Phi_j(s,t)\big)\,\mathrm{d}t, \tag{56}$$

then, with some $b_1 > 0$,

$$\begin{aligned} N_j^W[w_j,w_j] \ge {}& -\Big(\alpha^2 + \alpha H_j + \frac{H_j^2}{2}\Big)\|P_j u\|^2 + \big(1 - b_1\delta\big)\big\|(P_j u)'\big\|^2\\ & - b_1\big(\delta + \alpha^2 e^{-c\alpha\delta}\big)\|P_j u\|^2, \end{aligned}$$

and we recall that, using the Cauchy-Schwarz inequality and $\|\psi_j\|_{L^2(0,\delta)} = 1$,

$$\begin{aligned} \|P_j u\|^2 &= \int_{I_{j,\delta}} \Big(\int_0^\delta \psi(t)\sqrt{1 - H_j t}\, w_j\big(\Phi_j(s,t)\big)\,\mathrm{d}t\Big)^2 \mathrm{d}s\\ &\le \int_{I_{j,\delta}} \int_0^\delta (1 - H_j t) w_j\big(\Phi_j(s,t)\big)^2 \,\mathrm{d}t\,\mathrm{d}s = \iint_{W_{j,\delta}} w_j^2\,\mathrm{d}x = \|w_j\|^2. \end{aligned} \tag{57}$$

Now let us set $\delta := (c' \log\alpha)/\alpha$ with $c' \ge 3/c$, then the conditions (40) for the choice of δ are satisfied, and $\alpha^2 e^{-c\alpha\delta} = o(\delta)$, which implies $\alpha^2\delta^2 e^{-c\alpha\delta} = o(\delta^3)$ and $\alpha^3\delta^2 e^{-c\alpha\delta} = o(\alpha\delta^3)$. This simplifies the remainders in the above inequalities, and one can pick a sufficiently large $a > 0$ such that, for the same choice of ψ_j,

$$\|v_j\|^2 \le \frac{a\log^2\alpha}{\alpha^2}\Big(N_j^V[v_j,v_j] + (\alpha^2 + H_j\alpha)\|v_j\|^2\Big) + \frac{a\log^3\alpha}{\alpha^3}\|u\|^2, \tag{58}$$

$$\int_{\partial_{\text{ext}} V_{j,\delta}} v_j^2\,\mathrm{d}s \le \frac{a\log^2\alpha}{\alpha}\Big(N_j^V[v_j,v_j] + (\alpha^2 + H_j\alpha)\|v_j\|^2\Big) + \frac{a\log^3\alpha}{\alpha^2}\|u\|^2, \tag{59}$$

$$\begin{aligned} N_j^W[w_j,w_j] \ge {}& -\Big(\alpha^2 + \alpha H_j + \frac{H_j^2}{2}\Big)\|P_j u\|^2\\ & + \Big(1 - \frac{a\log\alpha}{\alpha}\Big)\big\|(P_j u)'\big\|^2 - \frac{a\log\alpha}{\alpha}\,\|P_j u\|^2. \end{aligned} \tag{60}$$

Consider the self-adjoint operators

$$B := R_\alpha^\Omega + \Big(\alpha^2 + \alpha H_* + \frac{H_*^2}{2}\Big) + \frac{(M+a)\log\alpha}{\alpha} \text{ viewed as an operator in } \mathcal{H},$$
$$B' := \bigoplus_{j\in\mathcal{J}_*} D_{j,\delta} \text{ in } \mathcal{H}',$$

with $\mathcal{Q}(B) = H^1(\Omega)\cap\mathcal{H}$ and $\mathcal{Q}(B') = \bigoplus_{j\in\mathcal{J}_*} H_0^1(I_{j,\delta})$. Recall that

$$\begin{aligned}
B[u,u] &= \iint_\Omega |\nabla u|^2\,\mathrm{d}x - \alpha\int_{\partial\Omega} u^2\,\mathrm{d}s + \Big(\alpha^2 + \alpha H_* + \frac{H_*^2}{2}\Big)\iint_\Omega u^2\,\mathrm{d}x\\
&\quad + \frac{(M+a)\log\alpha}{\alpha}\iint_\Omega u^2\,\mathrm{d}x\\
&= \sum_{j=1}^M \bigg(\iint_{V_{j,\delta}} |\nabla u|^2\,\mathrm{d}x - \alpha\int_{\partial_* V_{j,\delta}} u^2\,\mathrm{d}s + \Big(\alpha^2 + \alpha H_* + \frac{H_*^2}{2}\Big)\iint_{V_{j,\delta}} u^2\,\mathrm{d}x\bigg)\\
&\quad + \sum_{j=1}^M \bigg(\iint_{W_{j,\delta}} |\nabla u|^2\,\mathrm{d}x - \alpha\int_{\partial_* W_{j,\delta}} u^2\,\mathrm{d}s + \Big(\alpha^2 + \alpha H_* + \frac{H_*^2}{2}\Big)\iint_{W_{j,\delta}} u^2\,\mathrm{d}x\bigg)\\
&\quad + \iint_{\Omega_\delta^c} |\nabla u|^2\,\mathrm{d}x + \Big(\alpha^2 + \alpha H_* + \frac{H_*^2}{2}\Big)\iint_{\Omega_\delta^c} u^2\,\mathrm{d}x + \frac{(M+a)\log\alpha}{\alpha}\iint_\Omega u^2\,\mathrm{d}x.
\end{aligned}$$

