THE KINETIC FOKKER-PLANCK EQUATION WITH GENERAL FORCE

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Abstract. We consider the kinetic Fokker-Planck equation with a class of general force. We prove the existence and uniqueness of a positive normalized equilibrium (in the case of a general force) and establish some exponential rate of convergence to the equilibrium (and the rate can be explicitly computed). Our results improve similar results established by [26, 5, 6, 14, 10, 11, 1] to general force case, and improve the non-quantitative rate of convergence in [18] to quantitative explicit rate.

Contents

1. Introduction

In this paper, we consider the kinetic Fokker-Planck (KFP for short) equation with general force

$$\partial_t f = \mathcal{L} f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + \operatorname{div}_v(\nabla_v W(v) f), \tag{1.1}$$

for a density function $f = f(t, x, v)$, with $t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d$, with

$$V(x) = \frac{\langle x\rangle^\gamma}{\gamma}, \quad \gamma \geq 1, \quad W(v) = \frac{\langle v\rangle^\beta}{\beta}, \quad \beta \geq 2$$

where $\langle x\rangle^2 := 1 + |x|^2$, and the kinetic Fitzhugh-Nagumo equation

$$\partial_t f := \mathcal{L} f = \partial_x (A(x,v) f) + \partial_v (B(x,v) f) + \partial^2_{vv} f \tag{1.2}$$

with

$$A(x,v) = ax - bv, \quad B(x,v) = v(v-1)(v-\lambda) + x$$

for some $a, b, \lambda > 0$. The evolution equations are complemented with an initial datum

$$f(0,x,v) = f_0(x,v) \quad \text{on } \mathbb{R}^{2d}.$$

It's easily seen that both equations are mass conservative, that is

$$\mathcal{M}(f(t,\cdot)) = \mathcal{M}(f_0),$$

where we define the mass of f by

$$\mathcal{M}(f) = \int_{R^d\times R^d} f(x,v)dxdv.$$

When G satisfies

$$\mathcal{L}G = 0, \quad \mathcal{M}(G) = 1, \quad G \geq 0,$$

we say that G is a nonnegative normalized steady state.
For a given weight function m, we will denote $L^p(m) = \{f | fm \in L^p\}$ the associated Lebesgue space and $\|f\|_{L^p(m)} = \|fm\|_{L^p}$ the associated norm.
With these notations, we can introduce the main result of this paper.

Theorem 1.1. *(1) When $2 \leq \beta, 1 \leq \gamma$, there exist a weight function $m > 0$ and a nonnegative normalized steady state $G \in L^1(m)$ such that for any initial datum $f_0 \in L^1(m)$, the associated solution $f(t,\cdot)$ of the kinetic Fokker-Planck equation (1.1) satisfies*

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^1(m)} \leq Ce^{-\lambda t}\|f_0 - \mathcal{M}(f_0)G\|_{L^1(m)},$$

for some constant $C, \lambda > 0$.
(2) The same conclusion holds for the kinetic Fitzhugh-Nagumo equation (1.2).

In the results above the constants C and λ can be explicitly estimated in terms of the parameters appearing in the equation by following the calculations in the proofs. We do not give them explicitly since we do not expect them to be optimal, but they are nevertheless completely constructive.

Remark 1.2. Theorem 1.1 is also true when $V(x)$ behaves like $\langle x\rangle^\gamma$ and $W(v)$ behaves like $\langle v\rangle^\beta$, that is for any $V(x)$ satisfying

$$C_1\langle x\rangle^\gamma \leq V(x) \leq C_2\langle x\rangle^\gamma, \quad \forall x \in \mathbb{R}^d,$$

$$C_3|x|\langle x\rangle^{\gamma-1} \leq x\cdot\nabla_x V(x) \leq C_4|x|\langle x\rangle^{\gamma-1}, \quad \forall x \in B_R^c,$$

and

$$|D_x^n V(x)| \leq C_n\langle x\rangle^{\gamma-2}, \quad \forall x \in \mathbb{R}^d, \quad \forall n \geq 2,$$

for some constant $C_i > 0$, $R > 0$, and similar estimates holds for $W(v)$.

In fact, Theorem 1.1 is a special case of the following theorem.

Theorem 1.3. *Consider the following equation*

$$\partial_t f := \mathcal{L} f = \partial_x(A(x,v)f) + \partial_v(B(x,v)f) + \Delta_v f \tag{1.3}$$

with

$$A(x,v) = -v + \Phi(x)$$

where $\Phi(x)$ *is Lipschitz*

$$|\Phi(x) - \Phi(y)| \le M|x-y|,$$

for some $M > 0$*, define*

$$\begin{aligned} \phi_2(m) &= -v \cdot \frac{\nabla_x m}{m} - \Phi(x) \cdot \frac{\nabla_x m}{m} + \frac{1}{2} div_x \Phi(x) + \frac{|\nabla_v m|^2}{m^2} \\ &+ \frac{\Delta_v m}{m} - B(x,v) \cdot \frac{\nabla_v m}{m} + \frac{1}{2} div_v B(x,v) \end{aligned}$$

then if we can find a weight function m *and a function* $H \ge 1$ *such that*

$$\mathcal{L}^* m \le -\alpha m + b,$$

for some $\alpha, b > 0$

$$-C_1 H m \le \phi_2(m) \le -C_2 H m + C_3,$$

for some $C_1, C_2, C_3 > 0$*, and for any integer* $n \ge 2$ *fixed, for any* $\epsilon > 0$ *small, we can find a constant* $C_{\epsilon,n}$ *such that*

$$\sum_{k=2}^{n} |D_x^k \Phi(x)| + \sum_{k=2}^{n} |D_{x,v}^k B(x,v)| \le C_{n,\epsilon} + \epsilon H$$

and

$$\frac{\Delta_{x,v} m}{m} \ge -C_4$$

for some $C_4 > 0$*, then we have there exist a steady state* G *such that*

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^1(m)} \le C e^{-\lambda t} \|f_0 - \mathcal{M}(f_0)G\|_{L^1(m)}.$$

for some $C, \lambda > 0$*.*

Remark 1.4. In fact $\phi_2(m)$ satisfies

$$\int (f(\mathcal{L}g) + g(\mathcal{L}f))m^2 = -2\int \nabla_v f \cdot \nabla_v g m^2 + 2\int f g \phi_2(m) m^2.$$

the computation can be found in Appendix C.

Remark 1.5. For the kinetic Fokker-Planck equation with general force 1.1, we can take

$$m = e^{H_1}, \quad H_1 = |v|^2 + V(x) + \epsilon v \cdot \nabla_x \langle x \rangle, \quad H = \langle v \rangle^\beta + \langle x \rangle^{\gamma - 1},$$

for some $\epsilon > 0$ small, the computation can be found in Appendix B below. For the kinetic Fitzhugh-Nagumo equation (1.2), we can take

$$m = e^{\lambda(x^2+v^2)}, \quad H = |v|^4 + |x|^2,$$

for some constant $\lambda > 0$, the computation can be found in [18].

