AN OPEN MICROSCOPIC MODEL OF HEAT CONDUCTION: EVOLUTION AND NON-EQUILIBRIUM STATIONARY STATES

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Abstract. We consider a one-dimensional chain of coupled oscillators in contact at both ends with heat baths at different temperatures, and subject to an external force at one end. The Hamiltonian dynamics in the bulk is perturbed by random exchanges of the neighbouring momenta such that the energy is locally conserved. We prove that in the stationary state the energy and the volume stretch profiles, in large scale limit, converge to the solutions of a diffusive system with Dirichlet boundary conditions. As a consequence the macroscopic temperature stationary profile presents a maximum inside the chain higher than the thermostats temperatures, as well as the possibility of uphill diffusion (energy current against the temperature gradient).

1. Introduction

Non-equilibrium transport in one dimension presents itself to be an interesting phenomenon and in many models numerical simulations can be easily performed. Most of the attention has been focused on the study of the non-equilibrium stationary states (NESS), where the systems are subject to exterior heat baths at different temperatures and other external forces, so that the invariant measure is not the equilibrium Gibbs measure.

The most interesting models are those with various conserved quantities (energy, momentum, volume stretch...) whose transport is coupled. The densities of these quantities may evolve at different time scales, and their interaction can give rise to a superdiffusive energy behaviour, particularly when the spatial dimension of the system equals one. For example, in the Fermi-Pasta-Ulam (FPU) chain, volume stretch, mechanical energy and momentum all evolve in the hyperbolic time scale. Their evolution is governed by the Euler equations, see [4], while the thermal energy is expected to evolve at a superdiffusive time scale, with an

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autonomous evolution described by a fractional heat equation. This has been predicted [13], confirmed by many numerical experiments on the NESS [12, 11] and proved analytically for harmonic chains with random exchanges of momenta that conserve energy, momentum and volume stretch, see [7].

In the present paper we are interested in systems, for which conserved quantities evolve macroscopically in the same *diffusive* time scale, and their macroscopic evolution is governed by a system of *coupled* diffusive equations. One example is given by the chain of coupled rotors, whose dynamics conserves the energy and the angular momentum. In [5] the NESS of this chain is studied numerically, when Langevin thermostats are applied at both ends, while a constant force is applied to one end and the position of the rotor on the opposite side is kept fixed. While heat flows from the thermostats, work is performed by the torque, increasing the mechanical energy, which is then transformed into thermal energy by the dynamics of the rotors. The stationary temperature profiles observed numerically in [5] present a maximum inside the chain higher than the temperature of both thermostats. Furthermore, a negative linear response for the energy flux has been observed for certain values of the external parameters. These numerical results have been confirmed in [6], as well as an instability of the system when thermostats are at 0 temperature.

The present work aims at describing a similar phenomenon for the NESS, but for a different model. In particular, we are able to show rigorously that the maximum of the temperature profile occurs inside the system. According to our knowledge it is the first theoretical result concerning this kind of phenomena.

More specifically, we consider a chain of unpinned harmonic oscillators whose dynamics is perturbed by a random mechanism that conserves the energy and volume stretch: any two nearest neighbour particles exchange their momenta randomly in such a way that the total kinetic energy is conserved. Two Langevin thermostats are attached at the opposite ends of the chain and a constant force $\overline{\tau}_+$ acts on the last particle of the chain. This system has only two conserved quantities: total energy and volume. Since the random mechanism does not conserve the total momentum, the macroscopic behaviour of these two quantities is diffusive, and the non stationary hydrodynamic limit with periodic boundary conditions (no thermostats or exteriour force present) has been proven in [9].

The action of this constant force puts the system out of equilibrium, even when the temperatures of the thermostats are equal. As in the rotor chain described above, the exterior force performs positive work on the system, that increases the mechanical energy (concentrated on low frequency modes). The random mechanism transforms the mechanical energy into the thermal one (uniformly distributed in all frequencies, when the system is in a local equilibrium), which is eventually dissipated by the thermostats. This transfer of mechanical into thermal energy happens in the bulk of the system and is already completely predicted by the solution of the macroscopic diffusive system of equations obtained in the hydrodynamic limit [9].

In the present article we study the NESS of this dynamics. We prove that the energy and the volume stretch profiles converge to the stationary solution of the diffusive system, with the boundary conditions imposed by the thermostats and the external tension. It turns out that these stationary equations can be solved explicitly and the stretch profile is linear between 0 and $\overline{\tau}_+$, while the thermal energy (temperature) profile is a concave parabola with the boundary conditions coinciding with the temperatures of the thermostats. The curvature of the parabola is proportional to $\overline{\tau}_+^2$, i.e. the increase of the bulk temperature is not a linear response term. In the case $\overline{\tau}_+ = 0$, the NESS was studied in [1], where the temperature profile is proved to be linear. This *heating inside the system* phenomenon is similar to the ohmic loss, due to the diffusion of electricity in a resistive system (see e.g. [3]).

The NESS for our model also provides a simple example of an *uphill* energy diffusion (see [10]): if the force $\overline{\tau}_+$ is large enough and applied on the side, where the coldest thermostat is acting, the sign of the energy current can be equal to the one of the temperature gradient. It is not surprising after understanding that this is regulated by a system of two diffusive coupled equations. On the other hand, the model does not work as a *stationary refrigerator*: i.e. a system where the heat on the coldest thermostat flows into it.

Our results suggest that there is a universal behaviour of the temperature profiles in the NESS when there are at least two conserved quantities. This should be tested on a system with three conserved quantities that evolve in the diffusive scale, such as e.g. a non-acoustic harmonic chain with a random exchange of momentum as considered in [8], where the non-stationary hydrodynamic limit is proven.

Concerning an outline of the paper: in Section 2 we define the microscopic model under investigation and give the expected macroscopic system of equations, showing the phenomenon of uphill diffusion. In Section 3 we state the main results of the paper, namely the convergence of the non-equilibrium stationary profiles of elongation, current and energy. In order to prove them, we need precise computations on the averages and second order moments taken with respect to the NESS. Section 4 provides elements of the proofs and preliminary computations on the averages, while Section 5 provides all the remaining technical lemmas, concerning the second order moments.

2. Microscopic dynamics and macroscopic behaviour

2.1. **Open chain of oscillators.** Let $\mathbb{I}_n := \{1, \dots, n\}$, $\overline{\mathbb{I}}_n := \mathbb{I}_n \cup \{0\}$ and $\mathbb{I} := [0,1]$. The configuration space $(\mathbb{R}\times\mathbb{R})^{\overline{\mathbb{I}}_n}$ consists of all sequences $(\mathbf{q},\mathbf{p}) := \{q_x, p_x\}_{x\in\overline{\mathbb{I}}_n}$, where $p_x \in \mathbb{R}$ stands for the momentum of the oscillator at site x, and $q_x \in \mathbb{R}$ represents its position. The interaction between two particles x and $x+1$ is described by the quadratic potential energy $V(q_x - q_{x+1}) := \frac{1}{2}(q_x - q_{x+1})^2$ of a harmonic spring relying the particles. At the boundaries the system is connected to two Langevin heat baths at temperatures T_- and T_+. Furthermore, on the

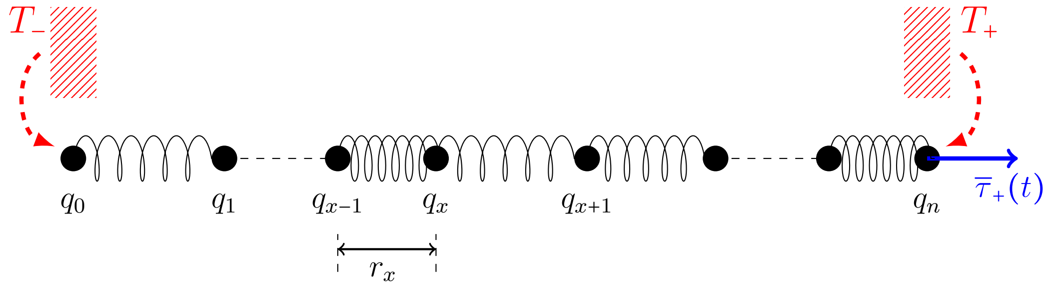

FIGURE 1. Oscillator chains with heat baths and one boundary force.

right boundary acts a force (tension) $\overline{\tau}_+$, eventually slowly changing in time at a scale t/n^2. Notice that the system is *unpinned*, i.e. there is no external potential binding the particles. Consequently, the absolute positions q_x do not have precise meaning, and the dynamics only depends on the interparticle elongations $r_x := q_x - q_{x-1}, x \in \mathbb{I}_n$. The configurations can then be described by

$$(\mathbf{r}, \mathbf{p}) = (r_1, \dots, r_n, p_0, \dots, p_n) \in \Omega_n := \mathbb{R}^{\mathbb{I}_n} \times \mathbb{R}^{\overline{\mathbb{I}}_n}. \tag{2.1}$$

The total energy of the system is defined by the Hamiltonian:

$$\mathcal{H}_n(\mathbf{r}, \mathbf{p}) := \sum_{x \in \mathbb{I}_n} \mathcal{E}_x + \frac{p_0^2}{2},$$

with

$$\mathcal{E}_x := \frac{p_x^2}{2} + \frac{r_x^2}{2}, \quad x \in \mathbb{I}_n.$$