For large α we estimate the portion in Ω_δ^c, uniformly in u,

$$\iint_{\Omega_\delta^c} |\nabla u|^2\,\mathrm{d}x + \Big(\alpha^2 + \alpha H_* + \frac{H_*^2}{2}\Big)\iint_{\Omega_\delta^c} u^2\,\mathrm{d}x \ge \frac{\alpha^2}{2}\iint_{\Omega_\delta^c} u^2\,\mathrm{d}x,$$

and then

$$\begin{aligned}
B[u,u] \ge &\sum_{j=1}^M \Big(N_j^V[v_j,v_j] + \Big(\alpha^2 + \alpha H_* + \frac{H_*^2}{2}\Big)\|v_j\|^2\Big)\\
&+ \sum_{j=1}^M \Big(N_j^W[w_j,w_j] + \Big(\alpha^2 + \alpha H_* + \frac{H_*^2}{2}\Big)\|w_j\|^2\Big)\\
&+ \frac{\alpha^2}{2}\|u_c\|^2 + \frac{(M+a)\log\alpha}{\alpha}\|u\|^2.
\end{aligned}\tag{61}$$

Let us estimate the terms containing v_j. Using (58) and (59) we obtain

$$\begin{aligned}
&N_j^V[v_j,v_j] + \Big(\alpha^2 + \alpha H_* + \frac{H_*^2}{2}\Big)\|v_j\|^2\\
&\qquad = \Big(N_j^V[v_j,v_j] + (\alpha^2 + H_j\alpha)\|v_j\|^2\Big) + \Big(\alpha(H_* - H_j) + \frac{H_*^2}{2}\Big)\|v_j\|^2\\
&\qquad \ge \frac{1}{2}\Big(\frac{\alpha^2}{a\log^2\alpha}\|v_j\|^2 + \frac{\alpha}{a\log^2\alpha}\int_{\partial_{\mathrm{ext}} V_{j,\delta}} v_j^2\,\mathrm{d}s\Big) - \frac{\log\alpha}{\alpha}\|u\|^2\\
&\qquad\quad + \Big(\alpha(H_* - H_j) + \frac{H_*^2}{2}\Big)\|v_j\|^2.
\end{aligned}$$

For each j one has $\alpha(H_* - H_j) + H_*^2/2 \ge 0$ for large α, therefore,

$$\sum_{j=1}^M \Big(N_j^V[v_j,v_j] + \Big(\alpha^2 + \alpha H_* + \frac{H_*^2}{2}\Big)\|v_j\|^2\Big)$$

$$\geq \frac{\alpha^2}{2a\log^2\alpha}\sum_{j=1}^{M}\|v_j\|^2 + \frac{\alpha}{2a\log^2\alpha}\sum_{j=1}^{M}\int_{\partial_{\text{ext}}V_{j,\delta}} v_j^2\,\mathrm{d}s - \frac{M\log\alpha}{\alpha}\|u\|^2. \quad (62)$$

On the other hand, with the help of (60) we estimate

$$\begin{aligned}
N_j^W[w_j,w_j] &+ \Big(\alpha^2+\alpha H_* + \frac{H_*^2}{2}\Big)\|w_j\|^2\\
&= N_j^W[w_j,w_j] + \Big(\alpha^2+\alpha H_j + \frac{H_j^2}{2}\Big)\|P_ju\|^2 + \Big(\alpha(H_*-H_j) + \frac{H_*^2-H_j^2}{2}\Big)\|P_ju\|^2\\
&\quad + \Big(\alpha^2+\alpha H_* + \frac{H_*^2}{2}\Big)\big(\|w_j\|^2 - \|P_ju\|^2\big)\\
&\geq \Big(1-\frac{a\log\alpha}{\alpha}\Big)\big\|(P_ju)'\big\|^2 - \frac{a\log\alpha}{\alpha}\|P_ju\|^2 + \Big(\alpha(H_*-H_j) + \frac{H_*^2-H_j^2}{2}\Big)\|P_ju\|^2\\
&\quad + \Big(\alpha^2+\alpha H_* + \frac{H_*^2}{2}\Big)\big(\|w_j\|^2 - \|P_ju\|^2\big)
\end{aligned}$$

and then, estimating $\alpha^2+\alpha H_* + H_*^2/2 \geq \alpha^2/2$ for large α as well as $H_* - H_j \geq a_0 > 0$ for $j \notin \mathfrak{J}_*$,

$$\begin{aligned}
\sum_{j=1}^{M}\Big(N_j^W[w_j,w_j] &+ \Big(\alpha^2+\alpha H_* + \frac{H_*^2}{2}\Big)\|w_j\|^2\Big)\\
&\geq \Big(1-\frac{a\log\alpha}{\alpha}\Big)\sum_{j\in\mathfrak{J}_*}\big\|(P_ju)'\big\|^2 - \frac{a\log\alpha}{\alpha}\|u\|^2\\
&\qquad + a_0\alpha\sum_{j\notin\mathfrak{J}_*}\|P_ju\|^2 + \frac{\alpha^2}{2}\sum_{j=1}^{M}\big(\|w_j\|^2-\|P_ju\|^2\big), \quad (63)
\end{aligned}$$

where we use (57) and the obvious inequality $\sum_{j=1}^{M}\|w_j\|^2 \leq \|u\|^2$. By substituting (62) and (63) into (61) we arrive at

$$\begin{aligned}
B[u,u] \geq{}& \frac{\alpha^2}{2a\log^2\alpha}\sum_{j=1}^{M}\|v_j\|^2 + \frac{\alpha}{2a\log^2\alpha}\sum_{j=1}^{M}\int_{\partial_{\text{ext}}V_{j,\delta}} v_j^2\,\mathrm{d}s\\
&+ a_0\alpha\sum_{j\notin\mathfrak{J}_*}\|P_ju\|^2 + \Big(1-\frac{a\log\alpha}{\alpha}\Big)\sum_{j\in\mathfrak{J}_*}\big\|(P_ju)'\big\|^2 \qquad (64)\\
&+ \frac{\alpha^2}{2}\sum_{j=1}^{M}\big(\|w_j\|^2-\|P_ju\|^2\big) + \frac{\alpha^2}{2}\|u_c\|^2.
\end{aligned}$$