For the kinetic Fitzhugh-Nagumo equation (1.2), an exponential convergence with non-quantitative rate to the convergence has already been proved in [18], our method improves the result to a quantitative rate.

If $\beta = 2$, the equation (1.1) will turns to the classical KFP equation

$$\partial_t f = \mathcal{L} f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + \mathrm{div}_v(vf),$$

This time we observe that

$$G = Z^{-1} e^{-W}, \quad W = \frac{v^2}{2} + V(x), \quad Z \in \mathbb{R}_+,$$

is an explicit steady state. There are many classical results on the case $\gamma \geq 1$, where there is an exponentially decay. We refer the interested readers to [26, 5, 6, 14, 10, 11, 1, 17], and for the weak confinement case $\gamma \in (0, 1)$, there are also some polynomial or sub-geometric convergence results proved in [1, 2, 7]. We also emphasize that our results for kinetic Fokker-Planck equation with general potentials are to our knowledge new.

We carry out all of our proofs using variations of Harris's Theorem for Markov semigroup. Harris's Theorem originated in the paper [12] where Harris gave conditions for existence and uniqueness of a steady state for Markov processes. It was then pushed forward by Meyn and Tweedie in [25] to show exponential convergence to equilibrium. [13] gives an efficient way of getting quantitative rates for convergence to equilibrium once the assumptions have been quantitatively verified. We give the precise statement in the next section.

One advantage of the Harris method is that it directly yields convergence for a wide range of initial conditions, while previous proofs of convergence to equilibrium mainly use some strongly weighted L^2 or H^1 norms (typically with a weight which is the inverse of a Gaussian). The Harris method also gives existence of stationary solutions under quite general conditions; in some cases these are explicit and easy to find, but in other cases such as the two models in our paper they can be nontrivial. Also the Harris method provides a quantitative rate of convergence to the steady state, which is better than non-quantitative type argument such as the consequence of Krein Rutman theorem.

Let us end the introduction by describing the plan of the paper. In Section 2, we introduce Harris Theorem. In section 3, we compute the Lyapunov function for the kinetic Fokker Planck equation. In Section 4 we present the proof of a regularization estimate on $S_{\mathcal{L}}$. In Section 5 we the Harris condition for the general kinetic Fokker- Planck equation. Finally in Appendix we present some

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2. Harris Theorem and existence of steady state

In this section we introduce Doeblin-Harris theorem and the existence of steady state.

Theorem 2.1. *(Harris- Doeblin Theorem) We consider a semigroup S_t with generator $\mathcal{L}$ and we assume that*
(H1)(Lyapunov condition) There exists some weight function $m : \mathbb{R}^d \to [1,\infty)$ satisfying $m(x) \to \infty$ as $x \to \infty$ and there exist some constants $\alpha > 0, b > 0$ such that

$$\mathcal{L}^* m \leq -\alpha m + b,$$

(H2)(Harris condition) For any $R > 0$ there exist a constant $T = T(R) > 0$ and a positive, nonzero measure $\mu = \mu(R)$ such that

$$S_T f \geq \mu \int_{B_R} f, \quad \forall f \in X, \quad f \geq 0.$$

where B_R denotes the ball centered at origin with radius R. There exist some constants $C \geq 1$ and $a < 0$ such that

$$\|S_t f\|_{L^1(m)} \leq C e^{at} \|f\|_{L^1(m)}, \quad \forall t \geq 0, \quad \forall f \in X, \quad \mathcal{M}(f) = 0.$$

Proof. See [23] Proposition 2.2 for instance. □

The Lyapunov condition also provides a sufficient condition for the existence of an invariant measure (for the dual semigroup).

Theorem 2.2. *Any mass conserving positive Markov semigroup (S_t) which fulfills the above Lyapunov condition has at least one invariant borelian measure $G \in M^1(m)$, where M^1 is the space of measures.*

Proof. Step 1. We prove that (S_t) is a bounded semigroup. For $f_0 \in M^1(m)$, we define $f_t := S_{\mathcal{L}}(t) f_0$, and we easily compute

$$\frac{d}{dt} \int |f_t| m \leq \int |f_t| \mathcal{L}^* m \leq \int |f_t| (-am + b).$$

Using the mass conservation and positivity, integrating the above differential inequality, we get

$$\begin{aligned} \int |f_t| m &\leq e^{-at} \int |f_0| m + \frac{b}{a}(1 - e^{-at}) \int |f_0| \\ &\leq max(1, \frac{b}{a}) \int |f_0| m, \forall t \geq 0, \end{aligned}$$

so that (S_t) is bounded in $M^1(m)$.
Step 2. We prove the existence of a steady state, more precisely, we start proving that there exists a positive and normalized steady state $G \in M^1(m)$. For the equivalent norm $|||\cdot|||$ defined on $M^1(m)$ by

$$|||f||| := \sup_{t>0} \|S_{\mathcal{L}}(t) f\|_{M^1(m)},$$

we have $|||S_{\mathcal{L}}(t)f||| \leq |||f|||$ for all $t \geq 0$, that is the semigroup $S_{\mathcal{L}}$ is a contraction semigroup on $(M^1(m), |||\cdot|||)$. There exists $R > 0$ large enough such that the intersection of the closed hyperplane $\{f \in M^1(m); \langle f \rangle = 1\}$ and the closed ball of radius R in $(M^1(m), |||\cdot|||)$ is a convex, non-empty subset. Then consider the closed, weakly * compact convex set

$$\mathbb{K} := \{f \in M^1(m); |||f||| \leq R, f \geq 0, \langle f \rangle = 1\},$$

Since $S_{\mathcal{L}}(t)$ is a linear, weakly * continuous, contraction in $(M^1(m), |||\cdot|||)$ and $\langle S_{\mathcal{L}}(t)f \rangle = \langle f \rangle$ for all $t \geq 0$, we see that $\mathbb{K}$ is stable under the action of the semigroup. Therefore we apply the Markov-Kakutani fixed point theorem and we conclude that there exists $G \in \mathbb{K}$ such that $S_{\mathcal{L}}(t)G = G$. Therefore we have in particular $G \in D(\mathcal{L})$ and $\mathcal{L}G = 0$. □

3. Regularization property of $S_{\mathcal{L}}$

The aim of this section is to establish the following regularization property. The proof closely follows the proof of similar results in [11, 17, 26]

Theorem 3.1. *Consider the weight function m as defined in Theorem 1.3, there exist $\eta, C > 0$ such that*

$$\|S_{\mathcal{L}}(t)f\|_{L^2(m)} \leq \frac{C}{t^{\frac{5d+1}{2}}}\|f\|_{L^1(m)}, \quad \forall t \in [0, \eta].$$

for some weight function m. In addition, for any integer $k > 0$ there exist we some $\alpha(k), C(k) > 0$ such that

$$\|S_{\mathcal{L}}(t)f\|_{H^k(m)} \leq \frac{C}{t^{\alpha}}\|f\|_{L^1(m)}, \quad \forall t \in [0, \eta].$$

as a consequence we have

$$\|S_{\mathcal{L}}(t)f\|_{C^{2,\delta}(m)} \leq \frac{C}{t^{\zeta}}\|f\|_{L^1(m)}, \quad \forall t \in [0, \eta],$$

for some $\delta \in (0,1), \zeta > 0$

We start with some elementary lemmas.

