

Domination and fractional domination in digraphs

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Abstract

In this paper, we investigate the relation between the (fractional) domination number of a digraph G and the independence number of its underlying graph, denoted by $\alpha(G)$. More precisely, we prove that every digraph G on n vertices has fractional domination number at most $2\alpha(G)$ and domination number at most $2\alpha(G) \cdot \log n$. Both bounds are sharp.

Mathematics Subject Classifications: 05C20, 05C69

1 Introduction

Every digraph in this paper is simple, loopless and finite, where a digraph G is *simple* if for every two vertices u and v of G , there is at most one arc with endpoints $\{u, v\}$. Given a digraph G , we denote by $V(G)$ and $E(G)$ the vertex set and arc set of G , respectively. The *independence number* $\alpha(G)$ of a digraph G is the independence number of the underlying (undirected) graph of G . A digraph with independence number 1 is called a *tournament*. An *Eulerian tournament* is a tournament that is in addition Eulerian (i.e., the indegree of each vertex equals its outdegree).

Given a digraph $G = (V, E)$, we say that a vertex of G *dominates* itself and all of its out-neighbors. A set of vertices $S \subseteq V$ is called a *dominating set* if every vertex $v \in V$ is either an element of S or is an out-neighbor of some element of S . The *domination number* $\gamma(G)$ is the cardinality of a minimum (by cardinality) dominating set of G . Domination

in tournaments has been well-studied [1], [6], [13], while little is known for domination in general digraphs. Recently, it was proved [4], [8], [9] that the domination number is closely related to the dichromatic number when the digraph is a tournament or a dense digraph. The topic of domination in undirected graphs (where $S \subseteq V$ is a *dominating set* if every vertex $v \in V$ is either an element of S or is a neighbor of some element of S) has also been studied extensively, see for example the monograph [10]. It is a well-known fact that in an undirected graph, any maximal independent set is also a dominating set.

Suppose we are given a digraph $G = (V, E)$, a subset S of V , and a function $g : V \rightarrow \{0, 1\}$ such that $g(v) = 1$ if $v \in S$ and $g(v) = 0$ otherwise. Then S is a dominating set of G if and only if $\sum_{x \in N^-(v) \cup \{v\}} g(x) \geq 1$ for every $v \in V$, where $N^-(v)$ is the set of in-neighbors of v . Thus, a natural linear relaxation of domination in digraphs arises as follows. A *fractional dominating function* of G is a function $g : V \rightarrow [0, 1]$ such that $\sum_{x \in N^-(v) \cup \{v\}} g(x) \geq 1$ for every $v \in V$. The *fractional domination number* $\gamma^*(G)$ is the smallest value of $\sum_{v \in V} g(v)$ over all fractional dominating functions g of G .¹ The fractional domination number of a tournament was the main tool to prove the long standing Erdős-Sands-Sauer-Woodrow conjecture in [2].

In this paper, we show that for any digraph, its fractional domination number is at most twice its independence number, and this bound is sharp.

Theorem 1.1. *For every digraph G , we have $\gamma^*(G) \leq 2\alpha(G)$.*

In contrast to the fractional domination number, it is not possible to bound the domination number of a digraph in terms of its independence number. Indeed, it was shown in [12] that almost surely a random tournament has domination number on the order of $\log n$, much larger than its independence number of 1. However, the upper bound of $\log n$ on the domination number of a tournament can be extended to general digraphs.

Theorem 1.2. *For every digraph G on n vertices, we have $\gamma(G) \leq \alpha(G) \cdot \log n$.*

Sometimes, it is in fact possible to bound the domination number of a digraph purely in terms of its independence number. We discuss this further in the last section.