We remark that each term on the right-hand side is non-negative and, hence, the left-hand side is an upper bound for each term on the right-hand side. It also implies that B is positive and then

$$E_1(B) \equiv E_{K+1}(R_\alpha^\Omega) + \Big(\alpha^2+\alpha H_* + \frac{H_*^2}{2}\Big) + \frac{(M+a)\log\alpha}{\alpha} \geq 0.$$

By combining with Proposition 48 we see that for any fixed $n\in\mathbb{N}$ one can choose $\lambda_n > 0$ which is independent of α and such that

$$0 \leq E_n(B) \leq \lambda_n, \quad \big(1+E_n(B)\big)^{-1} \geq (1+\lambda_n)^{-1}. \quad (65)$$

In order to construct a suitable identification map $J : \mathcal{Q}(B) \to \mathcal{Q}(B')$ we pick functions $\rho_j^\pm \in C^1\big([0, \ell_j]\big)$ such that

$$\rho_j^+ = \begin{cases} 1 & \text{in a neighborhood of } 0, \\ 0 & \text{in a neighborhood of } \ell_j, \end{cases} \quad \rho_j^- = \begin{cases} 0 & \text{in a neighborhood of } 0, \\ 1 & \text{in a neighborhood of } \ell_j, \end{cases} \quad j \in \mathcal{J}_*,$$

and then choose a constant $\rho_0 > 0$ such that

$$\|\rho_j^\pm\|_{L^\infty(0,\ell_j)} + \|(\rho_j^\pm)'\|_{L^\infty(0,\ell_j)} \le \rho_0 \text{ for all } j \in \mathcal{J}_*. \tag{66}$$

Recall that due to the definition we have $I_{j,\delta} := \big(\lambda_j^+(\delta), \ell_j - \lambda_{j+1}^-(\delta)\big) =: (\iota_j, \tau_j)$, and that due to $\lambda_j^\pm(\delta) = \mathcal{O}(\delta)$ we have $\iota_j = \mathcal{O}(\delta)$ and $\tau_j = \ell_j + \mathcal{O}(\delta)$, hence,

$$\rho^+(\iota_j) = 1, \quad \rho^+(\tau_j) = 0, \quad \rho^-(\iota_j) = 0, \quad \rho^-(\tau_j) = 1$$

as α is sufficiently large. Therefore, the map

$$J : \mathcal{Q}(B) \to \mathcal{Q}(B') \equiv \bigoplus_{j\in\mathcal{J}_*} H_0^1(I_{j,\delta}), \quad Ju = (J_j u),$$
$$(J_j u)(s) := (P_j u)(s) - (P_j u)(\iota_j)\rho_j^+(s) - (P_j u)(\tau_j)\rho_j^-(s)$$

is well-defined. In order to proceed with the estimates we remark that for large α one has $1 - H_j t \le 2$ for $t \in (0, \delta)$, and we can estimate, with the help of the Cauchy-Schwarz inequality and of the normalization of ψ_j,

$$\begin{aligned}
&\big|P_j u(\iota_j)\big|^2 + \big|P_j u(\tau_j)\big|^2 \\
&\quad = \Big(\int_0^\delta \psi_j(t)\sqrt{1 - H_j t}\, w_j\big(\Phi_j(\iota_j, t)\big)\,\mathrm{d}t\Big)^2 + \Big(\int_0^\delta \psi_j(t)\sqrt{1 - H_j t}\, w_j\big(\Phi_j(\tau_j, t)\big)\,\mathrm{d}t\Big)^2 \\
&\qquad \le \int_0^\delta (1 - H_j t) w_j\big(\Phi_j(\iota_j, t)\big)^2\, dt + \int_0^\delta (1 - H_j t) w_j\big(\Phi_j(\tau_j, t)\big)^2\,\mathrm{d}t \\
&\qquad\quad \le 2\Big(\int_0^\delta w_j\big(\Phi_j(\iota_j, t)\big)^2\, dt + \int_0^\delta w_j\big(\Phi_j(\tau_j, t)\big)^2\,\mathrm{d}t\Big) \equiv 2\int_{\partial_{\mathrm{ext}} W_{j,\delta}} w_j^2\,\mathrm{d}s.
\end{aligned} \tag{67}$$