The equations of the microscopic dynamics are given in the bulk by

$$\mathrm{d}r_x(t) = n^2 \left(p_x(t) - p_{x-1}(t)\right) \mathrm{d}t, \qquad x \in \mathbb{I}_n \tag{2.2}$$

$$\begin{aligned} \mathrm{d}p_x(t) &= n^2 \left(r_{x+1}(t) - r_x(t)\right) \mathrm{d}t - \gamma n^2 p_x(t) \mathrm{d}t \\ &\quad + n\sqrt{\gamma}\big(p_{x-1}(t)\mathrm{d}w_{x-1,x}(t) - p_{x+1}(t)\mathrm{d}w_{x,x+1}(t)\big), \quad x \in \{1, \dots, n-1\} \end{aligned} \tag{2.3}$$

and at the boundaries:

$$\mathrm{d}p_0(t) = n^2\, r_1(t)\mathrm{d}t - \frac{n^2}{2}(\gamma + \widetilde{\gamma})p_0(t)\mathrm{d}t - n\sqrt{\gamma}p_1(t)\mathrm{d}w_{0,1}(t) + n\sqrt{\widetilde{\gamma}T_-}\mathrm{d}\widetilde{w}_0(t) \tag{2.4}$$

$$\begin{aligned} \mathrm{d}p_n(t) &= -n^2\, r_n(t)\mathrm{d}t + n^2\, \overline{\tau}_+(t)\mathrm{d}t - \frac{n^2}{2}(\gamma + \widetilde{\gamma})p_n(t)\mathrm{d}t \\ &\quad + n\sqrt{\gamma}p_{n-1}(t)\mathrm{d}w_{n-1,n}(t) + n\sqrt{\widetilde{\gamma}T_+}\mathrm{d}\widetilde{w}_n(t) \end{aligned} \tag{2.5}$$

where $w_{x,x+1}(t)$, $x \in \{0, \dots, n-1\}$, $\widetilde{w}_0(t)$ and $\widetilde{w}_n(t)$ are independent, standard one dimensional Wiener processes, and $\gamma > 0$ (resp. $\widetilde{\gamma} > 0$) regulates the intensity of the random perturbation (resp. the Langevin thermostats). See Figure 1 for a representation of the chain.

It is useful to use the generator of the dynamics in order to compute time evolutions of various quantities. It is given by

$$L := n^2\Big(A + \frac{\gamma}{2} S + \frac{\widetilde{\gamma}}{2}\widetilde{S}\Big), \tag{2.6}$$

where

$$A := \sum_{x=1}^{n}(p_x - p_{x-1})\partial_{r_x} + \sum_{x=1}^{n-1}(r_{x+1} - r_x)\partial_{p_x} + r_1\partial_{p_0} + (\overline{\tau}_+(t) - r_n)\,\partial_{p_n}$$

$$Sf := \sum_{x=0}^{n-1}\mathcal{X}_x \circ \mathcal{X}_x(f),$$

where $\mathcal{X}_x$ is the exchange operator defined as

$$\mathcal{X}_x := p_{x+1}\partial_{p_x} - p_x\partial_{p_{x+1}},$$

and moreover the generator of the Langevin heat baths at the boundaries is given by

$$\widetilde{S} := T_-\partial^2_{p_0} - p_0\partial_{p_0} + T_+\partial^2_{p_n} - p_n\partial_{p_n}.$$

2.2. **Macroscopic equations.** Suppose that $r(t,u)$, $e_{\rm th}(t,u)$, $(t,u) \in \mathbb{R}_+ \times \mathbb{I}$, are the macroscopic profiles of *elongation* and *thermal energy* of the macroscopic system, obtained in the diffusive scaling limit. Precisely, define the thermal energy per particle as $\mathcal{E}^{\rm th}_x := \frac{1}{2}p_x^2 + \frac{1}{2}(r_x^2 - (\mathbb{E}[r_x])^2)$, where $\mathbb{E}[\cdot]$ is the expectation with respect to the law of the process. The profiles $r(t,\cdot), e_{\rm th}(t,\cdot)$ are the expected large n limits (in the weak formulation sense) of

$$\frac{1}{n}\sum_{x\in\mathbb{I}_n} r_x(t)\delta_{x/n}(\cdot), \qquad \text{and} \qquad \frac{1}{n}\sum_{x\in\overline{\mathbb{I}}_n}\mathcal{E}^{\rm th}_x(t)\delta_{x/n}(\cdot),$$

where $\delta_u(\cdot)$ is the Delta dirac function at point u. If both convergences do hold at time $t = 0$ to some given profiles $r_0(u)$ and $\mathcal{T}_0(u)$, then we expect (see Appendix A for a formal derivation), that they satisfy the following system of equations[1],

$$\partial_t r(t,u) = \gamma^{-1}\,\partial^2_{uu} r(t,u) \tag{2.7}$$

$$\partial_t e_{\rm th}(t,u) = \big(\gamma^{-1} + \gamma\big)\partial^2_{uu} e_{\rm th}(t,u) + 2\gamma^{-1}\Big(\partial_u r(t,u)\Big)^2, \qquad (t,u) \in \mathbb{R}_+ \times \mathbb{I}, \tag{2.8}$$

with boundary conditions

$$r(t,0) = 0, \qquad r(t,1) = \overline{\tau}_+(t),$$
$$e_{\rm th}(t,0) = T_-, \qquad e_{\rm th}(t,1) = T_+,$$

and with the initial condition

$$r(0,u) = r_0(u), \qquad e_{\rm th}(0,u) = \mathcal{T}_0(u).$$

[1]See also Theorems 3.7 and 3.8 of [9] for a similar model which gives a similar coupled diffusive system.

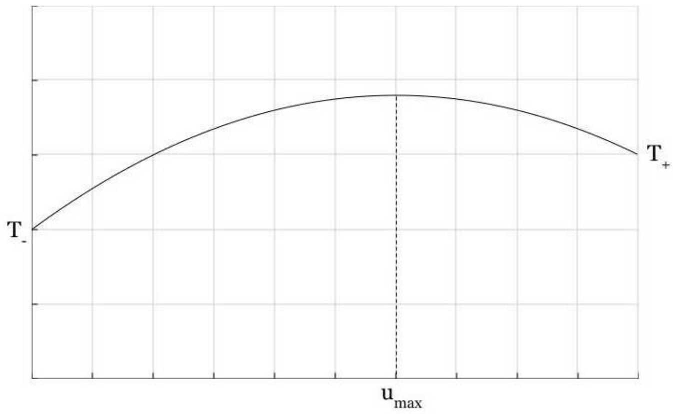

FIGURE 2. Temperature profile when $T_+ - T_- < 2\overline{\tau}_+^2$.

2.3. **Stationary non-equilibrium states.** From now on we assume $\overline{\tau}_+$ to be constant in time. If $\overline{\tau}_+ \neq 0$, the *stationary* profiles, denoted respectively by $r_{\rm ss}(u)$ and $e_{\rm th,ss}(u)$, solving equations (2.7) and (2.8), satisfy:

$$r_{\rm ss}(u) = \overline{\tau}_+ u \tag{2.9}$$

and

$$\left(\gamma^{-1} + \gamma\right)\partial_{uu}^2 e_{\rm th,ss}(u) + 2\gamma^{-1}\,\overline{\tau}_+^2 = 0,$$

with the boundary conditions

$$e_{\rm th,ss}(0) = T_-, \qquad e_{\rm th,ss}(1) = T_+.$$

In other words

$$e_{\rm th,ss}(u) = \frac{\overline{\tau}_+^2}{1+\gamma^2}\,u(1-u) + (T_+ - T_-)u + T_-. \tag{2.10}$$

Moreover, the *stationary energy current* is given by

$$J_{\rm ss} = -\frac{1}{2}(\gamma^{-1} + \gamma)(T_+ - T_-) - \frac{\overline{\tau}_+^2}{2\gamma}.$$

Observe that current can flow against the temperature gradient if $T_- > T_+$ and $|\overline{\tau}_+|$ is large enough (*uphill diffusion*). Besides, the chain heats up, reaching the maximum temperature $e_{\rm th,ss}^{\max}$ at

$$u_{\max} = \frac{1}{2} + \frac{T_+ - T_-}{\overline{\tau}_+^2},$$

which equals

$$e_{\rm th,ss}^{\max} = \frac{(T_- + T_+)}{2} + \frac{(T_+ - T_-)^2}{\overline{\tau}_+^2} + \frac{\overline{\tau}_+^2}{4(1+\gamma^2)}\left(1 - \frac{4(T_+ - T_-)^2}{\overline{\tau}_+^4}\right).$$

Note that this does not depend on the sign of $\overline{\tau}_+$, and moreover, if $T_+ - T_- < 2\overline{\tau}_+^2$, then the chain heats up inside ($u_{\max} < 1$), see Figure 2.

This phenomenon was observed by dynamical numerical simulations in [5] for the stationary states of the rotor model. It has attracted quite some interest from physicists, see [10] for a review. The present article is devoted to the proof of such a phenomenon, when $\gamma = 1$. According to our knowledge it is the first rigorous result of this fact in the existing literature.

3. Main results

From the microscopic point of view, there exists a unique stationary probability distribution $\mu_{\rm ss}$ on Ω_n (cf (2.1)) such that $\langle LF \rangle_s = 0$ for any function F in the domain of the operator L, given by (2.6). Hereafter,

$$\langle F \rangle_s := \int_{\Omega_n} F \, \mathrm{d}\mu_{\rm ss}.$$

The proof of existence and uniqueness of the stationary state follows from the same argument as the one used in [2].

Let us start with the following:

Theorem 3.1 (Stationary elongation profile). *For any continuous test function $G : \mathbb{I} \to \mathbb{R}$,*

$$\frac{1}{n} \sum_{x \in \mathbb{I}_n} G\Big(\frac{x}{n}\Big) \langle r_x \rangle_s \xrightarrow[n\to\infty]{} \int_{\mathbb{I}} G(u) r_{\rm ss}(u) \, \mathrm{d}u,$$

where $r_{\rm ss}(u) = \overline{\tau}_+ u$.