1.1 Notation

Let $G = (V, E)$ be a digraph; for every $v \in V$, we denote by $N_G^+(v), N^-(v)$ the set of out-neighbors and in-neighbors of v , respectively. Let $N_G^+[v] = N_G^+(v) \cup \{v\}$ and $N_G^-[v] = N_G^-(v) \cup \{v\}$. Given a subset S of V , we write $N_G^+(S) = \bigcup_{v \in S} N_G^+(v)$, and similarly for $N_G^-(S), N_G^+[S]$, and $N_G^-[S]$. Given two vertices u, v , if $uv, vu \notin E$, we say that u and v are *independent*. We denote by $N_G^o(v)$ the set of vertices that are independent with v (i.e., $N_G^o(v) = V \setminus (N_G^+[v] \cup N_G^-[v])$). When it is clear from the context, we may omit the subscript G . Given a subset X of V , we denote by $G[X]$ the induced subgraph of G on X . Given a digraph $G = (V, E)$ and a function g on V , we write $g(X) := \sum_{v \in X} g(v)$

¹We remark that when constructing a dominating function $g(\cdot)$ to upper bound $\gamma^*(G)$ by the value $g(V)$, it is sufficient to show that $g(v) \geq 0$ and $\sum_{x \in N^-(v) \cup \{v\}} g(x) \geq 1$ for every $v \in V$. If $g(v) > 1$ for some $v \in V$, then the function $g(\cdot)$ is not minimal (i.e., $g(v)$ can be decreased).

for short. We sometimes use n to denote $|V(G)|$. Finally, we mention a trivial but useful observation regarding the independence number of a digraph.

Observation 1.3. *If G is a digraph with independence number α and v is an arbitrary vertex in G , then $G[N^o(v)]$ has independence number at most $\alpha - 1$.*

2 Fractional domination in digraphs

In this section, we present two proofs of Theorem 1.1. The first proof uses the duality of linear programming, while the second proof is by induction. We first present some useful lemmas.

Lemma 2.1. *Given a digraph $G = (V, E)$ and a function $p : V \rightarrow [0, 1]$, there is a vertex $v \in V$ such that $p(N^-(v)) \leq p(N^+(v))$.*

Proof. Suppose that the lemma is false. Then for the function p , we have $p(N^-(v)) > p(N^+(v))$ for every $v \in V$ (i.e., $\sum_{x \in N^-(v)} p(x) > \sum_{y \in N^+(v)} p(y)$ for every v). Hence

$$\begin{aligned} \sum_{v \in V} p(v) \left(\sum_{x \in N^-(v)} p(x) \right) &> \sum_{v \in V} p(v) \left(\sum_{y \in N^+(v)} p(y) \right) \\ \implies \sum_{xv \in E} p(v)p(x) &> \sum_{vy \in E} p(v)p(y), \end{aligned}$$

a contradiction. □

Lemma 2.2. *Given a digraph $G = (V, E)$ and a function $p : V \rightarrow [0, 1]$ with $p(V) > 0$, there is a stable set $S \subseteq V$ such that $p(N^+[S]) \geq p(V)/2$.*

Proof. We prove the lemma by induction on $|V|$. The lemma clearly holds for $|V| = 1$. For $|V| > 1$, fix some function $p : V \rightarrow [0, 1]$ and apply Lemma 2.1 to obtain a vertex v such that $p(N^-(v)) \leq p(N^+(v))$. If $N^o(v) = \emptyset$, then

$$\begin{aligned} 2p(N^+[v]) &= 2p(N^+(v)) + p(v) \\ &\geq p(N^+(v)) + p(N^-(v)) + p(v) \\ &= p(V), \end{aligned}$$

which proves the lemma.

If $N^o(v) \neq \emptyset$, we apply induction on $G[N^o(v)]$ to obtain a stable set S' such that $p(N_{G[N^o(v)]}^+[S']) \geq p(N^o(v))/2$. Let $S = S' \cup \{v\}$. We have the following remarks.

- $N_{G[N^o(v)]}^+[S'] = N^+[S'] \cap N^o(v)$, and
- $N^+[v]$ and $N^+[S'] \cap N^o(v)$ are disjoint.

Thus, for the stable set S , we have

$$\begin{aligned}
 p(N^+[S]) &\geq p(N^+[v]) + p(N^+[S'] \cap N^o(v)) \\
 &= p(v) + p(N^+(v)) + p(N_{G[N^o(v)]}^+[S']) \\
 &\geq p(v) + (p(N^+(v)) + p(N^-(v)))/2 + p(N^o(v))/2 \\
 &= p(V)/2.
 \end{aligned}$$