Recall the inequality $(x + y)^2 \ge (1 - \varepsilon)x^2 - y^2/\varepsilon$ valid for any $x, y \in \mathbb{R}$ and $\varepsilon > 0$, then

$$\begin{aligned}
\|J_j u\|^2 &= \int_{I_{j,\delta}} \Big|(P_j u)(s) - (P_j u)(\iota_j)\rho_j^+(s) - (P_j u)(\tau_j)\rho_j^-(s)\Big|^2\,\mathrm{d}s \\
&\ge (1 - \varepsilon)\int_{I_{j,\delta}} \Big|(P_j u)(s)\Big|^2\,\mathrm{d}s - \frac{1}{\varepsilon}\int_{I_{j,\delta}} \Big|(P_j u)(\iota_j)\rho_j^+(s) + (P_j u)(\tau_j)\rho_j^-(s)\Big|^2\,\mathrm{d}s,
\end{aligned}$$

and, using (67) and the constant ρ_0 from (66) we have

$$\begin{aligned}
&\int_{I_{j,\delta}} \Big|(P_j u)(\iota_j)\rho_j^+(s) + (P_j u)(\tau_j)\rho_j^-(s)\Big|^2\,\mathrm{d}s \\
&\qquad \le 2\int_{I_{j,\delta}} \Big(\Big|(P_j u)(\iota_j)\rho_j^+(s)\Big|^2 + \Big|(P_j u)(\tau_j)\rho_j^-(s)\Big|^2\Big)\,\mathrm{d}s\Big) \\
&\qquad\qquad \le 2\ell_j\rho_0^2\Big(\big|P_j u(\iota_j)\big|^2 + \big|P_j u(\tau_j)\big|^2\Big) \le 4\ell_j\rho_0^2\int_{\partial_{\mathrm{ext}} W_{j,\delta}} w_j^2\,\mathrm{d}s,
\end{aligned}$$

hence,

$$\|J_j u\|^2 \ge (1 - \varepsilon)\|P_j u\|^2 - \frac{4\ell\rho_0^2}{\varepsilon}\int_{\partial_{\mathrm{ext}} W_{j,\delta}} w_j^2\,\mathrm{d}s, \quad \ell := \max_{j\in\mathcal{J}_*} \ell_j.$$

Therefore, using (55) and (57) we arrive at

$$
\begin{aligned}
\|u\|^2 - \|Ju\|^2 &= \sum_{j=1}^{M} \|v_j\|^2 + \sum_{j=1}^{M} \|w_j\|^2 + \|u_c\|^2 - \sum_{j\in\mathcal{J}_*} \|J_j u\|^2 \\
&\le \sum_{j=1}^{M} \|v_j\|^2 + \sum_{j=1}^{M} \|w_j\|^2 + \|u_c\|^2 \\
&\quad - (1-\varepsilon) \sum_{j\in\mathcal{J}_*} \|P_j u\|^2 + \frac{4\ell\rho_0^2}{\varepsilon} \sum_{j\in\mathcal{J}_*} \int_{\partial_{\mathrm{ext}} W_{j,\delta}} w_j^2 \, \mathrm{d}s \\
&= \sum_{j=1}^{M} \|v_j\|^2 + \sum_{j=1}^{M} \big(\|w_j\|^2 - \|P_j u\|^2\big) + \sum_{j\notin\mathcal{J}_*} \|P_j u\|^2 \\
&\quad + \varepsilon \sum_{j\in\mathcal{J}_*} \|P_j u\|^2 + \frac{4\ell\rho_0^2}{\varepsilon} \sum_{j\in\mathcal{J}_*} \int_{\partial_{\mathrm{ext}} V_{j,\delta}} v_j^2 \, \mathrm{d}s + \|u_c\|^2 \\
&\le \sum_{j=1}^{M} \|v_j\|^2 + \sum_{j=1}^{M} \big(\|w_j\|^2 - \|P_j u\|^2\big) + \sum_{j\notin\mathcal{J}_*} \|P_j u\|^2 \\
&\quad + \varepsilon \|u\|^2 + \frac{4\ell\rho_0^2}{\varepsilon} \sum_{j=1}^{M} \int_{\partial_{\mathrm{ext}} V_{j,\delta}} v_j^2 \, \mathrm{d}s + \|u_c\|^2.
\end{aligned}
$$

Using (64) we obtain an upper bound for all terms on the right-hand side except $\varepsilon\|u\|^2$, which gives

$$
\|u\|^2 - \|Ju\|^2 \le \Big(\frac{2a \log^2 \alpha}{\alpha^2} + \frac{4}{\alpha^2} + \frac{1}{a_0 \alpha} + \frac{8\ell\rho_0^2 a \log^2\alpha}{\varepsilon\alpha} \Big) B[u,u] + \varepsilon \|u\|^2.
$$

Taking $\varepsilon := \log\alpha/\sqrt{\alpha}$ and choosing $c_1 > 0$ sufficiently large we obtain

$$
\|u\|^2 - \|Ju\|^2 \le \frac{c_1 \log \alpha}{\sqrt{\alpha}} \Big(B[u,u] + \|u\|^2 \Big). \tag{68}
$$

To study the difference $B'[Ju,Ju] - B[u,u]$ recall that $B'[Ju,Ju] = \sum_{j\in\mathcal{J}_*} \|(J_j u)'\|^2$. Using the elementary inequality $(x+y)^2 \le (1+\varepsilon)x^2 + 2y^2/\varepsilon$ valid for all $x,y\in\mathbb{R}$ and $\varepsilon\in(0,1)$ we estimate

$$
\begin{aligned}
\big\|(J_j u)'\big\|^2 &= \int_{I_{j,\delta}} \Big| (P_j u)'(s) - (P_j u)(\iota_j)(\rho_j^+)'(s) - (P_j u)(\tau_j)(\rho_j^-)'(s) \Big|^2 \mathrm{d}s \\
&\le (1+\varepsilon) \int_{I_{j,\delta}} \Big| (P_j u)'(s) \Big|^2 \mathrm{d}s + \frac{2}{\varepsilon} \int_{I_{j,\delta}} \Big| (P_j u)(\iota_j)(\rho_j^+)'(s) + (P_j u)(\tau_j)(\rho_j^-)'(s) \Big|^2 \mathrm{d}s.
\end{aligned}
$$

