ASYMPTOTIC SCATTERING BY POISSONIAN THERMOSTATS

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Abstract. We consider an infinite chain of coupled harmonic oscillators with a Poisson thermostat at the origin. In the high frequency limit, we establish the reflection-transmission-scattering coefficients for the wave energy scattered off the thermostat. Unlike the case of the Langevin thermostat [5], in the macroscopic limit the Poissonian thermostat scattering generates a continuous cloud of waves of frequencies different from that of the incident wave.

1. Introduction

Heat reservoirs at some temperature T are usually modelled at the microscopic level by the Langevin stochastic dynamics, or by other random mechanisms such as the renewal of velocities at random times with Gaussian distributed velocities of variance T. The latter represents the interaction with an infinitely extended reservoir of independent particles in equilibrium at temperature T and uniform density.

When such reservoirs are in contact with the system boundary and if energy diffuses or superdiffuse on the macroscopic space-time scale, then it is expected that the thermostat enforces a local equilibrium at the boundary at the temperature T. The situation is much less clear for the kinetic (hyperbolic) space-time scales. For instance, if the bulk evolution is governed by a discrete nonlinear wave equation, then in the kinetic (high frequency-small non-linearity) limit the wave energy density is expected to be governed by a phonon Boltzmann equation [1, 8]. In this limit the thermostat generates non-trivial boundary scattering conditions.

In the present paper we consider an infinite one-dimensional chain of harmonic oscillators, where particles are labelled by the elements of the integer lattice $\mathbb{Z}$, characterized by its dispersion relation $\omega(k)$ (cf. (2.12)). The chain is coupled with a single thermostat acting on the particle labelled 0. The thermostat is modelled by a random mechanism depending on two parameters: $\gamma > 0$, describing its strength, and $\mu \ge 1/2$, whose role is more technical as it decribes an interpolation between Poisson and Gaussian mechanisms. At random times determined by a Poisson process of intensity $\gamma\mu$, the velocity p_0 of the particle 0 is changed to

$$p_0' = \left(1 - \frac{1}{\mu}\right) p_0 + \frac{\sqrt{2\mu - 1}}{\mu} \tilde{p},$$

where $\tilde{p}$ is a centered Gaussian random variable with variance T (the temperature of the thermostat). The case $\mu = 1/2$ corresponds to a velocity flip from $p_0 \mapsto -p_0$ at Poisson random times, $\mu = 1$ ensures complete renewal of p_0, replacing it at those times by a $\mathcal{N}(0, T)$ random variable $\tilde{p}$. Letting $\mu \to \infty$ the process described in the foregoing converges to the Langevin thermostat considered in [5](cf. (2.10)). In this sense the parameter μ allows to interpolate between various models of thermostats: starting from the random flip process ($\mu = 1/2$), through the simple complete Poisson renewal ($\mu = 1$) and ending up at the Langevin thermostat ($\mu = +\infty$).

In the case $\mu = 1/2$ (the random velocity flip) the energy of the chain is conserved and there is no thermalization. On the other hand, when $\mu > 1/2$ the Gaussian distribution $\mathcal{N}(0,T)$ is the only stationary measure that is asymptotically stable for the process associated with the thermostat and the thermalization of the chain at temperature T occurs.

An efficient way to localize the distribution of the energy both at the wave number k, belonging to the unit torus $\mathbb{T}$, and spatial location x is to use the Wigner distribution. In the space-time hyperbolic scaling, ignoring at first the thermostat, the Wigner distribution converges to the solution $W(t,x,k)$ of a simple transport equation, namely phonons of wavenumber k travel independently with the group velocity $\omega'(k)/2\pi$. Taking into account the presence of the thermostat the respective limit, see (2.51) below, can be decomposed into the parts that, besides the aforementioned free energy transport, correspond to the production, absorption, scattering, transmission and reflection of a phonon. More precisely, we show that when the dispersion relation is unimodal, see Section 2 for a precise definition, in the scaling limit, the thermostat at temperature $T > 0$ and corresponding to $\mu \ge 1/2$ enforces the following reflection-transmission (and production) conditions at $x = 0$: phonons of wave number ℓ are generated at the rate $p_{\rm abs}\,\mathfrak{g}(\ell)T$ and an incoming ℓ-phonon, arriving with velocity $\bar\omega'(\ell)$, is transmitted with probability $p_+(\ell)$, reflected with probability $p_-(\ell)$, scattered, as an k-phonon, with the outgoing velocity $\bar\omega'(k)$ according to the scattering kernel $\mathfrak{g}(\ell)p_{\rm sc}(k)$ and absorbed with probability $p_{\rm abs}\,\mathfrak{g}(\ell)$, see formulas (2.41) below. These coefficients are non-negative, depend on $\omega(\cdot)$, the parameters $\gamma > 0$ and $\mu \ge 1/2$, and satisfy

$$p_+(\ell) + p_-(\ell) + p_{\rm abs}\,\mathfrak{g}(\ell) + \mathfrak{g}(\ell)\int_{\mathbb{T}} p_{\rm sc}(k)dk = 1, \quad \ell \in \mathbb{T}.$$

Coefficients $p_\pm(\ell), \mathfrak{g}(\ell)$ do not depend on μ. The coefficient $p_{\rm abs}$ is independent of ℓ and for $\mu \to +\infty$, $p_{\rm abs} \to 1$ and $p_{\rm sc}(k) \to 0$. With such boundary conditions the thermal equilibrium Wigner function $W(t,x,k) = T$ is a stationary solution of the transport equation for any $\mu > 1/2$.

Our result covers also the random flip of sign of p_0, i.e. $\mu = 1/2$. In this case there is no absorbtion of phonons: $p_{\rm abs} = 0$, and $\int_{\mathbb{T}} p_{\rm sc}(k)dk = 1$, i.e. all the energy that is not transmitted or reflected at the same frequency is scattered at various frequencies.

The thermostat corresponding to a finite value of μ can be therefore viewed as a “scatterer" of a time-varying strength: at the macroscopic scale a wave incident on the thermostat would produce reflected and transmitted waves at all frequencies. This is in constrast with the case of the Langevin thermostat ($\mu = +\infty$) considered in [5], where, after the scaling limit, the reflected and transmitted waves are of the same frequency as the incident wave ($p_{\rm sc}(k) = 0$).

Similarly to [5] the presence of oscillatory integrals, responsible for the damping mechanism, presents the difficulty of the model and is dealt with using the Laplace transform of the Wigner distribution. An additional difficulty lies in the fact that, contrary to [5], the noise appearing in the dynamics (2.11) is multiplicative (rather than additive as in ibid.), which makes the computations much less explicit.

Introducing a rarefied random scattering in the bulk, in the same fashion as in [1], should lead to a similar transport equation with a linear scattering term, without modifying the conditions at the interface with the thermostat. Analogous case for the Langevin thermostat has been considered in [4].

Acknowledgement. TK was partially supported by the NCN grant 2016/23/B/ST1/00492, SO by the French Agence Nationale Recherche grant LSD ANR-15-CE40-0020-01.

2. Preliminaries and formulation of the main result

2.1. **Notation.** We use the notation $\mathbb{T}_a = [-a/2, a/2]$ for the torus of size $a > 0$, with identified endpoints. In particular for $a = 1$ we write $\mathbb{T}$ instead of $\mathbb{T}_1$.

The Fourier transform of a square integrable sequence (α_x) and the inverse Fourier transform of $\hat{\alpha} \in L^2(\mathbb{T})$ are defined as

$$\hat{\alpha}(k) = \sum_{x\in\mathbb{Z}} \alpha_x \exp\{-2\pi ixk\}, \ \alpha_x = \int_{\mathbb{T}} \hat{\alpha}(k) \exp\{2\pi ixk\} dk, \quad x \in \mathbb{Z}, \ k \in \mathbb{T}. \tag{2.1}$$

Suppose that $f, g \in L^1[0, +\infty)$. Their convolution, also belonging to $L^1[0, +\infty)$, is given by

$$f \star g(t) := \int_0^t f(t-s)g(s)ds, \quad t \in [0, +\infty)$$

By $f^{\star,k}$ we denote the k-times convolution of f with itself, i.e. $f^{\star,1} := f$, $f^{\star,k+1} := f \star f^{\star,k}$, $k \ge 1$. We let $f^{\star,0} \star g := g$. We denote by

$$\tilde{f}(\lambda) = \int_0^{+\infty} e^{-\lambda t} f(t)dt, \quad \operatorname{Re}\lambda > 0,$$

the Laplace transform of f.

For a function $G(x,k)$, we denote by $\tilde{G} : \mathbb{R}\times\mathbb{Z} \to \mathbb{C}$, $\hat{G} : \mathbb{R}\times\mathbb{T} \to \mathbb{C}$ the Fourier transforms of G in the k and x variables, respectively,

$$\tilde{G}(x,y) := \int_{\mathbb{T}} e^{-2\pi iky} G(x,k)dk, \quad (x,y) \in \mathbb{R}\times\mathbb{Z},$$
$$\hat{G}(\eta,k) := \int_{\mathbb{R}} e^{-2\pi i\eta x} G(x,k)dx, \quad (\eta,k) \in \mathbb{R}\times\mathbb{T}.$$

Let us denote by $\mathcal{A}$ the Banach space obtained as the completion of $\mathcal{S}(\mathbb{R}\times\mathbb{T})$ in the norm

$$\|G\|_{\mathcal{A}} := \int_{\mathbb{R}} \sup_{k\in\mathbb{T}} |\hat{G}(\eta,k)| d\eta \tag{2.2}$$

and by $\mathcal{A}'$ its dual.

2.2. **The infinite chain of harmonic oscillators.** We consider the evolution of an infinite particle system governed by the Hamiltonian

$$\mathcal{H}(\mathfrak{p},\mathfrak{q}) := \frac{1}{2}\sum_{y\in\mathbb{Z}} \mathfrak{p}_y^2 + \frac{1}{2}\sum_{y,y'\in\mathbb{Z}} \alpha_{y-y'}\mathfrak{q}_y\mathfrak{q}_{y'}. \tag{2.3}$$

Here, the particle label is $y \in \mathbb{Z}$, $(\mathfrak{p}_y, \mathfrak{q}_y)$ is the position and momentum of the y's particle, respectively, and $(\mathfrak{q},\mathfrak{p}) = \{(\mathfrak{p}_y,\mathfrak{q}_y),\ y \in \mathbb{Z}\}$ denotes the entire configuration of which we assume that is real valued and square summable. The coupling coefficients α_y are assumed to have exponential decay and chosen such that the energy is bounded from below.

2.3. **Poisson type thermostat.** The stochastic process describing a thermostat is a jump process, whose generator is given by

$$L_{\mu,\gamma}f(\mathfrak{p}) := \frac{\gamma\mu}{\sqrt{2\pi T}}\int_{\mathbb{R}}\left[f\left(\left(1-\frac{1}{\mu}\right)\mathfrak{p}+\rho(\mu)\tilde{\mathfrak{p}}\right)-f(\mathfrak{p})\right]\exp\left\{-\frac{\tilde{\mathfrak{p}}^2}{2T}\right\}d\tilde{\mathfrak{p}}, \quad f\in B_b(\mathbb{R}). \tag{2.4}$$

Here $B_b(\mathbb{R})$ denotes the space of all bounded and Borel measurable functions, $T,\gamma>0$, $\mu\ge 1/2$ and

$$\rho(\mu) := \frac{\sqrt{2\mu-1}}{\mu}. \tag{2.5}$$

It is easy to verify that the Gaussian measure $\mathcal{N}(0,T)$ is invariant under the dynamics of the process. In the case $\mu=1/2$ Gaussian measure $\mathcal{N}(0,T')$ is invariant for any $T'\ge 0$.

The process $(\mathfrak{p}_t)_{t\ge0}$ can be also described using the Itô stochastic differential equation, with a noise corresponding to a Poisson jump process, see e.g. [7, Chapter V],

$$\begin{aligned} d\mathfrak{p}(t) &= \left(\tilde{p}(t-)-\frac{1}{\mu}\mathfrak{p}(t-)\right)dN(\gamma\mu t), \quad t\ge0,\\ \mathfrak{p}(0) &= \tilde{p}_0. \end{aligned} \tag{2.6}$$

Here $(N(t))_{t\ge0}$ is a Poisson process of intensity 1 defined over some probability space $(\Omega,\mathcal{F},\mathbb{P})$ and $(\tilde{p}(t))_{t\ge0}$ is given by

$$\tilde{p}(t) := \rho(\mu)\tilde{p}_{N'(\gamma\mu t)}, \tag{2.7}$$

where $N'(t)=N(t)+1$. We suppose that $(\tilde{p}_j)_{j\ge0}$ are i.i.d. $\mathcal{N}(0,T)$ random variables over $(\Omega,\mathcal{F},\mathbb{P})$.

The process $(\tilde{p}(t))_{t\ge0}$ is Levy stationary and

$$\begin{aligned} &\mathbb{E}\tilde{p}(t)=0,\\ &\mathbb{E}\left[\tilde{p}(t)\tilde{p}(t')\right]=\frac{2\mu-1}{\mu^2}e^{-\gamma\mu|t-t'|}T, \quad t,t'\ge0. \end{aligned} \tag{2.8}$$

From equation (2.6) we can see that in case $\mu=1$ we have $\tilde{p}(t)=\tilde{p}_{N'(\gamma t)}$, $t\ge0$. On the other hand, after a simple calculation, from (2.4), we conclude that for any $f\in C^2(\mathbb{R})$

$$\lim_{\mu\to+\infty}L_{\mu,\gamma}f(\mathfrak{p})=L_{\infty,\gamma}f(\mathfrak{p})=\gamma T\exp\left\{\frac{\mathfrak{p}^2}{2T}\right\}\frac{d}{d\mathfrak{p}}\left(\exp\left\{-\frac{\mathfrak{p}^2}{2T}\right\}\frac{df(\mathfrak{p})}{d\mathfrak{p}}\right). \tag{2.9}$$

The termostat correspong to $\mu=+\infty$ can be therefore identified with the Langevin thermostat at temperature T, whose dynamics is described by the Itô stochastic differential equation, with an additive Gaussian white noise $dw(t)$:

$$\begin{aligned} d\mathfrak{p}(t) &= -\gamma\mathfrak{p}(t)dt+\sqrt{2\gamma T}dw(t), \quad t\ge0,\\ \mathfrak{p}(0) &= \tilde{p}_0. \end{aligned} \tag{2.10}$$

2.4. **Harmonic chain coupled with a point thermostat.** We couple the particle with label $y=0$ with a thermostat described in Section 2.3. Then the Hamiltonian dynamics with stochastic source is governed by

$$\begin{aligned} &\dot{\mathfrak{q}}_y(t)=\mathfrak{p}_y(t), \\ &d\mathfrak{p}_y(t)=-(\alpha\star\mathfrak{q}(t))_y dt+\delta_{0,y}\left(\rho(\mu)\tilde{p}_{N(\gamma\mu t)}-\frac{1}{\mu}\mathfrak{p}_y(t-)\right)dN(\gamma\mu t), \quad y\in\mathbb{Z}. \end{aligned} \tag{2.11}$$

We use the notation

$$(f\star g)_y=\sum_{y'\in\mathbb{Z}}f_{y-y'}g_{y'}$$

for the convolution of two functions on $\mathbb{Z}$. The coupling constants $\alpha = (\alpha_y)$ are even $\alpha_{-y} = \alpha_y$, $y \in \mathbb{Z}$, decay exponentially and

$$\hat{\alpha}(k) := \sum_y \alpha_y \exp\{-2\pi i k y\} > 0, \quad k \in \mathbb{T}_* := \mathbb{T} \smallsetminus \{0\}.$$

2.4.1. *Assumptions on the dispersion relation and its basic properties.* We assume, as in [5], that α_y is a real-valued even function of $y \in \mathbb{Z}$, and there exists $C > 0$ so that

$$|\alpha_y| \le C e^{-|y|/C}, \text{ for all } y \in \mathbb{Z},$$

thus $\hat{\alpha} \in C^\infty(\mathbb{T})$. Furthermore, we suppose that the dispersion relation

$$\omega(k) := \sqrt{\hat{\alpha}(k)}, \quad k \in \mathbb{T}, \tag{2.12}$$

is a continuous function on $\mathbb{T}$ belonging to $C^2(\mathbb{T} \smallsetminus \{0\})$, with two derivatives possessing one sided limits at $k = 0$. The typical examples are the *acoustic chains* where $\omega(k) \sim |k|$ for $k \sim 0$, and the *optical chains* where $\omega'(k) \sim k$ for $k \sim 0$. In general, we assume that ω is unimodal, i.e. it is increasing on $[0, 1/2]$. Denote its unique minimum attained at $k = 0$ by $\omega_{\min} \ge 0$, its unique maximum, attained at $k = 1/2$, by $\omega_{\max}$, and the two branches of the inverse of $\omega(\cdot)$ as $\omega_- : [\omega_{\min}, \omega_{\max}] \to [-1/2, 0]$ and $\omega_+ : [\omega_{\min}, \omega_{\max}] \to [0, 1/2]$. They satisfy $\omega_- = -\omega_+$, $\omega_+(\omega_{\min}) = 0$, $\omega_+(\omega_{\max}) = 1/2$ and in the case $\omega \in C^\infty(\mathbb{T})$:

$$\omega_\pm'(w) = \pm(w - \omega_{\min})^{-1/2} \rho_*(w), \; w - \omega_{\min} \ll 1, \tag{2.13}$$

and

$$\omega_\pm'(w) = \pm(\omega_{\max} - w)^{-1/2} \rho^*(w), \; \omega_{\max} - w \ll 1, \tag{2.14}$$

with $\rho_*, \rho^* \in C^\infty(\mathbb{T})$ that are strictly positive. When ω is not differentiable at 0 (the acoustic case) instead of (2.13) we assume

$$\omega_\pm'(w) = \pm \rho_*(w), \; w - \omega_{\min} \ll 1, \tag{2.15}$$

leaving condition (2.14) unchanged.

2.4.2. *The wave-function.* It is convenient to introduce the complex wave function

$$\psi_y(t) := (\tilde{\omega} \star \mathfrak{q}(t))_y + i \mathfrak{p}_y(t) \tag{2.16}$$

where $\{\tilde{\omega}_y, \, y \in \mathbb{Z}\}$ is the inverse Fourier transform of the dispersion relation $\omega(k)$.

Hence $|\psi_y(t)|^2$ is the local energy of the chain at time t. The Fourier transform of the wave function is given by

$$\hat{\psi}(t,k) := \omega(k) \hat{\mathfrak{q}}(t,k) + i \hat{\mathfrak{p}}(t,k), \tag{2.17}$$

so that

$$\hat{\mathfrak{p}}(t,k) = \frac{1}{2i} [\hat{\psi}(t,k) - \hat{\psi}^*(t,-k)], \; \mathfrak{p}_0(t) = \int_{\mathbb{T}} \operatorname{Im} \hat{\psi}(t,k) dk.$$

Using (2.11), it is easy to check that the wave function evolves according to

$$d\hat{\psi}(t,k) = -i\omega(k) \hat{\psi}(t,k) dt + i \left(\tilde{p}(t-) - \frac{1}{\mu} \mathfrak{p}_0(t-) \right) dN(\gamma \mu t). \tag{2.18}$$

2.4.3. *The initial conditions.* For simplicity sake we restrict ourselves to initial configurations of finite energy. In addition, we assume that the initial energy density $|\psi_y|^2$ is finite per unit length on the macroscopic scale $x \sim \varepsilon y$, where $\varepsilon > 0$ is the scaling parameter. More precisely, given $\varepsilon > 0$, the initial wave function is distributed randomly, according to a probability measure μ_ε, and

$$\sup_{\varepsilon\in(0,1)} \sum_{y\in\mathbb{Z}} \varepsilon\langle|\psi_y|^2\rangle_{\mu_\varepsilon} = \sup_{\varepsilon\in(0,1)} \varepsilon\langle\|\hat{\psi}\|^2_{L^2(\mathbb{T})}\rangle_{\mu_\varepsilon} < \infty, \tag{2.19}$$

where $\langle\cdot\rangle_{\mu_\varepsilon}$ denotes the expectation with respect to μ_ε. We will also assume that

$$\langle\hat{\psi}(k)\hat{\psi}(\ell)\rangle_{\mu_\varepsilon} = 0, \quad k,\ell\in\mathbb{T}, \tag{2.20}$$

Condition (2.20) can be replaced by $\langle\hat{\psi}(k)\hat{\psi}(\ell)\rangle_{\mu_\varepsilon} \sim 0$, as $\varepsilon\to 0$ at the expense of some additional calculations that we prefer not to perform in this article.