This proves the lemma. \square

In the first proof of Theorem 1.1, we will use the following linear program. Let \mathcal{S} be the set of all maximal stable sets in G , and let A be the matrix with $|V|$ rows and $|\mathcal{S}|$ columns, where for every $v \in V, S \in \mathcal{S}$,

$$A(v, S) = \begin{cases} 1 & \text{if } v \in N^+[S], \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Let us consider the following linear program

$$\begin{array}{ll}
 \text{(P)} & \text{Minimize } \mathbf{1}^T z \\
 & \text{Subject to } Az \geq \mathbf{1} \text{ and } z \geq \mathbf{0},
 \end{array}$$

and its dual

$$\begin{array}{ll}
 \text{(D)} & \text{Maximize } \mathbf{1}^T w \\
 & \text{Subject to } A^T w \leq \mathbf{1} \text{ and } w \geq \mathbf{0}.
 \end{array}$$

Lemma 2.3. *The value of an optimal solution for (P) is at most 2.*

Proof. We prove that an optimal solution to (P) is at most 2, by proving that the optimal solution to (D) is at most 2. Then we apply the Strong Duality Theorem (see [3], Theorem 17.2 for example) to complete the proof.

Suppose for a contradiction that the optimal solution of (D) is greater than 2. Then there is a function w such that $\mathbf{1}^T w > 2$ and $A^T w \leq \mathbf{1}$. Then for every $S \in \mathcal{S}$, $\sum_{v \in V} A(v, S)w(v) \leq 1$, and so by (1),

$$w(N^+[S]) = \sum_{v \in N^+[S]} w(v) \leq 1.$$

However, by Lemma 2.2, there is $S \in \mathcal{S}$ such that $w(N^+[S]) \geq w(V)/2 = (\mathbf{1}^T w)/2 > 1$. This proves the lemma. \square

We now restate Theorem 1.1.

Theorem 2.4. *For any digraph $G = (V, E)$, we can construct a fractional dominating function $g : V \rightarrow [0, 1]$ such that $g(V) \leq 2\alpha(G)$.*

First proof of Theorem 2.4. Invoking Lemma 2.3, there is a z such that $Az \geq \mathbf{1}$ and $\mathbf{1}^T z \leq 2$. Note that z is a vector of length $|\mathcal{S}|$ and w is a vector of length $|V|$. Let $g(v) := \sum_{S:v \in S} z(S)$ for every $v \in V$. Let $\alpha = \alpha(G)$.

Note that for every $S \in \mathcal{S}$, $|S| \leq \alpha$ since S is stable. We have

$$g(V) = \sum_{v \in V} \sum_{S:v \in S} z(S) = \sum_{S \in \mathcal{S}} \sum_{v \in S} z(S) = \sum_{S \in \mathcal{S}} |S| z(S) \leq \alpha \sum_{S \in \mathcal{S}} z(S) = \alpha(\mathbf{1}^T z) \leq 2\alpha.$$

Fix v , since $Az \geq \mathbf{1}$, we have $\sum_{S \in \mathcal{S}} A(v, S)z(S) \geq 1$. In other words,

$$\sum_{S \in \mathcal{S}: v \in N^+[S]} z(S) \geq 1.$$

Besides,

$$g(N^-[v]) = \sum_{x:v \in N^+[x]} g(x) = \sum_{x:v \in N^+[x]} \sum_{S:x \in S} z(S) \geq \sum_{S \in \mathcal{S}: v \in N^+[S]} z(S).$$

Thus, $g(N^-[v]) \geq 1$ for every $v \in V$ (i.e., $g(v)$ is a fractional dominating function of G). This proves the theorem. \square

In the second proof of Theorem 2.4, we will use the following consequence of Farkas' Lemma (we refer the reader to [11] for the proof of Lemma 2.5; see also Theorem 1 in [5]).

Lemma 2.5. *For any digraph $G = (V, E)$, there exists a function $p : V \rightarrow [0, 1]$ such that $p(V) = 1$ and $p(N^-(v)) \geq p(N^+(v))$ for every vertex v .*

Second Proof of Theorem 2.4. We prove the theorem by induction on $\alpha(G)$. If $\alpha(G) = 1$, then G is a tournament. Let p be a function satisfying Lemma 2.5. Let $g(v) = 2p(v)$ for every $v \in V$. Then $g(V) = 2$ and for every v , we have

$$g(N^-[v]) = 2p(N^-[v]) \geq 2p(v) + p(N^-(v)) + p(N^+(v)) \geq p(V) = 1.$$

Thus, g is a fractional domination function of G . We conclude that $\gamma^*(G) \leq g(V) = 2 = 2\alpha(G)$ for the case $\alpha(G) = 1$.