Lemma 3.2. *For $f_t = S_{\mathcal{L}}(t)f_0$, define an energy functional*

$$\begin{aligned} \mathcal{F}(t, f_t) \quad &:= \quad A\|f_t\|^2_{L^2(m)} + at^2\|\nabla_v f_t\|^2_{L^2(m)} \\ &+ \quad 2ct^4(\nabla_v f_t, \nabla_x f_t)^2_{L^2(m)} + bt^6\|\nabla_x f_t\|^2_{L^2(m)}, \end{aligned} \tag{3.1}$$

with $a, b, c > 0, c \leq \sqrt{ab}$ and A large enough. Then there exist $\eta > 0$ such that

$$\frac{d}{dt}\mathcal{F}(t, f_t) \leq -L(\|\nabla_v f_t\|^2_{L^2(m)} + t^4\|\nabla_x f_t\|^2_{L^2(m)}) + C\|f_t\|^2_{L^2(m)},$$

for all $t \in [0, \eta]$, for some $L > 0, C > 0$, as a consequence, we have

$$\|S_{\mathcal{L}}f_0\|_{H^1(m)} \leq Ct^{-6}\|f_0\|_{L^2(m)},$$

for all $t \in [0,\eta]$, iterating k times we get

$$\|S_{\mathcal{L}} f_0\|_{H^k(m)} \le C t^{-6k} \|f_0\|_{L^2(m)}.$$

Proof. We only prove the case $k=1$, for $k=2$, one need only replace f by $\partial_{x_i} f$ and $\partial_{v_i} f$, similarly for $k>2$. First by Theorem 1.3 and Remark 1.4 we have

$$(f,\mathcal{L}g)_{L^2(m)} + (g,\mathcal{L}f)_{L^2(m)} = -2(\nabla_v f, \nabla_v g)_{L^2(m)} + (f, g\phi_2(m))_{L^2(m)},$$

for any $f,g \in L^2(m)$. As a consequence, we have

$$\frac{d}{dt}\|f\|^2_{L^2(m)} = (f,\mathcal{L}f)_{L^2(m)} \le -\|\nabla_v f\|^2_{L^2(m)} - C_1\|f\|^2_{L^2(mH^{1/2})} + C_2\|f\|^2_{L^2(m)}.$$

By

$$\partial_{x_i}\mathcal{L}f = \mathcal{L}\partial_{x_i} f + \sum_{j=1}^d \partial^2_{x_i x_j} V \partial_{v_j} f, \tag{3.2}$$

and since

$$|\nabla_x^2 V(x)| \le C H_1,$$

for some $C>0$, we have

$$\begin{aligned}
&\frac{d}{dt}\|\partial_{x_i} f\|_{L^2(m)} \\
= \;& (\partial_{x_i} f, \mathcal{L}\partial_{x_i} f)_{L^2(m)} + (\partial_{x_i} f, \sum_{j=1}^d \partial^2_{x_i x_j} V \partial_{v_j} f)_{L^2(m)} \\
\le \;& -\|\nabla_v(\partial_{x_i} f)\|^2_{L^2(m)} - C_1\|\partial_{x_i} f\|^2_{L^2(mH^{1/2})} + C_2\|\partial_{x_i} f\|^2_{L^2(m)} + C(|\nabla_x f|, |\nabla_v f|)_{L^2(mH^{1/2})}.
\end{aligned}$$

Using Cauchy-Schwarz inequality and summing over $i = 1,2,3,...,n$, we get

$$\begin{aligned}
&\frac{d}{dt}\|\nabla_x f\|^2_{L^2(m)} \\
\le \;& -\sum_{i=1}^n \|\nabla_v(\partial_{x_i} f)\|^2_{L^2(m)} - \frac{C_1}{2}\|\nabla_x f\|^2_{L^2(mH^{1/2})} + C_2\|\nabla_x f\|^2_{L^2(m)} + C\|\nabla_v f\|^2_{L^2(mH^{1/2})}.
\end{aligned}$$

for some $C>0$. Similarly using

$$\partial_{v_i}\mathcal{L}f = \mathcal{L}\partial_{v_i} f - \partial_{x_i} f + \sum_{j=1}^d \partial^2_{v_i v_j} W \partial_{v_j} f, \tag{3.3}$$

and since

$$|\nabla_v^2 W(v)| \le \frac{C_1}{2d} H + C,$$

we have

$$\begin{aligned}
&\frac{d}{dt}\|\partial_{v_i} f\|^2_{L^2(m)} \\
= \quad & (\partial_{v_i} f, \mathcal{L}\partial_{v_i} f)_{L^2(m)} - (\partial_{x_i} f, \partial_{v_i} f)_{L^2(m)} + (\partial_{v_i} f, \sum_{j=1}^{d} \partial^2_{v_i v_j} W \partial_{v_j} f)_{L^2(m)} \\
\leq \quad & -\|\nabla_v(\partial_{v_i} f)\|^2_{L^2(m)} - C_1\|\partial_{v_i} f\|^2_{L^2(mH^{1/2})} + C_2\|\partial_{v_i} f\|^2_{L^2(m)} + \frac{C_1}{2d}\|\|\nabla_v f\|^2_{L^2(mH^{1/2})} \\
& + C\|\nabla_v f\|^2_{L^2(m)} - (\partial_{x_i} f, \partial_{v_i} f)_{L^2(m)}.
\end{aligned}$$

Using Cauchy-Schwarz inequality and summing over $i = 1, 2, ..., n$ we get

$$\begin{aligned}
&\frac{d}{dt}\|\nabla_v f\|^2_{L^2(m)} \\
\leq \quad & -\sum_{i=1}^{n} \|\nabla_v(\partial_{v_i} f)\|^2_{L^2(m)} - \frac{C_1}{2}\|\nabla_v f\|^2_{L^2(mH^{1/2})} \\
& + C\|\|\nabla_v f\|^2_{L^2(m)} - (\nabla_v f, \nabla_x f)_{L^2(m)}.
\end{aligned}$$