An additional hypothesis concerning the distribution of the initial configuration will be stated later on, see (2.26).

2.4.4. *The Wigner distributions.* To study the effect of the thermostat, we follow the evolution of the chain on the macroscopic time scale $t' \sim \varepsilon t$, and consider the rescaled wave function $\psi_y^{(\varepsilon)}(t) = \psi_y(t/\varepsilon)$, with its Fourier transform $\hat{\psi}^{(\varepsilon)}(t,k)$. A convenient tool to analyse the energy density are the Wigner distributions $W_\pm^{(\varepsilon)}(t)$ and $Y_\pm^{(\varepsilon)}(t)$ defined by its action on a test function $G\in\mathcal{S}(\mathbb{R}\times\mathbb{T})$ as

$$\begin{aligned}\langle G, W_\pm^{(\varepsilon)}(t)\rangle &= \int_{\mathbb{T}\times\mathbb{R}} \widehat{W}_{\varepsilon,\pm}(t,\eta,k)\hat{G}^*(\eta,k)d\eta dk,\\ \langle G, Y_\pm^{(\varepsilon)}(t)\rangle &= \int_{\mathbb{T}\times\mathbb{R}} \widehat{Y}_{\varepsilon,\pm}(t,\eta,k)\hat{G}^*(\eta,k)d\eta dk, \quad G\in\mathcal{S}(\mathbb{R}\times\mathbb{T}).\end{aligned} \tag{2.21}$$

Here, $\mathbb{E}$ is the expectation with respect to the product measure $\mu_\varepsilon\otimes\mathbb{P}$ and

$$\begin{aligned}\widehat{W}_{\varepsilon,\pm}(t,\eta,k) &:= \frac{\varepsilon}{2}\mathbb{E}\left[\left(\hat{\psi}^{(\varepsilon)}\right)^*\left(t,\pm k-\frac{\varepsilon\eta}{2}\right)\hat{\psi}^{(\varepsilon)}\left(t,\pm k+\frac{\varepsilon\eta}{2}\right)\right],\\ \widehat{Y}_{\varepsilon,+}(t,\eta,k) &:= \frac{\varepsilon}{2}\mathbb{E}\left[\hat{\psi}^{(\varepsilon)}\left(t,k+\frac{\varepsilon\eta}{2}\right)\hat{\psi}^{(\varepsilon)}\left(t,-k+\frac{\varepsilon\eta}{2}\right)\right],\\ \widehat{Y}_{\varepsilon,-}(t,\eta,k) &:= \frac{\varepsilon}{2}\mathbb{E}\left[\left(\hat{\psi}^{(\varepsilon)}\right)^\star\left(t,k-\frac{\varepsilon\eta}{2}\right)\left(\hat{\psi}^{(\varepsilon)}\right)^\star\left(t,-k-\frac{\varepsilon\eta}{2}\right)\right], \quad (\eta,k)\in\mathbb{T}_{2/\varepsilon}\times\mathbb{T}.\end{aligned} \tag{2.22}$$

are the (averaged) Fourier-Wigner functions.

To simplify the notation we shall also write $\widehat{W}_\varepsilon(t,\eta,k)$ instead of $\widehat{W}_{\varepsilon,+}(t,\eta,k)$.

A straightforward calculation shows that the macroscopic energy grows at most linearly in time. More precisely, using (2.18), we obtain

$$\frac{d}{dt}\int_{\mathbb{T}}\mathbb{E}\left|\hat{\psi}^{(\varepsilon)}(t,k)\right|^2 dk = \frac{\gamma}{\varepsilon}\left(2-\frac{1}{\mu}\right)\left(T-\mathbb{E}[\mathfrak{p}_0^{(\varepsilon)}(t)]^2\right)$$

with $\mathfrak{p}_0^{(\varepsilon)}(t) := \mathfrak{p}_0(t/\varepsilon)$. As a result we get

$$\varepsilon\int_{\mathbb{T}}\mathbb{E}\left|\hat{\psi}^{(\varepsilon)}(t,k)\right|^2 dk \le \varepsilon\int_{\mathbb{T}}\mathbb{E}\left|\hat{\psi}^{(\varepsilon)}(0,k)\right|^2 dk + \left(2-\frac{1}{\mu}\right)\gamma T t, \quad t\ge 0. \tag{2.23}$$

In particular, we conclude from (2.23) that (see [2])

$$\sup_{t\in[0,\tau]}\|W^{(\varepsilon)}(t)\|_{\mathcal{A}'} < \infty, \text{ for each } \tau > 0. \tag{2.24}$$

Hence $W^{(\varepsilon)}(\cdot)$ is sequentially weak-$\star$ compact over $(L^1([0,\tau];\mathcal{A}))^\star$ for any $\tau > 0$.

We will assume that the initial Wigner distribution

$$\widehat{W}_\varepsilon(\eta,k) := \widehat{W}_\varepsilon(0,\eta,k), \quad (\eta,k)\in\mathbb{T}_{2/\varepsilon}\times\mathbb{T} \tag{2.25}$$

is a family that converges weakly in $\mathcal{A}'$ to a non-negative function $W_0 \in L^1(\mathbb{R}\times\mathbb{T})\cap C(\mathbb{R}\times\mathbb{T})$. We will also assume that there exist $C,\kappa>0$ such that

$$|\widehat{W}_\varepsilon(\eta,k)| \le C\varphi(\eta), \quad (\eta,k)\in\mathbb{T}_{2/\varepsilon}\times\mathbb{T},\ \varepsilon\in(0,1], \tag{2.26}$$

where

$$\varphi(\eta) := \frac{1}{(1+\eta^2)^{3/2+\kappa}}. \tag{2.27}$$

Define the Fourier-Laplace-Wigner functions

$$\begin{aligned} \widehat{w}_{\pm,\varepsilon}(\lambda,\eta,k) &= \varepsilon\int_0^{+\infty} e^{-\lambda\varepsilon t}\widehat{W}_{\pm,\varepsilon}(t,\eta,k)dt, \\ \widehat{y}_{\pm,\varepsilon}(\lambda,\eta,k) &= \varepsilon\int_0^{+\infty} e^{-\lambda\varepsilon t}\widehat{Y}_{\pm,\varepsilon}(t,\eta,k)dt, \end{aligned} \tag{2.28}$$

where $\operatorname{Re}\lambda>0$, $(\eta,k)\in\mathbb{T}_{2/\varepsilon}\times\mathbb{T}$. We shall also write $\widehat{w}_\varepsilon(\lambda,\eta,k)$ instead of $\widehat{w}_{+,\varepsilon}(\lambda,\eta,k)$.

2.5. Some additional notation.

Define

$$J(t) = \int_{\mathbb{T}} \cos(\omega(k)t)\,dk, \quad t\in\mathbb{R}. \tag{2.29}$$

Its Laplace transform

$$\tilde{J}(\lambda) := \int_0^\infty e^{-\lambda t}J(t)dt = \int_{\mathbb{T}}\frac{\lambda}{\lambda^2+\omega^2(k)}dk, \quad \operatorname{Re}\lambda>0. \tag{2.30}$$

Let

$$\tilde{g}(\lambda) := (1+\gamma\tilde{J}(\lambda))^{-1}. \tag{2.31}$$

Note that $\operatorname{Re}\tilde{J}(\lambda)>0$ for $\lambda\in\mathbb{C}_+ := [\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0]$, therefore

$$|\tilde{g}(\lambda)|\le 1, \quad \lambda\in\mathbb{C}_+. \tag{2.32}$$

In addition, we have

$$(\tilde{g}\tilde{J})(\lambda) = \frac{1}{\gamma}(1-\tilde{g}(\lambda)) = \frac{\tilde{J}(\lambda)}{1+\gamma\tilde{J}(\lambda)} = \sum_{n=1}^{+\infty}(-\gamma)^{n-1}\tilde{J}(\lambda)^n, \quad \lambda\in\mathbb{C}_+. \tag{2.33}$$

Since $|J(t)|\le 1$ we have $|J^{\star,n}(t)|\le t^{n-1}/(n-1)!$, as the n-th convolution power involves the integration over an $n-1$-dimensional simplex of size t. One can show therefore, see [5, (3.12) and (3.13)], that the series

$$g_*(t) := \sum_{n=1}^{+\infty}(-\gamma)^n J^{\star,n}(t) \tag{2.34}$$

defines a C^∞ class function on $[0,+\infty)$ that satisfies the following growth condition: for any $\rho>0$ there exists $C>0$ such that $|g_*(t)|\le Ce^{\rho t}$, $t>0$. In addition

$$\tilde{g}_*(\lambda) = \tilde{g}(\lambda) - 1. \tag{2.35}$$

Therefore $\tilde{g}(\lambda)$, given by (2.31), is the Laplace transform of the signed measure $g(dt) := \delta_0(dt) + g_*(t)dt$. Combining (2.31), (2.34) and (2.35) we obtain

$$\gamma J\star g(t) = \sum_{n=1}^{+\infty}(-1)^{n-1}\gamma^n J^{\star,n}(t), \quad t\ge 0. \tag{2.36}$$

2.6. **Properties of functions $\tilde{g}$ and $\tilde{J}$.** The function $\tilde{g}(\cdot)$ is analytic on $\mathbb{C}_+$ so, by the Fatou theorem, see e.g. p. 107 of [6], we know that

$$\tilde{g}(i\beta) = \lim_{\varepsilon\to 0} \tilde{g}(\varepsilon + i\beta), \quad \beta \in \mathbb{R} \tag{2.37}$$

exists a.e. In Section 6.1 we show the following.

Lemma 2.1. *The holomorphic function $\tilde{J}\tilde{g}$ belongs to the Hardy space $H^p(\mathbb{C}_+)$ for any $p \in (1,+\infty)$. The limit*

$$(\tilde{J}\tilde{g})(i\beta) = \lim_{\varepsilon\to 0+} (\tilde{J}\tilde{g})(\varepsilon + i\beta), \quad \beta \in \mathbb{R} \tag{2.38}$$

exists both a.e. and in the $L^p(\mathbb{R})$ sense for $p \in (1,+\infty)$.

In addition, there exists

$$\nu(k) := \lim_{\varepsilon\to 0} \tilde{g}\big(\varepsilon + i\omega(k)\big), \quad k \in \Omega_*, \tag{2.39}$$

where $\Omega_* := [k \in \mathbb{T} : \omega'(k) = 0, \quad or \quad \omega(k) = 0]$. The function is continuous on $\mathbb{T} \setminus \Omega_*$. Moreover, for any $\delta > 0$ there exists $C > 0$ such that

$$\Big|\tilde{g}\big(\varepsilon + i\omega(k)\big) - \nu(k)\Big| \le C\varepsilon, \quad \operatorname{dist}\big(k, \Omega_*\big) \ge \delta. \tag{2.40}$$

To state our main result we need some additional notation. Define the group velocity

$$\bar{\omega}'(k) := \omega'(k)/(2\pi)$$

and

$$\wp(k) := \frac{\gamma\nu(k)}{2|\bar{\omega}'(k)|}, \quad \mathfrak{g}(k) := \frac{\gamma|\nu(k)|^2}{|\bar{\omega}'(k)|}, \quad p_+(k) := |1 - \wp(k)|^2, \quad p_-(k) := |\wp(k)|^2. \tag{2.41}$$

It has been shown in Section 10 of [5] that

$$\operatorname{Re}\nu(k) = \left(1 + \frac{\gamma}{2|\bar{\omega}'(k)|}\right)|\nu(k)|^2 \tag{2.42}$$

and

$$p_+(k) + p_-(k) = 1 - \mathfrak{g}(k) \le 1, \tag{2.43}$$

so that, in particular, we have

$$0 \le \mathfrak{g}(k) \le 1, \quad k \in \mathbb{T}. \tag{2.44}$$

In the model considered in [5] the coefficients $p_+(k)$, $p_-(k)$ and $\mathfrak{g}(k)$ have expressed, see [5, Theorem 2.1], the probabilities of a phonon being transmitted, reflected and absorbed at the interface $[x = 0]$.

In our present situation the absorption probability needs to be modified. In addition, the phonon can be also scattered at the interface with outgoing frequency ℓ with some scattering rate $r(k,\ell)$. To be more precise we introduce the following notation

$$p_{\rm abs} := \frac{1}{1 - \Gamma/\mu}\left(1 - \frac{1}{2\mu}\right), \quad p_{\rm sc}(\ell) := \frac{1}{2\mu(1 - \Gamma/\mu)}|\nu(\ell)|^2, \tag{2.45}$$

where

$$\Gamma := \frac{\gamma}{2\pi}\int_{\mathbb{R}} |\tilde{J}\tilde{g}(i\beta')|^2 d\beta'. \tag{2.46}$$

The following result holds.

Lemma 2.2. *For any $\gamma > 0$ we have*

$$\Gamma + \frac{1}{2}\int_{\mathbb{T}} |\nu(\ell)|^2 d\ell = \frac{1}{2}. \tag{2.47}$$

In addition, if $\mu \geq 1/2$, then

$$p_{\rm abs} + \int_{\mathbb{T}} p_{\rm sc}(\ell) d\ell = 1. \tag{2.48}$$

The proof of the lemma is contained in Section 6.2.

Remark 2.3. It turns out, see [3, Theorem 4. part iii)], that for any unimodal dispersion relation we have $|\nu(\ell)| > 0$, except possibly $\ell = 0$, or $1/2$. Thanks to the identity (2.47) below, we have then

$$\Gamma < \frac{1}{2} \leq \mu. \tag{2.49}$$

Therefore, in particular, the coefficients defined in (2.45) are strictly positive for $\mu > 1/2$ and $\ell \notin \{0, 1/2\}$.

2.7. **The main result.** Our main result is as follows. For brevity, we use the notation

$$[[0,a]] := \begin{cases} [0,a], & \text{if } a > 0 \\ [a,0], & \text{if } a < 0. \end{cases}$$

Theorem 2.4. *Suppose that the initial conditions and the dispersion relation satisfy the above assumptions. Then, for any $\tau > 0$ and $G \in L^1([0,\tau];\mathcal{A})$ we have*

$$\lim_{\varepsilon\to 0}\int_0^\tau \langle G(t), W_\varepsilon(t)\rangle dt = \int_0^\tau dt \int_{\mathbb{R}\times\mathbb{T}} G^*(t,x,k) W(t,x,k) dx dk, \tag{2.50}$$

where

$$\begin{aligned} W(t,x,k) &= W_0(x - \bar\omega'(k)t, k)\, 1_{[[0,\bar\omega'(k)t]]^c}(x) + p_+(k) W_0(x - \bar\omega'(k)t, k)\, 1_{[[0,\bar\omega'(k)t]]}(x) \\ &+ p_-(k) W_0(-x + \bar\omega'(k)t, -k)\, 1_{[[0,\bar\omega'(k)t]]}(x) \\ &+ \mathfrak{g}(k) 1_{[[0,\bar\omega'(k)t]]}(x) \int_{\mathbb{T}} W_0\left(\frac{\bar\omega'(\ell)}{\bar\omega'(k)}(x - \bar\omega'(k)t), \ell\right) p_{\rm sc}(\ell) d\ell \\ &+ p_{\rm abs}\, \mathfrak{g}(k) T 1_{[[0,\bar\omega'(k)t]]}(x). \end{aligned} \tag{2.51}$$

The proof of this result is given in Section 5.4.

The limit dynamics has an obvious interpretation. Namely, $W(t,x,k)$ describes the energy density in (x,k) at time t of the phonons initially distributed according to $W_0(x,k)$. The first term corresponds then to the ballistic transport of those phonons which did not cross $\{x = 0\}$ up to time t. The second and third terms correspond, respectively, to the transmission and reflection of the phonons at the boundary point $\{x = 0\}$ with probabilities $p_+(k)$ and $p_-(k)$, respectively. The fourth term describes the phonon scattering that occurs at the interface. The phonon with frequency ℓ, arriving at the interface with the velocity $\bar\omega'(\ell)$ is scattered with frequency k at the rate $\mathfrak{g}(\ell) p_{\rm sc}(k)$ and moves away from the interface with the velocity $\bar\omega'(k)$. Finally, the last term in the right side of (2.51) describes the k-phonon production of the thermostat at the rate $p_{\rm abs}\, \mathfrak{g}(k) T$. From (2.43) and (2.48) we conclude that

$$1 - p_+(\ell) - p_-(\ell) - \int_{\mathbb{T}} \mathfrak{g}(\ell) p_{\rm sc}(k) dk = p_{\rm abs}\, \mathfrak{g}(\ell), \quad \ell \in \mathbb{T}. \tag{2.52}$$

Therefore, the ℓ-phonon is absorbed by the thermostat with probability $p_{\rm abs}\, \mathfrak{g}(\ell)$.

Our result can be written as a boundary value problem. Note that $W(t,x,k)$ solves the homogeneous transport equation

$$\partial_t W(t,x,k) + \bar\omega'(k)\partial_x W(t,x,k) = 0, \tag{2.53}$$

away from the boundary point $[x=0]$.

At the boundary $[x=0]$ the outgoing phonons are related to the incoming phonons as follows. Let

$$W(t,0^\pm,k) := \lim_{x\to\pm 0} W(t,x,k).$$

If $k\in\mathbb{T}_+$, then

$$\begin{aligned} W(t,0^+,k) &= p_+(k)W(t,0^-,k) + p_-(k)W(t,0^+,-k) + p_{\rm abs}\,\mathfrak{g}(k)T \\ &+ \mathfrak{g}(k)\int_{\mathbb{T}_+} W(t,0^-,\ell)\,p_{\rm sc}(\ell)d\ell + \mathfrak{g}(k)\int_{\mathbb{T}_+} W(t,0^+,-\ell)\,p_{\rm sc}(\ell)d\ell. \end{aligned} \tag{2.54}$$

If, on the other hand, $k\in\mathbb{T}_-$, then

$$\begin{aligned} W(t,0^-,k) &= p_+(k)W(t,0^+,k) + p_-(k)W(t,0^-,-k) + p_{\rm abs}\,\mathfrak{g}(k)T \\ &+ \mathfrak{g}(k)\int_{\mathbb{T}_-} W(t,0^+,\ell)\,p_{\rm sc}(\ell)d\ell + \mathfrak{g}(k)\int_{\mathbb{T}_-} W(t,0^-,-\ell)\,p_{\rm sc}(\ell)d\ell. \end{aligned}$$

3. The solution of (2.18) and its Laplace-Fourier-Wigner distribution

In this section, we obtain an explicit expression for the solution of the wave function (2.18). The mild formulation of the equation reads as follows

$$\begin{aligned} \hat\psi(t,k) &= e^{-i\omega(k)t}\hat\psi(k) - \frac{i}{\mu}\int_0^t e^{-i\omega(k)(t-s)}\mathfrak{p}_0(s-)dN(\gamma\mu s) \\ &+ i\int_0^t e^{-i\omega(k)(t-s)}\tilde p(s-)dN(\gamma\mu s), \end{aligned} \tag{3.1}$$

where $\tilde p(t)$ is given by (2.7). Letting

$$\mathfrak{p}_0^0(t) := \operatorname{Im}\left(\int_{\mathbb{T}} e^{-i\omega(k)t}\hat\psi(k)dk\right) \tag{3.2}$$

we conclude the following closed equation on the momentum at $y=0$:

$$\mathfrak{p}_0(t) = \mathfrak{p}_0^0(t) - \frac{1}{\mu}\int_0^t J(t-s)\mathfrak{p}_0(s-)dN(\gamma\mu s) + \int_0^t J(t-s)\tilde p(s-)dN(\gamma\mu s). \tag{3.3}$$

Equation (3.1) is linear, so its solution can be written as the sum of the solution $\hat\psi_1(t,k)$ corresponding to the null initial data $\hat\psi(k)\equiv 0$ and the solution $\hat\psi_2(t,k)$ of the homogeneous equation corresponding to $\tilde p(t)\equiv 0$.