If $\alpha(G) > 1$, let p be a function satisfying Lemma 2.5. By Observation 1.3, we have $\alpha(G[N^o(v)]) \leq \alpha(G) - 1$ for every $v \in V$, and so $\gamma^*(G[N^o(v)]) \leq 2(\alpha(G) - 1)$ by induction. In the rest of the proof, we write $G_v := G[N^o(v)]$ for short. For each vertex v , let g_v be a minimum fractional dominating function of G_v . Set $g(x) = 2p(x) + \sum_{y \in N^o(x)} g_y(x)p(y)$ for every vertex x . We show that g is a fractional dominating function of G . Note that $x \in N^o(y)$ if and only if $y \in N^o(x)$, and for every v, y with $v \in G_y$,

$$\sum_{x \in N_{G_y}^-[v]} g_y(x) = g_y(N_{G_y}^-[v]) \geq 1$$

since g_y is a fractional dominating function of G_y . Fix v , we have (we omit the subscript G if applicable)

$$\begin{aligned}
g(N^-[v]) &= \sum_{x \in N^-[v]} g(x) = \sum_{x \in N^-[v]} \left(2p(x) + \sum_{y \in N^o(x)} g_y(x)p(y) \right) \\
&= 2 \sum_{x \in N^-[v]} p(x) + \sum_{x \in N^-[v]} \sum_{y \in N^o(x)} g_y(x)p(y) \\
&= 2p(N^-[v]) + \sum_{y \in V} p(y) \sum_{x \in N^-[v] \cap N^o(y)} g_y(x) \\
&\geq 2p(N^-[v]) + \sum_{y \in N^o(v)} p(y) \sum_{x \in N^-[v] \cap N^o(y)} g_y(x) \\
&= 2p(N^-[v]) + \sum_{y \in N^o(v)} p(y) \sum_{x \in N_{G_y}^-[v]} g_y(x) \\
&\geq 2p(N^-[v]) + \sum_{y \in N^o(v)} p(y) \cdot 1 \\
&= 2p(N^-[v]) + p(N^o(v)) \\
&\geq 2p(v) + p(N^-(v)) + p(N^+(v)) + p(N^o(v)) \\
&\geq p(V), \\
&= 1.
\end{aligned}$$

We conclude that g is a fractional dominating function of G .

Note that g_v is a minimum fractional dominating function of G_v , and so $g_v(N^o(v)) = \gamma^*(G_v) \leq 2\alpha(G_v) \leq 2(\alpha(G) - 1)$. Hence

$$\begin{aligned}
g(V) &= \sum_{v \in V} g(v) = 2 \sum_{v \in V} p(v) + \sum_{v \in V} \sum_{y \in N^o(v)} g_y(v)p(y) \\
&= 2 + \sum_{y \in V} p(y) \sum_{v \in N^o(y)} g_y(v) \\
&\leq 2 + \sum_{y \in V} p(y)(2\alpha(G) - 2) \\
&= 2\alpha(G).
\end{aligned}$$

Thus, $\gamma^*(G) \leq \sum_{v \in V} g(v) \leq 2\alpha(G)$. This completes the proof. \square

We can show that the bound in Theorem 2.4 is sharp.

Proposition 2.6. *Given an arbitrarily small positive real number ε , for any positive integer k , there exists a digraph G such that $\alpha(G) = k$ and $\gamma^*(G) > 2k - \varepsilon$.*

Proof. Let $r = \lceil 1/\varepsilon \rceil + 1$. For $k = 1$, let G be an Eulerian tournament with $V(G) = \{v_1, \dots, v_{2r-1}\}$ and $N^+(v_i) = \{v_j : 1 \leq j - i \pmod{2r-1} \leq r-1\}$. Let g be a

minimum fractional dominating function of G . Suppose that $g(v_r) = \min_i g(v_i)$. Note that $\sum_i g(v_i) = \gamma^*(G) \leq 2$, and r is chosen sufficiently large so that $g(v_r) < 2/(2r-1) < \varepsilon$. We have

$$\begin{aligned} \gamma^*(G) &= \sum_i g(v_i), \\ &= \sum_{1 \leq i \leq r} g(v_i) + \sum_{r \leq i \leq 2r-1} g(v_i) - g(v_r) \\ &= \sum_{v_i \in N^-[v_r]} g(v_i) + \sum_{v_i \in N^-[v_{2r-1}]} g(v_i) - g(v_r) \\ &> 2 - \varepsilon. \end{aligned}$$