For the crossing term, we split it also into two parts. Using (3.2) and (3.3), we have

$$\begin{aligned}
&\frac{d}{dt}2(\partial_{v_i} f, \partial_{x_i} f)_{L^2(m)} \\
= \quad & (\partial_{x_i} f, \mathcal{L}\partial_{v_i} f)_{L^2(m)} - (\partial_{x_i} f, \partial_{x_i} f)_{L^2(m)} + (\partial_{x_i} f, \sum_{j=1}^{d} \partial^2_{v_i v_j} W \partial_{v_j} f)_{L^2(m)} \\
& + (\partial_{v_i} f, \mathcal{L}\partial_{x_i} f)_{L^2(m)} + (\partial_{v_i} f, \sum_{j=1}^{d} \partial^2_{x_i x_j} V \partial_{v_j} f)_{L^2(m)} \\
\leq \quad & -2(\nabla_v(\partial_{x_i} f), \nabla(\partial_{v_i} f))_{L^2(m)} - \|\partial_{x_i} f\|^2_{L^2(m)} + C\|\nabla_v f\|^2_{L^2(mH^{1/2})} \\
+ \quad & C(|\nabla_v f|, |\nabla_x f|)_{L^2(mH^{1/2})},
\end{aligned}$$

Combining the two parts, using Cauchy-Schwarz inequality, and summing over i we get

$$\begin{aligned}
&\frac{d}{dt}2(\nabla_v f, \nabla_x f)_{L^2(m)} \\
\leq \quad & -2\sum_{i=1}^{n}(\nabla_v(\partial_{x_i} f), \nabla(\partial_{v_i} f))_{L^2(m)} - \|\nabla_x f\|^2_{L^2(m)} + C\|\nabla_v f\|^2_{L^2(mH^{1/2})} \\
+ \quad & C(|\nabla_v f|, |\nabla_x f|)_{L^2(mH^{1/2})},
\end{aligned}$$

For the very definition of $\mathcal{F}$ in (3.1), we easily compute

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t,f_t) &= A\frac{d}{dt}\|f_t\|^2_{L^2(m)} + at^2\frac{d}{dt}\|\nabla_v f_t\|^2_{L^2(m)} + 2ct^4\frac{d}{dt}\langle\nabla_v f_t,\nabla_x f_t\rangle^2_{L^2(m)}\\ &\quad +bt^6\frac{d}{dt}\|\nabla_x f_t\|^2_{L^2(m)} + 2at\|\nabla_v f_t\|^2_{L^2(m)} + 8ct^3\langle\nabla_v f_t,\nabla_x f_t\rangle^2_{L^2(m)}\\ &\quad +6bt^5\|\nabla_x f_t\|^2_{L^2(m)}.\end{aligned}$$

Gathering all the inequalities above together, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t,f_t) &\le (2at - A + Cat^2)\|\nabla_v f_t\|^2_{L^2(m)} + (6bt^5 - \frac{c}{2}t^4 + Cbt^6)\|\nabla_x f_t\|^2_{L^2(m)}\\ &+ (8ct^3 - Cat^2)(\nabla_v f_t,\nabla_x f_t)_{L^2(m_2)} + CA\|f_t\|^2_{L^2(m)},\\ &- \sum_{i=1}^d (at^2\|\nabla_v(\partial_{v_i} f_t)\|^2_{L^2(m)} + bt^6\|\nabla_v(\partial_{x_i} f_t)\|^2_{L^2(m)}\\ &+ 2ct^4(\nabla_v(\partial_{x_i} f_t),\nabla_v(\partial_{v_i} f_t))_{L^2(m)}) - \frac{C_1}{2}ct^6\|\nabla_x f\|^2_{L^2(mH^{1/2})}\\ &+ (-\frac{C_1}{2}at^2 + 2Cbt^6 + Cct^4)\|\nabla_v f\|^2_{L^2(mH^{1/2})} + 2bt^4C(|\nabla_v f|,|\nabla_x f|)_{L^2(mH^{1/2})},\end{aligned}$$

for some $C > 0$. We observe that

$$\begin{aligned}&|2ct^4(\nabla_v(\partial_{x_i} f_t),\nabla_v(\partial_{v_i} f_t))_{L^2(m)})|\\ \le\ & at^2\|\nabla_v(\partial_{v_i} f_t)\|^2_{L^2(m)} + bt^6\|\nabla_v(\partial_{v_i} f_t)\|^2_{L^2(m)},\end{aligned}$$

by our choice on a, b, c. So by taking A large and $0 < \eta$ small ($t \in [0,\eta]$), we conclude to

$$\frac{d}{dt}\mathcal{F}(t,f_t) \le -L(\|\nabla_v f_t\|^2_{L^2(m)} + t^4\|\nabla_x f_t\|^2_{L^2(m)}) + C\|f_t\|^2_{L^2(m)},$$

for some $L, C > 0$, and that ends the proof. □

Lemma 3.3. *We have*

$$\int |\nabla_{x,v}(fm)|^2 \le \int |\nabla_{x,v} f|^2 m^2 + C\int f^2 m^2,$$

Proof. We have

$$\begin{aligned}\int |\nabla(fm)|^2 &= \int |\nabla f m + \nabla m f|^2\\ &= \int |\nabla f|^2 m^2 + \int |\nabla m|^2 f^2 + \int 2f\nabla f m\nabla_x m\\ &= \int |\nabla f|^2 m^2 + \int (|\nabla m|^2 - \frac{1}{2}\Delta(m^2))f^2,\\ &= \int |\nabla f|^2 m^2 - \int \frac{\Delta m}{m} f^2 m^2,\end{aligned}$$

since

$$\frac{\Delta m}{m} \ge -C,$$

for some $C>0$, we are done. $\square$

Lemma 3.4. *Nash's inequality: for any $f\in L^1(\mathbb{R}^d)\cap H^1(\mathbb{R}^d)$,there exist a constant C_d such that:*

$$\|f\|_{L^2}^{1+\frac{2}{d}}\le C_d\|f\|_{L^1}^{\frac{2}{d}}\|\nabla_v f\|_{L^2},$$

For the proof of Nash's inequality, we refer to [16], Section 8.13 for instance. $\square$

Lemma 3.5. *There exist $\lambda>0$ such that*

$$\frac{d}{dt}\|f\|_{L^1(m)}\le\lambda\|f\|_{L^1(m)} \tag{3.4}$$

which implies

$$\|f_t\|_{L^1(m)}\le Ce^{\lambda t}\|f_0\|_{L^1(m)}$$

In particular we have

$$\|f_t\|_{L^1(m)}\le C\|f_0\|_{L^1(m)},\quad \forall t\in[0,\eta], \tag{3.5}$$

for some constant $C>0$.