Proof. The averages under the stationary state $\langle r_x \rangle_s$ and $\langle p_x \rangle_s$ are computable explicitly, see Proposition 4.1 in the next section. It turns out that $\langle p_x \rangle_s$ is constant for all $x \in \overline{\mathbb{I}}_n$ and equals to $\overline{p}_s = \overline{\tau}_+ / (\gamma n + \widetilde{\gamma})$ (see (4.1)). From (4.2) we also have

$$n\left(\langle r_{x+1} \rangle_s - \langle r_x \rangle_s\right) = 2n\gamma \overline{p}_s \xrightarrow[n\to\infty]{} \overline{\tau}_+, \qquad \text{for } x \in \{1, \dots, n-1\}$$

and

$$\langle r_1 \rangle_s \xrightarrow[n\to\infty]{} 0, \qquad \langle r_n \rangle_s \xrightarrow[n\to\infty]{} \overline{\tau}_+,$$

that implies directly,

$$\langle r_{[nu]} \rangle_s \xrightarrow[n\to\infty]{} r_{\rm ss}(u), \quad \text{for any } u \in \mathbb{I},$$

where $r_{\rm ss}$ is given by (2.9), which proves the theorem. □

We now want to investigate the *stationary energy flow* and check the validity of the Fourier law. For that purpose, let us start with some definitions. From the energy conservation law, there exist microscopic currents $j_{x,x+1}$ which satisfy

$$n^{-2} L\mathcal{E}_x = j_{x-1,x} - j_{x,x+1}, \qquad \text{for any } x \in \overline{\mathbb{I}}_n \tag{3.1}$$

and are given by

$$j_{x,x+1} := -p_x r_{x+1} + \frac{\gamma}{2}(p_x^2 - p_{x+1}^2), \qquad \text{if } x \in \{0, \dots, n-1\}, \tag{3.2}$$

while at the boundaries

$$j_{-1,0} := \frac{\widetilde{\gamma}}{2}\left(T_- - p_0^2\right), \qquad j_{n,n+1} := -\frac{\widetilde{\gamma}}{2}\left(T_+ - p_n^2\right) - \overline{\tau}_+ p_n. \tag{3.3}$$

Taking the average with respect to the stationary state in (3.1) we get

$$\langle j_{x,x+1}\rangle_s =: \overline{j}_s, \qquad \text{for any } x \in \{-1,\dots,n\}. \tag{3.4}$$

Theorem 3.2 (Stationary energy current and Fourier law)**.**

$$n\overline{j}_s \underset{n\to\infty}{\longrightarrow} -\frac{1}{2}(\gamma^{-1}+\gamma)(T_+ - T_-) - \frac{\overline{\tau}_+^2}{2\gamma}. \tag{3.5}$$

Note that Theorems 3.1 and 3.2 are valid for any $\gamma > 0$. We now state our last main result about the stationary energy profile, which we are able to prove only for $\gamma = 1$.

Theorem 3.3 (Stationary energy profile)**.** *Assume that $\gamma = 1$. For any continuous test function $G : \mathbb{I} \to \mathbb{R}$,*

$$\frac{1}{n}\sum_{x\in\mathbb{I}_n} G\Big(\frac{x}{n}\Big)\langle \mathcal{E}_x\rangle_s \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{I}} G(u)\Big(e_{\mathrm{th,ss}}(u) + \frac{1}{2}r_{\mathrm{ss}}^2(u)\Big)\,\mathrm{d}u, \tag{3.6}$$

where

$$r_{\mathrm{ss}}(u) = \overline{\tau}_+ u,$$
$$e_{\mathrm{th,ss}}(u) = \frac{\overline{\tau}_+^2}{2}u(1-u) + (T_+ - T_-)u + T_+.$$

The remaining part of the paper deals with the proof of Theorems 3.2 and 3.3.

4. The stationary state

Let us start with explicit computations for the average momenta and elongations with respect to the NESS.

4.1. **Elongation and momenta averages.**

Proposition 4.1. *The average stationary momenta are equal to*

$$\langle p_x\rangle_s = \overline{p}_s := \frac{\overline{\tau}_+}{\gamma n + \widetilde{\gamma}}, \qquad \text{for any } x \in \overline{\mathbb{I}}_n. \tag{4.1}$$

The average stationary elongations are equal to

$$\langle r_x\rangle_s = \frac{\overline{p}_s}{2}(\widetilde{\gamma} - \gamma + 2\gamma x) = \frac{\overline{\tau}_+(2\gamma x + \widetilde{\gamma} - \gamma)}{2(\gamma n + \widetilde{\gamma})} \qquad \text{for any } x \in \mathbb{I}_n. \tag{4.2}$$

Proof. We start with some useful relations that hold for the stationary state:

(1) since $\langle L r_x\rangle_s = 0$, applying (2.6), we conclude

$$\langle p_x\rangle_s = \langle p_{x-1}\rangle_s = \overline{p}_s, \qquad \text{for any } x \in \mathbb{I}_n;$$

(2) from $\langle L p_x \rangle_s = 0$ we get

$$\langle r_{x+1}\rangle_s - \langle r_x\rangle_s = \gamma \overline{p}_s, \qquad \text{for any } x \in \{2,\dots,n-2\}$$
$$\langle r_1\rangle_s = \frac{1}{2}(\gamma+\widetilde{\gamma})\overline{p}_s,$$
$$\langle r_n\rangle_s = -\frac{1}{2}(\gamma+\widetilde{\gamma})\overline{p}_s + \overline{\tau}_+.$$

These equations determine the average stationary momentum and elongation as given in formulas (4.1) and (4.2). □

4.2. **Elements of proof for Theorems 3.2 and 3.3.** One of the main characteristics of this model is the existence of a *fluctuation-dissipation relation*, which permits to write the stationary current $\overline{j}_s$ as a discrete gradient of some explicit local function, as given in the following:

Proposition 4.2 (Decomposition of the stationary current). *We can write $\overline{j}_s$ as a discrete gradient, namely*

$$\overline{j}_s = \nabla\phi(x) := \phi(x+1) - \phi(x), \qquad x \in \{1,\dots,n-1\}, \tag{4.3}$$

with

$$\phi(x) := -\frac{1}{2\gamma}\big(\langle r_x^2\rangle_s + \langle p_{x-1}p_x\rangle_s\big) - \frac{\gamma}{4}\big(\langle p_x^2\rangle_s + \langle p_{x-1}^2\rangle_s\big), \qquad x \in \mathbb{I}_n. \tag{4.4}$$

Remark 4.3. *Thanks to (4.3), the function $\phi(x)$ is a* harmonic *function, namely:*

$$\Delta\phi(x) := \phi(x+1) + \phi(x-1) - 2\phi(x) = 0, \qquad \text{for any } x \in \{2,\dots,n-1\}. \tag{4.5}$$

Proof of Proposition 4.2. By a direct calculation one can easily check that the energy currents $j_{x,x+1}$ satisfy the following *fluctuation-dissipation relation*:

$$j_{x,x+1} = n^{-2} L g_x - \frac{1}{2\gamma}\nabla\big(r_x^2 + p_{x-1}p_x\big) - \frac{\gamma}{4}\nabla\big(p_{x-1}^2 + p_x^2\big), \tag{4.6}$$

for any $x \in \{1,\dots,n-1\}$, with

$$g_x = -\frac{1}{4}p_x^2 + \frac{1}{2\gamma}p_x(r_x + r_{x+1}).$$

Therefore, (4.3) is obtained by taking the average in (4.6) with respect to the stationary state. □

We can now sketch the proof of Theorem 3.3: straightforward computations, using the definition (4.4) of ϕ, yield

$$\langle \mathcal{E}_x\rangle_s = \frac{1}{2}\big(\langle p_x^2\rangle_s + \langle r_x^2\rangle_s\big) = -\frac{2\gamma}{1+\gamma^2}\phi(x) + \frac{\gamma^2}{2(1+\gamma^2)}\big(\langle p_x^2\rangle_s - \langle p_{x-1}^2\rangle_s\big)$$
$$- \frac{1}{1+\gamma^2}\langle p_x p_{x-1}\rangle_s + \frac{1-\gamma^2}{2(1+\gamma^2)}\big(\langle p_x^2\rangle_s - \langle r_x^2\rangle_s\big). \tag{4.7}$$

Therefore, the microscopic energy profile can be decomposed as the sum of four terms:

$$\mathcal{H}_n(G) := \frac{1}{n}\sum_{x\in\mathbb{I}_n} G\Big(\frac{x}{n}\Big)\langle \mathcal{E}_x\rangle_s = \mathcal{H}_n^{\phi}(G) + \mathcal{H}_n^{\nabla}(G) + \mathcal{H}_n^{\mathrm{corr}}(G) + \mathcal{H}_n^{\mathrm{m}}(G),$$

where

$$\begin{aligned}
\mathcal{H}_n^{\phi}(G) &:= -\frac{2\gamma}{1+\gamma^2}\,\frac{1}{n}\sum_{x\in\mathbb{I}_n} G\Big(\frac{x}{n}\Big)\phi(x),\\
\mathcal{H}_n^{\nabla}(G) &:= \frac{\gamma^2}{2(1+\gamma^2)}\,\frac{1}{n}\sum_{x\in\mathbb{I}_n} G\Big(\frac{x}{n}\Big)\big(\langle p_x^2\rangle_s - \langle p_{x-1}^2\rangle_s\big),\\
\mathcal{H}_n^{\mathrm{corr}}(G) &:= -\frac{1}{1+\gamma^2}\,\frac{1}{n}\sum_{x\in\mathbb{I}_n} G\Big(\frac{x}{n}\Big)\langle p_{x-1}p_x\rangle_s,\\
\mathcal{H}_n^{\mathrm{m}}(G) &:= \frac{1-\gamma^2}{2(1+\gamma^2)}\,\frac{1}{n}\sum_{x\in\mathbb{I}_n} G\Big(\frac{x}{n}\Big)\big(\langle p_x^2\rangle_s - \langle r_x^2\rangle_s\big).
\end{aligned}$$

Note that, if $\gamma = 1$, then $\mathcal{H}_n^{\mathrm{m}} \equiv 0$. The limits of the other three terms will be obtained in the next section and are summarized in the following proposition:

Proposition 4.4. *For any continuous test function $G : \mathbb{I} \to \mathbb{R}$,*

$$\mathcal{H}_n^{\phi}(G) \xrightarrow[n\to\infty]{} \int_{\mathbb{I}} G(u)\Big(\frac{\overline{\tau}_+^2}{1+\gamma^2}u + (T_+ - T_-)u + T_-\Big)\,\mathrm{d}u, \tag{4.8}$$

$$\mathcal{H}_n^{\nabla}(G) \xrightarrow[n\to\infty]{} 0, \tag{4.9}$$

$$\mathcal{H}_n^{\mathrm{corr}}(G) \xrightarrow[n\to\infty]{} 0. \tag{4.10}$$

The complete proof of the proposition will be given further, let us first comment on the ideas used in the argument. The limit (4.8) will be concluded using the fact that ϕ is harmonic, the limit (4.9) is a consequence of the presence of a discrete gradient $\langle p_x^2\rangle_s - \langle p_{x-1}^2\rangle_s$ inside the sum, and the limit (4.10) will be shown thanks to the second order bounds, which are obtained in the next section.

