

On the analogy between real reductive groups and Cartan motion groups. II: Contraction of irreducible tempered representations

Alexandre Afgoustidis

CEREMADE, Université Paris-Dauphine

Abstract

Attached to any reductive Lie group G is a “Cartan motion group” G_0 – a Lie group with the same dimension as G , but a simpler group structure. A natural one-to-one correspondence between the irreducible tempered representations of G and the unitary irreducible representations of G_0 , whose existence had been suggested by Mackey in the 1970s, has recently been described by the author. In the present article, we use the existence of a family of groups interpolating between G and G_0 to realize the bijection as a deformation: for every irreducible tempered representation π of G , we build, in an appropriate Fréchet space, a family of subspaces and evolution operators that contract π onto the corresponding representation of G_0 .

Contents

1. Introduction.....	1
2. Preliminaries	4
3. Examples of Fréchet contractions: the discrete series and the spherical principal series.....	8
3.1. Contraction of discrete series representations	9
3.2. On the principal series and generic spherical representations	13
4. Real-infinitesimal-character representations.....	20
4.1. Reduction to the quasi-split case	20
4.2. The case of quasi-split groups and fine K -types.....	30
5. Contraction of an arbitrary tempered representation	33
5.1. Real-infinitesimal-character representations for disconnected groups	33
5.2. Parabolic induction and reduction to real infinitesimal character	37
References.....	40

1. Introduction

Suppose G is a reductive Lie group. Fix a maximal compact subgroup K in G and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the corresponding Cartan decomposition of the Lie algebra of G . The *Cartan motion group* attached to G (and K) is the semidirect product $G_0 = K \ltimes \mathfrak{p}$ associated with the adjoint action of K on \mathfrak{p} .

The Lie groups G and G_0 have the same dimension, but very different algebraic structures; their representation theories are, accordingly, quite different. The motion group G_0 is a split extension of a compact group by an abelian group: as a result, the classification of its unitary representations is among the simplest applications of Mackey’s theory of induced representations [27]. The representation theory of G is more difficult: in general, much about the unitary dual \widehat{G} remains mysterious, and the complete description of the tempered dual \widetilde{G} achieved by Knapp and Zuckerman [23, 24] is a very deep result which draws on much of Harish-Chandra’s work.

Nevertheless, Mackey remarked in the early 1970s that his own description of the unitary dual \widehat{G}_0 featured parameters formally similar to those that appeared in Harish-Chandra’s then-ongoing study of \widetilde{G} ; he went on to wonder whether the analogy could give rise to a natural one-to-one correspondence between large subsets of \widehat{G}_0 and \widetilde{G} .

His conjecture seems to have been met with skepticism. But later on, the Baum-Connes-Kasparov conjecture in operator algebra theory renewed interest in Mackey’s observations (see [11], Ch. II, or [7], §4); Nigel Higson eventually strengthened Mackey’s rather tentative suggestion into a precise guess that *exactly the same* parameters can be used to describe \widehat{G}_0 and \widetilde{G} [19]. That guess has recently turned out to be accurate: we constructed in [3] a natural bijection between \widehat{G}_0 and \widetilde{G} , using Vogan’s notion of lowest K -types (see §2.1).

The existence of a common parametrization for the reduced duals of such different groups would perhaps seem incomprehensible, were it not for the existence of a continuous family $(G_t)_{t \in [0,1]}$ of Lie groups that interpolates between G_0 and $G = G_1$. The group G can be viewed as the isometry group of the riemannian symmetric space G/K , while G_0 acts by affine isometries on the flat tangent space to G/K at the identity coset. The (constant) curvature of G/K is negative and can be set to (-1) with suitable normalizations; a change of scale in that metric (akin to zooming-in on a neighborhood of the origin in G/K) yields, for each $t > 0$, a symmetric space with curvature $-t^2$ whose isometry group G_t is isomorphic with G . The motion group G_0 appears as a limiting case of the construction as the curvature parameter t goes to zero (see §2.2).

The relationship between G and G_0 is thus of a kind often encountered in physics, when the structure constants determining the symmetries of a physical system are changed and brought to a limiting value (for instance when the speed of light becomes infinite): the motion group G_0 is a *contraction* of G , in the sense of İnönü and Wigner ([21]; see also [33], §6). Given a Lie group Γ and a contraction Γ_0 , it is a classical problem in physics to try to relate the unitary representations of Γ and Γ_0 (to be thought of as carrier Hilbert spaces for the states of quantum systems with different but related symmetries) through a limiting process, studying the behavior of representation spaces and operators as the physical parameter of interest becomes singular. Mackey’s conjecture had in fact been motivated by the fact that an infinite space with negative but very small curvature is as plausible a model for physical space as a flat one – whence the idea that the representations of the corresponding symmetry groups ought to “resemble” one another.