Proof. It' s an immediate consequence of the Lyapunov condition (H1). $\square$

Now we come to the proof of Theorem 3.1.

Proof. (**Proof of Theorem 3.1.**) We define

$$\mathcal{G}(t,f_t)=B\|f_t\|^2_{L^1(m)}+t^Z\mathcal{F}(t,f_t),$$

with $B,Z>0$ to be fixed and $\mathcal{F}$ defined in Lemma 3.2. We choose $t\in[0,\eta]$, η small enough such that $(a+b+c)Z\eta^{Z+1}\le\frac{1}{2}L\eta^Z$ $(a,b,c,L$ are also defined Lemma 3.2). By (3.4) and Lemma 3.2, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{G}(t,f_t)&\le\lambda B\|f_t\|^2_{L^1(m)}+Zt^{Z-1}\mathcal{F}(t,f_t)\\&\quad -Lt^Z(\|\nabla_v f_t\|^2_{L^2(m)}+t^4\|\nabla_x f_t\|^2_{L^2(m)})+Ct^Z\|f_t\|^2_{L^2(m)}\\&\le\lambda B\|f_t\|^2_{L^1(m)}+Ct^{Z-1}\|f_t\|^2_{L^2(m)}\\&\quad -\frac{L}{2}t^Z(\|\nabla_v f_t\|^2_{L^2(m)}+t^4\|\nabla_x f_t\|^2_{L^2(m)}),\end{aligned}$$

where λ is defined in Lemma 3.5. Nash's inequality and Lemma 3.2 imply

$$\|fm\|_{L^2}\le C\|fm\|_{L^1}^{\frac{2}{d+2}}\|\nabla_{x,v}(fm)\|_{L^2}^{\frac{d}{d+2}}\le C\|fm\|_{L^1}^{\frac{2}{d+2}}(\|\nabla_{x,v}fm\|_{L^2}+C\|fm\|_{L^2})^{\frac{d}{d+2}}.$$

Using Young's inequality, we have

$$\|f_t\|^2_{L^2(m)}\le C_\epsilon t^{-5d}\|f\|^2_{L^1(m)}+\epsilon t^5(\|\nabla_{x,v}f_t\|^2_{L^2(m)}+C\|f_t\|^2_{L^2(m)}).$$

Taking ϵ small such that $C\epsilon\eta^5\le\frac{1}{2}$, we deduce

$$\|f_t\|^2_{L^2(m)}\le 2C_\epsilon t^{-5d}\|f\|^2_{L^1(m)}+2\epsilon t^5\|\nabla_{x,v}f_t\|^2_{L^2(m)}.$$

Taking ϵ small we have

$$\frac{d}{dt}\mathcal{G}(t,f_t) \le dB\|f_t\|^2_{L^1(m)} + C_1 t^{Z-1-5d}\|f_t\|^2_{L^1(m)},$$

for some $C_1 > 0$. Choosing $Z = 1+5d$, and using (3.5), we deduce

$$\forall t \in [0,\eta], \quad \mathcal{G}(t,f_t) \le \mathcal{G}(0,f_0) + C_2\|f_0\|^2_{L^1(m)} \le C_3\|f_0\|^2_{L^1(m)},$$

which proves

$$\|S_{\mathcal{L}}(t)f\|_{L^2(m)} \le \frac{C}{t^{\frac{5d+1}{2}}}\|f\|_{L^1(m)}, \quad \forall t \in [0,\eta].$$

together with Lemma 3.2 ends the proof. □

4. Proof of Harris condition

In this section we prove the Harris condition (H2) for equation (1.3). Before the proof of the theorem, we first prove a useful lemma.

Lemma 4.1. *For any $R > 0$, there exist a λ, ρ such that for any $t > 0$, there exists $(x_0, v_0) \in B_\rho$ such that*

$$f(t,x_0,v_0) \ge \lambda \int_{B_R} f_0.$$

Proof. From conservation of mass, we classically show that

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x,v)dxdv = 0,$$

so we have

$$\|S_{\mathcal{L}}(t)\|_{L^1\to L^1} \le 1, \quad \forall t \ge 0, \tag{4.1}$$

Define the splitting of the KFP operator $\mathcal{L}$ by

$$\mathcal{B} = \mathcal{L} - \mathcal{A}, \quad \mathcal{A} = M\chi_R(x,v).$$

with $M, R > 0$ large, where χ is the cut-off function such that $\chi(x,v) \in [0,1]$, $\chi(x,v) \in C^\infty$, $\chi(x,v) = 1$ when $x^2+v^2 \le 1$, $\chi(x,v) = 0$ when $x^2+v^2 \ge 2$, and $\chi_R = \chi(x/R, v/R)$. From the Lyapunov function condition (H1) and taking M, R large, we have

$$\|S_{\mathcal{B}}(t)\|_{L^1(m)\to L^1(m)} \le Ce^{-\lambda t}, \quad \forall t \ge 0. \tag{4.2}$$

By Duhamel's formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{L}},$$

we directly deduce from (4.1) and 4.2 that

$$\|S_{\mathcal{L}}(t)\|_{L^1(m)\to L^1(m)} \le A, \quad \forall t \ge 0,$$

for some $A > 0$. We fix $R > 0$ and take $g_0 = f_0 1_{B_R} \in L^1(\mathbb{R}^d)$ with that supp $g_0 \subset B_R$, ,denote $g_t = S_{\mathcal{L}} g_0, f_t = S_{\mathcal{L}} f_0$, then we have

$$\int_{\mathbb{R}^d} g_t = \int_{\mathbb{R}^d} g_0 = \int_{B_R} g_0 = \int_{B_R} f_0.$$

Moreover, since there exists $A > 0$ such that

$$\int_{\mathbb{R}^d} g_t m \le A \int_{\mathbb{R}^d} g_0 m \le A m(R) \int_{B_R} g_0.$$

For any $\rho > 0$, we write

$$\begin{aligned} \int_{B_\rho} g_t &= \int_{\mathbb{R}^d} g_t - \int_{B_\rho^c} g_t \\ &\ge \int_{\mathbb{R}^d} g_0 - \frac{1}{\rho} \int_{\mathbb{R}^d} g_t m \\ &\ge \int_{\mathbb{R}^d} g_0 - \frac{A m(R)}{\rho} \int_{B_R} g_0 \ge \frac{1}{2} \int_{B_R} g_0, \end{aligned}$$

by taking $\rho = 2Am(R)$. As a consequence, for any $t > 0$, there exist a $(x_0, v_0) \in B_\rho$ which may depend on g_0 such that

$$g(t, x_0, v_0) \ge \frac{1}{|B_\rho|} \int_{B_\rho} g_t \ge \frac{1}{2|B_{2Am(R)}|} \int_{B_R} g_0 := \lambda \int_{B_R} g_0.$$