Proof of Theorem 3.3. With the help of Proposition 4.4 the proof of Theorem 3.3 becomes straightforward. Assume that $\gamma = 1$. From the decomposition (4.7) and Proposition 4.4, we get

$$\begin{aligned}
\frac{1}{n}\sum_{x\in\mathbb{I}_n} G\Big(\frac{x}{n}\Big)\langle \mathcal{E}_x\rangle_s &\xrightarrow[n\to\infty]{} \int_{\mathbb{I}} G(u)\Big(\frac{\overline{\tau}_+^2}{2}\,u + (T_+ - T_-)u + T_-\Big)\,\mathrm{d}u\\
&= \int_{\mathbb{I}} G(u)\Big(\frac{\overline{\tau}_+^2}{2}u(1-u) + (T_+ - T_-)u + T_- + \frac{(\overline{\tau}_+ u)^2}{2}\Big)\,\mathrm{d}u.
\end{aligned}$$

Recalling (2.9) and (2.10) we conclude that the right hand side equals

$$\int_{\mathbb{I}} G(u)\Big(e_{\mathrm{th,ss}}(u) + \frac{r_{\mathrm{ss}}^2(u)}{2}\Big)\,\mathrm{d}u.$$

Thus (3.6) follows. □

5. Moment bounds under the stationary state

In this section we present a complete proof of Proposition 4.4 (see Section 5.4) and we show Theorem 3.2 (see Proposition 5.9). Before presenting the proof, we need a few technical estimates on the entropy production (Section 5.1) and second order moments (Section 5.2 and Section 5.3). In the whole section we do not assume $\gamma = 1$, since our results hold for any $\gamma > 0$.

5.1. Entropy production of the stationary state.

Denote by $g_T(\mathbf{r},\mathbf{p})$ the product of Gaussian densities of null average and variance T:

$$g_T(\mathbf{r},\mathbf{p}) = \frac{e^{-p_0^2/2T}}{\sqrt{2\pi T}} \prod_{x\in\mathbb{I}_n} \frac{e^{-\mathcal{E}_x/T}}{\sqrt{2\pi T}}.$$

We will use

$$d\nu_T(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{p}) = g_T(\mathbf{r},\mathbf{p}) dp_0 \prod_{x\in\mathbb{I}_n} dp_x dr_x,$$

as the reference measure and denote its respective expectation by $\langle\!\langle \cdot \rangle\!\rangle_T$. Let f_s be the density of the stationary state μ_{ss} with respect to ν_{T_+}, i.e.

$$\langle F\rangle_s = \langle\!\langle F f_s \rangle\!\rangle_{T_+} = \int F f_s \, d\nu_{T_+}.$$

Proposition 5.1 (Entropy production)**.** *Denote $h := g_{T_-}/g_{T_+}$. The following formula holds*

$$\gamma \sum_{x=0}^{n-1} \mathcal{D}_x(f_s) + \widetilde{\gamma} T_- \left\langle\!\!\left\langle \frac{(\partial_{p_0}(f_s/h))^2}{(f_s/h)} \right\rangle\!\!\right\rangle_{T_-} + \widetilde{\gamma} T_+ \left\langle\!\!\left\langle \frac{(\partial_{p_n} f_s)^2}{f_s} \right\rangle\!\!\right\rangle_{T_+}$$
$$= \frac{\overline{\tau}_+^2}{T_+(\gamma n + \widetilde{\gamma})} + \widetilde{\gamma}(T_+^{-1} - T_-^{-1})\big(T_- - \langle p_0^2 \rangle_s\big), \quad (5.1)$$

where

$$\mathcal{D}_x(f_s) := \left\langle\!\!\left\langle \frac{(\mathcal{X}_x f_s)^2}{f_s} \right\rangle\!\!\right\rangle_{T_+}.$$

Proof. By stationarity we have

$$0 = -n^{-2} \langle L \log f_s \rangle_s = -n^{-2} \langle\!\langle f_s L \log f_s \rangle\!\rangle_{T_+}$$
$$= \gamma \sum_{x=0}^{n-1} \mathcal{D}_x(f_s) + \widetilde{\gamma} T_+ \left\langle\!\!\left\langle \frac{(\partial_{p_n} f_s)^2}{f_s} \right\rangle\!\!\right\rangle_{T_+} - \overline{\tau}_+ \langle\!\langle \partial_{p_n} f_s \rangle\!\rangle_{T_+} - \widetilde{\gamma} \left\langle \big(T_- \partial_{p_0}^2 - p_0 \partial_{p_0}\big) \log f_s \right\rangle_s.$$

From the definition $h = g_{T_-}/g_{T_+}$, the last term can be rewritten in the form:

$$-\big\langle \big(T_-\partial^2_{p_0} - p_0\partial_{p_0}\big)\log f_s\big\rangle_s = -\int \frac{f_s}{h}\big(T_-\partial^2_{p_0} - p_0\partial_{p_0}\big)\Big(\log\Big(\frac{f_s}{h}\Big)\Big)\, g_{T_-}dp_0\prod_{x=1}^n dp_x dr_x$$

$$-\int f_s\big(T_-\partial^2_{p_0} - p_0\partial_{p_0}\big)\big(\log h\big)\, g_{T_+}dp_0\prod_{x=1}^n dp_x dr_x$$

$$= T_-\left\langle\!\!\left\langle \frac{(\partial_{p_0}(f_s/h))^2}{(f_s/h)}\right\rangle\!\!\right\rangle_{T_-} + (T_-^{-1} - T_+^{-1})\big(T_- - \langle p_0^2\rangle_s\big).$$

Moreover, observe that by integration by parts and (4.1) we obtain

$$\langle\!\langle \partial_{p_n} f_s\rangle\!\rangle_{T_+} = T_+^{-1}\langle\!\langle p_n f_s\rangle\!\rangle_{T_+} = \frac{\overline{p}_s}{T_+} = \frac{\overline{\tau}_+}{T_+(\gamma n + \widetilde{\gamma})}$$

and (5.1) follows. □

5.2. **Bounds on second moments.** In the present section we obtain some bounds on the covariance functions of momenta and positions, with respect to the stationary states. In particular, we estimate the average current $|\overline{j}_s|$, see (3.4) and investigate the behaviour of $\phi(1)$ and $\phi(n)$ as $n \to \infty$ (see (4.4)).

Let us first state a rough estimate on the second moments at the boundaries, which is going to be refined further.

Proposition 5.2 (Second moments the boundaries: Part I)**.** *The following equality holds*

$$\langle p_0^2\rangle_s + \langle p_n^2\rangle_s = T_+ + T_- + \frac{2\overline{\tau}_+\overline{p}_s}{\widetilde{\gamma}}. \tag{5.2}$$

Moreover, there exists a constant $C = C(\gamma, \widetilde{\gamma}, \overline{\tau}_+, T_+, T_-) > 0$*, such that*

$$\langle r_1^2\rangle_s + \langle r_n^2\rangle_s \leqslant C. \tag{5.3}$$

Remark 5.3. We make a convention that the constants appearing in statements of the results below depend only on the parameters indicated in parentheses.

Proof of Proposition 5.2. The first identity (5.2) is an easy consequence of (3.3) and (3.4), which yields

$$\overline{j}_s = \frac{\widetilde{\gamma}}{2}\left(T_- - \langle p_0^2\rangle_s\right), \tag{5.4}$$

$$\overline{j}_s = -\overline{\tau}_+\overline{p}_s - \frac{\widetilde{\gamma}}{2}\left(T_+ - \langle p_n^2\rangle_s\right). \tag{5.5}$$

Identity (5.2) is obtained by adding sideways the above equalities. To show estimate (5.3) note that

$$n^{-2}L(p_0r_1) = (p_1 - p_0)p_0 + r_1^2 - \frac{1}{2}(\widetilde{\gamma} + \gamma)p_0r_1 \tag{5.6}$$

$$n^{-2}L(p_nr_n) = p_n(p_n - p_{n-1}) + (\overline{\tau}_+ - r_n)r_n - \frac{1}{2}(\widetilde{\gamma} + \gamma)p_nr_n. \tag{5.7}$$

After taking the average with respect to the stationary state from (5.6) we conclude

$$\langle r_1^2\rangle_s = \langle p_0^2\rangle_s - \langle p_1p_0\rangle_s + \frac{1}{2}(\widetilde{\gamma}+\gamma)\langle p_0r_1\rangle_s. \tag{5.8}$$

Recalling the definition of the current (3.2) and then invoking (5.4), we get

$$\begin{aligned}\langle r_1^2\rangle_s &= \langle p_0^2\rangle_s - \langle p_1p_0\rangle_s - \frac{1}{2}(\widetilde{\gamma}+\gamma)\Big(\langle j_{0,1}\rangle_s + \frac{\gamma}{2}\big(\langle p_1^2\rangle_s - \langle p_0^2\rangle_s\big)\Big)\\ &= \langle p_0^2\rangle_s - \langle p_1p_0\rangle_s - \frac{\widetilde{\gamma}}{4}(\widetilde{\gamma}+\gamma)\big(T-\langle p_0^2\rangle_s\big) - \frac{\gamma}{4}(\widetilde{\gamma}+\gamma)\big(\langle p_1^2\rangle_s - \langle p_0^2\rangle_s\big).\end{aligned}$$

Using Young's inequality

$$\big|\langle p_1p_0\rangle_s\big| \leqslant \frac{A}{2}\langle p_1^2\rangle_s + \frac{1}{2A}\langle p_0^2\rangle_s,$$

with $A = \frac{\gamma}{2}(\gamma+\widetilde{\gamma})$, we get

$$\langle r_1^2\rangle_s \leqslant \Big(\frac{1}{\gamma(\gamma+\widetilde{\gamma})} + 1 + \frac{1}{4}(\gamma+\widetilde{\gamma})^2\Big)\langle p_0^2\rangle_s.$$

From (5.2) we conclude that $\langle r_1^2\rangle_s$ is bounded.