More precisely, suppose that $\hat\psi_1(t,k)$ is the solution of

$$\begin{aligned} &d\hat\psi_1(t,k) = -i\omega(k)\hat\psi_1(t,k)dt + i\left(\tilde p(t-) - \frac{1}{\mu}\mathfrak{p}_{0,1}(t-)\right)dN(\gamma\mu t), \\ &\hat\psi_1(0,k) \equiv 0 \end{aligned} \tag{3.4}$$

and $\hat\psi_2(t,k)$ satifsies

$$\begin{aligned} &d\hat\psi_2(t,k) = -i\omega(k)\hat\psi_2(t,k) - \frac{i}{\mu}\mathfrak{p}_{0,2}(t-)dN(\gamma\mu t), \\ &\hat\psi_2(0,k) = \hat\psi(k). \end{aligned} \tag{3.5}$$

Here

$$\mathfrak{p}_{0,j}(t) := \operatorname{Im}\int_{\mathbb{T}}\hat\psi_j(t,k)dk, \quad j=1,2.$$

Then

$$\hat\psi(t,k) = \hat\psi_1(t,k) + \hat\psi_2(t,k). \tag{3.6}$$

The respective Fourier-Wigner functions are defined as

$$\widehat{W}_{\varepsilon}^{j_1,j_2}(t,\eta,k) := \frac{\varepsilon}{2}\mathbb{E}\left[\hat{\psi}_{j_1}^*\left(\frac{t}{\varepsilon},k-\frac{\varepsilon\eta}{2}\right)\hat{\psi}_{j_2}\left(\frac{t}{\varepsilon},k+\frac{\varepsilon\eta}{2}\right)\right], \quad j_1, j_2 \in \{1,2\}.$$

Since the process $\left(\tilde{p}(t)\right)_{t\geq 0}$ is independent of the initial data field $\left(\hat{\psi}(k)\right)_{k\in\mathbb{T}}$ we conclude easily that

$$\widehat{W}_{\varepsilon}^{j_1,j_2}(t,\eta,k) \equiv 0, \quad \text{if } j_1 \neq j_2.$$

Therefore,

$$\widehat{W}_{\varepsilon}(t,\eta,k) := \frac{\varepsilon}{2}\mathbb{E}\left[\hat{\psi}^*\left(\frac{t}{\varepsilon},k-\frac{\varepsilon\eta}{2}\right)\hat{\psi}\left(\frac{t}{\varepsilon},k+\frac{\varepsilon\eta}{2}\right)\right] = \widehat{W}_{\varepsilon}^{1,1}(t,\eta,k) + \widehat{W}_{\varepsilon}^{2,2}(t,\eta,k). \tag{3.7}$$

Accordingly, the respective Laplace-Fourier-Wigner transforms satisfy

$$\widehat{w}_{\varepsilon}(\lambda,\eta,k) = \widehat{w}_{\varepsilon}^{1,1}(\lambda,\eta,k) + \widehat{w}_{\varepsilon}^{2,2}(\lambda,\eta,k), \tag{3.8}$$

where

$$\widehat{w}_{\varepsilon}(\lambda,\eta,k) = \int_0^{+\infty} e^{-\lambda t}\widehat{W}_{\varepsilon}(t,\eta,k)dt, \quad (\eta,k) \in \mathbb{T}_{2/\varepsilon}\times\mathbb{T}$$

and $\operatorname{Re}\lambda > 0$. The definitions of $\widehat{w}_{\varepsilon}^{j,j}$, corresponding to $\widehat{W}_{\varepsilon}^{j,j}(t,\eta,k)$, $j = 1,2$ are analogous.

3.1. **Solving (2.18) for the null initial data.** We suppose that $\hat{\psi}(0,k) \equiv 0$. Let $s_0 := t$, $\Delta_1(t) := [0,t]$ and

$$\Delta_n(t) := [(s_1,\ldots,s_n) : t > s_1 > s_2 > \ldots > s_n > 0], \quad n \geq 2.$$

Iterating (3.3) and remembering that $\mathfrak{p}_0^0(t) \equiv 0$ we can write

$$\mathfrak{p}_{0,1}(t) = \sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^{n-1}\int_{\Delta_n(t)}\prod_{j=1}^{n} J(s_{j-1}-s_j)\tilde{p}(s_n-)dN(\gamma\mu s_1)\ldots dN(\gamma\mu s_n), \tag{3.9}$$

with $s_0 := t$. Therefore, substituting for the momentum into the respective form of (3.1) we get

$$\hat{\psi}_1(t,k) = i\int_0^t e^{-i\omega(k)(t-s)}\left(\tilde{p}(s) - \frac{1}{\mu}\mathfrak{p}_0(s-)\right)dN(\gamma\mu s) = \sum_{n=1}^{+\infty}\hat{\psi}_{1,n}(t,k), \tag{3.10}$$

where

$$\begin{aligned} \hat{\psi}_{1,1}(t,k) &:= i\int_0^t e^{-i\omega(k)(t-s)}\tilde{p}(s-)dN(\gamma\mu s),\\ \hat{\psi}_{1,n}(t,k) &:= \left(-\frac{1}{\mu}\right)^{n-1} i\int_{\Delta_n(t)} e^{-i\omega(k)(t-s_1)}\\ &\qquad \times\prod_{j=1}^{n-1} J(s_j-s_{j+1})\tilde{p}(s_n-)dN(\gamma\mu s_1)\ldots dN(\gamma\mu s_n), \quad n\geq 2. \end{aligned} \tag{3.11}$$

3.2. **The case $T = 0$ and non-zero initial data.** The mild formulation of (3.5) is as follows

$$\hat{\psi}_2(t,k) = e^{-i\omega(k)t}\hat{\psi}(k) - \frac{i}{\mu}\int_0^t e^{-i\omega(k)(t-s)}\mathfrak{p}_{0,2}(s-)dN(\gamma\mu s). \tag{3.12}$$

From here we conclude the following closed equation on the momentum at $y = 0$:

$$\mathfrak{p}_{0,2}(t) = \mathfrak{p}_0^0(t) - \frac{1}{\mu}\int_0^t J(t-s)\mathfrak{p}_{0,2}(s-)dN(\gamma\mu s), \tag{3.13}$$

where $\mathfrak{p}_0^0(t)$ is given by (3.2). The solution of (3.13) is given by

$$\mathfrak{p}_{0,2}(t)=\mathfrak{p}_0^0(t)+\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n\int_{\Delta_n(t)}J(t-s_1)\dots J(s_{n-1}-s_n)\\ \times\mathfrak{p}_0^0(s_n)dN(\gamma\mu s_1)\dots dN(\gamma\mu s_n). \tag{3.14}$$

Substituting into (3.12) we get

$$\hat{\psi}_2(t,k)=\sum_{n=0}^{+\infty}\hat{\psi}_{2,n}(t,k), \tag{3.15}$$

where

$$\hat{\psi}_{2,1}(t,k):=-\frac{i}{\mu}\int_0^t e^{-i\omega(k)(t-s)}\mathfrak{p}_0^0(s)dN(\gamma\mu s)\\ \hat{\psi}_{2,n}(t,k):=-i\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^{n+1}\int_{\Delta_n(t)}e^{-i\omega(k)(t-s_1)}\prod_{j=1}^{n-1}J(s_j-s_{j+1})\\ \times\mathfrak{p}_0^0(s_n)dN(\gamma\mu s_1)\dots dN(\gamma\mu s_n),\quad n\ge 2. \tag{3.16}$$

4. The phonon creation term

Consider first the case when the null initial data, i.e. $\hat{\psi}_2(t,k)\equiv 0$. Then,

$$\widehat{w}_\varepsilon(\lambda,\eta,k)=\widehat{w}_\varepsilon^{(1,1)}(\lambda,\eta,k). \tag{4.1}$$

We wish to use the chaos expansion, corresponding to the Poisson process $(N(t))_{t\ge0}$ to represent the Laplace-Fourier-Wigner function $\widehat{w}_\varepsilon(\lambda,\eta,k)$.

Lemma 4.1. *The following formula holds*

$$\widehat{w}_\varepsilon(\lambda,\eta,k)=\frac{\varepsilon T\gamma}{\lambda}\left(1-\frac{1}{2\mu}\right)\int_0^{+\infty}e^{-\lambda\varepsilon s}\mathbb{E}\left[\hat{\chi}^*\left(s,k-\frac{\varepsilon\eta}{2}\right)\hat{\chi}\left(s,k+\frac{\varepsilon\eta}{2}\right)\right]ds \tag{4.2}$$

for any $\lambda\in\mathbb{C}_+$, $(\eta,k)\in\mathbb{T}_{2/\varepsilon}\times\mathbb{T}$ *and* $\varepsilon>0$. *Here*

$$\hat{\chi}(t,k):=\exp\{-i\omega(k)t\}\\ +\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n\int_{\Delta_n(t)}\exp\{-i\omega(k)(t-s_1)\}\prod_{j=1}^{n}J(s_j-s_{j+1})dN(\gamma\mu s_1)\dots dN(\gamma\mu s_n),$$

with $s_{n+1}:=0$.

Proof. Substituting from (3.11) we get

$$\widehat{w}_\varepsilon(\lambda,\eta,k)=\sum_{n,m=1}^{+\infty}\widehat{w}_{\varepsilon,n,m}(\lambda,\eta,k), \tag{4.3}$$

where

$$\widehat{w}_{\varepsilon,n,m}(\lambda,\eta,k):=\frac{\varepsilon}{2}\int_0^{+\infty}e^{-\varepsilon\lambda t}\mathbb{E}\left[\hat{\psi}_{1,n}^\star\left(t,k-\frac{\varepsilon\eta}{2}\right)\hat{\psi}_{1,m}\left(t,k+\frac{\varepsilon\eta}{2}\right)\right]dt\quad n,m\ge1.$$

Note that for $s>s'$

$$\mathbb{E}\Big[\tilde{p}(s-)\tilde{p}(s'-),\ N(\gamma\mu s-)-N(\gamma\mu s')\ge1\Big]=0.$$

The above implies that

$$
\begin{aligned}
&\widehat{w}_{\varepsilon,n,m}(\lambda,\eta,k) = \frac{\varepsilon^2}{2}\left(-\frac{1}{\mu}\right)^{n+m}\int_0^{+\infty} e^{-\lambda\varepsilon t}dt\\
&\times \mathbb{E}\Bigg[\int_{\Delta_n(t)} dN(\gamma\mu s_1)\dots dN(\gamma\mu s_n)\int_{\Delta_m(t)} dN(\gamma\mu s_1')\dots dN(\gamma\mu s_m')\exp\left\{i\omega\left(k-\frac{\varepsilon\eta}{2}\right)(t-s_1)\right\}\\
&\times \exp\left\{-i\omega\left(k+\frac{\varepsilon\eta}{2}\right)(t-s_1')\right\}\prod_{j=1}^{n-1}J(s_j-s_{j+1})\prod_{j=1}^{m-1}J(s_j'-s_{j+1}')\tilde{p}(s_n-)\tilde{p}(s_m'-)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \varepsilon^2 T\gamma\left(1-\frac{1}{2\mu}\right)\left(-\frac{1}{\mu}\right)^{n+m}\int_0^{+\infty} e^{-\lambda\varepsilon t}dt\int_0^t ds\\
&\times \mathbb{E}\Bigg[\int_{\Delta_{n-1}(t-s)} dN(\gamma\mu s_1)\dots dN(\gamma\mu s_{n-1})\int_{\Delta_{m-1}(t-s)} dN(\gamma\mu s_1')\dots dN(\gamma\mu s_{m-1}')\\
&\times \exp\left\{i\omega\left(k-\frac{\varepsilon\eta}{2}\right)(t-s-s_1)\right\}\exp\left\{-i\omega\left(k+\frac{\varepsilon\eta}{2}\right)(t-s-s_1')\right\}\prod_{j=1}^{n-1}J(s_j-s_{j+1})\prod_{j=1}^{m-1}J(s_j'-s_{j+1}')\Bigg].
\end{aligned}
$$

Here $s_n = s_m' := 0$. Integrating out the t variable we get

$$
\begin{aligned}
&\widehat{w}_{\varepsilon,n,m}(\lambda,\eta,k) = \frac{\varepsilon\gamma T}{\lambda}\left(1-\frac{1}{2\mu}\right)\left(-\frac{1}{\mu}\right)^{n+m}\int_0^{+\infty} e^{-\lambda\varepsilon s}\exp\left\{i\left[\omega\left(k-\frac{\varepsilon\eta}{2}\right)-\omega\left(k+\frac{\varepsilon\eta}{2}\right)\right]s\right\}ds\\
&\times \mathbb{E}\Bigg[\int_{\Delta_{n-1}(s)} dN(\gamma\mu s_1)\dots dN(\gamma\mu s_{n-1})\int_{\Delta_{m-1}(s)} dN(\gamma\mu s_1')\dots dN(\gamma\mu s_{m-1}')\\
&\times \exp\left\{-i\omega\left(k-\frac{\varepsilon\eta}{2}\right)s_1\right\}\exp\left\{i\omega\left(k+\frac{\varepsilon\eta}{2}\right)s_1'\right\}\prod_{j=1}^{n-1}J(s_j-s_{j+1})\prod_{j=1}^{m-1}J(s_j'-s_{j+1}')\Bigg]
\end{aligned}
$$

for $n,m \ge 1$. Summing out over n,m we conclude (4.2). □

Next, we write the Poisson chaos decomposition of the random field $\hat{\chi}(t,k)$. Let

$$\phi(t,k) := \int_0^t e^{-i\omega(k)(t-s)}g(ds). \tag{4.4}$$

Define, the cadlag martingale

$$\tilde{N}(t) := N(t) - t, \quad t \ge 0. \tag{4.5}$$

Lemma 4.2. *The following expansion holds*

$$\hat{\chi}(t,k) = \sum_{n=0}^{+\infty}\hat{\chi}_n(t,k), \tag{4.6}$$

where

$$
\begin{aligned}
&\hat{\chi}_0(t,k) := \phi(t,k),\\
&\hat{\chi}_n(t,k) := \left(-\frac{1}{\mu}\right)^n\int_{\Delta_n(t)}\phi(t-s_1,k)\prod_{j=1}^n J\star g(s_j-s_{j+1})\\
&\qquad\times d\tilde{N}(\gamma\mu s_1)\dots d\tilde{N}(\gamma\mu s_n), \quad n\ge 1.
\end{aligned} \tag{4.7}
$$

Proof. Writing $N(\gamma\mu t) = \tilde{N}(\gamma\mu t) + \gamma\mu t$, where $\big(\tilde{N}(\gamma\mu t)\big)_{t\ge 0}$ is a cadlag martingale we obtain

$$\begin{aligned}
\hat{\chi}(t,k) &= \exp\{-i\omega(k)t\} + \sum_{n=1}^{+\infty}(-\gamma)^n \int_{\Delta_n(t)} \exp\{-i\omega(k)(t-s_1)\}\prod_{j=1}^{n} J(s_j - s_{j+1})ds_1\dots ds_n \\
&+ \sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n \sum_{k=1}^{n-1}(\gamma\mu)^k \sum_{\mathbf{i}\in\mathcal{I}_k^n} \int_{\Delta_n(t)} \exp\{-i\omega(k)(t-s_1)\}\prod_{j=1}^{n} J(s_j - s_{j+1})d\mathbf{s}_{\mathbf{i}} \prod_{j\notin\mathbf{i}} d\tilde{N}(\gamma\mu s_j) \\
&+ \sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n \int_{\Delta_n(t)} \exp\{-i\omega(k)(t-s_1)\}\prod_{j=1}^{n} J(s_j - s_{j+1})d\tilde{N}(\gamma\mu s_1)\dots d\tilde{N}(\gamma\mu s_n).
\end{aligned} \tag{4.8}$$

For $1\le k\le n$ we denote by $\mathcal{I}_k^n$ the set of all ordered k-indices $\mathbf{i}: 1\le i_1<\dots<i_k\le n$. We shall also use the abbreviation $d\mathbf{s}_{\mathbf{i}} := \prod_{j\in\mathbf{i}} ds_j$.

Using (2.36) we can combine the first two terms in the right hand side of (4.8) and obtain that they are equal to $\phi(t,k)$ (cf (4.4))

Changing the order of summation in the remaining two expressions in the right hand side of (4.8) we conclude that their sum equals

$$\begin{aligned}
&\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n \sum_{r_1=0}^{+\infty}\sum_{r_2,\dots,r_n=1}^{+\infty} \int_{\Delta_n(t)} \exp\{-i\omega(k)(t-s_1)\} \\
&\times \prod_{j=1}^{n}(-\gamma)^{r_j-1} J^{\star,r_j}(s_j - s_{j+1}) d\tilde{N}(\gamma\mu s_1)\dots d\tilde{N}(\gamma\mu s_n)
\end{aligned}$$

Using formula (2.36) the above expression can be rewritten in the form

$$\begin{aligned}
&\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n \int_{\Delta_n(t)} \left(\int_0^{t-s_1} \exp\{-i\omega(k)(t-s_1-\sigma)\} g(d\sigma)\right) \\
&\times \prod_{j=1}^{n} J\star g(s_j - s_{j+1}) d\tilde{N}(\gamma\mu s_1)\dots d\tilde{N}(\gamma\mu s_n) \\
&= \sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n \int_{\Delta_n(t)} \phi(t-s_1,k)\prod_{j=1}^{n} J\star g(s_j - s_{j+1}) d\tilde{N}(\gamma\mu s_1)\dots d\tilde{N}(\gamma\mu s_n)
\end{aligned}$$

and (4.6), with (4.7) follow. □

Coming back to calculation of the asymptotics of $\widehat{w}_\varepsilon(\lambda,\eta,k)$ given by (4.1) we have the following result. Recall that Γ is defined by (2.46).