For $k > 1$, let $r = \lceil k/\varepsilon \rceil + 1$. Let G be a disjoint union of k tournaments G_1, \dots, G_k , each is constructed as in the case $k = 1$. Since the k tournaments are disjoint, $\gamma^*(G) = \sum_i \gamma^*(G_i) \geq k(2 - \varepsilon/k) = 2k - \varepsilon$. \square

We can also show that almost surely a random tournament has a fractional domination number close to the upper bound of 2. First, we need the following proposition.

Proposition 2.7. *Let $G = (V, E)$ be a digraph of maximum out-degree d , then $\gamma^*(G) \geq n/(d+1)$.*

Proof. Suppose that the statement was false, then there is a function $g : V \rightarrow [0, 1]$ such that $g(V) < n/(d+1)$ and $g(N^-[x]) \geq 1$ for every x . Then

$$\begin{aligned} n &\leq \sum_{x \in V} g(N^-[x]) = g(V) + \sum_{x \in V} g(N^-(x)) \\ &= g(V) + \sum_{ux \in E} g(u) \\ &= g(V) + \sum_{u \in V} g(u) |N^+(u)| \\ &\leq g(V) + d \cdot g(V) \\ &= (d+1)g(V) < n, \end{aligned}$$

a contradiction. \square

We also need Chernoff's inequality (see for example [14]).

Proposition 2.8 (Chernoff's Inequality). *Let X be a binomial random variable consisting of n Bernoulli trials, each with probability of success p . Then, for all $0 < \epsilon < 1$,*

$$\Pr[|X - np| > \epsilon np] \leq 2e^{-\epsilon^2 np/3}.$$

Proposition 2.9. *For any $\varepsilon > 0$, $\mathbb{P}[\gamma^*(T_n) > 2 - \varepsilon] = 1 - o(1)$ (i.e., $\gamma^*(T_n) > 2 - \varepsilon$ almost surely).*

Proof. By Chernoff's bound, the probability that a given vertex has out-degree more than $n/2 + 10\sqrt{n \log n}$ is $O(n^{-C})$ for some constant $C > 1$. Thus, the probability that there is a vertex with out-degree more than $n/2 + 10\sqrt{n \log n}$ is, by the union bound, at most $n \cdot O(n^{-C}) = o(1)$. Thus, almost surely all the vertices of T_n have out-degree at most $n/2 + 10\sqrt{n \log n}$. It follows that almost surely $\gamma^*(T_n) \geq \frac{n}{n/2 + 10\sqrt{n \log n}}$. \square

3 Dominating sets in digraphs

In the previous section, we showed that the fractional domination number of a digraph can be bounded from above by twice its independence number. In general, we cannot bound the (integral) domination number of a digraph in terms of its independence number, as mentioned towards the end of the introduction section. Nevertheless, these two quantities can be related.

It is well known that a tournament has a dominating set of size at most $\log n$ [13]. Analogously, we can show that a digraph $G = (V, E)$ has a dominating set of size at most $\alpha(G) \cdot \log n$. For $S \subseteq V$, let $\chi(S)$ denote the chromatic number of the underlying (undirected) graph of digraph $G[S]$.

Lemma 3.1. *Every digraph $G = (V, E)$ with $n = |V|$ has a dominating set $D \subseteq V$ such that $\chi(D) \leq \log n$.*

Proof. We can assign each vertex a value $p(v) = 1$ and apply Lemma 2.2 to find a stable set S such that $p(N^+[S]) \geq p(V)/2$. We add the stable set S to the dominating set and recurse on the induced subgraph $G[V \setminus N^+[S]]$. Performing this routine $\log n$ times results in the bound. \square

Since each stable set S has cardinality at most $\alpha(G)$, Lemma 3.1 implies the following Theorem.