To estimate $\langle r_n^2\rangle_s$, note that from (5.7) we write

$$\langle r_n^2\rangle_s = \langle p_n^2\rangle_s - \langle p_np_{n-1}\rangle_s + \overline{\tau}_+\langle r_n\rangle_s - \frac{1}{2}(\widetilde{\gamma}+\gamma)\langle p_nr_n\rangle_s. \tag{5.9}$$

We use again Young's inequality

$$\big|\langle p_np_{n-1}\rangle_s\big| \leqslant \frac{A}{2}\langle p_n^2\rangle_s + \frac{1}{2A}\langle p_{n-1}^2\rangle_s,$$

with $A = 1/(2\gamma)$ and we get

$$\langle r_n^2\rangle_s \leqslant \Big(1+\frac{1}{4\gamma}\Big)\langle p_n^2\rangle_s + \gamma\langle p_{n-1}^2\rangle_s + \tau\langle r_n\rangle_s - \frac{1}{2}(\widetilde{\gamma}+\gamma)\langle p_nr_n\rangle_s. \tag{5.10}$$

To replace $\langle p_{n-1}^2\rangle_s$, note that

$$n^{-2}L(p_n^2) = 2(\overline{\tau}_+ - r_n)p_n + \gamma(p_{n-1}^2 - p_n^2) + \widetilde{\gamma}(T_+ - p_n^2).$$

Taking the average with respect to the stationary state, we obtain:

$$\gamma\langle p_{n-1}^2\rangle_s = 2\langle r_np_n\rangle_s - 2\overline{\tau}_+\langle p_n\rangle_s + (\gamma+\widetilde{\gamma})\langle p_n^2\rangle_s - \widetilde{\gamma}T_+, \tag{5.11}$$

which, in (5.10), gives

$$\langle r_n^2\rangle_s \leqslant \Big(1+\frac{1}{4\gamma}+\gamma+\widetilde{\gamma}\Big)\langle p_n^2\rangle_s + \overline{\tau}_+\,\big(\langle r_n\rangle_s - 2\langle p_n\rangle_s\big) + \frac{1}{2}(4-\widetilde{\gamma}-\gamma)\langle p_nr_n\rangle_s - \widetilde{\gamma}T_+.$$

Using again Young's inequality

$$\big|\langle p_nr_n\rangle_s\big| \leqslant \frac{A}{2}\langle p_n^2\rangle_s + \frac{1}{2A}\langle r_n^2\rangle_s$$

with $A = \frac{1}{2}|4-\gamma-\widetilde{\gamma}|$, we finally arrive at

$$\frac{1}{2}\langle r_n^2\rangle_s \leqslant \Big(1+\frac{1}{4\gamma}+\gamma+\widetilde{\gamma}+\frac{1}{4}(4-\gamma-\widetilde{\gamma})^2\Big)\langle p_n^2\rangle_s + \overline{\tau}_+\,\big(\langle r_n\rangle_s - 2\langle p_n\rangle_s\big) - \widetilde{\gamma}T_+.$$

We now invoke (5.2), (4.1) and (4.2) to conclude the bound on $\langle r_n^2\rangle_s$, which combined with the already obtained bound on $\langle r_1^2\rangle_s$ yields (5.3). $\square$

Corollary 5.4 (Second moments at the boundaries: Part II)**.** *There exists* $C = C(\gamma, \widetilde{\gamma}, \overline{\tau}_+, T_+, T_-) > 0$*, such that*

$$\langle p_1^2\rangle_s + \langle p_{n-1}^2\rangle_s \leqslant C. \tag{5.12}$$

Proof. To bound $\langle p_{n-1}^2\rangle_s$ we use formula (5.11). From an elementary inequality $\langle r_n p_n\rangle_s \leqslant 2\langle p_n^2\rangle_s + 2\langle r_n^2\rangle_s$ and Proposition 5.2, we easily conclude that $\langle p_{n-1}^2\rangle_s$ is bounded.

The bound for $\langle p_1^2\rangle_s$ is obtained similarly. First, note that

$$n^{-2}L(p_0^2) = 2r_1p_0 + \gamma(p_1^2 - p_0^2) + \widetilde{\gamma}(T_- - p_0^2). \tag{5.13}$$

Taking the average with respect to the stationary state, using the inequality $\langle r_1p_0\rangle_s \leqslant 2\langle r_1^2\rangle_s + 2\langle p_0^2\rangle_s$, and invoking Proposition 5.2, we conclude the desired bound on $\langle p_1^2\rangle_s$. Thus (5.12) follows. $\square$

In the next proposition we provide a bound on the energy current under the stationary state, which will be refined further in Proposition 5.9:

Proposition 5.5 (The stationary current: Part I)**.** *There exists a constant* $C = C(\gamma, \widetilde{\gamma}, \overline{\tau}_+, T_+, T_-) > 0$*, such that the stationary current satisfies*

$$\left|\overline{j}_s\right| \leqslant \frac{C}{n}. \tag{5.14}$$

Proof of Proposition 5.5. We sum the identity (4.3) from $x = 1$ to $n-1$ and apply (4.4) to express $\phi(n)$ and $\phi(1)$. In this way we get

$$\begin{aligned}(n-1)\overline{j}_s &= \phi(n) - \phi(1)\\ &= \left\langle -\frac{p_{n-1}p_n + r_n^2}{2\gamma} - \frac{\gamma(p_{n-1}^2 + p_n^2)}{4}\right\rangle_s + \left\langle \frac{p_1p_0 + r_1^2}{2\gamma} + \frac{\gamma(p_1^2 + p_0^2)}{4}\right\rangle_s. \end{aligned}\tag{5.15}$$

Therefore, (5.14) follows from elementary inequalities $|\langle p_{n-1}p_n\rangle_s| \leqslant 2\langle p_n^2\rangle_s + 2\langle p_{n-1}^2\rangle_s$ and $|\langle p_1p_0\rangle_s| \leqslant 2\langle p_0^2\rangle_s + 2\langle p_1^2\rangle_s$, and an application of the bounds obtained in Proposition 5.2 and Corollary 5.4. $\square$

Proposition 5.5 permits to get a better bound on the entropy production. Namely, combining (5.1), (5.4) and (5.14) we conclude the following.

Corollary 5.6. *There exists* $C = C(\gamma, \widetilde{\gamma}, \overline{\tau}_+, T_+, T_-) > 0$*, such that*

$$\gamma\sum_{x=0}^{n-1}\mathcal{D}_x(f_s) + \widetilde{\gamma}T_-\left\langle\!\!\left\langle \frac{(\partial_{p_n}(f_s/h))^2}{(f_s/h)}\right\rangle\!\!\right\rangle_{T_-} + \widetilde{\gamma}T_+\left\langle\!\!\left\langle \frac{(\partial_{p_n}f_s)^2}{f_s}\right\rangle\!\!\right\rangle_{T_+} \leqslant \frac{C}{n}. \tag{5.16}$$

Thanks to Proposition 5.5 we are now able to estimate the covariances of momenta and stretches at the boundaries as follows:

Proposition 5.7 (Second moment at the boundaries: Part III)**.** *There exists* $C = C(\gamma, \widetilde{\gamma}, \overline{\tau}_+, T_+, T_-) > 0$*, such that, at the left boundary point*

$$\left|\langle p_0 p_1\rangle_s\right| + \left|\langle r_1 p_1\rangle_s\right| + \left|\langle r_1 p_0\rangle_s\right| \leqslant \frac{C}{\sqrt{n}}, \tag{5.17}$$

and at the right boundary point

$$\left|\langle p_n p_{n-1}\rangle_s\right| + \left|\langle r_n p_n\rangle_s\right| + \left|\langle r_n p_{n-1}\rangle_s\right| \leqslant \frac{C}{\sqrt{n}}. \tag{5.18}$$

Proof. Integration by parts yields

$$\langle p_0 p_1\rangle_s = -T_- \big\langle\!\big\langle p_1 (f_s/g_{T_-}) \partial_{p_0} g_{T_-} \big\rangle\!\big\rangle_{T_+} = T_- \big\langle\!\big\langle p_1 \partial_{p_0}(f_s/h) \big\rangle\!\big\rangle_{T_-}.$$

We use the entropy production bound (5.16) and the bound (5.12) on $\langle p_1^2\rangle_s$, to estimate the right hand side. As a result we get

$$\left|\langle p_0 p_1\rangle_s\right| = T_- \left|\big\langle\!\big\langle p_1 \partial_{p_0}(f_s/h) \big\rangle\!\big\rangle_{T_-}\right| \leqslant T_- \big\langle p_1^2 \big\rangle_s^{\frac12} \left\langle\!\!\left\langle \frac{(\partial_{p_0}(f_s/h))^2}{(f_s/h)} \right\rangle\!\!\right\rangle_{T_-}^{\frac12} \leqslant \frac{C}{\sqrt{n}}.$$

Similarly,

$$\left|\langle p_n p_{n-1}\rangle_s\right| = T_+ \left|\big\langle\!\big\langle p_{n-1} \partial_{p_n} f_s \big\rangle\!\big\rangle_{T_+}\right| \leqslant T_+ \big\langle p_{n-1}^2 \big\rangle_s^{\frac12} \left\langle\!\!\left\langle \frac{(\partial_{p_n} f_s)^2}{f_s} \right\rangle\!\!\right\rangle_{T_+}^{\frac12} \leqslant \frac{C}{\sqrt{n}}.$$