Proposition 4.3. *For any $\gamma>0$ the parameter Γ, defined by (2.46), belongs to $(0,1/2]$. In addition, for any $\mu>1/2$, $\gamma>0$, $\lambda\in\mathbb{C}_+$ and $(\eta,k)\in\mathbb{R}\times\mathbb{T}$ we have*

$$\lim_{\varepsilon\to 0+}\widehat{w}_\varepsilon(\lambda,\eta,k) = \frac{\gamma T|\nu(k)|^2}{(1-\Gamma/\mu)\lambda(\lambda+i\omega'(k)\eta)}\left(1-\frac{1}{2\mu}\right). \tag{4.9}$$

Proof. We can use the $L^2(\mathbb{P})$ orthogonality of the terms of the expansion (4.6), with (4.7). We get

$$\widehat{w}_\varepsilon(\lambda,\eta,k) = \sum_{n=0}^{+\infty}\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k), \tag{4.10}$$

where

$$\widehat{w}_{\varepsilon}^{(0)}(\lambda,\eta,k) := \frac{\varepsilon T\gamma}{\lambda}\left(1-\frac{1}{2\mu}\right)\int_0^{+\infty} e^{-\varepsilon\lambda t}\phi^\star\left(t,k-\frac{\varepsilon\eta}{2}\right)\phi\left(t,k+\frac{\varepsilon\eta}{2}\right)dt,$$

$$\widehat{w}_{\varepsilon}^{(n)}(\lambda,\eta,k) := \frac{\varepsilon T\gamma}{\lambda}\left(1-\frac{1}{2\mu}\right)\left(\frac{\gamma}{\mu}\right)^n\int_0^{+\infty} e^{-\varepsilon\lambda t}dt\int_{\Delta_n(t)}\phi^\star\left(t-s_1,k-\frac{\varepsilon\eta}{2}\right)\phi\left(t-s_1,k+\frac{\varepsilon\eta}{2}\right) \tag{4.11}$$

$$\times\prod_{j=1}^n\left(J\star g(s_j-s_{j+1})\right)^2 ds_1\dots ds_n,\quad n\ge 1$$

Computation of $\widehat{w}_{\varepsilon}^{(0)}(\lambda,\eta,k)$. Thanks to (4.7) and (4.11) we have

$$\widehat{w}_{\varepsilon}^{(0)}(\lambda,\eta,k) = \frac{\varepsilon T\gamma}{\lambda}\left(1-\frac{1}{2\mu}\right)\int_0^{+\infty}\int_0^{+\infty} dtdt' e^{-\varepsilon\lambda(t+t')/2}\delta(t-t')\phi^\star\left(t,k-\frac{\varepsilon\eta}{2}\right)\phi\left(t',k+\frac{\varepsilon\eta}{2}\right)$$

Using

$$\delta(t-t') = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\beta(t-t')}d\beta, \tag{4.12}$$

we can write

$$\widehat{w}_{\varepsilon}^{(0)}(\lambda,\eta,k) = \frac{\varepsilon T\gamma}{(2\pi)\lambda}\left(1-\frac{1}{2\mu}\right)\int_{\mathbb{R}} d\beta\int_0^{+\infty} e^{-(\varepsilon\lambda/2-i\beta)t}dt\int_0^t \exp\left\{i\omega\left(k-\frac{\varepsilon\eta}{2}\right)(t-s)\right\}g(ds)$$

$$\times\int_0^{+\infty} e^{-(\varepsilon\lambda/2+i\beta)t'}dt'\int_0^{t'}\exp\left\{-i\omega\left(k+\frac{\varepsilon\eta}{2}\right)(t'-s')\right\}g(ds').$$

Integrating out s,t and s',t' variables we obtain

$$\widehat{w}_{\varepsilon}^{(0)}(\lambda,\eta,k) = \frac{\varepsilon T\gamma}{(2\pi)\lambda}\left(1-\frac{1}{2\mu}\right)\int_{\mathbb{R}}\left\{\varepsilon\lambda/2-i\omega\left(k-\frac{\varepsilon\eta}{2}\right)-i\beta\right\}^{-1}\left\{i\omega\left(k+\frac{\varepsilon\eta}{2}\right)+\varepsilon\lambda/2+i\beta\right\}^{-1}$$

$$\times\tilde{g}(\varepsilon\lambda/2-i\beta)\tilde{g}(\varepsilon\lambda/2+i\beta)d\beta.$$

Change variables $\varepsilon\beta' := \beta+\omega\left(k-\frac{\varepsilon\eta}{2}\right)$ and obtain, cf (2.39),

$$\widehat{w}_{\varepsilon}^{(0)}(\lambda,\eta,k) = \frac{T\gamma}{(2\pi)\lambda}\left(1-\frac{1}{2\mu}\right)\int_{\mathbb{R}}\left\{\lambda/2-i\beta\right\}^{-1}\left\{i\delta_\varepsilon\omega(k;\eta)+\lambda/2+i\beta\right\}^{-1}$$

$$\times\tilde{g}\left(\varepsilon\lambda/2-i\varepsilon\beta+i\omega\left(k-\frac{\varepsilon\eta}{2}\right)\right)\tilde{g}\left(\varepsilon\lambda/2+i\varepsilon\beta-i\omega\left(k+\frac{\varepsilon\eta}{2}\right)\right)d\beta.$$

Here

$$\delta_\varepsilon\omega(k;\eta) := \varepsilon^{-1}\Big[\omega\left(k+\frac{\varepsilon\eta}{2}\right)-\omega\left(k-\frac{\varepsilon\eta}{2}\right)\Big]. \tag{4.13}$$

Therefore

$$\lim_{\varepsilon\to0+}\widehat{w}_{\varepsilon}^{(0)}(\lambda,\eta,k) = \frac{T\gamma|\nu(k)|^2}{(2\pi)\lambda}\left(1-\frac{1}{2\mu}\right)\int_{\mathbb{R}}\left\{\lambda/2-i\beta\right\}^{-1}\left\{i\omega'(k)\eta+\lambda/2+i\beta\right\}^{-1}d\beta. \tag{4.14}$$

To integrate out the β variable we use the Cauchy integral formula that in our context reads

$$\frac{1}{2\pi}\int_{\mathbb{R}}\frac{f(i\beta)d\beta}{z-i\beta} = f(z),\quad z\in\mathbb{C}_+. \tag{4.15}$$

It is valid for any holomorphic function f on the right half-plane $\mathbb{C}_+$ that belongs to the Hardy class $H^p(\mathbb{C}_+)$ for some $p\ge1$, see e.g. [6, p. 113]. Applying the formula we get

$$\lim_{\varepsilon\to0+}\widehat{w}_{\varepsilon}^{(0)}(\lambda,\eta,k) = \frac{\gamma T|\nu(k)|^2}{\lambda(\lambda+i\omega'(k)\eta)}\left(1-\frac{1}{2\mu}\right). \tag{4.16}$$

Computation of $\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k)$ for $n\ge 1$. Change variables

$$\tau_0 := t - s_1, \dots, \tau_n := s_n - s_{n+1} (= s_n)$$

in (4.11). As a result we get

$$\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k) = \frac{\varepsilon T\gamma}{\lambda}\left(1-\frac{1}{2\mu}\right)\left(\frac{\gamma}{\mu}\right)^n \int_0^{+\infty} e^{-\varepsilon\lambda t/2}dt \int_{[0,+\infty)^{n+1}} d\tau_{0,n} \exp\left\{-\varepsilon\lambda\left(\tau_0+\ldots+\tau_n\right)/2\right\}$$
$$\times\,\delta\left(t-\tau_0-\ldots-\tau_n\right)\phi^\star\left(\tau_0,k-\frac{\varepsilon\eta}{2}\right)\phi\left(\tau_0,k+\frac{\varepsilon\eta}{2}\right)\prod_{j=1}^n\left(J\star g(\tau_j)\right)^2.$$

Here $d\tau_{0,n} := d\tau_0\ldots d\tau_n$. Using (4.12) for each variable t and τ_j, $j=0,\dots,n$, we can further write

$$\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k) = \frac{\varepsilon T\gamma}{(2\pi)^{n+2}\lambda}\left(1-\frac{1}{2\mu}\right)\left(\frac{\gamma}{\mu}\right)^n \int_{\mathbb{R}} d\beta\int_{\mathbb{R}^{n+1}} d\beta_{0,n}\int_{[0,+\infty)^{2n+2}} d\tau_{0,n}d\tau'_{0,n}$$
$$\times\int_0^{+\infty} e^{-(\varepsilon\lambda/2-i\beta)t}dt\prod_{j=0}^n \exp\left\{-(\varepsilon\lambda/4+i\beta/2+i\beta_j)\tau_j\right\}\prod_{j=0}^n\exp\left\{-(\varepsilon\lambda/4+i\beta/2-i\beta_j)\tau'_j\right\}$$
$$\times\,\phi^\star\left(\tau_0,k-\frac{\varepsilon\eta}{2}\right)\phi\left(\tau'_0,k+\frac{\varepsilon\eta}{2}\right)\prod_{j=1}^n\left(J\star g(\tau_j)\right)\prod_{j=1}^n\left(J\star g(\tau'_j)\right).$$

To abbreviate we have used the notation $d\beta_{0,n} := d\beta_0\ldots d\beta_n$ and analogously for the remaining variables.

Integrating the t, τ variables and its primed counter-parts we get

$$\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k) = \frac{\varepsilon T\gamma}{(2\pi)^{n+2}\lambda}\left(1-\frac{1}{2\mu}\right)\left(\frac{\gamma}{\mu}\right)^n\int_{\mathbb{R}}\frac{d\beta}{\varepsilon\lambda/2-i\beta}\int_{\mathbb{R}^{n+1}}d\beta_{0,n}$$
$$\times\,\frac{\tilde g(\varepsilon\lambda/4+i\beta_0+i\beta/2)}{\varepsilon\lambda/4+i\Big(\beta_0+\beta/2-\omega\left(k-\frac{\varepsilon\eta}{2}\right)\Big)}\times\frac{\tilde g(\varepsilon\lambda/4-i\beta_0+i\beta/2)}{\varepsilon\lambda/4+i\Big(\beta/2-\beta_0+\omega\left(k+\frac{\varepsilon\eta}{2}\right)\Big)}$$
$$\times\prod_{j=1}^n\tilde J\tilde g(\varepsilon\lambda/4+i\beta/2+i\beta_j)\prod_{j=1}^n\tilde J\tilde g(\varepsilon\lambda/4+i\beta/2-i\beta_j).$$

We integrate the β variable using the Cauchy integral formula (4.15) and get

$$\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k) = \frac{\varepsilon T\gamma}{(2\pi)^{n+1}\lambda}\left(1-\frac{1}{2\mu}\right)\left(\frac{\gamma}{\mu}\right)^n\int_{\mathbb{R}^{n+1}}d\beta_{0,n}\prod_{j=1}^n\tilde J\tilde g(\varepsilon\lambda/2+i\beta_j)\prod_{j=1}^n\tilde J\tilde g(\varepsilon\lambda/2-i\beta_j)$$
$$\times\,\frac{\tilde g(\varepsilon\lambda/2+i\beta_0)}{\varepsilon\lambda/2+i\Big(\beta_0-\omega\left(k-\frac{\varepsilon\eta}{2}\right)\Big)}\times\frac{\tilde g(\varepsilon\lambda/2-i\beta_0)}{\varepsilon\lambda/2+i\Big(-\beta_0+\omega\left(k+\frac{\varepsilon\eta}{2}\right)\Big)}.$$

Change of variables $\varepsilon\beta'_0 := \beta_0-\omega\left(k-\frac{\varepsilon\eta}{2}\right)$ and obtain

$$\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k) = \frac{T\gamma}{(2\pi)^{n+1}\lambda}\left(1-\frac{1}{2\mu}\right)\left(\frac{\gamma}{\mu}\right)^n\int_{\mathbb{R}^{n+1}}d\beta_{0,n}\prod_{j=1}^n\tilde J\tilde g(\varepsilon\lambda/2+i\beta_j)\prod_{j=1}^n\tilde J\tilde g(\varepsilon\lambda/2-i\beta_j)$$
$$\times\,\frac{\tilde g(\varepsilon\lambda/2+i\varepsilon\beta_0+i\omega\left(k-\frac{\varepsilon\eta}{2}\right))}{\lambda/2+i\beta_0}\times\frac{\tilde g(\varepsilon\lambda/2-i\varepsilon\beta_0-i\omega\left(k-\frac{\varepsilon\eta}{2}\right))}{\lambda/2+i\left(-\beta_0+\delta_\varepsilon\omega(k;\eta)\right)}.$$

Thus,

$$\widehat{w}^{(n)}(\lambda,\eta,k) := \lim_{\varepsilon\to0+}\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k)$$
$$= \frac{\gamma T}{\lambda(2\pi)}\left(\frac{\Gamma}{\mu}\right)^n|\nu(k)|^2\left(1-\frac{1}{2\mu}\right)\int_{\mathbb{R}}\frac{d\beta_0}{(\lambda/2+i\beta_0)\big\{\lambda/2+i\big[-\beta_0+\omega'(k)\eta\big]\big\}}.$$

Here Γ is given by (2.46). Integrating the β_0 variable out, using again (4.15), we get

$$\widehat{w}^{(n)}(\lambda,\eta,k) = \frac{\gamma T|\nu(k)|^2}{\lambda(\lambda+i\omega'(k)\eta)}\left(\frac{\Gamma}{\mu}\right)^n\left(1-\frac{1}{2\mu}\right).$$

To prove that $\Gamma \in (0,1/2]$ note first that for $\eta = 0$ the terms of the expansion (4.10) are non-negative for each $\varepsilon > 0$ and $\lambda > 0$. Thanks to (2.23) and the Fatou lemma (applied in the context of series) we can write

$$+\infty > \liminf_{\varepsilon\to 0+} \widehat{w}_\varepsilon(\lambda,0,k) \ge \sum_{n=0}^{+\infty} \liminf_{\varepsilon\to 0+} \widehat{w}_\varepsilon^{(n)}(\lambda,0,k) = \sum_{n=0}^{+\infty} \frac{\gamma T|\nu(k)|^2}{\lambda^2}\left(\frac{\Gamma}{\mu}\right)^n\left(1-\frac{1}{2\mu}\right), \tag{4.17}$$

which proves that $\Gamma < \mu$ for any $\mu > 1/2$. This in turn implies that $\Gamma \le 1/2$.

Since the function $\tilde{J}\tilde{g}$ belongs to $H^2(\mathbb{C}_+)$, see Lemma 2.1, we can write

$$\tilde{J}\tilde{g}(\varepsilon+i\beta) = \frac{1}{2\pi}\int_{-\infty}^0 e^{(\varepsilon+i\beta)\eta}\widehat{\tilde{J}\tilde{g}}(\eta)d\eta, \quad \varepsilon > 0,\ \beta \in \mathbb{R},$$

where $\widehat{\tilde{J}\tilde{g}}$ is the Fourier transform of $\tilde{J}\tilde{g}(i\beta)$, that is supported in $(-\infty,0)$. In particular, we obtain then

$$\frac{1}{2\pi}\int_{\mathbb{R}} |\tilde{J}\tilde{g}(\varepsilon\lambda/2+i\beta)|^2 d\beta < \Gamma.$$

Therefore, by the dominated convergence theorem, we conclude that

$$\widehat{w}(\lambda,\eta,k) = \lim_{\varepsilon\to 0+} \widehat{w}_\varepsilon(\lambda,\eta,k) = \sum_{n=0}^{+\infty} \lim_{\varepsilon\to 0+} \widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k) \tag{4.18}$$

and formula (4.10) follows. □

In the case $\mu = 1/2$ we can use identity (2.47), whose proof (presented in Section 6.2) uses the conclusion of Proposition 4.3 for $\mu = 1$. We obtain then $\Gamma < 1/2 = \mu$. Therefore, we have the following extension of the proposition.

Proposition 4.4. *For any $\gamma > 0$, $\mu = 1/2$, $\lambda \in \mathbb{C}_+$ and $(\eta,k) \in \mathbb{R}\times\mathbb{T}$ we have*

$$\lim_{\varepsilon\to 0+} \widehat{w}_\varepsilon(\lambda,\eta,k) = 0. \tag{4.19}$$

5. The case $T = 0$ and non-zero initial data

Here, as in Section 3.2, we assume that $T = 0$ and the initial data need not be null, and satisfies the assumptions made in Sections 2.4.3 and 2.4.4. The solution $\hat{\psi}(t,k)$ is then described by the expansion (3.14) and (3.16). Using the same argument as in the proof of Lemma 4.2 we obtain the following Poisson chaos expansion for the momentum at $x = 0$ and the Fourier transform of the wave function

$$\mathfrak{p}_0(t) = g\star\mathfrak{p}_0^0(t) + \sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n \int_{\Delta_n(t)} \prod_{j=1}^n J\star g(s_{j-1}-s_j) g\star\mathfrak{p}_0^0(s_n) d\tilde{N}(\gamma\mu s_1)\dots d\tilde{N}(\gamma\mu s_n), \tag{5.1}$$

and

$$\hat{\psi}(t,k) = e^{-i\omega(k)t}\hat{\psi}(0,k) - i\gamma\int_0^t \phi(t-s,k)\mathfrak{p}_0^0(s)ds$$
$$+ i\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n \int_{\Delta_n(t)} \phi(t-s_1,k)\prod_{j=1}^{n-1} J\star g(s_j-s_{j+1}) g\star\mathfrak{p}_0^0(s_n) d\tilde{N}(\gamma\mu s_1)\dots d\tilde{N}(\gamma\mu s_n), \tag{5.2}$$

where $\mathfrak{p}_0^0(\cdot)$ is given by (3.2). In light of (2.49) both of these expansions are valid for any $\mu \ge 1/2$.

On the other hand from (2.18), with $\tilde{p}(t) \equiv 0$, we obtain the following equation on the Fourier-Wigner function $W_\varepsilon(t,\eta,k)$

$$\partial_t \widehat{W}_\varepsilon(t,\eta,k) + i\delta_\varepsilon\omega(k;\eta)\widehat{W}_\varepsilon(t,\eta,k) = \frac{\gamma}{2\mu}\mathbb{E}\left[\mathfrak{p}_0^2\left(\frac{t}{\varepsilon}\right)\right] + \frac{i\gamma}{2}\left\{\mathbb{E}\left[\hat{\psi}\left(\frac{t}{\varepsilon}, k+\frac{\varepsilon\eta}{2}\right)\mathfrak{p}_0\left(\frac{t}{\varepsilon}\right)\right] - \mathbb{E}\left[\hat{\psi}^*\left(\frac{t}{\varepsilon}, k-\frac{\varepsilon\eta}{2}\right)\mathfrak{p}_0\left(\frac{t}{\varepsilon}\right)\right]\right\} \tag{5.3}$$

Taking the Laplace transform on both sides we arrive at

$$(\lambda + i\delta_\varepsilon\omega(k;\eta))\,\widehat{w}_\varepsilon(\lambda,\eta,k) = W_\varepsilon(0,\eta,k) + \frac{\gamma}{\mu}\mathfrak{e}_\varepsilon(\lambda) - \frac{\gamma}{2}\left[\mathfrak{d}_\varepsilon\left(\lambda, k-\frac{\varepsilon\eta}{2}\right) + \mathfrak{d}_\varepsilon^\star\left(\lambda, k+\frac{\varepsilon\eta}{2}\right)\right], \tag{5.4}$$

where

$$\mathfrak{e}_\varepsilon(\lambda) := \frac{\varepsilon}{2}\int_0^{+\infty} e^{-\lambda\varepsilon t}\mathbb{E}\left[\mathfrak{p}_0^2(t)\right]dt \quad \text{and}$$
$$\mathfrak{d}_\varepsilon(\lambda,k) := i\varepsilon\int_0^{+\infty} e^{-\lambda\varepsilon t}\mathbb{E}\left[\hat{\psi}^*(t,k)\,\mathfrak{p}_0(t)\right]dt. \tag{5.5}$$

In the present section we show the following.

Proposition 5.1. *For any $G \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$ and $\operatorname{Re}\lambda > 0$ we have*

$$\int_{\mathbb{R}}\int_{\mathbb{T}} \widehat{w}(\lambda,\eta,k)G^*(\eta,k)d\eta dk = \lim_{\varepsilon\to 0+}\int_{\mathbb{R}}\int_{\mathbb{T}} \widehat{w}_\varepsilon(\lambda,\eta,k)G^*(\eta,k)d\eta dk,$$

where

$$\widehat{w}(\lambda,\eta,k) = \frac{\widehat{W}(0,\eta,k)}{\lambda + i\omega'(k)\eta} + \frac{\gamma|\nu(k)|^2}{2(1-\Gamma/\mu)(\lambda+i\omega'(k)\eta)}\int_{\mathbb{R}}\int_{\mathbb{T}}\frac{\widehat{W}(0,\eta,\ell)|\nu(\ell)|^2}{\lambda+i\omega'(\ell)\eta}d\eta d\ell - \frac{\gamma\operatorname{Re}[\nu(k)]}{\lambda+i\omega'(k)\eta}\int_{\mathbb{R}\times\mathbb{T}}\frac{\widehat{W}(0,\eta',k)}{\lambda+i\omega'(k)\eta'}d\eta' + \frac{\gamma\mathfrak{g}(k)}{4(\lambda+i\omega'(k)\eta)}\int_{\mathbb{R}\times\mathbb{T}}\frac{\widehat{W}(0,\eta',k)d\eta'}{\lambda+i\omega'(k)\eta'} + \frac{\gamma\mathfrak{g}(k)}{4(\lambda+i\omega'(k)\eta)}\int_{\mathbb{R}\times\mathbb{T}}\frac{\widehat{W}(0,\eta',-k)d\eta'}{\lambda-i\omega'(k)\eta'}. \tag{5.6}$$

The proof of the proposition is carried out throughout Sections 5.1 - 5.3.