Using the estimate (67) for the last term and the constant ρ_0 from (66) we have

$$
\begin{aligned}
&\int_{I_{j,\delta}} \Big| (P_j u)(\iota_j)(\rho_j^+)'(s) + (P_j u)(\tau_j)(\rho_j^-)'(s) \Big|^2 \mathrm{d}s \\
&\qquad \le 2 \int_{I_{j,\delta}} \Big(\Big| (P_j u)(\iota_j)(\rho_j^+)'(s) \Big|^2 + \Big| (P_j u)(\tau_j)(\rho_j^-)'(s) \Big|^2 \Big) \mathrm{d}s \\
&\qquad\qquad \le 2\ell_j \rho_0^2 \Big(\big| P_j u(\iota_j) \big|^2 + \big| P_j u(\tau_j) \big|^2 \Big) \le 4\ell\rho_0^2 \int_{\partial_{\mathrm{ext}} W_{j,\delta}} w_j^2 \, \mathrm{d}s.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
B'[Ju,Ju] &\le (1+\varepsilon)\sum_{j\in\mathcal{J}_*}\big\|(P_ju)'\big\|^2 + \frac{8\ell\rho_0^2}{\varepsilon}\sum_{j\in\mathcal{J}_*}\int_{\partial_{\rm ext}W_{j,\delta}} w_j^2\,\mathrm{d}s\\
&\le (1+\varepsilon)\sum_{j\in\mathcal{J}_*}\big\|(P_ju)'\big\|^2 + \frac{8\ell\rho_0^2}{\varepsilon}\sum_{j=1}^{M}\int_{\partial_{\rm ext}W_{j,\delta}} w_j^2\,\mathrm{d}s.
\end{aligned}
$$

Recall that due to (64) we have

$$
\begin{aligned}
B[u,u] &\ge \Big(1-\frac{a\log\alpha}{\alpha}\Big)\sum_{j\in\mathcal{J}_*}\big\|(P_ju)'\big\|^2,\\
B[u,u] &\ge \frac{\alpha}{2a\log^2\alpha}\sum_{j=1}^{M}\int_{\partial_{\rm ext}V_{j,\delta}} v_j^2\,\mathrm{d}s \equiv \frac{\alpha}{2a\log^2\alpha}\sum_{j=1}^{M}\int_{\partial_{\rm ext}W_{j,\delta}} w_j^2\,\mathrm{d}s,
\end{aligned}
$$

where we used (55) on the last step. Therefore,

$$
\begin{aligned}
B'[Ju,Ju]-B[u,u] &\le \Big(\varepsilon+\frac{a\log\alpha}{\alpha}\Big)\sum_{j\in\mathcal{J}_*}\big\|(P_ju)'\big\|^2 + \frac{8\ell\rho_0^2}{\varepsilon}\sum_{j=1}^{M}\int_{\partial_{\rm ext}W_{j,\delta}} w_j^2\,\mathrm{d}s\\
&\le \left(\frac{\varepsilon+\dfrac{a\log\alpha}{\alpha}}{1-\dfrac{a\log\alpha}{\alpha}}+\frac{8\ell\rho_0^2}{\varepsilon}\cdot\frac{2a\log^2\alpha}{\alpha}\right)B[u,u].
\end{aligned}
$$

Therefore, by setting $\varepsilon=\log\alpha/\sqrt{\alpha}$ and by choosing $c_2>0$ sufficiently large we arrive at

$$
B'[Ju,Ju]-B[u,u]\le\frac{c_2\log\alpha}{\sqrt{\alpha}}\,B[u,u]\le\frac{c_2\log\alpha}{\sqrt{\alpha}}\Big(B[u,u]+\|u\|^2\Big). \tag{69}
$$

Due to the inequalities (68) and (69) we are in the situation of Proposition 7 with $\varepsilon_j := c_j\log\alpha/\sqrt{\alpha}$, $j\in\{1,2\}$, while for each fixed n the assumption $\varepsilon_1<1/\big(1+E_n(B)\big)$ is satisfied due to the estimate (65). Hence, for each fixed n we have

$$
E_n\Big(\bigoplus\nolimits_{j\in\mathcal{J}_*}D_{j,\delta}\Big)\equiv E_n(B')\le E_n(B)+\frac{\log\alpha}{\sqrt{\alpha}}\,\frac{\big(c_1E_n(B)+c_2\big)\big(1+E_n(B)\big)}{1-c_1\big(1+E_n(B)\big)\log\alpha/\sqrt{\alpha}}. \tag{70}
$$

By (65) we have $E_n(B)=\mathcal{O}(1)$ for each fixed n, and the substitution into (70) gives

$$
E_{K+n}(R_\alpha^\Omega)\ge-\alpha^2-\alpha H_*-\frac{H_*^2}{2}+E_n\Big(\bigoplus\nolimits_{j\in\mathcal{J}_*}D_{j,\delta}\Big)+\mathcal{O}\Big(\frac{\log\alpha}{\sqrt{\alpha}}\Big),
$$

and it remains to note that for fixed n and j one has

$$
E_n(D_{j,\delta})=E_n(D_j)+\mathcal{O}(\delta)=E_n(D_j)+\mathcal{O}\Big(\frac{\log\alpha}{\alpha}\Big)=E_n(D_j)+o\Big(\frac{\log\alpha}{\sqrt{\alpha}}\Big),
$$

which implies

$$
E_n\Big(\bigoplus\nolimits_{j\in\mathcal{J}_*}D_{j,\delta}\Big)=E_n\Big(\bigoplus\nolimits_{j\in\mathcal{J}_*}D_j\Big)+o\Big(\frac{\log\alpha}{\sqrt{\alpha}}\Big)
$$

and finishes the proof of Proposition 51. □

The combination of Propositions 48 and 51 gives a proof of Theorem 1.