In most cases of physical interest, the unitary representations of Γ and Γ_0 do not correspond exactly under the contraction – far from it. The most famous example is obtained in the transition from the Poincaré group to the Galilei group [21]: the “physically meaningful” unitary irreducible representations do not in any reasonable sense “contract” to unitary representations of the Galilei group.

From this point of view, the existence of the Mackey-Higson bijection appears to involve a very unusual rigidity phenomenon in the deformation from G to G_0 . The fact that the representations paired by the bijection can look either very similar (as in the case of minimal principal series representations, where the rigidity has long been known [13, 15]) or quite different (as in the case where G admits discrete series representations, whose counterparts in \widehat{G}_0 are then much more trivial finite-dimensional representations), and the importance for several aspects of the representation theory of G of the large-scale geometry of G/K (while G_0 is a first-order approximation of G near K), add to the mystery.

The mere existence of the deformation $(G_t)_{t \in [0,1]}$ does not, then, make the Mackey-Higson bijection any less unexpected. Yet, one can wonder whether the persistence phenomenon now established at the level of parameters is reflected in the behavior of representation spaces as the contraction is performed, and it is natural to ask whether the “coincidence of parametrizations” discovered by Mackey can be established by a deformation procedure.

That is the problem to be addressed here: when G is a linear connected reductive group, we will indicate how every tempered irreducible representation π of G can be “contracted” – at the level of geometrical realizations – onto the representation π_0 of G_0 that corresponds to it in the Mackey-Higson bijection.



Of course if one is to associate with π a family $(\pi_t)_{t > 0}$, where π_t is a representation of G_t for each $t > 0$, and to try to prove that “ π_t goes to π_0 as t goes to zero”, one needs to give a meaning to the convergence. There does not seem to be a universally acknowledged method for doing so; the various notions of “contraction of representations” that have been used in mathematical physics are somewhat disparate in nature, and often formulated at the infinitesimal (Lie algebra) level; see [20, 18, 30, 28] for classical examples.

Our strategy can be described, in somewhat imprecise terms, as follows (we postpone a precise statement of our results to §2.3: we will then have introduced enough notation to give a more formal description).

Suppose π is an irreducible tempered representation of G . Drawing from the existing constructions of geometrical realizations for π , we will attach to π a Fréchet space \mathbf{E} , together with a family $(\pi_t)_{t>0}$ of representations in which each $\pi_t, t > 0$, acts on a subspace \mathbf{V}_t of \mathbf{E} .

This will generate a natural notion of *evolution* in \mathbf{E} . Since each $G_t, t > 0$, is isomorphic with G , the vectors in subspaces \mathbf{V}_t that correspond to different values of t will lie in distinct spaces that carry irreducible representations of G . These G -representations will be equivalent, or at least can be turned into equivalent representations if an appropriate (and classical) renormalization of their continuous parameters is performed (see §2.3). Schur’s lemma will determine a way of following a vector in the subspace \mathbf{V}_1 , and in fact any vector of \mathbf{E} , through the contraction.

We will then be able to watch vectors and operators deform according to that natural evolution. What we will prove is that as t goes to zero, the initial carrier space and operators for π do “converge”, in a precise sense, to a subspace of \mathbf{E} and operators yielding a linear representation of G_0 on that subspace; furthermore, the equivalence class of the G_0 -representation obtained in this way corresponds to π in the Mackey-Higson bijection.

We should mention that except in the case of minimal principal series, the settings we will use (for instance the description of discrete series representations using Dirac induction, or the realization of real-infinitesimal-character representations in Dolbeault cohomology spaces) force upon us a choice of Fréchet space whose topology is quite different from the Hilbert space topology inherited from the unitary structure of the representations. Thus, our geometrical analysis of Mackey’s analogy is made possible only by the deep work of many mathematicians (among whom Helgason, Parthasarathy, Atiyah, Schmid, Vogan, Zuckerman, Wong), ranging from the early 1970s to the late 1990s, on the geometric realization of irreducible tempered (\mathfrak{g}, K) -modules.

The most suggestive and important examples are technically much simpler to deal with than the general case: before embarking on the contraction of an arbitrary tempered representation, we will take the time to inspect the contraction of a discrete series representation to its (unique) lowest K -type (see §3.1), as well as the contraction of minimal principal series representations (see §3.2) to generic representations of G_0 – for the latter case, our results are closely related to earlier treatments in the physical literature and in [13]. Our general procedure, relying on successive reductions of the general case to that of real-infinitesimal character representations of connected semisimple groups, then to that of irreducible constituents of real-infinitesimal-character principal series representations of quasi-split groups, is more technical (especially in §4.1); but most of the important ideas and many useful simple lemmas are already present in the two extreme examples. Section 4 contains our description of the contraction process for real-infinitesimal-character representations. Section 5 reduces the general case to that one.