Finally, note that, for any $x \in \mathbb{I}_n$

$$n^{-2} L(r_x^2) = 2(p_x - p_{x-1}) r_x.$$

Therefore, upon averaging with respect to the NESS, we get

$$\langle p_x r_x\rangle_s = \langle p_{x-1} r_x\rangle_s, \quad x \in \mathbb{I}_n. \tag{5.19}$$

In particular, applying (5.19) for $x = 1$ and $x = n$, we remark that the only quantities we need to yet estimate are $|\langle r_1 p_0\rangle_s|$ and $|\langle r_n p_n\rangle_s|$. This is done using the entropy production bound (5.16) in the same manner as before, namely:

$$\left|\langle r_1 p_0\rangle_s\right| = T_- \left|\big\langle\!\big\langle r_1 \partial_{p_0}(f_s/h) \big\rangle\!\big\rangle_{T_-}\right| \leqslant T_- \big\langle r_1^2 \big\rangle_s^{\frac12} \left\langle\!\!\left\langle \frac{(\partial_{p_0}(f_s/h))^2}{(f_s/h)} \right\rangle\!\!\right\rangle_{T_-}^{\frac12} \leqslant \frac{C}{\sqrt{n}},$$

from (5.3) and (5.1). We leave the last estimate for the reader. ☐

We now have all the ingredients necessary to prove moments convergences at the boundaries:

Corollary 5.8 (Second moments at the boundaries: Part IV)**.** *The following limits hold: at the left boundary point,*

$$\langle p_x^2\rangle_s \xrightarrow[n\to\infty]{} T_- \qquad \text{for } x \in \{0,1\}, \tag{5.20}$$

$$\langle r_1^2\rangle_s \xrightarrow[n\to\infty]{} T_-, \tag{5.21}$$

$$\langle r_1 r_2\rangle_s \xrightarrow[n\to\infty]{} 0, \tag{5.22}$$

and at the right boundary point,

$$\langle p_x^2\rangle_s \xrightarrow[n\to\infty]{} T_+ \qquad \textit{for } x \in \{n-1, n\}, \tag{5.23}$$

$$\langle r_n^2\rangle_s \xrightarrow[n\to\infty]{} T_+ + \overline{\tau}_+^2, \tag{5.24}$$

$$\langle r_{n-1} r_n\rangle_s \xrightarrow[n\to\infty]{} \overline{\tau}_+^2. \tag{5.25}$$

Proofs of (5.20) and (5.23). From (5.4) and Proposition 5.5 we get $\langle p_0^2\rangle_s \to T_-$. Thanks to (5.13) we deduce $\langle p_1^2\rangle_s \to T_-$, which in turn proves (5.20). A similar argument proves (5.23). Indeed, from (5.5), Proposition 5.5, and (5.18), we get $\langle p_n^2\rangle_s \to T_+$ and from (5.11) we deduce $\langle p_{n-1}^2\rangle_s \to T_+$.

Proofs of (5.21) and (5.24). The limit (5.21) follows directly from (5.8) and Proposition 5.7. From (4.2) we conclude that $\langle r_n\rangle_s \to \overline{\tau}_+$. Using then (5.9) together with Proposition 5.7 we conclude (5.24).

Proofs of (5.21) and (5.22). Note that

$$n^2 L(r_1 p_1) = (p_1 - p_0)p_1 + (r_2 - r_1)r_1 - \gamma r_1 p_1$$
$$n^{-2} L(r_n p_{n-1}) = (p_n - p_{n-1})p_{n-1} + (r_n - r_{n-1})r_n - \gamma r_n p_{n-1}.$$

Taking the average with respect to the stationary state, and using Proposition 5.7 together with (5.20) proved above, we get

$$\langle r_1^2\rangle_s - \langle r_1 r_2\rangle_s \xrightarrow[n\to\infty]{} T_-, \tag{5.26}$$

and

$$\langle r_n^2\rangle_s - \langle r_{n-1} r_n\rangle_s \xrightarrow[n\to\infty]{} T_+. \tag{5.27}$$

Using the already proved limits (5.21) and (5.24) we conclude (5.22) and (5.25). □

Proposition 5.9 (The stationary current: Part II)**.** *The following limits hold:*

$$\phi(1) \xrightarrow[n\to\infty]{} -\frac{1}{2}(\gamma^{-1} + \gamma)T_- \tag{5.28}$$

$$\phi(n) \xrightarrow[n\to\infty]{} -\frac{1}{2}(\gamma^{-1} + \gamma)T_+ - \frac{\overline{\tau}_+^2}{2}. \tag{5.29}$$

In consequence, (3.5) *holds and Theorem 3.2 is proved.*

Proof. Limits in (5.28) and (5.29) follow from formula (4.4), Proposition 5.7 and the limits computed in Corollary 5.8. The limit (3.5) is a consequence of (5.28), (5.29) and formula (5.15). □

5.3. **Energy bounds.** We now provide bounds on the total energy under the stationary state:

Proposition 5.10 (Energy bounds)**.** *There exists* $C = C(\gamma, \widetilde{\gamma}, \overline{\tau}_+, T_+, T_-) > 0$*, such that*

$$\frac{1}{n}\sum_{x=1}^{n}\langle p_x^2\rangle_s \leqslant C \qquad \textit{and} \qquad \frac{1}{n}\sum_{x=1}^{n}\langle r_x^2\rangle_s \leqslant C, \quad n \geqslant 1. \tag{5.30}$$

Proof. From the current decomposition given by (4.3), we easily get that

$$\phi(x) = (x-1)\overline{j}_s + \phi(1), \qquad \text{for any } x \in \mathbb{I}_n.$$

Summing over x, this gives

$$\frac{1}{n}\sum_{x=1}^{n}\phi(x) = \frac{1}{n}\sum_{x=2}^{n}(x-1)\overline{j}_s + \phi(1) = \frac{n(n-1)}{2n}\,\overline{j}_s + \phi(1).$$

Therefore, recalling (5.28) and (3.5), we get

$$\frac{1}{n}\sum_{x=1}^{n}\phi(x) \xrightarrow[n\to\infty]{} -\frac{1}{4}\big(\gamma^{-1}+\gamma\big)(T_+ + T_-) - \frac{\overline{\tau}_+^2}{4\gamma}. \tag{5.31}$$

From(4.4), we have

$$\frac{1}{n}\sum_{x=1}^{n}\phi(x) = -\frac{1}{2\gamma n}\sum_{x=1}^{n}\langle r_x^2\rangle_s - \frac{1}{2\gamma n}\sum_{x=1}^{n}\langle p_x p_{x-1}\rangle_s - \frac{\gamma}{2n}\sum_{x=1}^{n}\langle p_x^2\rangle_s + \frac{\gamma}{4n}\big(\langle p_n^2\rangle_s - \langle p_0^2\rangle_s\big). \tag{5.32}$$

To compute the limit of the second sum in the right hand side of (5.32), we first write:

$$n^{-2}L(p_{x-1}p_x) = (r_{x+1}-r_x)p_{x-1} + (r_x - r_{x-1})p_x - 2\gamma p_x p_{x-1}, \qquad x \in \{2,\dots,n-1\}.$$

Thus, taking the average with respect to the stationary state and subsequently using (5.19), we obtain

$$\begin{aligned} 2\gamma\langle p_x p_{x-1}\rangle_s &= \langle p_x r_x\rangle_s + \langle p_{x-1} r_{x+1}\rangle_s - \langle p_x r_{x-1}\rangle_s - \langle p_{x-1} r_x\rangle_s \\ &= \langle p_{x-1} r_{x+1}\rangle_s - \langle p_x r_{x-1}\rangle_s. \end{aligned} \tag{5.33}$$

On the other hand

$$n^{-2}L(r_x r_{x+1}) = (p_x - p_{x-1})r_{x+1} + (p_{x+1}-p_x)r_x, \qquad x \in \{1,\dots,n-1\}.$$

Hence, taking the average and using again (5.19), we get

$$\begin{aligned} 0 &= \langle p_{x+1} r_x\rangle_s + \langle p_x r_{x+1}\rangle_s - \langle p_x r_x\rangle_s - \langle p_{x-1} r_{x+1}\rangle_s \\ &= \langle p_{x+1} r_x\rangle_s + \langle p_{x+1} r_{x+1}\rangle_s - \langle p_x r_x\rangle_s - \langle p_{x-1} r_{x+1}\rangle_s, \end{aligned}$$

which yields

$$\langle p_{x+1} r_{x+1}\rangle_s - \langle p_x r_x\rangle_s = \langle p_{x-1} r_{x+1}\rangle_s - \langle p_{x+1} r_x\rangle_s$$

for any $x \in \{2,\dots,n-1\}$. Combining with (5.33) we get

$$2\gamma\langle p_x p_{x-1}\rangle_s = \langle p_{x+1} r_{x+1}\rangle_s - \langle p_x r_x\rangle_s + \langle p_{x+1} r_x\rangle_s - \langle p_x r_{x-1}\rangle_s, \tag{5.34}$$

for any $x \in \{2, \ldots, n-1\}$. Summing over x, one gets:

$$\sum_{x=2}^{n-1} \langle p_x p_{x-1} \rangle_s = \frac{1}{2\gamma}\Big(\langle p_n r_n \rangle_s - \langle p_2 r_2 \rangle_s + \langle p_n r_{n-1} \rangle_s - \langle p_2 r_1 \rangle_s \Big). \tag{5.35}$$