By the maximum principle we have

$$f(t, x_0, v_0) \ge g(t, x_0, v_0) \ge \lambda \int_{B_R} g_0 = \lambda \int_{B_R} f_0.$$

□

Theorem 4.2. *The equation (1.3) satisfies the Harris condition.*

Proof. By Theorem 3.1 we know take for $t > \frac{\eta}{2}$, we have

$$\Delta_v f, \nabla_x f, \nabla_v f \in C^{0,\alpha},$$

and by equation

$$\partial_t f := \mathcal{L} f = \partial_x (A(x,v) f) + \partial_v (B(x,v) f) + \Delta_v f,$$

we have

$$|\partial_t f| + |\partial_x f| + |\partial_v f| \le C \quad on \ [\frac{\eta}{2}, \eta] \times B_{2R},$$

for some constant $C > 0$. By continuity for every $R > 0$, there exist $t_1, t_2, r_0, \rho, \lambda > 0$ which do not depend on f and $(x_0, v_0) \in B_\rho$ which may depend on f, such that for all $t \in (t_1, t_2)$, we have

$$f(t, x, v) \ge \frac{\lambda}{2} 1_{B_{r_0}(x_0, v_0)} \int_{B_R} f_0,$$

where $B_{r_0}(x_0, v_0)$ denotes the ball centered at (x_0, v_0) with radius r_0, to make x_0, v_0 f independent we use

Theorem 4.3. *Let $f(t,x,v)$ be a classical nonnegative solution of*

$$\partial_t f - \Delta_v f = -(v+\Phi(x))\cdot\nabla_x f + A(t,x,v)\cdot\nabla_v f + B(t,x,v)f,$$

in $[0,T)\times\Omega$, where $\Phi(x)$ is Lipschitz

$$|\Phi(x)-\Phi(y)| \le M|x-y|, \quad \forall x,y\in\Omega,$$

Ω is an open subset of $\mathbb{R}^2$, and $A, B : [0,T)\times\mathbb{R}^d$ and bounded continuous functions. Let $(x_0,v_0)\in\Omega$, let $V\ge|v_0+\Phi(x_0)|$,then for any $r,\tau>0$ there are constants $\lambda, K>0$, only depending on $\bar{A}=\|A\|_{L^\infty(\Omega)}, \bar{B}=\|B\|_{L^\infty(\Omega)}$ and r^2/τ, , such that the following holds: If $B_{\lambda r}(x_0,v_0)\in\Omega, \tau< min(1/2, r^3/4V)$ and $f\ge\delta>0$ in $[0,\tau)\times B_r(x_0,v_0)$, then $f\ge K\delta$ in $[\tau/2,\tau)\times B_{2r}(x_0,v_0)$.

Proof. See Appendix A. □

Coming back to the proof of Theorem 4.2. Define

$$T=\min(t_2-t_1, 1/2, r_0^3/4R),$$

iterate n times we have for any $t\in(t_2-\frac{T}{2^n}, t_2)$

$$f(t,x,v)\ge\frac{\lambda}{2}\prod_{i=1}^n K_i 1_{B_{2^n r_0}(x_0,v_0)}\int_{B_R} f_0,$$

take n large such that $2^n r_0>2\rho$, since $(x_0,v_0)\in B_\rho$ implies that $B_\rho\subset B_{2\rho}(x_0,v_0)$, we have

$$f(t,x,v)\ge\frac{\lambda}{2}\prod_{i=1}^n K_i 1_{B_\rho}\int_{B_R} f_0,$$

for any $t\in(t_2-\frac{T}{2^n}, t_2)$, which is just Harris condition. □

Appendix A. Proof of spreading of positivity

Theorem A.1. *Let $f(t,x,v)$ be a classical nonnegative solution of*

$$\partial_t f - \Delta_v f = -(v+\Phi(x))\cdot\nabla_x f + A(t,x,v)\cdot\nabla_v f + B(t,x,v)f,$$

in $[0,T)\times\Omega$, where $\Phi(x)$ is Lipschitz

$$|\Phi(x)-\Phi(y)| \le M|x-y|, \quad \forall x,y\in\Omega,$$

Ω is an open subset of $\mathbb{R}^2$, and $A, B : [0,T)\times\mathbb{R}^d$ and bounded continuous functions. Let $(x_0,v_0)\in\Omega$, let $V\ge|v_0+\Phi(x_0)|$,then for any $r,\tau>0$ there are constants $\lambda, K>0$, only depending on $\bar{A}=\|A\|_{L^\infty(\Omega)}, \bar{B}=\|B\|_{L^\infty(\Omega)}$ and r^2/τ, , such that the following holds: If $B_{\lambda r}(x_0,v_0)\in\Omega, \tau< min(1/2, r^3/4V)$ and $f\ge\delta>0$ in $[0,\tau)\times B_r(x_0,v_0)$, then $f\ge K\delta$ in $[\tau/2,\tau)\times B_{2r}(x_0,v_0)$.

Proof. This proof is similar to the proof in [26] Appendix A. 22. Let $g(t,x,v) = e^{\bar{B}t} f(t,x,v)$, then $g \geq f$ and $\mathcal{L}g \geq 0$ in $(0,T) \times \Omega$, where

$$\mathcal{L} = \frac{\partial}{\partial t} + (v + \Phi(x)) \cdot \nabla_x - \Delta_v - A(t,x,v) \cdot \nabla_v,$$

Let us construct a particular subsolution for $\mathcal{L}$. In the sequel, B_r will stand for $B_r(x_0, v_0)$. For $t \in (0, \tau]$ and $(x,v) \in \Omega \setminus B_r$ let

$$Q(t,x,v) = a\frac{|v - v_0|^2}{2t} - b\frac{\langle v - v_0, x - X_t(x_0,v_0)\rangle}{t^2} + c\frac{|x - X_t(x_0,v_0)|^2}{2t^3},$$

where $X_t(x_0, v_0) = x_0 + t(v_0 + \Phi(x_0))$ (abbreviated X_t in the sequel) is the position at time t of the geodesic flow starting from (x_0, v_0), and $a, b, c > 0$ will be chosen later on. Let further

$$\phi(t,x,v) = \delta e^{-\mu Q(t,x,v)} - \epsilon,$$

where $\mu, \epsilon > 0$ will be chosen later on. Let us assume $b^2 < ac$, so that Q is a positive definite quadratic form in the two variables $v - v_0$ and $x - X_t$. Then

$$\mathcal{L}\phi = \mu\delta e^{-\mu Q} \mathcal{A}(Q),$$

where

$$\mathcal{A}(Q) = \partial_t Q + (v + \Phi(x)) \cdot \nabla_x Q - \Delta_v Q + \mu|\nabla_v Q|^2 - A(t,x,v) \cdot \nabla_v Q.$$