The analytical methods to be used here are hardly the final word on the relationship between Mackey’s analogy and deformation theory. In the time elapsed since part of these results (corresponding roughly to §3 and §5) began circulating in preprint form in 2015, Bernstein, Higson and Subag have suggested that the methods of algebraic geometry may provide a framework for studying (algebraic) deformations of groups and families of representations in a broader context. Their recent work [9, 8, 36, 35, 5], as well as Shilin Yu’s recent attempts to obtain an algebraic-geometric realization of Mackey’s correspondence using deformations of algebraic \mathcal{D} -modules [37, 45], make it reasonable to hope that insight will be gained in this way on Mackey’s analogy. To mention only the simplest related issue, it will soon be obvious that the analytic methods and evolution operators in Fréchet space of the present article, in which no passage to subquotients is needed at any stage, cannot be well-suited to studying the relationship between deformation theory and the natural correspondence between the admissible duals introduced in the last section of [3]: it seems reasonable to turn to more algebraic methods for that study.

As one referee pointed out, it would certainly bring further light on our subject to determine whether there exists a deformation procedure that can be constructed independently of the extensive knowledge of the tempered dual, geometrical realizations, and lowest K -types, on which this paper entirely rests. The algebraic methods mentioned above may prove useful in that direction.

Acknowledgments. Most of the material in §2, 3 and 5 appeared in preliminary form in my Ph.D. thesis [2]. I am profoundly indebted to Daniel Bennequin’s guidance and support. I am happy to thank Michel Duflo, Nigel Higson and Michèle Vergne for their help and advice at various stages of this investigation, as well as François Rouvière and David Vogan, whose reading of my thesis led to major corrections and improvements. The ideas in §4.1 took form only after Nicolas Prudhon pointed out [44] to me, and I may not have completed the present paper without the frequent conversations I had with Jeremy Daniel.

With the exception of §4, the results below were conceived around 2015; it was a happy time at Université Paris-7 and the Institut de Mathématiques de Jussieu – Paris Rive Gauche. Section 4, which is crucial to this paper, is much more recent: I am grateful in many ways to my colleagues at Université Paris-Dauphine and CEREMADE.

I thank the referees for their careful comments, which helped improve this paper.

2. Preliminaries

2.1. The Mackey-Higson bijection

In [3], §2.2, we called *Mackey parameter* any pair (χ, μ) where χ is an element of the vector space dual \mathfrak{p}^* of \mathfrak{p} , and μ is an irreducible representation of the stabilizer K_χ of χ in K (for the coadjoint action of K on \mathfrak{p}^*).

Attached to any Mackey parameter (χ, μ) is a unitary irreducible representation of G_0 : we form the centralizer $L_\chi^\times = K_\chi \ltimes \mathfrak{p}$ of χ in G_0 and consider the induced representation

$$\mathbf{M}_0(\chi, \mu) = \text{Ind}_{L_\chi^\times}^{G_0}(\sigma) = \text{Ind}_{K_\chi \ltimes \mathfrak{p}}^{G_0}(\mu \otimes e^{i\chi}). \quad (2.1)$$

It will be useful to record a construction of $\mathbf{M}_0(\chi, \mu)$ and three elementary remarks.

Construction 2.1. Fix a carrier space V for μ and a $\mu(K)$ -invariant inner product on V . Equip the Hilbert space

$$\mathbf{H} = \{f \in \mathbf{L}^2(K, V), \forall m \in K_\chi, \forall k \in K, f(km) = \mu(m^{-1})f(k)\} \quad (2.2)$$

with the G_0 -action in which

$$g_0 = (k, v) \text{ acts through } \pi_0(k, v) : f \mapsto \left[u \mapsto e^{i(\text{Ad}^*(u)\chi, v)} f(k^{-1}u) \right]. \quad (2.3)$$

Then the representation (\mathbf{H}, π_0) of G_0 is equivalent with $\mathbf{M}_0(\chi, \mu)$.

Remark 2.2. Suppose χ is zero. Then K_χ equals K , μ is an irreducible representation of K , and the representation $\mathbf{M}_0(\chi, \mu)$ of G_0 is the trivial extension of μ where \mathfrak{p} acts by the identity on V . Thus $\mathbf{M}_0(\chi, \mu)$ is finite-dimensional in that case.

Remark 2.3. For arbitrary χ , suppose μ is the trivial representation of K_χ . Then another geometric realization for $\mathbf{M}_0(\chi, \mu)$ may be obtained by considering the space \mathcal{E} of tempered distributions on \mathfrak{p} whose Fourier transform (a tempered distribution on \mathfrak{p}^*) has support contained in the orbit $\text{Ad}^*(K) \cdot \chi$; since that orbit is compact, \mathcal{E} is a space of smooth functions on \mathfrak{p} . The functions in \mathcal{E} are ‘‘combinations’’ of Euclidean plane waves on \mathfrak{p} whose wavevectors lie in the orbit $\text{Ad}^*(K) \cdot \chi$. The realization (2.2)-(2.3) of G_0 can be viewed as transferred from the quasi-regular action of G_0 on \mathcal{E} by taking Fourier transforms.