7. Concluding remarks

7.1. **Resonant angles: equilateral triangle.** We are going to show that there are some angles which do not satisfy the non-resonance condition. This will be done in an indirect way. First, remark that if Ω is a convex polygon (with straight sides) with non-resonant vertices, K corner-induced eigenvalues, and side lengths ℓ_j, then

$$\lim_{\alpha\to+\infty}\big(E_{K+1}(R^\Omega_\alpha)+\alpha^2\big)=E_1\big(\bigoplus\nolimits_j D_j\big)\equiv \pi^2/\ell^2>0, \tag{71}$$

where D_j is the Dirichlet Laplacian on $(0,\ell_j)$ and $\ell:=\max\ell_j$. Let us show that this can be violated for some particular polygons Ω and lead to a different eigenvalue asymptotics.

The paper by McCartin [42] contains a rather detailed analysis of the operator R^Ω_α for the case when Ω is an equilateral triangle using a separation of variables in a suitably chosen coordinate system. To be more precise, we assume that the side length of the triangle is 1. Let us give a short account of the results of [42] concerning the behavior of the eigenvalues as $\alpha\to+\infty$ (which corresponds to $\sigma\to-\infty$ in the reference).

One constructs first a complete orthogonal system of eigenfunctions, noted $T^{m,n}_s$ with $n\ge m\ge 0$ and $T^{m,n}_a$ with $n>m\ge 0$ and $m,n\in\mathbb{N}\cup\{0\}$, and $R^\Omega_\alpha T^{m,n}_\circ=E_\alpha(m,n)T^{m,n}_\circ$ for $\circ\in\{s,a\}$, i.e. $T^{m,n}_s$ and $T^{m,n}_a$ share the same eigenvalue for $n>m$. It is then shown that $E_\alpha(m,n)\ge 0$ for $m\ge 2$, therefore, only $m\in\{0,1\}$ contribute to the negative spectrum. One shows then the following asymptotics for $\alpha\to+\infty$ (we cite the respective equations in Subsection 7.2 of [42]):

$$\begin{aligned}
&E_\alpha(0,0)=-4\alpha^2+o(1), && \text{Eq. (37)},\\
&E_\alpha(0,1)=-4\alpha^2+o(1), && \text{Eq. (50)},\\
&E_\alpha(0,n)=-\alpha^2+\frac{4}{27}\Big[\frac{\pi}{r}\Big(n-\frac{3}{2}\Big)\Big]^2+o(1) \text{ for } n\ge 2, && \text{Eq. (53)},\\
&E_\alpha(1,1)=-\alpha^2+o(1), && \text{Eq. (67)},\\
&E_\alpha(1,n)=-\alpha^2+\frac{4}{27}\Big[\frac{\pi}{r}(n-1)\Big]^2+o(1) \text{ for } n\ge 2, && \text{Eq. (80)},
\end{aligned}$$

where $r:=1/(2\sqrt{3})$ is the inradius. The eigenvalues $E_\alpha(0,0)$ and $E_\alpha(0,1)$ (twice) are corner-induced: the half-angle at each corner is $\pi/6$, and $\kappa(\pi/6)=1$ (see Subsection 2.6), hence $K=3$ (we remark that a more precise remainder for the first three eigenvalues was obtained in [28]). Furthermore, by inspecting the above expressions and by taking into account the multiplicities one sees that for any fixed $n\in\mathbb{N}$ one has the asymptotics $E_{K+n}(R^\Omega_\alpha)=-\alpha^2+z_n+o(1)$, where z_n is the nth element (when enumerated in the non-decreasing order) of the *multiset* $Z:=\big\{\big(2\pi m/3\big)^2:\ m\in\mathbb{Z}\big\}$. In particular, one has $z_1=0$ and $E_{K+1}(R^\Omega)=-\alpha^2+o(1)$, which is in contradiction to (71). Hence, the half-angle $\pi/6$ is resonant. In fact, in the above multiset Z one easily recognizes the spectrum of the Laplacian on a circle of length 3, i.e. on the three sides of the triangles glued to each other without any obstacle at the vertices. This operator can be then viewed as the effective operator on the boundary.

We remark that the text of the paper [42] is included into McCartin's book [43] as Chapter 7, but due to a typesetting error some of the important formulas are missing on page 105, which complicates the understanding of the eigenvalue asymptotics. An interested reader should better refer to the original paper [42] for full details.