5.1. **Asymptotics of $\mathfrak{e}_\varepsilon(\lambda)$.**

Proposition 5.2. *Under the assumption about the initial data made in Sections 2.4.3 and 2.4.4 we have*

$$\lim_{\varepsilon\to 0+}\mathfrak{e}_\varepsilon(\lambda) = \frac{1}{2(1-\Gamma/\mu)}\int_{\mathbb{R}}\int_{\mathbb{T}}\frac{\widehat{W}(0,\eta,\ell)|\nu(\ell)|^2}{\lambda+i\omega'(\ell)\eta}d\eta d\ell. \tag{5.7}$$

Proof. From (5.1) we get

$$\mathbb{E}\left[\mathfrak{p}_0^2(t)\right] = \mathbb{E}[g\star\mathfrak{p}_0^0(t)]^2 + \sum_{n=1}^{+\infty}\left(\frac{\gamma}{\mu}\right)^n\int_{\Delta_n(t)}\prod_{j=1}^n (J\star g)^2(s_{j-1}-s_j)\mathbb{E}[g\star\mathfrak{p}_0^0(s_n)]^2 ds_1\dots ds_n. \tag{5.8}$$

Accordingly, we have

$$\begin{aligned}
&\mathfrak{e}_\varepsilon(\lambda) = \sum_{n=0}^{+\infty} E_n^{(\varepsilon)}(\lambda), \quad \text{where} \\
&E_0^{(\varepsilon)}(\lambda) := \frac{\varepsilon}{2}\int_0^{+\infty} e^{-\lambda\varepsilon t}\mathbb{E}[g\star\mathfrak{p}_0^0(t)]^2 dt, \\
&E_n^{(\varepsilon)}(\lambda) := \frac{\varepsilon}{2}\left(\frac{\gamma}{\mu}\right)^n \int_0^{+\infty} e^{-\lambda\varepsilon t}dt \int_{\Delta_n(t)} (J\star g)^2(t-s_1)\prod_{j=1}^{n-1}(J\star g)^2(s_j-s_{j+1}) \\
&\qquad\qquad\qquad\qquad\qquad \times \mathbb{E}[g\star\mathfrak{p}_0^0(s_n)]^2 ds_1\dots ds_n.
\end{aligned} \tag{5.9}$$

Asymtotics of $E_0^{(\varepsilon)}(\lambda)$. Using (4.12) we can write

$$E_0^{(\varepsilon)}(\lambda) = -\frac{\varepsilon}{2^4\pi}\int_{\mathbb{T}^2} dkdk'\int_0^{+\infty} dt\int_0^{+\infty} dt' e^{-\lambda\varepsilon(t+t')/2}\int_0^t\int_0^{t'} g(d\sigma)g(d\sigma')\int_{\mathbb{R}} d\beta e^{i\beta(t-t')}$$
$$\times\,\mathbb{E}\left\{\left\{e^{-i\omega(k)(t-\sigma)}\hat\psi(k) - e^{i\omega(k)(t-\sigma)}\hat\psi^*(k)\right\}\left\{e^{-i\omega(k')(t'-\sigma')}\hat\psi(k') - e^{i\omega(k')(t'-\sigma')}\hat\psi^*(k')\right\}\right\}.$$

Thanks to (2.20) we can write

$$E_0^{(\varepsilon)}(\lambda) = \frac{\varepsilon}{2^3\pi}\int_{\mathbb{T}^2} dkdk'\int_0^{+\infty} dt\int_0^{+\infty} dt' e^{-\lambda\varepsilon(t+t')/2}\int_0^t\int_0^{t'} g(d\sigma)g(d\sigma')\int_{\mathbb{R}} d\beta e^{i\beta(t-t')}$$
$$\times \exp\left\{i\omega(k')(t'-\sigma') - i\omega(k)(t-\sigma)\right\}\mathbb{E}\left\{\hat\psi(k)\hat\psi^*(k')\right\}.$$

Integrating out the t and t' variables we get

$$E_0^{(\varepsilon)}(\lambda) = \frac{\varepsilon}{2^3\pi}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\frac{|\tilde g(\lambda\varepsilon/2 - i\beta)|^2}{\lambda\varepsilon/2 - i\beta + i\omega(k)}\cdot\frac{\mathbb{E}\left\{\hat\psi(k)\hat\psi^*(k')\right\}}{\lambda\varepsilon/2 + i\beta - i\omega(k')}dkdk'd\beta.$$

Next we change variables $\varepsilon\beta' := \beta - \omega(k')$, which leads to

$$E_0^{(\varepsilon)}(\lambda) = \frac{1}{2^3\pi}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\frac{|\tilde g(\lambda\varepsilon/2 - i\varepsilon\beta + i\omega(k))|^2}{\lambda/2 - i\beta + i\varepsilon^{-1}[\omega(k)-\omega(k')]}\cdot\frac{\mathbb{E}\left\{\hat\psi(k)\hat\psi^*(k')\right\}}{\lambda/2 + i\beta}dkdk'd\beta. \tag{5.10}$$

Change variables $(k,k')\mapsto(\eta,\ell)$, by letting

$$k := \ell + \frac{\varepsilon\eta}{2}, \qquad k' := \ell - \frac{\varepsilon\eta}{2}. \tag{5.11}$$

The image of $\mathbb{T}^2$ under this mapping is

$$T_\varepsilon^2 := \left[(\eta,\ell) : |\eta| \le \frac{1}{\varepsilon},\ |\ell| \le \frac{1-\varepsilon|\eta|}{2}\right] \subset \mathbb{T}_{2/\varepsilon}\times\mathbb{T}. \tag{5.12}$$

Then, cf (4.13),

$$E_0^{(\varepsilon)}(\lambda) = \frac{1}{2^2\pi}\int_{T_\varepsilon^2}\widehat{W}_\varepsilon(0,\eta,\ell)d\eta d\ell\int_{\mathbb{R}} d\beta\frac{\left|\tilde g\left(\lambda\varepsilon/2 - i\varepsilon\beta + i\omega(\ell+\frac{\varepsilon\eta}{2})\right)\right|^2}{(\lambda/2 - i\beta + i\delta_\varepsilon\omega(\ell,\eta))(\lambda/2+i\beta)}.$$

Using estimates (2.26), (2.27) and the Cauchy formula (4.15), we obtain

$$\begin{aligned}
\lim_{\varepsilon\to0+} E_0^{(\varepsilon)}(\lambda) &= \frac{1}{2^2\pi}\int_{\mathbb{R}} d\eta\int_{\mathbb{T}} d\ell\int_{\mathbb{R}} d\beta\frac{|\nu(\ell)|^2\,\widehat{W}(0,\eta,\ell)}{(\lambda/2 - i\beta + i\omega'(\ell)\eta)(\lambda/2+i\beta)} \\
&= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{T}}\frac{\widehat{W}(0,\eta,\ell)\,|\nu(\ell)|^2}{\lambda + i\omega'(\ell)\eta}d\eta d\ell.
\end{aligned} \tag{5.13}$$

Asymptotics of $E_n^{(\varepsilon)}(\lambda)$ for $n\ge 1$. Using (3.2) and (2.20) we get

$$E_n^{(\varepsilon)}(\lambda)=\frac{\varepsilon}{2^2}\left(\frac{\gamma}{\mu}\right)^n\int_{\mathbb T}dk\int_{\mathbb T}dk'\mathbb E\left\{\hat\psi(k)\hat\psi^*(k')\right\}\int_0^{+\infty}e^{-\lambda\varepsilon t}dt\int_{\Delta_n(t)}(J\star g)^2(t-s_1)dt$$
$$\times\prod_{j=1}^{n-1}(J\star g)^2(s_j-s_{j+1})\int_0^{s_n}\int_0^{s_n}g(d\sigma_1)g(d\sigma_1')\exp\left\{i\omega(k')(s_n-\sigma_1')-i\omega(k)(s_n-\sigma_1)\right\}.$$

We substitute $\tau_j:=s_j-s_{j+1}$, $j=0,\dots,n$, with $s_0:=t$ and $s_{n+1}:=0$, and then use (4.12) to double variables τ_j and τ_j'. In this way we obtain

$$E_n^{(\varepsilon)}(\lambda)=\frac{\varepsilon}{2^2(2\pi)^{n+3}}\left(\frac{\gamma}{\mu}\right)^n\int_{\mathbb R^2}d\beta d\beta'\int_{\mathbb T^2}dkdk'\int_{(0,+\infty)^2}dtdt'e^{-\lambda\varepsilon(t+t')/4}\mathbb E\left\{\hat\psi(k)\hat\psi^*(k')\right\}$$
$$\times\int_{(0,+\infty)^{n+1}}d\tau_{0,n}\int_{(0,+\infty)^{n+1}}d\tau_{0,n}'\int_{\mathbb R^{n+1}}d\beta_{0,n}\prod_{j=0}^n e^{i\beta_j(\tau_j-\tau_j')}$$
$$\times\exp\left\{-\lambda\varepsilon\left(\sum_{j=0}^n\tau_j\right)/4\right\}\exp\left\{-\lambda\varepsilon\left(\sum_{j=0}^n\tau_j'\right)/4\right\}\exp\left\{i\beta\left(t-\sum_{j=0}^n\tau_j\right)\right\}\exp\left\{i\beta'\left(t'-\sum_{j=0}^n\tau_j'\right)\right\}$$
$$\times\prod_{j=0}^{n-1}(J\star g)(\tau_j)\prod_{j=0}^{n-1}(J\star g)(\tau_j')\int_0^{\tau_n}\int_0^{\tau_n'}g(d\sigma)g(d\sigma')\exp\left\{i\omega(k')(\tau_n'-\sigma')-i\omega(k)(\tau_n-\sigma)\right\}.$$

To abbreviate we have used the notation $d\tau_{0,n}=d\tau_0\dots d\tau_n$, $d\beta_{0,n}=d\beta_0\dots d\beta_n$ and similarly for the prime variables. Integrating out the t, τ variables and their prime counterparts we get

$$E_n^{(\varepsilon)}(\lambda)=\frac{\varepsilon}{2^2(2\pi)^{n+3}}\left(\frac{\gamma}{\mu}\right)^n\int_{\mathbb R^2}d\beta d\beta'\int_{\mathbb T^2}dkdk'\int_{\mathbb R^{n+1}}d\beta_{0,n}\mathbb E\left\{\hat\psi(k)\hat\psi^*(k')\right\}$$
$$\times\prod_{j=0}^{n-1}(\tilde J\tilde g)\left(\lambda\varepsilon/4-i\beta_j+i\beta\right)\prod_{j=0}^{n-1}(\tilde J\tilde g)\left(\lambda\varepsilon/4+i\beta_j+i\beta'\right)$$
$$\frac{1}{\lambda\varepsilon/4-i\beta}\cdot\frac{1}{\lambda\varepsilon/4-i\beta'}\cdot\frac{\tilde g(\lambda\varepsilon/4-i\beta_n+i\beta)}{\lambda\varepsilon/4-i\beta_n+i\beta+i\omega(k)}\cdot\frac{\tilde g(\lambda\varepsilon/4+i\beta_n+i\beta')}{\lambda\varepsilon/4+i\beta_n+i\beta'-i\omega(k')}.\tag{5.14}$$

Change variables k,k' according to (5.11) and

$$\varepsilon\beta_n:=\beta_n-\omega(k'),\quad \varepsilon\beta:=\beta,\quad \varepsilon\beta':=\beta'$$

we obtain

$$E_n^{(\varepsilon)}(\lambda)=\frac{1}{2(2\pi)^{n+3}}\left(\frac{\gamma}{\mu}\right)^n\int_{\mathbb R^2}\frac{d\beta d\beta'}{(\lambda/4-i\beta)(\lambda/4-i\beta')}\int_{T_\varepsilon^2}\widehat W_\varepsilon(0,\eta,\ell)d\eta d\ell$$
$$\times\int_{\mathbb R^{n+1}}d\beta_{0,n}\prod_{j=0}^{n-1}(\tilde J\tilde g)\left(\lambda\varepsilon/4-i\beta_j+i\varepsilon\beta\right)\prod_{j=0}^{n-1}(\tilde J\tilde g)\left(\lambda\varepsilon/4+i\beta_j-i\varepsilon\beta'\right)$$
$$\times\frac{\tilde g(\lambda\varepsilon/4-i\varepsilon\beta_n-i\omega(\ell-\varepsilon\eta/2)+i\varepsilon\beta)}{\lambda/4-i\beta_n+i\beta+i\delta_\varepsilon\omega(\ell,\eta)}\cdot\frac{\tilde g(\lambda\varepsilon/4+i\varepsilon\beta_n+i\omega(\ell-\varepsilon\eta/2)+i\varepsilon\beta')}{\lambda/4+i\beta_n+i\beta'}.$$

Hence

$$\lim_{\varepsilon\to0+}E_n^{(\varepsilon)}(\lambda)=\frac{\Gamma^n}{2\mu^n}\int_{\mathbb R}\int_{\mathbb T}\frac{\widehat W(0,\eta,\ell)|\nu(\ell)|^2}{\lambda+i\omega'(\ell)\eta}d\eta d\ell.\tag{5.15}$$

The conclusion of the proposition then follows from an application of the dominated convergence theorem to the series appearing in (5.9), as $\Gamma/\mu\in(0,1)$. ☐

5.2. **Asymptotics of the term involving $\mathfrak d_\varepsilon(\lambda)$.** Invoking (5.4) we wish to calculate the limit $\lim_{\varepsilon\to0+}\mathfrak L_\varepsilon$, where

$$\mathfrak L_\varepsilon:=\int_{\mathbb T}\int_{\mathbb R}\left[\mathfrak d_\varepsilon\left(\lambda,k-\frac{\varepsilon\eta}{2}\right)+\mathfrak d_\varepsilon^\star\left(\lambda,k+\frac{\varepsilon\eta}{2}\right)\right]\frac{G^\star(\eta,k)d\eta dk}{\lambda+i\delta_\varepsilon\omega(k;\eta)},\tag{5.16}$$

for any $G\in\mathcal S(\mathbb R\times\mathbb T)$.

Taking into account (5.1) and (5.2) we get

$$\mathfrak{d}_\varepsilon(\lambda,k) = \sum_{n=0}^{+\infty} D_n^\varepsilon(\lambda,k), \tag{5.17}$$

where

$$D_0^\varepsilon(\lambda,k) = D_{0,1}^\varepsilon(\lambda,k) + D_{0,2}^\varepsilon(\lambda,k) \tag{5.18}$$

and

$$\begin{aligned} D_{0,1}^\varepsilon(\lambda,k) &:= i\varepsilon \int_0^{+\infty} e^{-\lambda\varepsilon t} e^{i\omega(k)t} \mathbb{E}\left[\hat{\psi}^\star(0,k) g \star \mathfrak{p}_0^0(t)\right] dt, \\ D_{0,2}^\varepsilon(\lambda,k) &:= -\varepsilon\gamma \int_0^{+\infty} e^{-\lambda\varepsilon t} dt \int_0^t \phi^\star(t-s,k) \mathbb{E}\left[\mathfrak{p}_0^0(s) g \star \mathfrak{p}_0^0(t)\right] ds, \\ D_n^\varepsilon(\lambda,k) &:= \varepsilon \left(\frac{\gamma}{\mu}\right)^n \int_0^{+\infty} e^{-\lambda t} dt \int_{\Delta_n(t)} \phi^\star(t-s_1,k)(J\star g)(t-s_1) \\ &\times \prod_{j=1}^{n-1} (J\star g)^2(s_j - s_{j+1}) \mathbb{E}\left[\left(g\star \mathfrak{p}_0^0(s_n)\right)^2\right] ds_1 \dots ds_n,\ n\ge 1. \end{aligned} \tag{5.19}$$

Accordingly we can write $\mathfrak{L}_\varepsilon = \sum_{n=0}^{+\infty} \mathfrak{L}_\varepsilon^{(n)}$, where

$$\mathfrak{L}_\varepsilon^{(n)} := \int_{\mathbb{R}} \int_{\mathbb{T}} \left[D_n^\varepsilon\left(\lambda, k - \frac{\varepsilon\eta}{2}\right) + (D_n^\varepsilon)^* \left(\lambda, k + \frac{\varepsilon\eta}{2}\right)\right] \frac{G^\star(\eta,k)}{\lambda + i\delta_\varepsilon\omega(k;\eta)} d\eta dk. \tag{5.20}$$

5.2.1. *Computation of $D_{0,1}^\varepsilon(\lambda,k)$.* The term $D_{0,1}^\varepsilon(\lambda,k)$ coincides with $\mathfrak{d}_\varepsilon^1(\lambda,k)$ defined in [5, formulas (5.6) and (5.7)]. Therefore, see [5, Lemma 5.1], we have the following result.

Lemma 5.3. *For any test function $G \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$ and $\lambda > 0$ we have*

$$\begin{aligned} &-\frac{\gamma}{2} \lim_{\varepsilon\to 0+} \int_{\mathbb{R}\times\mathbb{T}} \frac{G^\star(\eta,k)}{\lambda + i\delta_\varepsilon\omega(k,\eta)} \left\{ D_{0,1}^\varepsilon\left(\lambda, k - \frac{\varepsilon\eta}{2}\right) + \left(D_{0,1}^\varepsilon\right)^*\left(\lambda, k + \frac{\varepsilon\eta}{2}\right)\right\} d\eta dk \\ &= -\gamma \int_{\mathbb{R}\times\mathbb{T}} \mathrm{Re}[\nu(k)] \frac{\widehat{W}(0,\eta',k)}{\lambda + i\omega'(k)\eta'} \left\{ \int_{\mathbb{R}} \frac{G^\star(\eta,k)}{\lambda + i\omega'(k)\eta} d\eta \right\} dk d\eta'. \end{aligned} \tag{5.21}$$

5.2.2. *Asymptotics of $D_{0,2}^\varepsilon(\lambda,k)$.* Using (4.4) we can write

$$D_{0,2}^\varepsilon(\lambda,k) = -\varepsilon\gamma \int_0^{+\infty} e^{-\lambda\varepsilon t} dt \int_0^t ds \exp\{i\omega(k)(t-s)\} \mathbb{E}\left[g\star\mathfrak{p}_0^0(s) g \star \mathfrak{p}_0^0(t)\right]$$

The expression for $D_{0,2}^\varepsilon(\lambda,k)$ is therefore identical with $\mathfrak{d}_\varepsilon^2(\lambda,k)$ defined by [5, formulas (5.6) and (5.7)]. We have therefore, see [5, Lemma 5.2].

Lemma 5.4. *For any $\lambda > 0$ and $G \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$ we have*

$$\begin{aligned} &-\frac{\gamma}{2} \lim_{\varepsilon\to 0} \int_{\mathbb{R}\times\mathbb{T}} \left[D_{0,2}^\varepsilon\left(\lambda, k - \frac{\varepsilon\eta}{2}\right) + \left(D_{0,2}^\varepsilon\right)^*\left(\lambda, k + \frac{\varepsilon\eta}{2}\right)\right] \frac{\hat{G}^\star(\eta,k) d\eta dk}{\lambda + i\delta_\varepsilon\omega(k,\eta)} \\ &= \frac{\gamma}{4} \int_{\mathbb{R}\times\mathbb{T}} \frac{\mathfrak{g}(k)\widehat{W}(0,\eta',k) d\eta' dk}{\lambda + i\omega'(k)\eta'} \int_{\mathbb{R}} \frac{\hat{G}^\star(\eta,k) d\eta}{\lambda + i\omega'(k)\eta} \\ &+ \frac{\gamma}{4} \int_{\mathbb{R}\times\mathbb{T}} \frac{\mathfrak{g}(k)\widehat{W}(0,\eta',-k) d\eta' dk}{\lambda - i\omega'(k)\eta'} \int_{\mathbb{R}} \frac{\hat{G}^\star(\eta,k) d\eta}{\lambda + i\omega'(k)\eta}. \end{aligned} \tag{5.22}$$

Summarizing, taking into account definitions (2.41), we have

$$-\frac{\gamma}{2} \lim_{\varepsilon\to 0} \mathfrak{L}_\varepsilon^{(0)} = \frac{\left(p_+(k) - 1\right)|\bar{\omega}'(k)|}{\lambda + i\omega'(k)\eta} \int_{\mathbb{R}\times\mathbb{T}} \frac{\widehat{W}(0,\eta',k) d\eta'}{\lambda + i\omega'(k)\eta'} + \frac{p_-(k)|\bar{\omega}'(k)|}{\lambda + i\omega'(k)\eta} \int_{\mathbb{R}} \frac{\widehat{W}(0,\eta',-k) d\eta'}{\lambda - i\omega'(k)\eta'}. \tag{5.23}$$

5.2.3. *Asymptotics of* $\sum_{n=1}^{+\infty} D_n^\varepsilon(\lambda,k)$. We prove the following.