Remark 2.4. Fix a maximal abelian subspace \mathfrak{a} of \mathfrak{p} , and view the vector space dual \mathfrak{a}^* as a subspace of \mathfrak{p}^* . Then every Mackey parameter (χ, μ) is conjugate under K to one in which χ lies in \mathfrak{a}^* . If χ lies in \mathfrak{a}^* and is a regular element of \mathfrak{a}^* ([25], §V.3), then K_χ is equal to the group $M = Z_K(\mathfrak{a})$.

We can also attach to any Mackey parameter (χ, μ) a tempered irreducible representation of G . Form the centralizer L_χ of χ in G (for the coadjoint action). There exists a parabolic subgroup $P_\chi = L_\chi N_\chi$ of G with Levi factor L_χ . Write $P_\chi = M_\chi A_\chi N_\chi$ for the Langlands decomposition of P_χ : the group M_χ admits K_χ as a maximal compact subgroup, is usually disconnected, but is linear reductive with abelian Cartan subgroups. The representation μ of K_χ determines, through Vogan’s theory of lowest K -types [39], a unique irreducible tempered representation of M_χ : among the representations of M_χ with real infinitesimal character, there is exactly one that admits μ as a lowest K_χ -type (and then μ is its only lowest K_χ -type): see [40], Theorem 1.3, or [3], §3.1. We write $\mathbf{V}_{M_\chi}(\mu)$ for it and define a representation of G as

$$\mathbf{M}(\chi, \mu) = \text{Ind}_{M_\chi A_\chi N_\chi}^G(\mathbf{V}_{M_\chi}(\mu) \otimes e^{i\chi} \otimes \mathbf{1}) \quad (2.4)$$

(for a construction, see §3.2).

It is a classical result of Mackey [27] that every unitary irreducible representation of G_0 is equivalent with

$\mathbf{M}_0(\chi, \mu)$ for some Mackey parameter (χ, μ) ; the corresponding tempered representation $\mathbf{M}(\chi, \mu)$ is always irreducible, and its class in \widetilde{G} depends only on the class of $\mathbf{M}_0(\chi, \mu)$ in \widehat{G}_0 . The above thus determines a map $\mathcal{M} : \widehat{G}_0 \rightarrow \widetilde{G}$.

Theorem 2.5 ([3], Theorem 3.2). *The map $\mathcal{M} : \widehat{G}_0 \rightarrow \widetilde{G}$ is a bijection between the unitary dual of G_0 and the tempered dual of G .*

2.2. Deformation to the motion group

2.2.1. A family of Lie groups

For every nonzero real number t , we can use the global diffeomorphism

$$\begin{aligned} \varphi_t : K \times \mathfrak{p} &\rightarrow G \\ (k, v) &\mapsto \exp_G(tv)k \end{aligned}$$

to endow $K \times \mathfrak{p}$ with a group structure which makes it isomorphic with G ; the idea dates back to [13]. We write G_t for that group, so that the underlying set of G_t is $K \times \mathfrak{p}$, and the product law reads

$$(k_1, v_1) \cdot_t (k_2, v_2) = \varphi_t^{-1}(\varphi_t(k_1, v_1) \cdot \varphi_t(k_2, v_2)).$$

The Cartan motion group G_0 also has underlying set $K \times \mathfrak{p}$, and its product law reads $(k_1, v_1) \cdot_0 (k_2, v_2) = (k_1 k_2, v_1 + \text{Ad}(k_1)v_2)$. The product law for G_t goes to the product law for G_0 as t goes to zero:

Lemma 2.6. *Suppose (k_1, v_1) and (k_2, v_2) are two elements of $K \times \mathfrak{p}$; then $(k_1, v_1) \cdot_t (k_2, v_2)$ goes to $(k_1, v_1) \cdot_0 (k_2, v_2)$ when t goes to zero. The convergence is uniform on compact subsets of $K \times \mathfrak{p}$.*

This is quite well-known, but I have not been able to locate a precise justification in the literature. Given that the very existence of the present article depends on Lemma 2.6, it is perhaps reasonable to include a proof: this one was communicated to me by François Rouvière.