7.2. **Variable curvature.** We were not able to obtain an analogous result for general domains with variable curvatures (i.e. without assuming that the curvatures of the sides are constant). By analogy with the works on smooth domains, see e.g. [53] one might expect the following asymptotics to be valid: if all corners are concave or convex non-resonant, then $E_{K+n}(R^\Omega_\alpha)=-\alpha^2+E_n\big(\bigoplus_j(D_j-\alpha H_j)\big)+r(\alpha)$ with a suitable error term $r(\alpha)$. Some steps of the above

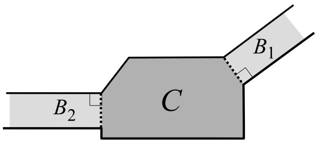


FIGURE 16. A star waveguide Λ with two branches and a dark-shaded center.

scheme are still easily transferable, but the whole machinery appears to fail when trying to prove the lower bound. The main obstacles, when projected to the proof of Proposition 51, are that the eigenvalues of the comparison operator $B' = \bigoplus_j (D_j - \alpha H_j) + \alpha H_* + \rho(\alpha)$ with suitably chosen constants $\rho(\alpha)$ and $H_* := \max_j \max H_j$, may become infinitely large for large α, and much smaller value of ε_j are needed to satisfy the initial assumption of Proposition 7 and to have a non-trivial resulting estimate. In a sense, the machinery we use implicitly aims at showing that the eigenfunctions are suitably small near vertices by controlling their norms and traces using the values in the rest of the domain (Lemma 50). For variable curvature, one expects similarly to the smooth case [26] that the eigenfunctions are localized near the points of maximal curvature. In particular, if the curvature takes its maximum at one of the corners, then the respective eigenfunctions should be localized near the corner, so the strategy of showing that it asymptotically vanishes at the corners (which then gives an effective operator with the Dirichlet boundary conditions) becomes contradictory. One might expect that a more precise analysis in this case can be done under explicit hypotheses on the curvatures (e.g. an isolated maximum at a corner) by showing first some semiclassical localization properties for the eigenfunctions, which might be a task of a higher complexity.

7.3. **Resonance and non-resonance conditions.** Our non-resonance condition introduced in Definition 29 and used in the proof is a slightly naive adaptation of a condition appearing in the spectral analysis of Laplacians on domain collapsing onto a graph. The topic is presented in a systematic way e.g. in the papers by Grieser [23], Molchanov and Vainberg [44], and in the monograph by Post [55]. Let us recall some basic notions of the theory, mostly following the short presentation given in the paper [51] by Pankrashkin.

Let $d \geq 2$ and $\omega \subset \mathbb{R}^{d-1}$ be a bounded connected Lipschitz domain. We denote by μ the first Dirichlet eigenvalue of ω. By a *star waveguide* we mean a connected Lipschitz domain $\Lambda \subset \mathbb{R}^d$ for which one can find n non-intersecting half-infinite cylinders $B_1, \dots, B_n \subset \Lambda$, all isometric to $(0, \infty) \times \omega$, such that Λ coincides with the union $B_1 \cup \ldots \cup B_n$ outside a compact set, see Figure 16. The cylinders B_j will be called *branches*, the connected bounded domain $C := \Lambda \setminus \overline{B_1 \cup \ldots \cup B_n}$ will be called *center*, which is also assumed Lipschitz. We call such a domain Λ a *star waveguide*. Remark that centers of star waveguides are not defined uniquely: one can attach finite pieces of B_j to a given center to obtain a new center.

For small $\varepsilon > 0$, let $\Omega_\varepsilon \subset \mathbb{R}^d$ be a domain composed of finite cylinders $B_{j,\varepsilon}$ isometric to $I_j \times (\varepsilon\omega)$ with $I_j := (0, \ell_j)$, $\ell_j < 0$, $j \in \{1, \dots, J\}$, connected to each other through some bounded Lipschitz domains $C_{k,\varepsilon}$, see Figure 17(a). In the context of the problem, it is natural to refer to $B_{j,\varepsilon}$ as to *edges* and to $C_{k,\varepsilon}$ as to *vertices*. We assume that the vertices $C_{k,\varepsilon}$ are isometric to εC_k with some ε-independent domains C_k, $k \in \{1, \dots, K\}$, and that if one considers a vertex $C_{k,\varepsilon}$ and extends the attached cylindrical edges to infinity, then one obtains a domain isometric to $\varepsilon \Lambda_k$ with some ε-independent star waveguide Λ_k having C_k as its center.

In various applications one is interested in the eigenvalues of the Dirichlet laplacian $-\Delta_D^{\Omega_\varepsilon}$ in Ω_ε as $\varepsilon \to 0^+$. As the domain Ω_ε collapses onto its one-dimensional skeleton X composed from the segments I_j coupled at the vertices, see Figure 17(b), it is natural to expect that the behavior of the eigenvalues should be determined by an effective operator associated with X. The results of [23, Theorems 2 and 3] can be summarized informally as follows. Consider the Dirichlet Laplacians $-\Delta_D^{\Lambda_k}$ in the star waveguides Λ_k associated with each vertex as described above: the spectrum consists of the essential part $[\mu, +\infty)$ and of discrete eigenvalues $E_j(-\Delta_D^{\Lambda_k})$,

$j \in \{1, \dots, N(\Lambda_k)\}$, $k \in \{1, \dots, K\}$. Then with some $N \geq N(\Lambda_1) + \cdots + N(\Lambda_K)$, $a_n \in (0, \mu]$ and $b > 0$ there holds, as $\varepsilon \to 0^+$:

- for $n \in \{1, \dots, N\}$ there holds $E_n(-\Delta_D^{\Omega_\varepsilon}) = a_n/\varepsilon^2 + \mathcal{O}(e^{-b/\varepsilon})$,
- for any fixed $n \in \mathbb{N}$ there holds $E_{N+n}(-\Delta_D^{\Omega_\varepsilon}) = \mu/\varepsilon^2 + E_n(L) + \mathcal{O}(\varepsilon)$, where L is a self-adjoint operator in $L^2(X) \simeq \bigoplus_{j=1}^J L^2(0, \ell_j)$ acting as $(f_j) \mapsto (-f_j'')$ with suitable self-adjoint boundary conditions determined by the scattering matrices of $-\Delta_D^{\Lambda_k}$ at the threshold energy μ (see e.g. the paper [25] by Guiloppé for the definition and properties of the scattering matrices).