Lemma 5.5. *For any* $\lambda > 0$ *we have*

$$-\frac{\gamma}{2}\lim_{\varepsilon\to 0}\sum_{n=1}^{+\infty}\int_{\mathbb{R}\times\mathbb{T}}\left[D_n^\varepsilon\left(\lambda,k-\frac{\varepsilon\eta}{2}\right)+(D_n^\varepsilon)^*\left(\lambda,k+\frac{\varepsilon\eta}{2}\right)\right]\frac{\hat{G}^*(\eta,k)d\eta dk}{\lambda+i\delta_\varepsilon\omega(k,\eta)} \tag{5.24}$$
$$=-\frac{\gamma}{2\mu(1-\Gamma/\mu)}\int_{\mathbb{R}\times\mathbb{T}}\frac{G^*(\eta,k)[1-|\nu(k)|^2]d\eta dk}{\lambda+i\omega'(k)\eta}\int_{\mathbb{R}\times\mathbb{T}}\frac{|\nu(\ell)|^2\widehat{W}(0,\eta',\ell)d\eta' d\ell}{\lambda+i\omega'(\ell)\eta'}.$$

The proof of the lemma is presented in Section 5.2.5. It requires some auxiliary calculations that are done in Section 5.2.4.

5.2.4. *Auxiliary calculations.* We suppose that $n\ge 1$. Using the change of variables $\tau_j := s_j - s_{j+1}$, $j=0,\dots,n$, with $s_0 := t$ and $s_{n+1} := 0$ in the last formula of (5.19) and then (4.12) we get

$$D_n^\varepsilon(\lambda,k)=\frac{\varepsilon}{2\pi}\left(\frac{\gamma}{\mu}\right)^n\int_0^{+\infty}e^{-\lambda\varepsilon t/2}dt\int_{\mathbb{R}}d\beta\int_{(0,+\infty)^{n+1}}d\tau_{0,n}\exp\left\{i\beta\left(t-\sum_{j=0}^n\tau_j\right)\right\} \tag{5.25}$$
$$\times\exp\left\{-\lambda\varepsilon\left(\sum_{j=0}^n\tau_j\right)/2\right\}\phi^*(\tau_0,k)(J\star g)(\tau_0)\prod_{j=1}^{n-1}(J\star g)^2(\tau_j)\mathbb{E}\left[\left(g\star\mathfrak{p}_0^0(\tau_n)\right)^2\right],\ n\ge 1.$$

Doubling the τ_j variables, via (4.12), we get

$$D_n^\varepsilon(\lambda,k)=\frac{\varepsilon}{(2\pi)^{n+2}}\left(\frac{\gamma}{\mu}\right)^n\int_0^{+\infty}dt\int_{\mathbb{R}^{n+2}}d\beta_{0,n}d\beta\int_{(0,+\infty)^{n+1}}d\tau_{0,n}\int_{(0,+\infty)^{n+1}}d\tau'_{0,n}$$
$$\times e^{-\lambda\varepsilon t/2}\prod_{j=0}^n\exp\left\{i\beta_j(\tau_j-\tau'_j)\right\}\exp\left\{i\beta\left(t-\frac{1}{2}\sum_{j=0}^n\tau_j-\frac{1}{2}\sum_{j=0}^n\tau'_j\right)\right\}$$
$$\times\exp\left\{-\lambda\varepsilon\left(\sum_{j=0}^n\tau_j\right)/4\right\}\exp\left\{-\lambda\varepsilon\left(\sum_{j=0}^n\tau'_j\right)/4\right\}$$
$$\times\phi^*(\tau'_0,k)(J\star g)(\tau_0)\prod_{j=1}^{n-1}(J\star g)(\tau_j)\prod_{j=1}^{n-1}(J\star g)(\tau'_j)\mathbb{E}\left[\left(g\star\mathfrak{p}_0^0(\tau_n)\right)\left(g\star\mathfrak{p}_0^0(\tau'_n)\right)\right].$$

Integrating out the t, τ and τ' variables we get

$$D_n^\varepsilon(\lambda,k)=\frac{\varepsilon}{(2\pi)^{n+2}}\left(\frac{\gamma}{\mu}\right)^n\int_{\mathbb{R}}\frac{d\beta}{\lambda\varepsilon/2-i\beta}\int_{\mathbb{R}^{n+1}}d\beta_{0,n}(\tilde{J}\tilde{g})(\lambda\varepsilon/4-i\beta_0+i\beta/2)\,\tilde{\phi}^*(\lambda\varepsilon/4-i\beta_0-i\beta/2,k)$$
$$\times\prod_{j=1}^{n-1}(\tilde{J}\tilde{g})(\lambda\varepsilon/4-i\beta_j+i\beta/2)\prod_{j=1}^{n-1}(\tilde{J}\tilde{g})(\lambda\varepsilon/4+i\beta_j+i\beta/2)\,\tilde{g}(\lambda\varepsilon/4-i\beta_n+i\beta/2)\tilde{g}(\lambda\varepsilon/4+i\beta_n+i\beta/2) \tag{5.26}$$
$$\times\mathbb{E}\left[\tilde{\mathfrak{p}}_0^0(\lambda\varepsilon/4-i\beta_n+i\beta/2)\tilde{\mathfrak{p}}_0^0(\lambda\varepsilon/4+i\beta_n+i\beta/2)\right].$$

Here

$$\tilde{\phi}(\lambda,k)=\frac{\tilde{g}(\lambda)}{\lambda+i\omega(k)}$$

and

$$\tilde{\mathfrak{p}}_0^0(\lambda)=\frac{1}{2i}\int_{\mathbb{T}}\left\{\frac{\hat{\psi}(\ell)}{\lambda+i\omega(\ell)}-\frac{\hat{\psi}^*(\ell)}{\lambda-i\omega(\ell)}\right\}d\ell$$

are the Laplace transforms of $\phi(t,k)$ and $\mathfrak{p}_0^0(t)$, respectively.

Thanks to (2.20) we have

$$\mathbb{E}\big[\tilde{\mathfrak{p}}_0^0(\lambda_1)\tilde{\mathfrak{p}}_0^0(\lambda_2)\big] = \frac{1}{2^2}\int_{\mathbb{T}} d\ell \int_{\mathbb{T}} d\ell' \left\{ \frac{\mathbb{E}\big[\hat{\psi}(\ell)\hat{\psi}^*(\ell')\big]}{\big(\lambda_1 + i\omega(\ell)\big)\big(\lambda_2 - i\omega(\ell')\big)} + \frac{\mathbb{E}\big[\hat{\psi}(\ell')\hat{\psi}^*(\ell)\big]}{\big(\lambda_1 - i\omega(\ell)\big)\big(\lambda_2 + i\omega(\ell')\big)} \right\}$$

Substituting into (5.26) we get

$$\begin{aligned} D_n^\varepsilon(\lambda,k) &= \frac{\varepsilon}{2^2(2\pi)^{n+2}}\left(\frac{\gamma}{\mu}\right)^n \int_{\mathbb{T}^2} d\ell d\ell' \int_{\mathbb{R}} \frac{d\beta}{\lambda\varepsilon/2 - i\beta} \int_{\mathbb{R}^{n+1}} d\beta_{0,n} (\tilde{J}\tilde{g})\,(\lambda\varepsilon/4 - i\beta_0 + i\beta/2) \\ &\times \frac{\tilde{g}(\lambda\varepsilon/4 + i\beta_0 + i\beta/2)}{\lambda\varepsilon/4 + i\beta_0 + i\beta/2 - i\omega(k)} \tilde{g}(\lambda\varepsilon/4 - i\beta_n + i\beta/2)\tilde{g}(\lambda\varepsilon/4 + i\beta_n + i\beta/2) \\ &\times \prod_{j=1}^{n-1} (\tilde{J}\tilde{g})\,(\lambda\varepsilon/4 - i\beta_j + i\beta/2) \prod_{j=1}^{n-1} (\tilde{J}\tilde{g})\,(\lambda\varepsilon/4 + i\beta_j + i\beta/2) \\ &\times \left\{ \frac{\mathbb{E}\big[\hat{\psi}(\ell)\hat{\psi}^*(\ell')\big]}{[\lambda\varepsilon/4 - i\beta_n + i\beta/2 + i\omega(\ell)][\lambda\varepsilon/4 + i\beta_n + i\beta/2 - i\omega(\ell')]} \right. \\ &\left. + \frac{\mathbb{E}\big[\hat{\psi}^*(\ell)\hat{\psi}(\ell')\big]}{[\lambda\varepsilon/4 - i\beta_n + i\beta/2 - i\omega(\ell)][\lambda\varepsilon/4 + i\beta_n + i\beta/2 + i\omega(\ell')]} \right\}. \end{aligned}$$

Change variables $\beta_j' := \beta_j + \beta/2,\ j = 0,\dots,n$ and integrate out the β variable, using (4.15). We can write then

$$D_n^\varepsilon(\lambda,k) = \frac{1}{4\mu^n}\left(\frac{\gamma}{2\pi}\right)^{n-1} I_\varepsilon II_\varepsilon \int_{\mathbb{R}^n} d\beta_{1,n} \prod_{j=1}^{n-1} (\tilde{J}\tilde{g})\,(3\lambda\varepsilon/4 - i\beta_j) \prod_{j=1}^{n-1} (\tilde{J}\tilde{g})\,(\lambda\varepsilon/4 + i\beta_j)\,, \tag{5.27}$$

where

$$I_\varepsilon := \frac{\gamma}{2\pi}\int_{\mathbb{R}} (\tilde{J}\tilde{g})\,(3\lambda\varepsilon/4 - i\beta_0)\,\frac{\tilde{g}(\lambda\varepsilon/4 + i\beta_0)}{\lambda\varepsilon/4 + i\beta_0 - i\omega(k)} d\beta_0 \tag{5.28}$$

and

$$\begin{aligned} II_\varepsilon := \frac{\varepsilon}{2\pi}\int_{\mathbb{T}^2} d\ell d\ell' \int_{\mathbb{R}} &\tilde{g}(3\lambda\varepsilon/4 - i\beta_n)\tilde{g}(\lambda\varepsilon/4 + i\beta_n) \\ \times &\left\{ \frac{\mathbb{E}\big[\hat{\psi}(\ell)\hat{\psi}^*(\ell')\big]}{[3\lambda\varepsilon/4 - i\beta_n + i\omega(\ell)][\lambda\varepsilon/4 + i\beta_n - i\omega(\ell')]} \right. \\ &\left. + \frac{\mathbb{E}\big[\hat{\psi}^*(\ell)\hat{\psi}(\ell')\big]}{[3\lambda\varepsilon/4 - i\beta_n - i\omega(\ell)][\lambda\varepsilon/4 + i\beta_n + i\omega(\ell')]} \right\} d\beta_n. \end{aligned} \tag{5.29}$$

5.2.5. *The end of the proof of Lemma 5.5.* Using formula (5.27) we conclude, cf (5.20) and (2.46), that

$$\lim_{\varepsilon\to 0+} \mathfrak{L}_\varepsilon^{(n)} = \lim_{\varepsilon\to 0+} \bar{\mathfrak{L}}_\varepsilon^{(n)}, \tag{5.30}$$

where

$$\bar{\mathfrak{L}}_\varepsilon^{(n)} := 2\int_{\mathbb{R}}\int_{\mathbb{T}} \operatorname{Re} \tilde{D}_n^\varepsilon\,(\lambda,k)\,\frac{G^\star(\eta,k)}{\lambda + i\omega'(k)\eta} d\eta dk. \tag{5.31}$$

Here

$$\tilde{D}_n^\varepsilon(\lambda,k) := \frac{\Gamma^{n-1}}{4\mu^n} I_\varepsilon\, II_\varepsilon.$$

The calculation of the limit (5.30) reduces therefore to computing the limits of I_ε and II_ε.

Computation of $\lim_{\varepsilon\to0+} I_\varepsilon$. Since $\tilde g(\lambda)=1-\gamma\tilde J\tilde g(\lambda)$ we can write $I_\varepsilon=I_\varepsilon^1+I_\varepsilon^2$, where

$$I_\varepsilon^1:=\frac{\gamma}{2\pi}\int_{\mathbb R}\frac{(\tilde J\tilde g)\,(3\lambda\varepsilon/4-i\beta_0)}{\lambda\varepsilon/4+i\beta_0-i\omega(k)}d\beta_0$$
$$I_\varepsilon^2:=-\frac{\gamma^2}{2\pi}\int_{\mathbb R}(\tilde J\tilde g)\,(3\lambda\varepsilon/4-i\beta_0)\,\frac{(\tilde J\tilde g)(\lambda\varepsilon/4+i\beta_0)}{\lambda\varepsilon/4+i\beta_0-i\omega(k)}d\beta_0.$$

Using (4.15) we get

$$I_\varepsilon^1=\frac{\gamma}{2\pi}\int_{\mathbb R}\frac{(\tilde J\tilde g)\,(3\lambda\varepsilon/4-i\beta_0)}{\lambda\varepsilon/4+i\beta_0-i\omega(k)}d\beta_0=\gamma(\tilde J\tilde g)\,(\lambda\varepsilon-i\omega(k))\,.$$

Therefore

$$\lim_{\varepsilon\to0+}I_\varepsilon^1=1-\nu(k).\tag{5.32}$$

On the other hand

$$\lim_{\varepsilon\to0}(\tilde J\tilde g)\,(3\lambda\varepsilon/4-i\beta_0)\,(\tilde J\tilde g)(\lambda\varepsilon/4+i\beta_0)=|(\tilde J\tilde g)|^2(i\beta_0)$$

in any $L^p(\mathbb R)$, $p\in(1,+\infty)$ and pointwise. Therefore,

$$\lim_{\varepsilon\to0+}I_\varepsilon^2=-\frac{\gamma^2}{2\pi}\lim_{\varepsilon\to0+}\left\{\int_{\mathbb R}\frac{|(\tilde J\tilde g)\,(i\beta_0)\,|^2d\beta_0}{\lambda\varepsilon/4+i\beta_0-i\omega(k)}\right\}.$$

Since $j(\beta):=|(\tilde J\tilde g)\,(i\beta_0)\,|^2$ belongs to any $L^p(\mathbb R)$ for $p\in[1,+\infty)$, by the multiplier theorem, see e.g. [9, Corollary of Theorem 3, p. 96]

$$\lim_{\varepsilon\to0+}\int_{\mathbb R}\frac{j(\beta_0)d\beta_0}{\lambda\varepsilon/4+i\beta_0-i\beta}=\mathfrak j(\eta):=2\pi\int_{-\infty}^0 e^{2\pi i\eta\beta}\hat j(\eta)d\eta,$$

in the $L^p(\mathbb R)$ sense, for any $p\in(1,+\infty)$. Here

$$\hat j(\eta):=\int_{\mathbb R}e^{-2\pi i\eta\beta}j(\beta)d\beta$$

is the Fourier transform of j. Therefore

$$\lim_{\varepsilon\to0+}I_\varepsilon^2=-\frac{\gamma^2}{2\pi}\mathfrak j(\omega(k))\tag{5.33}$$

in the $L^p(\mathbb T)$ sense for any $p\in[1,2)$. We have shown therefore that

$$\lim_{\varepsilon\to0+}I_\varepsilon=I:=1-\nu(k)-\frac{\gamma^2}{2\pi}\mathfrak j(\omega(k))\tag{5.34}$$

in the $L^p(\mathbb T)$ sense for any $p\in[1,2)$. Since j is real valued we have

$$\frac{1}{2\pi}\mathfrak j(\eta)=\frac12 j(\beta)=\frac12|(\tilde J\tilde g)\,(i\omega(k))\,|^2.\tag{5.35}$$

Thus, using the relation

$$\gamma(\tilde J\tilde g)(\lambda)=1-\tilde g(\lambda),$$

we conclude that

$$\begin{aligned}\operatorname{Re}I&:=1-\operatorname{Re}\nu(k)-\frac{\gamma^2}{2}|(\tilde J\tilde g)\,(i\omega(k))\,|^2\\&=1-\operatorname{Re}\nu(k)-\frac12|1-\nu(k)|^2=\frac12\left(1-|\nu(k)|^2\right).\end{aligned}\tag{5.36}$$

Computation of $\lim_{\varepsilon\to0+} II_\varepsilon$. We have $II_\varepsilon = II^1_\varepsilon + II^2_\varepsilon$, where

$$II^1_\varepsilon := \frac{\varepsilon}{2\pi}\int_{\mathbb{T}^2} d\ell d\ell' \int_{\mathbb{R}} d\beta_n \frac{\tilde g(3\lambda\varepsilon/4 - i\beta_n)\tilde g(\lambda\varepsilon/4 + i\beta_n)\mathbb{E}\left[\hat\psi(\ell)\hat\psi^*(\ell')\right]}{[3\lambda\varepsilon/4 - i\beta_n + i\omega(\ell)][\lambda\varepsilon/4 + i\beta_n - i\omega(\ell')]}$$

$$II^2_\varepsilon := \frac{\varepsilon}{2\pi}\int_{\mathbb{T}^2} d\ell d\ell' \int_{\mathbb{R}} d\beta_n \frac{\tilde g(3\lambda\varepsilon/4 - i\beta_n)\tilde g(\lambda\varepsilon/4 + i\beta_n)\mathbb{E}\left[\hat\psi^*(\ell)\hat\psi(\ell')\right]}{[3\lambda\varepsilon/4 - i\beta_n - i\omega(\ell)][\lambda\varepsilon/4 + i\beta_n + i\omega(\ell')]}.$$

Changing variables $\varepsilon\beta_n' := \beta_n - \omega(\ell')$ we obtain

$$II^1_\varepsilon := \frac{1}{2\pi}\int_{\mathbb{T}^2} d\ell d\ell' \int_{\mathbb{R}} d\beta_n \frac{\tilde g(3\lambda\varepsilon/4 - i\varepsilon\beta_n - i\omega(\ell))\tilde g(\lambda\varepsilon/4 + i\varepsilon\beta_n + i\omega(\ell))\mathbb{E}\left[\hat\psi(\ell)\hat\psi^*(\ell')\right]}{[3\lambda/4 - i\beta_n + i\varepsilon^{-1}\big(\omega(\ell) - \omega(\ell')\big)][\lambda/4 + i\beta_n]}.$$

Therefore

$$\lim_{\varepsilon\to0+} II^1_\varepsilon = \frac{1}{2\pi}\lim_{\varepsilon\to0+}\int_{\mathbb{T}^2} \frac{|\nu(\ell)|^2\mathbb{E}\left[\hat\psi(\ell)\hat\psi^*(\ell')\right]}{\lambda + i\varepsilon^{-1}\big(\omega(\ell) - \omega(\ell')\big)} d\ell d\ell'.$$

Changing again variables

$$\ell = \tilde\ell + \frac{\varepsilon\eta}{2}, \quad \ell' = \tilde\ell - \frac{\varepsilon\eta}{2}$$

we conclude that

$$\lim_{\varepsilon\to0+} II^1_\varepsilon = 2\int_{\mathbb{R}\times\mathbb{T}} \frac{|\nu(\ell)|^2\widehat W(0,\eta,\ell)}{\lambda + i\omega'(\ell)\eta} d\eta d\ell. \tag{5.37}$$

A similar calculation proves that also

$$\lim_{\varepsilon\to0+} II^2_\varepsilon = 2\int_{\mathbb{R}\times\mathbb{T}} \frac{|\nu(\ell)|^2\widehat W(0,\eta,\ell)}{\lambda + i\omega'(\ell)\eta} d\eta d\ell. \tag{5.38}$$

We conclude therefore

$$II = \lim_{\varepsilon\to0+} II_\varepsilon = 4\int_{\mathbb{R}\times\mathbb{T}} \frac{|\nu(\ell)|^2\widehat W(0,\eta,\ell)}{\lambda + i\omega'(\ell)\eta} d\eta d\ell. \tag{5.39}$$