To compute the limit as $n \to \infty$, we need to estimate the covariances appearing in the right hand side. The covariance $\langle p_n r_n \rangle_s$ can be estimated thanks to Proposition 5.7. We still need the bounds on the covariances $\langle p_2 r_2 \rangle_s$, $\langle p_n r_{n-1} \rangle_s$ and $\langle p_2 r_1 \rangle_s$. To deal with it write

$$n^{-2} L(p_0 p_1) = (r_2 - r_1) p_0 + r_1 p_1 - \frac{1}{2}(3\gamma + \widetilde{\gamma}) p_0 p_1 \tag{5.36}$$

$$n^{-2} L(r_1 r_2) = (p_1 - p_0) r_2 + (p_2 - p_1) r_1 \tag{5.37}$$

$$n^{-2} L(p_{n-1} p_n) = (\overline{\tau}_+ - r_n) p_{n-1} + (r_n - r_{n-1}) p_n - \frac{1}{2}(3\gamma + \widetilde{\gamma}) p_{n-1} p_n. \tag{5.38}$$

Taking the averages and summing (5.36) and (5.37) sideways gives (using $\langle p_2 r_2 \rangle_s = \langle p_1 r_2 \rangle_s$ from (5.19))

$$\langle p_2 r_2 \rangle_s + \langle p_2 r_1 \rangle_s = \langle p_0 r_1 \rangle_s + \frac{1}{2}(3\gamma + \widetilde{\gamma}) \langle p_0 p_1 \rangle_s \xrightarrow[n\to\infty]{} 0, \tag{5.39}$$

from Proposition 5.7. Moreover, (5.38) gives (using $\langle p_n r_n \rangle_s = \langle p_{n-1} r_n \rangle_s$)

$$\langle r_{n-1} p_n \rangle_s = \overline{\tau}_+ \langle p_{n-1} \rangle_s - \frac{1}{2}(3\gamma + \widetilde{\gamma}) \langle p_{n-1} p_n \rangle_s \xrightarrow[n\to\infty]{} 0, \tag{5.40}$$

from (4.1) and Proposition 5.7. Therefore, we have proved that (5.35) vanishes as $n \to \infty$. In fact, due to the estimates obtained in Proposition 5.7 we have even proved that there exists a constant $C = C(\gamma, \widetilde{\gamma}, \overline{\tau}_+, T_+, T_-) > 0$, such that

$$\Big| \sum_{x=1}^{n} \langle p_x p_{x-1} \rangle_s \Big| \leqslant \frac{C}{\sqrt{n}}, \qquad n \geqslant 1. \tag{5.41}$$

From (5.32) it follows that

$$\begin{aligned} \frac{1}{2\gamma n} \sum_{x=1}^{n} \langle r_x^2 \rangle_s + \frac{\gamma}{2n} \sum_{x=1}^{n} \langle p_x^2 \rangle_s &= -\frac{1}{2\gamma n} \sum_{x=1}^{n} \langle p_x p_{x-1} \rangle_s - \frac{1}{n} \sum_{x=1}^{n} \phi(x) + \frac{\gamma}{4n}\Big(\langle p_n^2 \rangle_s - \langle p_0^2 \rangle_s \Big) \\ &\xrightarrow[n\to\infty]{} \frac{1}{4}\big(\gamma^{-1} + \gamma\big)(T_+ + T_-) + \frac{\overline{\tau}_+^2}{4\gamma}, \end{aligned}$$

due to (5.31) and (5.41). This in particular implies estimate (5.30). □

Thanks to the energy bounds, we are finally able to prove one further convergence, which will be essential in establishing Proposition 4.4.

Proposition 5.11. *For any continuous test function $G : \mathbb{I} \to \mathbb{R}$ we have*

$$\frac{1}{n} \sum_{x=1}^{n-1} G\Big(\frac{x}{n}\Big) \langle p_x p_{x+1} \rangle_s \xrightarrow[n\to\infty]{} 0. \tag{5.42}$$

Proof. Assume first that $G \in C^1(\mathbb{I})$. For the brevity sake we denote $G_x := G(x/n)$ for any $x \in \mathbb{I}_n$ and $\psi(x) = \langle p_{x+1} r_{x+1} \rangle_s + \langle p_{x+1} r_x \rangle_s$. Then (5.34) says that

$$\langle p_x p_{x+1} \rangle_s = \frac{1}{2\gamma}\big(\psi(x+1) - \psi(x)\big), \qquad \text{for any } x \in \{1, \dots, n-2\}.$$

Therefore, by an application of summation by parts formula, we get

$$\frac{1}{n}\sum_{x=1}^{n-1} G\Big(\frac{x}{n}\Big)\langle p_x p_{x+1}\rangle_s = \frac{1}{2\gamma n^2}\sum_{x=2}^{n-2} n(G_{x-1} - G_x)\psi(x) \tag{5.43}$$

$$+ \frac{1}{n}\langle p_n p_{n-1}\rangle_s G_{n-1} + \frac{1}{2\gamma n}\big(\langle p_n r_n\rangle_s + \langle p_n r_{n-1}\rangle_s\big) \tag{5.44}$$

$$- \frac{1}{2\gamma n}\big(\langle p_2 r_2\rangle_s + \langle p_2 r_1\rangle_s\big)G_1. \tag{5.45}$$

The boundary terms (5.44) and (5.45) vanish, as $n \to \infty$, thanks to (5.18), (5.39) and (5.40). To deal with the sum in the right hand side of (5.43) note that, since $G \in C^1(\mathbb{I})$, we have

$$\sup_{x \in \{2, \dots, n-2\}} n|G_{x-1} - G_x| \leqslant \|G'\|_\infty. \tag{5.46}$$

Since

$$|\psi(x)| \leqslant 2\left(2\langle p_{x+1}^2\rangle_s + \langle r_x^2\rangle_s + \langle r_{x+1}^2\rangle_s\right), \qquad x \in \{1, \dots, n-1\}$$

we conclude that

$$\frac{1}{2\gamma n^2}\left|\sum_{x=2}^{n-2} n(G_{x-1} - G_x)\psi(x)\right| \leqslant \frac{C}{n}\frac{1}{n}\sum_{x=1}^{n}\big(\langle p_x^2\rangle_s + \langle r_x^2\rangle_s\big), \tag{5.47}$$

which vanishes, as $n \to +\infty$, thanks to the energy bound (5.30). This proves (5.42) for any test function $G \in C^1(\mathbb{I})$. The result can be extended to all continuous functions by the standard density argument and the energy bound (5.30). □

5.4. **Proof of Proposition 4.4.** We now have all in hands to prove Proposition 4.4 and therefore conclude the proof of Theorem 3.3. There are three convergences to prove:

Proof of (4.8). From Proposition 4.2 (in particular (4.5)) and Proposition 5.9, we easily obtain

$$\frac{1}{n}\sum_{x \in \mathbb{I}_n} G\Big(\frac{x}{n}\Big)\phi(x) \xrightarrow[n\to\infty]{} \int_{\mathbb{I}} G(u)\bigg(-\frac{\overline{\tau}_+^2}{2\gamma}\,u - \frac{1}{2}(\gamma^{-1} + \gamma)\Big((T_+ - T_-)u + T_-\Big)\bigg)\,\mathrm{d}u.$$

which proves directly (4.8).

Proof of (4.9). Concerning $\mathcal{H}_n^{\nabla}(G)$ we use a summation by parts formula (with the notation $G_x = G(x/n)$), which leads to:

$$\mathcal{H}_n^{\nabla}(G) = \frac{\gamma^2}{2(1+\gamma^2)}\left(\frac{G_n}{n}\langle p_n^2\rangle_s - \frac{G_1}{n}\langle p_0^2\rangle_s + \frac{1}{n^2}\sum_{x=1}^{n-1} n(G_x - G_{x+1})\langle p_x^2\rangle_s\right).$$

The boundary terms in the right hand side vanish, as $n \to +\infty$, since $\langle p_n^2\rangle_s$ and $\langle p_0^2\rangle_s$ are bounded, due to (5.2). To deal with the limit of the last sum in the right hand side, we can repeat the argument made in (5.46)-(5.47), which shows that the expression vanishes. Thus (4.9) holds.

Proof of (4.10). This is a consequence of (5.42). □

Appendix A. Non-Stationary behaviour

In this section we formally derive (2.7) and (2.8).

Equation (2.7) can be formulated in a weak form as:

$$\int_0^1 \mathrm{d}u\, G(u)\big(r(t,u) - r(0,u)\big) = \frac{1}{2\gamma}\int_0^t \mathrm{d}s \int_0^1 \mathrm{d}u\, G''(u) r(s,u) - \frac{1}{2\gamma} G'(1)\overline{\tau}_+(t), \tag{A.1}$$

for any smooth test function G on $[0,1]$ such that $G(0) = G(1) = 0$. Existence and uniqueness of such weak solutions are standard exercises.

Now consider such test function and denote as before $G_x = G(x/n)$. By the microscopic evolution equations (2.2) we have

$$\begin{aligned}\frac{1}{n}\sum_{x=1}^{n} G_x\left(r_x(t) - r_x(0)\right) &= \int_0^t \mathrm{d}s\; n\sum_{x=1}^{n} G_x\left(p_x(s) - p_{x-1}(s)\right)\\ &= \int_0^t \mathrm{d}s\left\{-\sum_{x=1}^{n-1}(\nabla_n G)_x\, p_x(s) + nG_n p_n(s) - nG_1 p_0(s)\right\}\\ &= \int_0^t \mathrm{d}s\left\{-\sum_{x=1}^{n-1}(\nabla_n G)_x\, p_x(s) - nG_1 p_0(s)\right\} \end{aligned}\tag{A.2}$$

where $(\nabla_n G)_x := n(G_{x+1} - G_x)$. Using (2.3) we can write (A.2) as

$$-\int_0^t \mathrm{d}s\left\{\sum_{x=1}^{n-1}\frac{1}{\gamma}(\nabla_n G)_x\left(r_{x+1}(s) - r_x(s)\right) + \frac{1}{2(\gamma+\widetilde{\gamma})} nG_1 r_1(s)\right\} \tag{A.3}$$

$$+\frac{1}{\gamma n^2}\sum_{x=1}^{n-1}(\nabla_n G)_x\left(p_x(t) - p_x(0)\right) + \frac{1}{2(\gamma+\widetilde{\gamma})n^2} nG_1\left(p_0(t) - p_0(0)\right) \tag{A.4}$$

$$-\frac{1}{\sqrt{\gamma}n}\sum_{x=1}^{n-1}(\nabla_n G)_x \int_0^t \big(p_{x-1}(s)\mathrm{d}w_{x-1,x}(s) - p_{x+1}(s)\mathrm{d}w_{x,x+1}(s)\big) \tag{A.5}$$

$$+\frac{\sqrt{\gamma}}{2(\gamma+\widetilde{\gamma})n} nG_1\int_0^t p_1(s)\mathrm{d}w_{0,1}(s) - \frac{\sqrt{\widetilde{\gamma}T_-}}{2(\gamma+\widetilde{\gamma})n} nG_1\int_0^t p_0(s)\mathrm{d}\widetilde{w}_0(s). \tag{A.6}$$