Write $(k_1, v_1) \cdot_t (k_2, v_2)$ as $(k(t), v(t))$, where $k(t)$ is in K and $v(t)$ in \mathfrak{p} ; what we have to prove is that when t goes to zero, $k(t)$ goes to $k_1 k_2$ and $v(t)$ goes to $v_1 + \text{Ad}(k_1)v_2$. We start from the fact that

$$e^{tv(t)}k(t) = (e^{tv_1}k_1)(e^{tv_2}k_2) = e^{tv_1}e^{t\text{Ad}(k_1)v_2}k_1k_2. \quad (2.5)$$

• Write

$$\begin{aligned} u : \mathfrak{p} &\rightarrow G/K \\ X &\mapsto \exp_G(X)K \end{aligned} \quad (2.6)$$

for the global diffeomorphism from the Cartan decomposition, and $(g, X) \mapsto g \cdot x = u^{-1}(g \exp_G(X)K)$ for the corresponding action of G on \mathfrak{p} ; then we deduce from (2.5) that $u[tv(t)] = e^{tv_1} \cdot u[t\text{Ad}(k_1)v_2]$. When t goes to zero, the right-hand side of the last equality goes to the origin $1_G K$ in G/K , so that $tv(t)$ must go to the origin of \mathfrak{p} ; taking up (2.5) we deduce that $k(t)$ goes to $k_1 k_2$.

• Let us turn to the convergence of $v(t)$. Apply the Cartan involution θ with fixed-point-set K to both sides of (2.5); we get $e^{-tv(t)}k(t) = e^{-v_1}e^{-t\text{Ad}(k_1)v_2}k_1k_2$; taking inverses and multiplying by (2.5) yields

$$e^{2tv(t)} = e^{tv_1}e^{2t\text{Ad}(k_1)v_2}e^{tv_1}. \quad (2.7)$$

From the Campbell-Hausdorff formula, we know that for small enough t , the right-hand side can be written as $e^{2t(v_1 + \text{Ad}(k_1)v_2 + r(t))}$, where $r(t)$ is an element of \mathfrak{g} (the sum of a convergent Lie series) and $r(t) = O(t)$ in the Landau “big O ” notation. If t is close enough to zero, then both $2tv(t)$ and $2t(v_1 + \text{Ad}(k_1)v_2 + r(t))$ lie in a neighborhood of the origin \mathfrak{g} over which \exp_G is injective, so that (2.7) yields $v(t) = v_1 + \text{Ad}(k_1)v_2 + r(t)$. We deduce that for small enough t , $r(t)$ lies in \mathfrak{p} , and of course since $r(t) = O(t)$ we see that $v(t)$ goes to $v_1 + \text{Ad}(k_1)v_2$ as t goes to zero. The last statement (the fact that the convergence is uniform on compact subsets) is now clear from the

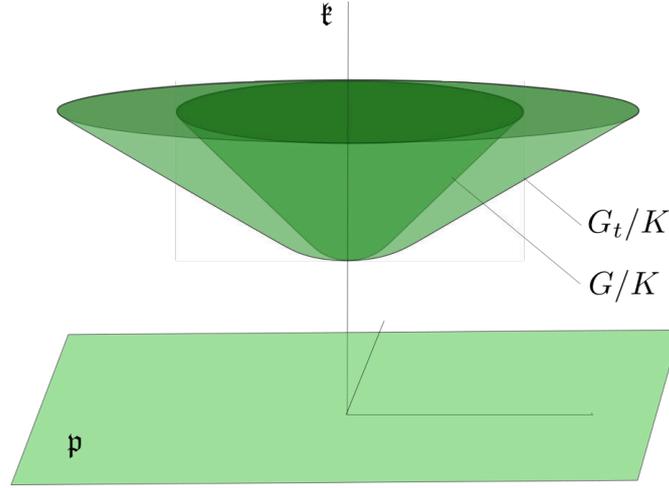


Figure 1 – This is a picture of (co)-adjoint orbits for G and G_t when G is $SL(2, \mathbb{R})$. The horizontal plane is the space \mathfrak{p} of trace-zero symmetric matrices, the vertical axis is the line \mathfrak{k} of antisymmetric matrices; the drawn manifolds are the G -adjoint orbit and the G_t -coadjoint orbit of the point on the vertical axis. It is tempting, but incorrect, to interpret the diffeomorphism u_t^{-1} from (2.8) as the vertical projection; however, the vertical projection can be recovered as $u_t^{-1} \circ \tau$, where $\tau : x \mapsto \frac{\sinh(\|x\|_B)}{\|x\|_B} \text{Rotation}_{\frac{\pi}{2}}(x)$. The map τ accounts for the fact that the geodesics in (\mathfrak{p}, η_t) and (\mathfrak{p}, η_0) do not spread at the same speed.

existence of explicit formulae for $r(t)$ and $k(t)(k_1 k_2)^{-1}$. □

2.2.2. A family of actions and metrics on \mathfrak{p}

In this section, we describe a family of actions and a family of metrics on \mathfrak{p} . We shall use them in §3 to describe the contraction process for *discrete series* and *spherical principal series* representations: these two cases will introduce us to the main ideas and lemmas to be used for the general case in sections 4 and 5.