The operator L, which is the so-called quantum graph laplacian on X (see the monograph [6] by Berkolaiko and Kuchment for an introduction and a review), represents the sought "effective operator" on X, and the associated boundary conditions describe the way how the branches of the network interact through the vertices in the limit $\varepsilon \to 0$. At the same time, finding explicitly the boundary condition in the general case represents a very difficult task.

The above general construction admits an important particular case, which can be formulated in simpler terms. One says that a star waveguide Λ admits a *threshold resonance* if there exists a non-zero function $\Phi \in L^\infty(\Lambda)$ satisfying $-\Delta\Phi = \mu\Phi$ in Λ and $\Phi = 0$ at $\partial\Lambda$, then the following result holds [23, Section 8]:

Proposition 52. *Assume that none of Λ_k admits a threshold resonance, then for $\varepsilon \to 0^+$ the following asymptotics are valid:*

- *Denote $N := N(\Lambda_1) + \cdots + N(\Lambda_K)$ and let $a_1, \dots, a_N$ be the family of the eigenvalues $E_j(-\Delta_D^{\Lambda_k})$, $j \in \{1, \dots, N(\Lambda_k)\}$, $k \in \{1, \dots, K\}$, enumerated in the non-decreasing order, then for $n \in \{1, \dots, N\}$ one has $E_n(-\Delta_D^{\Omega_\varepsilon}) = a_n/\varepsilon^2 + \mathcal{O}(e^{-b/\varepsilon})$, with some fixed $b > 0$,*
- *For any fixed $n \geq 1$ there holds $E_{N+n}(-\Delta_D^{\Omega_\varepsilon}) = \mu/\varepsilon^2 + E_j(\oplus_{j=1}^J D_j) + \mathcal{O}(\varepsilon)$ with D_j being the Dirichlet Laplacians on $(0, \ell_j)$.*

In other word, in the absence of threshold resonances the effective operator L is decoupled and corresponds to the Dirichlet boundary conditions at the vertices. In view of this result, it is important to be able to identify if star waveguides admits no threshold resonance. The following sufficient condition was obtained in [51], which was in turn motivated by the analysis of particular configurations carried out by Bakharev, Nazarov, Matveenko [2], Nazarov [45, 46], Nazarov, Ruotsalainen, Uusitalo [47]. For a star waveguide Λ with a center C we denote by $-\Delta_{DN}^C$ the Laplacian in C with the Dirichlet boundary condition of $\partial C \cap \partial\Lambda$ and the Neumann boundary condition at the remaining boundary, then if for some center C one has the strict inequality

$$E_{N(\Lambda)+1}(-\Delta_{DN}^C) > \mu, \tag{72}$$

then Λ has no threshold resonance. In the recent preprint [3] Bakharev and Nazarov prove that the condition (72) for some center C is also necessary for the absence of threshold resonance (hence, it is a necessary and sufficient condition).

By comparing Proposition 52 with our main Theorem 1 one may see that that role of the star waveguides attached to the vertices is quite similar to the role of the infinite sectors for

FIGURE 17. (a) An example of a domain Ω_ε with dark shaded vertices. (b) The associated one-dimensional skeleton X.

the Robin laplacians. In fact our condition of non-resonance (Definition 29) is a translation of the condition (72) into the framework of Robin sectors. Namely, one may rewrite (72) using the center εC of the scaled waveguide $\varepsilon\Lambda$ as $E_{N(\Lambda)+1}(-\Delta^{\varepsilon C}_{DN}) = \mu/\varepsilon^2 + c/\varepsilon^2$ with $c := E_{N(\Lambda)+1}(-\Delta^{C}_{DN}) - \mu > 0$ and remark that μ/ε^2 is the bottom of the essential spectrum of the Dirichlet laplacian on $\varepsilon\Lambda$. This should be compared with the scaled form of the non-resonance condition $E_{\kappa(\theta)+1}(N^{\delta}_{\theta,\alpha}) \geq -\alpha^2 + c/\delta^2$, $c > 0$, as $\alpha\delta$ is large, by noting that $-\alpha^2$ is the bottom of the spectrum of the α-Robin laplacian in the infinite sector. We also remark that the result of Proposition 52 was obtained earlier by Post [54] under the assumption that each Λ_k admits a center C_k such that $E_1(-\Delta^{C_k}_{DN}) > \mu$, which is exactly the condition (72) for $N(\Lambda) = 0$. In fact, the final steps of our proof (especially the construction of the identification map J) are an adaption of those from [54]. In view of the preceding analogies with the waveguides, it would be interesting to find alternative reformulations of our non-resonance condition e.g. in terms of generalized eigenfunctions at the bottom of the essential spectrum, which might help to extend our result to a larger range of angles. It would also be of interest to understand the eigenvalue asymptotics for general angles (i.e. without assuming that the angles are non-resonant), which might involve a development of the scattering theory in infinite sectors similar to the one for waveguides.

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