Since G is smooth, $nG_1 \to G'(0)$ as $n \to +\infty$, and using the orthogonality of the Wiener processes, it is easy to see that (A.4), (A.5) and (A.6) converge to 0 in quadratic expectation. We are left with the first term (A.3). Moreover, using the notation $(\Delta_n G)_x := n^2(G_{x+1} + G_{x-1} - 2G_x)$ and recalling $G(0) = 0$, the first term (A.3) can be rewritten as

$$\int_0^t \frac{1}{\gamma}\left\{\frac{1}{n}\sum_{x=2}^{n-1}(\Delta_n G)_x\, r_x(s) - (\nabla_n G)_{n-1}\, r_n(s)\right\}\mathrm{d}s \\ - \left\{\frac{1}{2(\gamma+\widetilde{\gamma})}(\nabla_n G)_0 - \frac{1}{\gamma}(\nabla_n G)_1\right\}\int_0^t r_1(s)\mathrm{d}s. \quad \text{(A.7)}$$

The following lemma can easily be proved from (2.4) and (2.5):

Lemma A.1. *The following convergences hold: at the left boundary point,*

$$\mathbb{E}\left[\left(\int_0^t r_1(s)\mathrm{d}s\right)^2\right] \xrightarrow[n\to\infty]{} 0,$$

and at the right boundary point,

$$\mathbb{E}\left[\left(\int_0^t (\overline{\tau}_+(s) - r_n(s))\,\mathrm{d}s\right)^2\right] \xrightarrow[n\to\infty]{} 0.$$

Then by a compactness argument, any limit point of the empirical distribution of the $\{r_x\}_{x\in\mathbb{I}_n}$ is a weak solution in the sense that it satisfies (A.1).

The weak formulation of equation (2.8) is the following: for any smooth test function G on $[0,1]$ such that $G(0) = G(1) = 0$ we have

$$\int_0^1 G(u)\big(e_{\mathrm{th}}(t,u) - e_{\mathrm{th}}(0,u)\big)\,\mathrm{d}u = (\gamma^{-1}+\gamma)\int_0^t ds\int_0^1 G''(u)e_{\mathrm{th}}(s,u)\,du \\ + \frac{1}{2\gamma}\int_0^t ds\int_0^1 G(u)\left(\partial_u r(s,u)\right)^2\,du - t(\gamma^{-1}+\gamma)\left(G'(1)T_+ - G'(0)T_-\right).$$

The profile of the *mechanical energy* of the system is defined as

$$\mathcal{E}_x^{\mathrm{mech}}(t) = \frac{1}{2}\overline{r}_x(t)^2$$

where $\overline{r}_x(t)$ is the expectation of $r_x(t)$. By the first hydrodynamic limit already proven, we known that $\overline{r}_{[nu]}(t) \longrightarrow r(u,t)$ weakly, so that $e_{[nu]}^{\mathrm{mech}}(t) \longrightarrow e_{\mathrm{mech}}(t,u)$, where $e_{\mathrm{mech}}(t,u)$ satisfy the equation

$$\partial_t e_{\mathrm{mech}}(t,u) = \frac{1}{\gamma}\left(\partial_{uu}^2 e_{\mathrm{mech}}(t,u) - (\partial_u r(t,u))^2\right)$$

Recall that the total energy per particle is defined as $\mathcal{E}_x(t) = \frac{1}{2}\left(p_x^2 + r_x^2\right)(t)$ and the thermal energy is given by

$$\mathcal{E}_x^{\mathrm{th}}(t) = \mathcal{E}_x(t) - \mathcal{E}_x^{\mathrm{mech}}(t).$$

We are going to use the microscopic energy currents given in (3.2). Consider now a smooth test function G on $[0,1]$ such that $G(0) = G(1) = 0$. Then

$$\frac{1}{n}\sum_{x=0}^{n} G_x\left(\mathcal{E}_x^{\rm th}(t) - \mathcal{E}_x^{\rm th}(0)\right) = \frac{1}{n}\sum_{x=1}^{n-1} G_x\left(\mathcal{E}_x^{\rm th}(t) - \mathcal{E}_x^{\rm th}(0)\right)$$

$$= \int_0^t {\rm d}s \left\{\sum_{x=1}^{n-2}(\nabla_n G)_x\, j_{x,x+1}(s) - nG_{n-1}j_{n-1,n}(s) + nG_1 j_{0,1}(s)\right\} \tag{A.8}$$

$$-\frac{1}{n}\sum_{x=1}^{n-1} G_x\left(\mathcal{E}_x^{\rm mech}(t) - \mathcal{E}_x^{\rm mech}(0)\right). \tag{A.9}$$

We have already seen that the last term (A.9) converges to

$$-\frac{1}{\gamma}\int_0^t {\rm d}s \int_0^1 {\rm d}u\, \left(G''(u)\frac{r(s,u)^2}{2} - G(u)\left(\partial_u r(s,u)\right)^2\right) + \frac{1}{2\gamma}G'(1)\overline{\tau}_+^2(t).$$

In order to treat (A.8) we use another fluctuation-dissipation relation satisfied by the energy currents $j_{x,x+1}$ (similar to (4.6)): one can easily check that

$$j_{x,x+1} = n^{-2}Lh_x - \frac{1}{2\gamma}\nabla\left(p_x^2 + r_x r_{x+1}\right) - \frac{\gamma}{2}\nabla\left(p_x^2\right), \qquad x = 1,\dots,n-2 \tag{A.10}$$

with

$$h_x = \frac{1}{4}r_{x+1}^2 + \frac{1}{2\gamma}r_{x+1}\left(p_{x+1} + p_x\right), \qquad x = 1,\dots,n-2.$$

Using these relations we can rewrite the term (A.8) as

$$\int_0^t {\rm d}s \left\{\frac{1}{n}\sum_{x=2}^{n-2}(\Delta_n G)_x\left(\frac{1}{2}(\gamma^{-1}+\gamma)p_x^2(s) + \frac{1}{2\gamma}r_x(s)r_{x+1}(s)\right)\right. \tag{A.11}$$

$$-(\nabla_n G)_{n-2}\left(\frac{1}{2}(\gamma^{-1}+\gamma)p_{n-1}^2(s) + \frac{1}{2\gamma}r_{n-1}(s)r_n(s)\right) \tag{A.12}$$

$$\left. + (\nabla_n G)_1\left(\frac{1}{2}(\gamma^{-1}+\gamma)p_1^2(s) + \frac{1}{2\gamma}r_1(s)r_2(s)\right)\right\}. \tag{A.13}$$

Similarly to Lemma A.1, it is not hard to prove the following convergences in quadratic expectation:

Lemma A.2. *The following convergences hold: at the left boundary point,*

$$\mathbb{E}\left[\left(\int_0^t \left(p_1^2(s) - p_0^2(s)\right){\rm d}s\right)^2\right] \xrightarrow[n\to\infty]{} 0, \qquad \mathbb{E}\left[\left(\int_0^t \left(p_0^2(s) - T_-\right){\rm d}s\right)^2\right] \xrightarrow[n\to\infty]{} 0,$$

$$\mathbb{E}\left[\left(\int_0^t r_1(s)r_2(s)\,{\rm d}s\right)^2\right] \xrightarrow[n\to\infty]{} 0, \qquad \mathbb{E}\left[\left(\int_0^t j_{0,1}(s)\,{\rm d}s\right)^2\right] \xrightarrow[n\to\infty]{} 0,$$

and at the right boundary point,

$$\mathbb{E}\left[\left(\int_0^t \big(p_{n-1}^2(s) - p_n^2(s)\big)\,\mathrm{d}s\right)^2\right] \xrightarrow[n\to\infty]{} 0,$$

$$\mathbb{E}\left[\left(\int_0^t \big(p_n^2(s) - T_+\big)\,\mathrm{d}s\right)^2\right] \xrightarrow[n\to\infty]{} 0,$$

$$\mathbb{E}\left[\left(\int_0^t \big(r_{n-1}(s)r_n(s) - \overline{\tau}_+^2(s)\big)\,\mathrm{d}s\right)^2\right] \xrightarrow[n\to\infty]{} 0,$$

$$\mathbb{E}\left[\left(\int_0^t j_{n-1,n}(s)\,\mathrm{d}s\right)^2\right] \xrightarrow[n\to\infty]{} 0.$$

This lemma permits to take care of all the boundary terms (A.12), (A.13). For the bulk (A.11) we expect the two following terms to vanish as $n \to +\infty$:

$$\int_0^t \mathrm{d}s\, \frac{1}{n}\sum_{x=2}^{n-2} (\Delta_n G)_x\, \big(p_x^2(s) - \mathcal{E}_x^{\mathrm{th}}(s)\big)$$

and

$$\int_0^t \mathrm{d}s\, \frac{1}{n}\sum_{x=2}^{n-2} (\Delta_n G)_x \left(r_x(s)r_{x+1}(s) - r\Big(s, \frac{x}{n}\Big)^2\right),$$

as can be guessed easily by local equilibrium considerations. Unfortunately in order to prove the last limits one needs some higher moment bounds that are not available from relative entropy considerations. One prospective work could be to proceed in an analogous way as in the periodic case [9], by studying the evolution of the Wigner distribution of the thermal energy in Fourier coordinates.

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