Suppose t is a real number (we do not prohibit $t = 0$ here). Then G_t acts on the symmetric space G_t/K ; because of the Cartan decomposition, the map

$$u_t : \mathfrak{p} \xrightarrow{\exp_{G_t}} G_t \xrightarrow{\text{quotient}} G_t/K \quad (2.8)$$

is a global diffeomorphism. This yields a transitive action of G_t on \mathfrak{p} : if γ is an element of G_t , we write the action of γ on \mathfrak{p} as $x \mapsto \gamma \cdot x \stackrel{\text{def}}{=} u_t^{-1} [\text{mult}_{G_t}(\gamma, \exp_{G_t}(x)) K]$, where $\text{mult}_{G_t} : G_t \times G_t \rightarrow G_t$ is the group law for G_t .

The action of G_t on \mathfrak{p} induces a G_t -invariant riemannian metric on \mathfrak{p} : we choose a fixed K -invariant inner product B on \mathfrak{p} and write η_t for the (uniquely determined) G_t -invariant metric on \mathfrak{p} which coincides with B at the origin of \mathfrak{p} . The metric η_t has constant scalar curvature, equal to $-t^2$: thus (\mathfrak{p}, η_t) is negatively curved for every nonzero t , while (\mathfrak{p}, η_0) is Euclidean.

In the proof of Lemma 2.6, we also used the Cartan decomposition of G to define an action $(g, x) \mapsto g \cdot x$ of G on \mathfrak{p} . Since each G_t , $t \neq 0$, is obtained from G by a simple rescaling in the Cartan decomposition, there must be a simple relationship between actions and metrics we just defined on \mathfrak{p} . The precise relationship can be phrased using the isomorphism $\phi_t : G_t \rightarrow G$ from §2.2.1, and the dilation

$$\begin{aligned} z_t : \mathfrak{p} &\rightarrow \mathfrak{p} \\ x &\mapsto x/t. \end{aligned}$$

Lemma 2.7. (i) For every x in \mathfrak{p} and g in G , we have $\phi_t^{-1}(g) \cdot_t z_t(x) = z_t(g \cdot x)$.

(ii) The metrics η_t and η_1 are related through $z_t^* \eta_t = t^{-2} \cdot \eta_1$.

Proof. Fix (g, x) in $G \times \mathfrak{p}$ and observe the image of both sides of (i) by the diffeomorphism u_t : on the left-hand side,

$$u_t(\varphi_t^{-1}(g) \cdot_t z_t(x)) = \varphi_t^{-1}(g) \exp_{G_t}(z_t(x))K = \varphi_t^{-1}(g \exp_G(x))K;$$

on the right-hand side, we see from the definitions of the actions that

$$u_t(z_t(g \cdot x)) = \exp_{G_t}[z_t(g \cdot x)]K = \varphi_t^{-1}(\exp_G(g \cdot x))K = \varphi_t^{-1}(g \exp_G(x))K.$$

This proves (i). For (ii), we know from (i) that $z_t^* \eta_t$ is a G_1 -invariant metric on \mathfrak{p} . But so is η_1 : since the derivative of z_t at zero is multiplication by t^{-1} , we know that $z_t^* \eta_t$ and $t^{-2} \cdot \eta_1$ coincide at zero, so they must be equal. \square

We now give a precise statement for the idea that as t goes to zero, the G_t -action on \mathfrak{p} “goes to” the natural action of G_0 on \mathfrak{p} by rigid motions. Each G_t , $t \in \mathbb{R}$, has underlying set $K \times \mathfrak{p}$, so the action is encoded by a map

$$\begin{aligned} \mathcal{A}_t : (K \times \mathfrak{p}) \times \mathfrak{p} &\rightarrow \mathfrak{p} \\ ((k, v), x) &\mapsto (k, v) \cdot_t x. \end{aligned}$$

Lemma 2.8. *For the topology of uniform convergence on compact subsets, \mathcal{A}_t goes to \mathcal{A}_0 when t goes to zero.*

Proof. For every $((k, v), x)$ in $(K \times \mathfrak{p}) \times \mathfrak{p}$ and t in \mathbb{R} , we first write

$$(k, v) \cdot_t x = (\varphi_t^{-1}[\varphi_1(k, tv)]) \cdot_t x = \frac{1}{t} [\varphi_1(k, tv)] \cdot (tx) \text{ by lemma 2.7(i).}$$

What we need to prove is that this goes to $v + \text{Ad}(k)x$ when t goes to zero. Apply the diffeomorphism u : we get

$$\begin{aligned} u\left(\frac{1}{t} \varphi_1(k, tv) \cdot (tx)\right) &= \exp_G\left(\frac{1}{t} u^{-1}[\varphi_1(k, tv) \exp_G(tx)K]\right)K \\ &= \exp_G\left(\frac{1}{t} u^{-1}[\exp_G(tv)k \exp_G(tx)K]\right)K \\ &= \exp_G\left(\frac{1}{t} u^{-1}[\exp_G(tv) \exp_G(t \text{Ad}(k)[x])K]\right)K. \end{aligned}$$