By computation,

$$\begin{aligned}
\mathcal{A}(Q) &= -a\frac{|v-v_0|^2}{2t^2} + 2b\frac{\langle v - v_0, x - X_t\rangle}{t^3} - 3c\frac{|x - X_t(x_0,v_0)|^2}{2t^4} \\
&+ b\frac{\langle v - v_0, v_0 + \Phi(x_0)\rangle}{t^2} - c\frac{\langle x - X_t, v_0 + \Phi(x_0)\rangle}{t^3} \\
&- b\frac{\langle v - v_0, v + \Phi(x)\rangle}{t^2} + c\frac{\langle x - X_t, v + \Phi(x)\rangle}{t^3} - a\frac{n}{t} \\
&+ \mu|a\frac{v - v_0}{t} - b\frac{x - X_t}{t^2}|^2 - a\frac{\langle A, v - v_0\rangle}{t} + b\frac{\langle A, x - X_t\rangle}{t^2} \\
&= \mathcal{B}(\frac{v - v_0}{t}, \frac{x - X_t}{t^2}) - a\frac{\langle A, v - v_0\rangle}{t} + b\frac{\langle A, x - X_t\rangle}{t^2} - a\frac{d}{t} \\
&- b\frac{\langle v - v_0, \Phi(x) - \Phi(x_0)\rangle}{t^2} + c\frac{\langle x - X_t, \Phi(x) - \Phi(x_0)\rangle}{2t^3},
\end{aligned}$$

where $\mathcal{B}$ is a quadratic form on $\mathbb{R}^n\mathbb{R}^n$ with matrix $M \otimes I_n$,

$$M = \begin{pmatrix} \mu a^2 - \frac{a}{2} + b & -\mu ab + b + \frac{c}{2} \\ -\mu ab + b + \frac{c}{2} & \mu b^2 - \frac{3c}{2} \end{pmatrix}$$

If a, b, c are given, then as $\mu \to \infty$

$$\begin{cases} \mathrm{tr}M = \mu(a^2 + b^2) + O(1), \\ \det M = \mu(\frac{3ab^2}{2} + abc - b^3 - \frac{3a^2c}{2}) + O(1). \end{cases}$$

Both quantities are positive if $b \geq a$ and $ac \geq b^2$, for example we can take $b = 2a, c = 8a$, then as $\mu \to \infty$ the eigenvalues of M are of order μb^2 and $ac/b > b$. So for any fixed C we may choose a, b, c and μ so that

$$\mathcal{B}(\frac{v - v_0}{t}, \frac{x - X_t}{t^2}) \geq Cb(\frac{|v - v_0|^2}{t^2} + \frac{|x - X_t|^2}{t^4}).$$

where C is arbitrarily large. If $t \in (0, \min\{\frac{1}{M}, 1\})$, we have

$$\epsilon(Q) \geq -8b\frac{|x - X_t|^2}{t^4} - 3b\frac{|v - v_0|^2}{t^2} - 3b\bar{A}^2 - \frac{\beta d}{2t},$$

gathering the two terms, we have

$$\mathcal{A}(Q) \geq const. \frac{b}{t}[C(\frac{|v - v_0|^2}{t} + \frac{|x - X_t|^2}{t^3}) - 1].$$

with C arbitrarily large. Recall that $(x, v) \notin B_r$, so
- either $|v - v_0| \geq r$, and then $\mathcal{A}(Q) \geq const.(b/t)[Cr^2/\tau - 1]$, which is positive if $C > \tau/r^2$;
- or $|x - x_0| \geq r^3$, and then, for any $\tau \leq r^3/(4V)$, then we have

$$\frac{|x - X_t|^2}{t^2} \geq \frac{|x - x_0|^2}{2t^2} 2|v_0|^2 \geq \frac{r^6}{2\tau^2} - 2V^2 \geq \frac{r^6}{4\tau^2},$$

so $\mathcal{A}(Q) \geq const.(b/t)[Cr^6/4\tau^3 - 1]$, which is positive as soon as $C > 4(\tau/r^2)^3$.

To summarize: under our assumptions there is a way to choose the constants a, b, c, μ, depending only on $d, \bar{A}, r^2/\tau$, satisfying $c > b > a > 1$ and $ac > b^2$, so that

$$\mathcal{L}\phi \leq 0, \quad \text{in } [0, \tau) \times (B_r \setminus B_r),$$

as soon as $\tau \leq min(1, r^3/(4V), \frac{1}{M})$. We now wish to enforce $\phi \leq g$ for $t = 0$ and for $(x, v) \in \partial(B_r \setminus B_r)$; then the classical maximum principle will imply $g \geq \phi$, in $[0, \tau)(B_r \setminus B_r)$. The boundary condition at $t = 0$ is obvious since ϕ vanishes identically there (more rigorously, ϕ can be extended by continuity by 0 at $t = 0$). The condition is also true on ∂B_r since $\phi \leq \delta$ and $g \geq \delta$. It remains to impose it on ∂B_r . For that we estimate Q from below: as soon as ac/b^2 is large enough, it's easily easy to seen that for any $(x, v) \in \partial B_{\lambda r}$

$$Q(t, x, v) \geq \frac{a}{4}(\frac{|vv_0|^2}{t} + \frac{|x - X_t|^2}{t^3}) \geq \frac{a}{4}\min(\frac{\lambda^2 r^2}{\tau}, \frac{\lambda^6 r^6}{4\tau^3}) \geq \frac{\alpha\lambda^2}{16}\min(\frac{r^2}{\tau}, \frac{r^6}{\tau^3}),$$

Thus if we choose

$$\epsilon = \delta\ exp(-\frac{\mu\alpha\lambda^2}{16}\min(\frac{r^2}{\tau}, \frac{r^6}{\tau^3})),$$

we make sure that $\phi = \delta e^{-\mu Q} - \epsilon \leq 0$ on $\partial B_{\lambda r}$, a fortiori $\phi \leq g$ on this set, and then we can apply the maximum principle.