The right hand side of (5.39) is real valued. Gathering all the facts proven above we conclude that

$$\lim_{\varepsilon\to0+} \bar{\mathfrak{L}}^{(n)}_\varepsilon = \frac{\Gamma^{n-1}}{2\mu^n}\int_{\mathbb{R}\times\mathbb{T}} II \,\mathrm{Re}\, I \frac{G^\star(\eta,k)}{\lambda + i\omega'(k)\eta} d\eta dk \tag{5.40}$$

$$= \frac{\Gamma^{n-1}}{\mu^n}\int_{\mathbb{R}\times\mathbb{T}} \frac{\big(1 - |\nu(k)|^2\big)G^\star(\eta,k)}{\lambda + i\omega'(k)\eta} d\eta dk \int_{\mathbb{R}\times\mathbb{T}} \frac{|\nu(\ell)|^2\widehat W(0,\eta',\ell)}{\lambda + i\omega'(\ell)\eta'} d\eta' d\ell.$$

Combining this with formula (5.31) we conclude the proof of Lemma 5.5. □

5.3. **Proof of Proposition 5.1.** According to (5.4) for any we have

$$\int_{\mathbb{R}\times\mathbb{T}} \widehat{w}_\varepsilon(\lambda,\eta,k)G^*(\eta,k)d\eta dk = \sum_{j=1}^{3}\mathcal{W}^{(\varepsilon)}_j, \quad \text{where}$$

$$\mathcal{W}^{(\varepsilon)}_1 := \int_{\mathbb{R}\times\mathbb{T}} \frac{W_\varepsilon(0,\eta,k)G^*(\eta,k)}{\lambda + i\delta_\varepsilon\omega(k;\eta)} d\eta dk$$

$$\mathcal{W}^{(\varepsilon)}_2 := \frac{\gamma\mathfrak{e}_\varepsilon(\lambda)}{\mu}\int_{\mathbb{R}\times\mathbb{T}} \frac{G^*(\eta,k)}{\lambda + i\delta_\varepsilon\omega(k;\eta)} d\eta dk \tag{5.41}$$

$$\mathcal{W}^{(\varepsilon)}_3 := -\frac{\gamma}{2}\int_{\mathbb{R}\times\mathbb{T}} \frac{G^*(\eta,k)}{\lambda + i\delta_\varepsilon\omega(k;\eta)}\left[\mathfrak{d}_\varepsilon\left(\lambda, k - \frac{\varepsilon\eta}{2}\right) + \mathfrak{d}^\star_\varepsilon\left(\lambda, k + \frac{\varepsilon\eta}{2}\right)\right] d\eta dk.$$

It is easy to see that the limit of $\mathcal{W}^{(\varepsilon)}_1$, as $\varepsilon\to0+$, corresponds to the first term in the right hand side of (5.6). Using Proposition 5.2 we conclude that the limit of $\mathcal{W}^{(\varepsilon)}_2$ matches the second term there. Finally $\mathcal{W}^{(\varepsilon)}_3 = -\frac{\gamma}{2}\sum_{n=0}^{+\infty}\mathfrak{L}^{(n)}_\varepsilon$ and the respective limit is a consequence of Lemmas 5.3, 5.4 and 5.5. This ends the proof of the proposition. □

5.4. **The end of the proof of Theorem 2.4.** Using the equality (3.8) and the results of Proposition 4.3 (for $\mu > 1/2$), Proposition 4.4 (for $\mu = 1/2$) and Proposition 4.3, together with formula (5.6) we conclude that for any $\lambda \in \mathbb{C}_+$ the Laplace-Fourier-Wigner functions $\widehat{w}_\varepsilon(\lambda,\eta,k)$ converge, as $\varepsilon \to 0+$, in $\mathcal{A}'$, in the $\star$-weak topology to

$$\widehat{w}(\lambda,\eta,k) = \frac{\widehat{W}(0,\eta,k)}{\lambda + i\omega'(k)\eta} + \frac{\gamma T|\nu(k)|^2}{(1-\Gamma/\mu)\lambda(\lambda + i\omega'(k)\eta)}\left(1 - \frac{1}{2\mu}\right) \tag{5.42}$$
$$+ \frac{\gamma|\nu(k)|^2}{2\mu[\lambda + i\omega'(k)\eta](1-\Gamma/\mu)}\int_{\mathbb{R}\times\mathbb{T}} \frac{|\nu(\ell)|^2\widehat{W}(0,\eta',\ell)}{\lambda + i\omega'(\ell)\eta'}d\eta' d\ell - \frac{\gamma \mathrm{Re}[\nu(k)]}{\lambda + i\omega'(k)\eta}\int_{\mathbb{R}} \frac{\widehat{W}(0,\eta',k)}{\lambda + i\omega'(k)\eta'}d\eta'$$
$$+ \frac{\gamma\, \mathfrak{g}(k)}{4(\lambda + i\omega'(k)\eta)}\int_{\mathbb{R}} \frac{\widehat{W}(0,\eta',k)d\eta'}{\lambda + i\omega'(k)\eta'} + \frac{\gamma\, \mathfrak{g}(k)}{4(\lambda + i\omega'(k)\eta)}\int_{\mathbb{R}} \frac{\widehat{W}(0,\eta',-k)d\eta'}{\lambda - i\omega'(k)\eta'}.$$

Inverting both the Laplace transform in t and Fourier transform in x we obtain (2.51), which ends the proof of the theorem. □

6. Proofs of Lemmas 2.1 and 2.2

6.1. **Proof of Lemma 2.1.** We have

$$\tilde{J}(\lambda) = G(\lambda) + H(\lambda), \tag{6.1}$$

where

$$G(\lambda) := \frac{1}{2}\int_{\mathbb{T}_+} \frac{d\ell}{\lambda + i\omega(\ell)} \qquad H(\lambda) := \frac{1}{2}\int_{\mathbb{T}_+} \frac{d\ell}{\lambda - i\omega(\ell)}. \tag{6.2}$$

Thanks to (6.1) and (2.32) we conclude that

$$|(\tilde{g}\tilde{J})(\lambda)| \le \frac{1}{|\lambda| - \omega_{\max}}, \quad |\lambda| > \omega_{\max},\ \mathrm{Re}\,\lambda > 0. \tag{6.3}$$

On the other hand, thanks to (2.32) and (2.33), we have also

$$|(\tilde{g}\tilde{J})(\lambda)| \le \frac{2}{\gamma}, \quad \mathrm{Re}\,\lambda > 0. \tag{6.4}$$

As a result $\tilde{g}\tilde{J} \in H^p(\mathbb{C}_+)$ for any $p \in (1,+\infty)$. The limits in (2.38) and (2.39) can be substantiated by the results of Sections A and B of Chapter 6 of [6].

Recall that $\omega_+^{-1}(\cdot)$ is the inverse of the restriction $\omega_{[0,1/2]}$. From (6.2) we get

$$G(\varepsilon + i\omega(k)) = \frac{1}{2}\int_{\omega_{\min}}^{\omega_{\max}} \frac{dv}{\omega'(\omega_+^{-1}(v))[\varepsilon + i(v + \omega(k))]}$$

To simplify assume that $k \in [0,1/2]$. It is clear that

$$\lim_{\varepsilon\to 0+} G(\varepsilon + i\omega(k)) = G(i\omega(k)) = -\frac{i}{2}\int_{\omega_{\min}}^{\omega_{\max}} \frac{dv}{\omega'(\omega_+^{-1}(v))(v + \omega(k))}$$

and there exists $C > 0$ such that

$$\Big|G(\varepsilon + i\omega(k)) - G(i\omega(k))\Big| \le C\varepsilon, \quad k \in \Omega_*^{(\delta)}, \tag{6.5}$$

where $\Omega_*^{(\delta)} := [k \in \mathbb{T} : \mathrm{dist}(k,\Omega_*) \ge \delta]$. Concerning $H(\cdot)$ we have

$$H(\varepsilon + i\omega(k)) = \frac{1}{2}\int_{\omega_{\min}}^{\omega_{\max}} \frac{dv}{\omega'(\omega_+^{-1}(v))[\varepsilon + i(\omega(k) - v)]}$$

A simple calculation leads to

$$H(i\omega(k)) := \lim_{\varepsilon\to 0+} H(\varepsilon + i\omega(k)) = \frac{1}{2\omega'(k)}\left[\pi + i\log\left(\frac{\omega_{\max}-\omega(k)}{\omega(k)-\omega_{\min}}\right)\right]$$
$$+\frac{i}{2}\int_{\omega_{\min}}^{\omega_{\max}} \frac{[\omega'(k)-\omega'(\omega_+^{-1}(v))]dv}{\omega'(\omega_+^{-1}(v))\omega'(k)(\omega(k)-v)}.$$

Since $\omega'(\cdot)$ is Lipschitz the integral in the right hand side makes sense. A straightforward calculation implies the existence of $C>0$ such that

$$\Big|H(\varepsilon + i\omega(k)) - H(i\omega(k))\Big| \le C\varepsilon, \qquad k\in\Omega_*^{(\delta)}. \tag{6.6}$$

From (6.5) and (6.6) we conclude (2.40). In addition we infer also the continuity of ν on $\mathbb{T}\smallsetminus\Omega_*$. □

6.2. **Proof of Lemma 2.2.**

6.2.1. *Some preliminaries.* With some abuse of notation we denote by $\mathcal{A}$ the Banach space of all matrix valued functions obtained by the completion of functions of the form

$$\mathbf{F}(y,k) = \begin{bmatrix} G(y,k) & H(y,k) \\ H^\star(y,k) & G(y,-k)\end{bmatrix}, \quad (y,k)\in\mathbb{R}\times\mathbb{T}, \tag{6.7}$$

with C^∞ smooth entries satisfying G is real valued and H is complex valued and even in k. The completion is taken in the norm given by the maximum of the $\mathcal{A}$ norms of the entries, see (2.2). The Fourier transform in the y variable shall be denoted by

$$\widehat{\mathbf{F}}(y,k) = \begin{bmatrix} \widehat{G}_+(\eta,k) & \widehat{H}_+(\eta,k) \\ \widehat{H}_-(\eta,k) & \widehat{G}_-(\eta,k)\end{bmatrix}, \quad (\eta,k)\in\mathbb{R}\times\mathbb{T}. \tag{6.8}$$

We have $\widehat{G}_-(\eta,k) = \widehat{G}_+(\eta,-k) = \widehat{G}_+^\star(-\eta,-k)$ and $\widehat{H}_-(\eta,k) = \widehat{H}_+^\star(-\eta,k)$.

The Wigner distribution, corresponding to the wave function $\psi^{(\varepsilon)}(t)$, is a 2×2-matrix tensor $\mathbf{W}_\varepsilon(t)$, whose entries are distributions belonging to $\mathcal{A}'$ - the dual to $\mathcal{A}$, given by their respective Fourier transforms

$$\widehat{\mathbf{W}}_\varepsilon(t,\eta,k) := \begin{bmatrix} \widehat{W}_{\varepsilon,+}(t,\eta,k) & \widehat{Y}_{\varepsilon,+}(t,\eta,k) \\ \widehat{Y}_{\varepsilon,-}(t,\eta,k) & \widehat{W}_{\varepsilon,-}(t,\eta,k)\end{bmatrix}, \quad (\eta,k)\in\mathbb{T}_\varepsilon\times\mathbb{T}, \tag{6.9}$$

with the entries defined in (2.22).

Using (2.18) with $\mu=1$ we obtain the following system of equations for the evolution of the tensor $\widehat{\mathbf{W}}_\varepsilon(t)$:

$$\frac{d}{dt}\mathbf{W}_\varepsilon(t) = (\mathfrak{H}_\varepsilon + \gamma\mathfrak{T}'_\varepsilon + \gamma\mathfrak{T}''_\varepsilon)\,\mathbf{W}_\varepsilon(t) + \frac{\gamma T}{2}\mathbf{J}. \tag{6.10}$$

Here

$$\mathbf{J} := \begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix}.$$

Operator $\mathfrak{H}_\varepsilon : \mathcal{A}'\to\mathcal{A}'$ is given by the action on the Fourier transform in the spatial variable

$$\widehat{\mathfrak{H}}_\varepsilon\widehat{\mathbf{W}}(\eta,k) := \begin{bmatrix} -i\delta_\varepsilon\omega(k;\eta)\widehat{W}_+(\eta,k) & -\dfrac{2i}{\varepsilon}\bar{\omega}(k,\varepsilon\eta)\widehat{Y}_+(\eta,k) \\ \dfrac{2i}{\varepsilon}\bar{\omega}(k,\varepsilon\eta)\widehat{Y}_-(\eta,k) & i\delta_\varepsilon\omega(k;\eta)\widehat{W}_-(\eta,k)\end{bmatrix}, \quad \mathbf{W}\in\mathcal{A}'. \tag{6.11}$$

Here $\delta_\varepsilon\omega(k;\eta)$ is given by (4.13) and

$$\bar\omega(k,\varepsilon\eta) := \frac{1}{2}\left[\omega\left(k-\frac{\varepsilon\eta}{2}\right)+\omega\left(k+\frac{\varepsilon\eta}{2}\right)\right].$$

Moreover, $\mathfrak{T}'_\varepsilon, \mathfrak{T}''_\varepsilon : \mathcal{A}' \to \mathcal{A}'$ act via the formulas

$$\widehat{\mathfrak{T}}'_\varepsilon\widehat{\mathbf{W}}(\eta,k) := \widehat{W}'(\eta,k)\mathbf{J}, \qquad \widehat{\mathfrak{T}}''_\varepsilon\widehat{\mathbf{W}}(\eta,k) := \begin{bmatrix} \widehat{W}''_+(\eta,k) & \widehat{Y}''_+(\eta,k) \\ \widehat{Y}''_-(\eta,k) & \widehat{W}''_-(\eta,k) \end{bmatrix}, \tag{6.12}$$

with

$$\widehat{W}'(\eta,k) = \frac{1}{2}\int_{T^2_\varepsilon}\left\{2\widehat{W}_+(\eta',\ell) - \widehat{Y}_+(\eta',\ell) - \widehat{Y}_-(\eta',\ell)\right\}d\eta' d\ell, \tag{6.13}$$

T^2_ε defined in (5.12), and

$$\begin{aligned} \widehat{W}''_\pm(\eta,k) &= \frac{1}{2\varepsilon}\int_{\mathbb{T}}\left[\widehat{Y}_\pm\left(\eta-\frac{2k'}{\varepsilon},k+k'\right)+\widehat{Y}_\mp\left(\eta+\frac{2k'}{\varepsilon},k+k'\right)\right. \\ &\left.-\widehat{W}_\pm\left(\eta-\frac{2k'}{\varepsilon},k+k'\right)-\widehat{W}_\pm\left(\eta+\frac{2k'}{\varepsilon},k+k'\right)\right]dk', \\ \widehat{Y}''_\pm(\eta,k) &= -\frac{1}{2\varepsilon}\int_{\mathbb{T}}\left[\widehat{Y}_\pm\left(\eta+\frac{2k'}{\varepsilon},k+k'\right)+\widehat{Y}_\pm\left(\eta-\frac{2k'}{\varepsilon},k+k'\right)\right. \\ &\left.-\widehat{W}_\mp\left(\eta+\frac{2k'}{\varepsilon},k+k'\right)-\widehat{W}_\pm\left(\eta-\frac{2k'}{\varepsilon},k+k'\right)\right]dk'. \end{aligned} \tag{6.14}$$

All the operators $\mathfrak{H}_\varepsilon$, $\mathfrak{T}'_\varepsilon$ and $\mathfrak{T}''_\varepsilon$ are bounded on $\mathcal{A}'$ for a fixed $\varepsilon > 0$. Therefore equation (6.10) has a unique solution for a given initial data $\mathbf{W}_\varepsilon(0) \in \mathcal{A}'$. A direct calculations show that $\mathbf{W}_\varepsilon(t)$ given by the entries

$$\widehat{W}_{\varepsilon,\pm}(t,\eta,k) \equiv T\sum_{\ell\in\mathbb{Z}}\delta_0\left(\eta-\frac{\ell}{\varepsilon}\right), \quad \widehat{Y}_{\varepsilon,\pm}(t,\eta,k) \equiv 0 \tag{6.15}$$

is a stationary solution of (6.10).

6.2.2. *Equilibrium initial data.* We suppose that $(\psi_x)_{x\in\mathbb{Z}}$ is an i.i.d. zero mean, complex Gaussian field with

$$\mathbb{E}[\psi_x\psi_{x'}] = 0 \quad \text{and} \quad \mathbb{E}[\psi_x\psi^*_{x'}] = 2\delta_{x,x'}T, \quad x,x' \in \mathbb{Z}. \tag{6.16}$$

Then

$$\hat\psi(k) = \sum_{x\in\mathbb{Z}} e^{-2\pi i k x}\psi_x$$

is the complex valued Gaussian white noise field on $\mathbb{T}$ with

$$\mathbb{E}[\hat\psi(k)\hat\psi(k')] = 0 \quad \text{and} \quad \mathbb{E}[\hat\psi(k)\hat\psi^*(k')] = \delta_0(k-k'), \quad k,k' \in \mathbb{T}, \tag{6.17}$$

where $\delta_0(\eta)$ is the Dirac delta distribution.

Suppose that $\hat\psi(t,k)$ is the solution of (2.18) with the initial data described above. Let $\hat\psi^{(\varepsilon)}(t,k) := \hat\psi(t/\varepsilon,k)$ and let $\mathbf{W}_\varepsilon(t)$ be the Wigner distribution tensor given by (6.9) with the entries given by the formula (2.22). It satisfies the system (6.10). As before we denote by $w_{\varepsilon,\pm}(\lambda,\eta,k)$, $y_{\varepsilon,\pm}(\lambda,\eta,k)$ the Laplace-Fourier-Wigner functions corresponding to the entries $\mathbf{W}_\varepsilon(t)$.

Note that $\widehat{Y}_{\varepsilon,\pm}(0,\eta,k) = 0$ and, by the Poisson summation formula,

$$\widehat{W}_{\varepsilon,\pm}(0,\eta,k) = \frac{\varepsilon}{2}\mathbb{E}\left[\hat\psi\left(\pm k+\frac{\varepsilon\eta}{2}\right)\hat\psi^*\left(\pm k-\frac{\varepsilon\eta}{2}\right)\right] = \varepsilon T\sum_{x\in\mathbb{Z}}e^{-2\pi i\varepsilon\eta x} = T\sum_{\ell\in\mathbb{Z}}\delta_0\left(\eta-\frac{\ell}{\varepsilon}\right).$$

6.2.3. *Proof of the identity* (2.48). Due to the uniqueness of solutions of the system (6.10) we have

$$\widehat{W}_{\varepsilon,\pm}(t,\eta,k) \equiv \widehat{W}_{\varepsilon,+}(0,\eta,k) = T\sum_{\ell\in\mathbb{Z}}\delta_0\left(\eta-\frac{\ell}{\varepsilon}\right), \quad \widehat{Y}_{\varepsilon,\pm}(t,\eta,k)\equiv 0. \tag{6.18}$$

In particular, we have

$$\lim_{\varepsilon\to 0}\widehat{w}_{\varepsilon,+}(\lambda,\eta,k) = \frac{T\delta_0(\eta)}{\lambda} \tag{6.19}$$

in the $\star$-weak sense in $\mathcal{A}'$.

On the other hand, from (3.7) and Proposition 4.10 (recall that $\mu=1$), we have

$$\lim_{\varepsilon\to 0}\widehat{w}_{\varepsilon,+}(\lambda,\eta,k) = \frac{\gamma T|\nu(k)|^2}{2\lambda(\lambda+i\omega'(k)\eta)} + \lim_{\varepsilon\to 0}\widehat{w}'_{\varepsilon,+}(\lambda,\eta,k), \tag{6.20}$$

where

$$\widehat{w}'_{\varepsilon}(\lambda,\eta,k) = \frac{\varepsilon}{2}\int_0^{+\infty}e^{-\lambda t}\mathbb{E}\left[\hat{\psi}'\left(t,k+\frac{\varepsilon\eta}{2}\right)(\hat{\psi}')^*\left(t,k-\frac{\varepsilon\eta}{2}\right)\right]dt, \quad (\eta,k)\in\mathbb{T}_{2/\varepsilon}\times\mathbb{T}$$

and $\hat{\psi}'(t,k)$ is given by (5.2), corresponding to the initial data described by (6.16) and $\mu=1$. It satisfies (5.4). The calculation concerning the asymptotics of $\widehat{w}'_{\varepsilon}(\lambda,\eta,k)$ are similar to those carried out in Section 5. However the latter cannot be directly applied, as in the case of initial data (6.16) the assumption (2.26) is not satisfied.