In the proof of Lemma 2.6, we saw that $e^{tv} e^{t \text{Ad}(k)[x]}$ reads $e^{t\beta(t)} \kappa(t)$, where $\kappa(t)$ lies in K and $\beta(t)$ in \mathfrak{p} , with $\beta(t) = v + \text{Ad}(k)[x] + r(t)$, $r(t) \in \mathfrak{p}$, $r(t) = O(t)$. We deduce that $\exp_G(tv) \exp_G(t \text{Ad}(k)[x])K = \exp_G(t\beta(t))K = u[t\beta(t)]$ and that

$$u\left(\frac{1}{t} \varphi_1(k, tv) \cdot (tx)\right) = \exp_G\left(\frac{1}{t} [t\beta(t)]\right)K = u(v + \text{Ad}(k)[x] + r(t)),$$

so $\frac{1}{t} \varphi_1(k, tv) \cdot (tx)$ does go to $v + \text{Ad}(k)x$ as t goes to zero. The fact that the convergence is uniform on compact subsets is immediate from Lemma 2.6. \square

2.3. Fréchet contractions and a program for §3-5

Suppose π is an irreducible representation of G and $\pi_0 = \mathcal{M}^{-1}(\pi)$ is the corresponding representation of G_0 in the Mackey-Higson bijection of §2.1. We wish to give a meaning to the phrase: “associated to the deformation $(G_t)_{t \in \mathbb{R}}$ is a continuous family $(\pi_t)_{t \in \mathbb{R}}$ of representations, so that $\pi_1 = \pi$ and π_t goes to π_0 as t goes to zero”. Of course a difficulty is that as t varies, the representations π_t usually will act on different spaces, and we need to choose an appropriate notion of “deformation of representations”.

For every irreducible tempered representation π of G , we shall describe

- a Fréchet space \mathbf{E} , (A)
- a family $(\mathbf{V}_t)_{t>0}$ of linear subspaces of \mathbf{E} , (B)
- for each $t > 0$, an irreducible representation $\pi_t : G_t \rightarrow \text{End}(\mathbf{V}_t)$ acting on \mathbf{V}_t , (B')
- a family $(\mathbf{C}_t)_{t>0}$ of endomorphisms of \mathbf{E} , to be called the *contraction operators*, (C)

with the following properties.

- For every f in \mathbf{E} and every $t > 0$, write f_t for $\mathbf{C}_t f$.
 - If f lies in \mathbf{V}_1 , then for every $t > 0$, f_t lies in \mathbf{V}_t . (Vect1)
 - For every f in \mathbf{E} , there is a limit (in \mathbf{E}) to f_t as t goes to zero. (Vect2)
- Set $\mathbf{V}_0 = \left\{ \lim_{t \rightarrow 0} (f_t), f \in \mathbf{V}_1 \right\}$.
 - Fix (k, v) in $K \times \mathfrak{p}$. Fix F in \mathbf{V}_0 and $f \in \mathbf{V}_1$ such that $F = \lim_{t \rightarrow 0} (f_t)$. (Op1)
Then there is a limit to $\pi_t(k, v) \cdot f_t$ as t goes to zero; the limit depends only on F .
 - Write $\pi_0(k, v)$ for the map from \mathbf{V}_0 to itself obtained from (Op1). (Op2)
Then $\pi_0(k, v)$ is linear for every (k, v) in $K \times \mathfrak{p}$; the map $\pi_0 : K \times \mathfrak{p} \rightarrow \text{End}(\mathbf{V}_0)$ defines a morphism from G_0 to $\text{End}(\mathbf{V}_0)$. The G_0 -representation (\mathbf{V}_0, π_0) is unitary irreducible. Its equivalence class is that which corresponds to π in the Mackey-Higson bijection.

For each $t > 0$, our contraction operator \mathbf{C}_t will be uniquely determined (up to a multiplicative constant) by a natural equivariance condition, together with Schur's lemma. To state the equivariance condition, we simplify the notation by identifying G with G_1 through the isomorphism $\varphi_1 : G_1 \rightarrow G$ of §2.2, and π with the representation $\pi_1 \circ \varphi_1^{-1}$ of G acting on \mathbf{V}_1 . Our requirement is most easily phrased in a special case:

If π has real infinitesimal character, we require that \mathbf{C}_t intertwine π and $\pi_t \circ \varphi_t^{-1}$. (Nat_{RIC})

If π does not have real infinitesimal character, we require that \mathbf{C}_t intertwine $\pi_t \circ \varphi_t^{-1}$, not quite with π , but with a representation obtained from π by renormalizing the imaginary part of the infinitesimal character. We bring in the renormalization map $\mathcal{R}_G^{1/t} : \tilde{G} \rightarrow \tilde{G}$ of [3], §4: starting from an irreducible tempered representation of π of G , we use the Knapp-Zuckerman classification to write π as $\text{Ind}_{M_P A_P N_P}^G (\sigma \otimes e^{i\nu})$ where $M_P A_P N_P$ is a cuspidal parabolic subgroup of G , σ is a discrete series or nondegenerate limit of discrete series of G , and ν is an element of \mathfrak{a}_P^* ; then we define $\mathcal{R}_G^{1/t}(\pi)$ as the equivalence class of the irreducible representation $\text{Ind}_{M_P A_P N_P}^G (\sigma \otimes e^{i\nu/t})$.