Theorem 3.2. *Every digraph $G = (V, E)$ with $n = |V|$ has a dominating set of size at most $\alpha(G) \cdot \log n$.*

When is it possible to bound the domination number of a digraph purely in terms of its independence number? For example, Theorem 3.2 implies that this can be done when the independence number of a digraph is sufficiently large.

Theorem 3.3. *For every digraph $G = (V, E)$ with $n = |V|$, if $\alpha(G) \geq \log n$, then $\gamma(G) \leq (\alpha(G))^2$.*

Another case in which the domination number of a digraph can be bounded in terms of its independence number is when the digraph is directed-triangle-free. For example, a directed-triangle-free digraph has independence number bounded by $\alpha(G) \cdot \alpha(G)!$ (see Theorem 3 in [7]). Moreover, when $\alpha(G) = 2$, this bound can be improved to $\gamma(G) \leq 3$ (see Theorem 4 in [7]).

In [8], we conjectured that the dichromatic number of a directed-triangle-free digraph can be bounded as a polynomial function of $\alpha(G)$. Let $\vec{\chi}(G)$ denote the dichromatic number of a digraph. Then for any digraph G , $\gamma(G)$ and $\vec{\chi}(G)$ can be related as follows.

Observation 3.4. For any digraph G , we have $\gamma(G) \leq \alpha(G) \cdot \bar{\chi}(G)$.

This follows from the fact that in a legal coloring, each color class forms an induced acyclic digraph and every acyclic digraph has a kernel (i.e., an independent dominating set). Thus, if the aforementioned conjecture holds, then the following must also hold.

Conjecture 3.5. There is an integer ℓ such that if G is a directed-triangle-free digraph with $\alpha(D) = \alpha$, then $\gamma(G) \leq \alpha^\ell$.

Moreover, as pointed out in [7], the best possible upper bound on the domination number of directed-triangle-free digraph G in terms of its independence number is $\gamma(G) \leq \frac{3}{2}\alpha(G)$, as demonstrated by a disjoint union of cyclically oriented pentagons.

References

- [1] N. Alon, G. Brightwell, H.A. Kierstead, A.V. Kostochka and P. Winkler, Dominating sets in k -majority tournaments, *Journal of Combinatorial Theory, Series B*, **96**, (2006), 374–387.
- [2] N. Bousquet, W. Lochet, and S. Thomassé, A proof of the Erdős-Sands-Sauer-Woodrow conjecture, *preprint*.
- [3] E. Chong and S. Žak, *An introduction to optimization*. Fourth edition. John Wiley and Sons, 2013.
- [4] M. Chudnovsky, R. Kim, C.-H. Liu, P. Seymour, and S. Thomassé, Domination in tournaments, *Journal of Combinatorial Theory, Series B*, **130**, (2018), 98–113.
- [5] D. C. Fisher, Squaring a tournament: A proof of Dean’s conjecture, *Journal of Graph Theory*, **23**(1), (1996), 43–48.
- [6] A. Gyárfás and D. Pálvölgyi, Domination in transitive colorings of tournaments, *Journal of Combinatorial Theory, Series B*, **107**, (2014), 1–11.
- [7] A. Gyárfás, G. Simonyi, and Á. Tóth, Gallai colorings and domination in multipartite digraphs, *Journal of Graph Theory*, **71**, (2012), 278–292.
- [8] A. Harutyunyan, T.-N. Le, A. Newman, and S. Thomassé, Coloring dense digraphs, *preprint*.
- [9] A. Harutyunyan, T.-N. Le, S. Thomassé, and H. Wu, Coloring tournaments: from local to global, *preprint*.
- [10] T. W. Haynes, S. Hedetniemi, P. Slater, *Fundamentals of domination in graphs*. CRC Press, 1998.
- [11] A. Lagoutte, Quasi-P versus P, *Master Thesis, ENS de Lyon*, 2012.
- [12] C. Lee, The domination number of a tournament, *Korean Journal of Mathematics*, **9**(1), (2001), 21–28.
- [13] N. Megiddo and U. Vishkin, On finding a minimum dominating set in a tournament, *Theoretical Computer Science*, **61**, (1988), 307–316.
- [14] M. Molloy and B. Reed. *Graph colouring and the probabilistic method*. Springer, 2002.