Note that

$$\begin{aligned}
&\widehat{w}'_{\varepsilon}(\lambda,\eta,k) = \sum_{j=1}^{3}\widehat{w}'_{\varepsilon,j}(\lambda,\eta,k), \quad \text{where}\\
&\widehat{w}'_{\varepsilon,1}(\lambda,\eta,k) := \frac{W_\varepsilon(0,\eta,k)}{\lambda+i\delta_\varepsilon\omega(k;\eta)}\\
&\widehat{w}'_{\varepsilon,2}(\lambda,\eta,k) := \frac{\gamma\mathfrak{e}_\varepsilon(\lambda)}{(\lambda+i\delta_\varepsilon\omega(k;\eta))},\\
&\widehat{w}'_{\varepsilon,3}(\lambda,\eta,k) := -\frac{\gamma}{2(\lambda+i\delta_\varepsilon\omega(k;\eta))}\left[\mathfrak{d}_\varepsilon\left(\lambda,k-\frac{\varepsilon\eta}{2}\right)+\mathfrak{d}^\star_\varepsilon\left(\lambda,k+\frac{\varepsilon\eta}{2}\right)\right],
\end{aligned} \tag{6.21}$$

with $W_\varepsilon(0,\eta,k)$ given by (6.18) and $\mathfrak{e}_\varepsilon(\lambda)$, $\mathfrak{d}_\varepsilon(\lambda,k)$ given by (5.5). Obviously,

$$\lim_{\varepsilon\to 0+}\widehat{w}'_{\varepsilon,1}(\lambda,\eta,k) = \frac{T\delta_0(\eta)}{\lambda} \tag{6.22}$$

in the $\star$-weak sense in $\mathcal{A}'$.

We can repeat the calculations leading to (5.7) and obtain that in this case

$$\lim_{\varepsilon\to 0+}\mathfrak{e}_\varepsilon(\lambda) = \frac{T}{2(1-\Gamma)}\int_{\mathbb{R}\times\mathbb{T}}\frac{\delta(\eta)|\nu(\ell)|^2}{\lambda+i\omega'(\ell)\eta}d\eta d\ell = \frac{T}{2\lambda(1-\Gamma)}\int_{\mathbb{T}}|\nu(\ell)|^2d\ell. \tag{6.23}$$

Accordingly,

$$\lim_{\varepsilon\to 0+}\widehat{w}'_{\varepsilon,2}(\lambda,\eta,k) = \frac{T}{2\lambda(1-\Gamma)(\lambda+i\omega'(k))}\int_{\mathbb{T}}|\nu(\ell)|^2d\ell, \tag{6.24}$$

boundedly pointwise.

On the other hand $\mathfrak{d}_\varepsilon(\lambda,k)$ is given by the series (5.17) with the accompanying equations (5.18) and (5.19).

Using the first formula of (5.19) together with (6.17) we conclude that

$$D^\varepsilon_{0,1}(\lambda,k) = \frac{\varepsilon}{2}\int_{\mathbb{T}}\frac{\langle\hat{\psi}^\star(k)\hat{\psi}(\ell)\rangle_{\mu_\varepsilon}d\ell}{\lambda\varepsilon+i(\omega(\ell)-\omega(k))}\tilde{g}(\varepsilon\lambda-i\omega(k)) = \frac{T}{\lambda}\tilde{g}(\varepsilon\lambda-i\omega(k)). \tag{6.25}$$

To compute $D^{\varepsilon}_{0,2}(\lambda,k)$, note that by (3.2).

$$\mathbb{E}\left[\mathfrak{p}^0_0(\sigma)\mathfrak{p}^0_0(\sigma')\right]=\frac{T}{2}\int_{\mathbb{T}}\left(e^{i\omega(k)(\sigma'-\sigma)}+e^{i\omega(k)(\sigma-\sigma')}\right)dk,\quad \sigma,\sigma'\ge 0. \tag{6.26}$$

Therefore,

$$D^{\varepsilon}_{0,2}(\lambda,k)=D^{\varepsilon,1}_{0,2}(\lambda,k)+D^{\varepsilon,2}_{0,2}(\lambda,k),$$

where

$$D^{\varepsilon,1}_{0,2}(\lambda,k):=-\frac{\varepsilon\gamma T}{2}\int_{\mathbb{T}}d\ell\int_0^{+\infty}e^{-\lambda\varepsilon t}dt\int_0^t g(d\sigma)\int_0^t ds \times\exp\left\{i(\omega(k)-\omega(\ell))(t-s)\right\}\int_0^s g(d\sigma')e^{i\omega(\ell)(\sigma-\sigma')}$$

and

$$D^{\varepsilon,2}_{0,2}(\lambda,k):=-\frac{\varepsilon\gamma T}{2}\int_{\mathbb{T}}d\ell\int_0^{+\infty}e^{-\lambda\varepsilon t}dt\int_0^t g(d\sigma)\int_0^t ds \times\exp\left\{i(\omega(k)+\omega(\ell))(t-s)\right\}\int_0^s g(d\sigma')e^{i\omega(\ell)(\sigma'-\sigma)}.$$

Using (4.12) we get

$$D^{\varepsilon,2}_{0,2}(\lambda,k)=-\frac{\varepsilon\gamma T}{2}\int_{\mathbb{T}}d\ell\int_0^{+\infty}\int_0^{+\infty}e^{-\lambda\varepsilon(t+t')/2}dtdt'\int_{\mathbb{R}}e^{i\beta(t-t')}d\beta\int_0^t e^{-i\omega(\ell)\sigma}g(d\sigma)$$
$$\times\int_0^{t'}ds\exp\left\{i(\omega(k)+\omega(\ell))(t'-s)\right\}\int_0^s g(d\sigma')e^{i\omega(\ell)\sigma'}.$$

After straightforward calculation we obtain

$$D^{\varepsilon,2}_{0,2}(\lambda,k)=-\frac{\varepsilon\gamma T}{2}\int_{\mathbb{R}}\frac{d\beta}{(\lambda\varepsilon/2)^2+\beta^2}\int_{\mathbb{T}}\tilde g\big(\lambda\varepsilon/2+i\omega(\ell)-i\beta\big)\frac{\tilde g\big(\lambda\varepsilon s/2+i\beta-i\omega(\ell)\big)}{\lambda\varepsilon/2+\beta-i(\omega(k)+\omega(\ell))}d\ell.$$

Therefore

$$\lim_{\varepsilon\to0+}D^{\varepsilon,2}_{0,2}(\lambda,k)=-\lim_{\varepsilon\to0+}\frac{\varepsilon\gamma T}{2}\int_{\mathbb{R}}\frac{d\beta}{(\lambda\varepsilon/2)^2+\beta^2}\int_{\mathbb{T}}\frac{|\nu(\ell)|^2d\ell}{\lambda\varepsilon/2+\beta-i(\omega(k)+\omega(\ell))}$$
$$=\frac{i\gamma T}{\lambda}\int_{\mathbb{T}}\frac{|\nu(\ell)|^2d\ell}{\omega(k)+\omega(\ell)}, \tag{6.27}$$

in the $L^p(\mathbb{T})$ sense for any $p\in[1,2)$.

Next, using again (4.12), we get

$$D^{\varepsilon,1}_{0,2}(\lambda,k)=-\frac{\varepsilon\gamma T}{2(2\pi)}\int_{\mathbb{T}}d\ell\int_0^{+\infty}\int_0^{+\infty}e^{-\lambda\varepsilon(t+t')/2}dtdt'\int_{\mathbb{R}}e^{i\beta(t-t')}d\beta\int_0^t g(d\sigma)\int_0^{t'}ds$$
$$=-\frac{\varepsilon\gamma T}{2(2\pi)}\int_{\mathbb{R}}\frac{d\beta}{(\varepsilon\lambda/2)^2+\beta^2}\int_{\mathbb{T}}\frac{\tilde g\big(\lambda\varepsilon/2-i\beta-i\omega(\ell)\big)}{\lambda\varepsilon/2+i\beta-i(\omega(k)-\omega(\ell))}\tilde g\big(\lambda\varepsilon/2+i\beta+i\omega(\ell)\big)d\ell.$$

Applying Lemma 2.1 we conclude that

$$\lim_{\varepsilon\to0+}D^{\varepsilon,1}_{0,2}(\lambda,k)=-\lim_{\varepsilon\to0+}\frac{\gamma T}{\lambda(2\pi)}\int_{\mathbb{R}}\frac{(\varepsilon\lambda/2)d\beta}{(\varepsilon\lambda/2)^2+\beta^2}\int_{\mathbb{T}}\frac{|\nu(\ell)|^2d\ell}{\lambda\varepsilon/2+i\beta-i(\omega(k)-\omega(\ell))}$$

in the $L^p(\mathbb{T})$ sense for $p\in[1,2)$. Suppose that $\rho\in(0,1)$. Observe that

$$\lim_{\varepsilon\to0+}D^{\varepsilon,1}_{0,2}(\lambda,k)=-\lim_{\varepsilon\to0+}\frac{\gamma T}{\lambda(2\pi)}\int_{|\beta|\le\varepsilon^\rho}\frac{(\varepsilon\lambda/2)d\beta}{(\varepsilon\lambda/2)^2+\beta^2}\int_{\mathbb{T}}\frac{|\nu(\ell)|^2d\ell}{\lambda\varepsilon/2+i\beta-i(\omega(k)-\omega(\ell))}$$
$$=-\frac{\pi\gamma T}{\lambda}\int_{-\infty}^0\hat h(\eta)e^{2\pi i\eta\omega(k)}d\eta,$$

where, cf Section 2.4.1

$$h(v):=\frac{2|\nu(\omega_+^{-1}(v))|^2}{\omega'(\omega_+^{-1}(v))}1_{[\omega_{\min},\omega_{\max}]}(v).$$

Summarizing we have shown that

$$\lim_{\varepsilon\to 0+} D_0^\varepsilon(\lambda,k) = D_0(\lambda,k), \quad \text{in the } L^p(\mathbb{T}) \text{ sense, } p\in[1,2), \tag{6.28}$$

where, cf (2.41),

$$\operatorname{Re} D_0(\lambda,k) = \frac{T}{\lambda}\operatorname{Re}\nu(k) - \frac{\gamma T|\nu(k)|^2}{2\lambda|\bar\omega'(k)|} = \frac{T}{\lambda}\operatorname{Re}\nu(k) - \frac{T\,\mathfrak{g}(k)}{2\lambda}. \tag{6.29}$$

We use formula (5.25) to express $D_n^\varepsilon(\lambda,k)$. Doubling the τ_j variables with the help of (4.12) we get

$$D_n^\varepsilon(\lambda,k) = \frac{\varepsilon\gamma_1^n T_0}{(2\pi)^{n+2}}\int_0^{+\infty}dt\int_{\mathbb{R}^{n+1}}d\beta_{0,n}\int_{[0,+\infty)^{2n+2}}d\tau_{0,n}d\tau'_{0,n}\int_{\mathbb{R}}d\beta\int_{\mathbb{T}}d\ell$$
$$\times e^{-\lambda\varepsilon t/2}\phi^*(\tau'_0,k)(J\star g)(\tau_0)\prod_{j=0}^n\exp\left\{i\beta_j(\tau_j-\tau'_j)\right\}$$
$$\times\exp\left\{i\beta\left(t-\frac12\sum_{j=0}^n\tau_j-\frac12\sum_{j=0}^n\tau'_j\right)\right\}\exp\left\{-\lambda\varepsilon\left(\sum_{j=0}^n\tau_j\right)/4\right\}$$
$$\times\exp\left\{-\lambda\varepsilon\left(\sum_{j=0}^n\tau'_j\right)/4\right\}\int_0^{\tau_n}e^{-i\omega(\ell)\sigma}g(d\sigma)\int_0^{\tau'_n}e^{i\omega(\ell)\sigma'}g(d\sigma'),\quad n\ge 1.$$

Integrating the τ and τ' variables, we get

$$D_n^\varepsilon(\lambda,k) = \frac{\varepsilon\gamma^n T}{(2\pi)^{n+2}}\int_{\mathbb{T}}d\ell\int_{\mathbb{R}}\frac{d\beta}{\lambda\varepsilon/2-i\beta}\int_{\mathbb{R}^{n+1}}d\beta_{0,n}(\tilde J\tilde g)\left(\lambda\varepsilon/4-i\beta_0+i\beta/2\right)\frac{\tilde g(\lambda\varepsilon/4+i\beta_0+i\beta/2)}{\lambda\varepsilon/4+i\beta_0+i\beta/2-i\omega(k)}$$
$$\times\prod_{j=1}^{n-1}(\tilde J\tilde g)\left(\lambda\varepsilon/4-i\beta_j+i\beta/2\right)\prod_{j=1}^{n-1}(\tilde J\tilde g)\left(\lambda\varepsilon/4+i\beta_j+i\beta/2\right)$$
$$\times\frac{\tilde g(\lambda\varepsilon/4-i\beta_n+i\beta/2+i\omega(\ell))}{\lambda\varepsilon/4-i\beta_n+i\beta/2}\cdot\frac{\tilde g(\lambda\varepsilon/4+i\beta_n+i\beta/2-i\omega(\ell))}{\lambda\varepsilon/4+i\beta_n+i\beta/2},\quad n\ge1.$$

Change variables $\beta'_j := \beta_j+\beta/2$, $j=0,\dots,n$. We can write then

$$D_n^\varepsilon(\lambda,k) = \frac{\varepsilon\gamma^n T}{(2\pi)^{n+2}}\int_{\mathbb{T}}d\ell\int_{\mathbb{R}}\frac{d\beta}{\lambda\varepsilon/2-i\beta}\int_{\mathbb{R}^{n+1}}d\beta_{0,n}(\tilde J\tilde g)\left(\lambda\varepsilon/4-i\beta_0+i\beta\right)\frac{\tilde g(\lambda\varepsilon/4+i\beta_0)}{\lambda\varepsilon/4+i\beta_0-i\omega(k)}$$
$$\times\prod_{j=1}^{n-1}(\tilde J\tilde g)\left(\lambda\varepsilon/4-i\beta_j+i\beta\right)\prod_{j=1}^{n-1}(\tilde J\tilde g)\left(\lambda\varepsilon/4+i\beta_j\right)$$
$$\times\frac{\tilde g(\lambda\varepsilon/4-i\beta_n+i\beta+i\omega(\ell))}{\lambda\varepsilon/4-i\beta_n+i\beta}\cdot\frac{\tilde g(\lambda\varepsilon/4+i\beta_n-i\omega(\ell))}{\lambda\varepsilon/4+i\beta_n},\quad n\ge1.$$

We integrate the β variable using the Cauchy formula (4.15) and obtain

$$\lim_{\varepsilon\to0+}D_n^\varepsilon(\lambda,k) = \lim_{\varepsilon\to0+}\frac{\varepsilon\gamma\Gamma^{n-1}T}{(2\pi)^2}\int_{\mathbb{T}}d\ell\int_{\mathbb{R}}d\beta_0\int_{\mathbb{R}}d\beta_n$$
$$\times(\tilde J\tilde g)\left(3\lambda\varepsilon/4-i\beta_0\right)\frac{\tilde g(\lambda\varepsilon/4+i\beta_0)}{\lambda\varepsilon/4+i\beta_0-i\omega(k)}$$
$$\times\frac{\tilde g(3\lambda\varepsilon/4-i\beta_n+i\omega(\ell))}{3\lambda\varepsilon/4-i\beta_n}\cdot\frac{\tilde g(\lambda\varepsilon/4+i\beta_n-i\omega(\ell))}{\lambda\varepsilon/4+i\beta_n} = \lim_{\varepsilon\to0+}\Gamma^{n-1}TI_\varepsilon II'_\varepsilon,\quad n\ge1,$$

where I_ε is given by (5.28) and

$$II'_\varepsilon := \frac{\varepsilon}{2\pi}\int_{\mathbb{T}}d\ell\int_{\mathbb{R}}d\beta_n\frac{\tilde g(3\lambda\varepsilon/4-i\beta_n+i\omega(\ell))}{3\lambda\varepsilon/4-i\beta_n}\cdot\frac{\tilde g(\lambda\varepsilon/4+i\beta_n-i\omega(\ell))}{\lambda\varepsilon/4+i\beta_n}. \tag{6.30}$$

We have shown that $\lim_{\varepsilon\to0+}I_\varepsilon = I$, see (5.34).

In the case of II'_ε we perform the change variables $\varepsilon\beta'_n := \beta_n$ and get

$$\lim_{\varepsilon\to 0+} II'_\varepsilon = \frac{1}{2\pi}\lim_{\varepsilon\to 0+}\int_{\mathbb{T}} d\ell \int_{\mathbb{R}} d\beta_n \frac{\tilde{g}(3\lambda\varepsilon/4 - i\varepsilon\beta_n + i\omega(\ell))}{3\lambda/4 - i\beta_n}\cdot\frac{\tilde{g}(\lambda\varepsilon/4 + i\varepsilon\beta_n - i\omega(\ell))}{\lambda/4 + i\beta_n} \tag{6.31}$$

$$= \lim_{\varepsilon\to 0+}\frac{1}{2\pi}\int_{\mathbb{T}}|\nu(\ell)|^2 d\ell \int_{\mathbb{R}}\frac{d\beta_n}{(3\lambda/4 - i\beta_n)(\lambda/4 + i\beta_n)} = \frac{1}{\lambda}\int_{\mathbb{T}}|\nu(\ell)|^2 d\ell \tag{6.32}$$

Hence,

$$\lim_{\varepsilon\to 0+}\sum_{n=1}^{+\infty} D_n^\varepsilon(\lambda,k) = \frac{T\big(1-|\nu(k)|^2\big)}{2\lambda(1-\Gamma)}\int_{\mathbb{T}}|\nu(\ell)|^2 d\ell. \tag{6.33}$$

As a result for any $G\in\mathcal{S}(\mathbb{R}\times\mathbb{T})$ we can write

$$\begin{aligned}\lim_{\varepsilon\to 0+}\int_{\mathbb{R}\times\mathbb{T}}\widehat{w}'_{\varepsilon,3}(\lambda,\eta,k)G^*(\eta,k)d\eta dk &= -\int_{\mathbb{R}\times\mathbb{T}}\frac{\gamma G^*(\eta,k)d\eta dk}{\lambda + i\omega'(k)\eta}\mathrm{Re}\,D_0(\lambda,k)\\ &- \sum_{n=1}^{+\infty}\lim_{\varepsilon\to 0+}\int_{\mathbb{R}\times\mathbb{T}}\frac{\gamma G^*(\eta,k)d\eta dk}{\lambda + i\omega'(k)\eta}\mathrm{Re}\,D_n(\lambda,k).\end{aligned} \tag{6.34}$$

Substituting from (6.29) and (6.33) into (6.34) we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0+}\int_{\mathbb{R}\times\mathbb{T}}\widehat{w}'_{\varepsilon,3}(\lambda,\eta,k)G^*(\eta,k)d\eta dk &= \int_{\mathbb{R}\times\mathbb{T}}\Big\{-\frac{\gamma T\mathrm{Re}\,\nu(k)}{\lambda(\lambda + i\omega'(k)\eta)}\\ &+\frac{\gamma\,\mathfrak{g}(k)T}{2\lambda(\lambda+i\omega'(k)\eta)} - \frac{\gamma T\big(1-|\nu(k)|^2\big)}{2(\lambda+i\omega'(k)\eta)\lambda(1-\Gamma)}\int_{\mathbb{T}}|\nu(\ell)|^2 d\ell\Big\}G^*(\eta,k)d\eta dk.\end{aligned} \tag{6.35}$$

Combining (6.20) with (6.22), (6.24) and (6.35) we get, after some elementary computation,

$$1 = \frac{1}{2(1-\Gamma)} + \frac{1}{2(1-\Gamma)}\int_{\mathbb{T}}|\nu(\ell)|^2 d\ell, \tag{6.36}$$

which in turn yields (2.48). Note that (6.36) is in fact equivalent with (2.47). □

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