So now we have $\phi \leq g$, and this will yield a lower bound for g in $[\tau/2, \tau) \times (B_{2r} \setminus B_r)$: indeed, if $t \geq \tau/2$ and $(x, v) \in B_{2r} \setminus B_r$ then

$$Q(t, x, v) \leq 2c(\frac{|v - v_0|^2}{t} + \frac{|x - X_t|^2}{t^3}) \leq 2c(8\frac{r^2}{\tau} + \frac{1026r^6}{\tau^3}) \leq 2068c\max(\frac{r^2}{\tau}, \frac{r^6}{t^3})$$

For λ large enough we find $K_0 > 0$ such that

$$\phi(t,x,v) \geq \delta[exp(-2068\mu c \max(\frac{r^2}{\tau}, \frac{r^6}{\tau^3})) - exp(-\frac{-\mu a \lambda^2}{16} \min(\frac{r^2}{\tau}, \frac{r^6}{\tau^3}))] \geq K_0\delta,$$

because $c = 8a$, to find such λ it suffices that

$$2068 \times 16 \times 8 \max(\frac{r^2}{\tau}, \frac{r^6}{\tau^3}) \leq \lambda \min(\frac{r^2}{\tau}, \frac{r^6}{\tau^3}),$$

by consequence λ depends only on r^2/τ.
Finally we find $K, \lambda > 0$ depending on $\bar{A}, \bar{C}$ and r^2/τ such that

$$f \geq K_0 \delta e^{-\tau \bar{C}} \quad \text{on} \quad [\tau/2, \tau) \times (B_{2\lambda} \setminus B_r),$$

□

Appendix B. Lyapunov function for the KFP equation

In this section we will give Lyapunov condition for the kinetic Fokker-Planck equation.

Theorem B.1. *Denote $\mathcal{L}$ the operator of the kinetic Fokker-Planck equation (1.1), then there exist a weight function m satisfies Theorem 1.3.*

Proof. First we have

$$\mathcal{L}^* f = v \cdot \nabla_v f - \nabla_x V(x) \cdot \nabla_v f + \Delta_v f - \nabla_v W(v) \cdot \nabla_v f,$$

we compute

$$\mathcal{L}^*(v^2 + V(x)) = d - v \cdot \nabla_v W(v),$$

and

$$\mathcal{L}^*(v \cdot \nabla_x \langle x \rangle) = v \nabla_x^2 \langle x \rangle v - \nabla_x V(x) \cdot \nabla_x \langle x \rangle + \nabla_v W(v) \cdot \nabla_x \langle x \rangle,$$

since

$$\nabla_x^2 \langle x \rangle \leq CI,$$

where I is the $d \times d$ identity matrix, combine the two terms together we have

$$\mathcal{L}^*(|v|^2 + V(x) + \epsilon v \cdot \nabla_x \langle x \rangle) \leq C - C(\langle v \rangle^\beta + \langle x \rangle^{\gamma - 1}),$$

with $\epsilon > 0$ small, denote

$$H = |v|^2 + V(x) + \epsilon v \cdot \nabla_x \langle x \rangle,$$

and since

$$\frac{\mathcal{L}^* e^{\lambda H}}{e^{\lambda H}} = \lambda(v \cdot \nabla_x H - \nabla_x V(x) \cdot \nabla_v H + \Delta_v H + \lambda |\nabla_v H|^2 - \nabla_v W(v) \cdot \nabla_v H),$$

take $\lambda > 0$ small, we have

$$\mathcal{L}^*(e^{\lambda H}) \leq -C_1 H_1 e^{\lambda H} + C_2,$$

for some constant $C_1, C_2 > 0$, with $H_1 = \langle v\rangle^\beta + \langle x\rangle^{\gamma-1}$, then the Lyapunov condition follows. For the second inequality, by Lemma C.1 we have

$$\begin{aligned}\phi_2(e^{\lambda H}) &= \lambda(v\cdot\nabla_x H + \nabla_x V(x)\cdot\nabla_v H + \frac{1}{2}\Delta_v W(v)\\ &+ \Delta_v H + (\lambda^2+\lambda)|\nabla_v H|^2 - \nabla_v W(v)\cdot\nabla_v H),\end{aligned}$$

and we still have

$$\phi_2(e^{\lambda H}) \le -C_1 H_1 e^{\lambda H} + C_2,$$

for some constant $C_1, C_2 > 0$, thus the theorem is proved. □

Appendix C. Computation for $\phi_2(m)$

Lemma C.1. *Define*

$$\partial_t f := \mathcal{L} f = \partial_x(A(x,v)f) + \partial_v(B(x,v)f) + \Delta_v f, \tag{C.1}$$

with

$$A(x,v) = -v + \Phi(x),$$

Then for any weight function m we have

$$\int (f(\mathcal{L}g) + g(\mathcal{L}f))m^2 = -2\int \nabla_v f\cdot\nabla_v g m^2 + 2\int fg\phi_2(m)m^2, \tag{C.2}$$

with

$$\begin{aligned}\phi_2(m) &= -v\cdot\frac{\nabla_x m}{m} - \Phi(x)\cdot\frac{\nabla_x m}{m} + \frac{1}{2}div_x\Phi(x) + \frac{|\nabla_v m|^2}{m^2}\\ &+ \frac{\Delta_v m}{m} - B(x,v)\cdot\frac{\nabla_v m}{m} + \frac{1}{2}div_v B(x,v),\end{aligned}$$

where we use $\int f$ in place of $\int_{\mathbb{R}^d\times\mathbb{R}^d} f dxdv$ for short.

Proof. Define

$$\mathcal{T} f = -v\cdot\nabla_x f,$$

we have

$$\int f(\mathcal{T}g)m^2 + \int(\mathcal{T}f)gm^2 = \int \mathcal{T}(fg)m^2 = -\int fg\mathcal{T}(m^2) = -2\int fgm^2\frac{\mathcal{T}m}{m},$$

for the term with operator Δ we have

$$\begin{aligned}\int(f\Delta_v g + \Delta_v f g)m^2 &= -\int \nabla_v(fm^2)\cdot\nabla_v g + \nabla_v(gm^2)\cdot\nabla_v f\\ &= -2\int\nabla_v f\cdot\nabla_v g m^2 + \int fg\Delta_v(m^2)\\ &= -2\int\nabla_v f\cdot\nabla_v g m^2 + 2\int fg|\nabla_v m|^2 + \Delta_v m m,\end{aligned}$$

using integration by parts

$$\begin{aligned}&\int f\mathrm{div}_v(B(x,v)g)m^2+g\mathrm{div}_v(B(x,v)f)m^2\\=&\int fB(x,v)\cdot\nabla_v gm^2+gB(x,v)\cdot\nabla_v fm^2+2\mathrm{div}_vB(x,v)fgm^2\\=&-\int fg\nabla_v\cdot(B(x,v)m^2)+2\mathrm{div}_vB(x,v)fgm^2\\=&\int -2fgB(x,v)\cdot\frac{\nabla_v m}{m}m^2+\mathrm{div}_vB(x,v)fgm^2,\end{aligned}$$

similarly

$$\begin{aligned}&\int f\mathrm{div}_x(\Phi(x)g)m^2+g\mathrm{div}_x(\Phi(x)f)m^2\\=&\int -2fg\Phi(x)\cdot\frac{\nabla_v m}{m}m^2+\mathrm{div}_x\Phi(x)fgm^2,\end{aligned}$$

so (C.2) are proved by combining the terms above. □

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