For arbitrary π , we require that \mathbf{C}_t intertwine $\mathcal{R}_G^{1/t}(\pi)$ and $\pi_t \circ \varphi_t^{-1}$. (Nat_{gen})

Of course we shall explain why the renormalization in (Nat_{gen}) is natural: see Proposition 3.11, Remark 3.14 and Remark 3.20.

In §3-5 below, we will implement this program for every irreducible tempered representation of G .

When we will have succeeded in carrying it out for a representation π , we shall say that the quadruple $(\mathbf{E}, (\mathbf{V}_t)_{t>0}, (\pi_t)_{t>0}, (\mathbf{C}_t)_{t>0})$ describes a *Fréchet contraction of π onto π_0* .

3. Examples of Fréchet contractions: the discrete series and the spherical principal series

We here illustrate the program outlined in §2.3 by implementing it for two particular classes of irreducible tempered representations of G .

There are at least three good reasons for beginning with these extreme and opposite examples before turning to more general results in sections 4 and 5, even though the later results will imply those of this section:

- All of the most important ideas in §4 and §5 will appear in these two examples, but they will be easier to discern without the technicalities to be met later.
- In our two examples, the representations admit geometric realizations as spaces of functions (or sections of a finite-rank vector bundle) on \mathfrak{p} : the elementary results of §2.2.2 will make the proofs particularly simple.
- Most of the technical lemmas to be proven along the way will be used again in §4 and §5.

3.1. Contraction of discrete series representations

Throughout §3.1, we assume that G is a linear connected semisimple Lie group with finite center and that G and K have equal ranks, so that G has a nonempty discrete series. We fix a discrete series representation π of G and recall that the representation of G_0 that corresponds to π in the Mackey-Higson bijection is the (unique) lowest K -type of π . We thus aim at describing a Fréchet contraction of π onto its lowest K -type μ .

3.1.1. Square-integrable solutions of the Dirac equation

We here recall results of Parthasarathy, Atiyah and Schmid [4, 31] which provide a geometrical realization for π .

Fix a maximal torus T in K . We will write $\Lambda \subset i\mathfrak{t}^*$ for the set of linear functionals that arise as the derivative of a unitary character of T , Δ_c for the set of roots of $(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, and Δ for the set of roots of $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$; of course $\Delta_c \subset \Delta$. If Δ^+ is a system of positive roots for Δ , we can consider the associated half-sum ρ of positive roots, write ρ_c for the half-sum of positive compact roots, and ρ_n for the half-sum of positive noncompact roots, so that $\rho_c + \rho_n = \rho$. We will denote the Δ_c^+ -highest weight of μ by $\bar{\mu}$.

We now call in the spin double cover $\text{Spin}(\mathfrak{p})$ of the special orthogonal group $SO(\mathfrak{p})$ associated with the inner product B chosen in §2.2.2, and its spin representation S . Recall that the condition that G and K have equal ranks guarantees that $\dim(G/K)$ is an even integer, say $2q$; the $\text{Spin}(\mathfrak{p})$ -module S has dimension 2^q and splits into two irreducible 2^{q-1} -dimensional $\text{Spin}(\mathfrak{p})$ -submodules S^+ and S^- , with ρ_n a weight of S^+ (for the action induced by the natural map from \mathfrak{k} to $\text{Spin}(\mathfrak{p})$).

Suppose V_{μ^b} is the carrier space for an irreducible $\mathfrak{k}_\mathbb{C}$ -module with highest weight $\mu^b = \bar{\mu} - \rho_n$. Then we can consider the tensor product $V_{\mu^b} \otimes S^\pm$, and although neither V_{μ^b} nor S^\pm need be a K -module if G is not simply connected, the action of \mathfrak{k} on $V_{\mu^b} \otimes S^\pm$ does lift to K (the half-integral ρ_n -shifts in the weights do compensate). So we can consider the equivariant bundle $\mathcal{E} = G \otimes_K (V_{\mu^b} \otimes S)$ over G/K , and its sub-bundles $\mathcal{E}^\pm = G \otimes_K (V_{\mu^b} \otimes S^\pm)$.

Now, the G -invariant metric that G/K inherits from the Killing form of \mathfrak{g} , together with the built-in spin structure of \mathcal{E} , make it possible to define a first-order differential operator D acting on smooth sections of \mathcal{E} , the Dirac operator. We will need a few immediate consequences of its definition in the next subsection, so let us briefly spell it out, referring to [31] for details.

We need the Clifford multiplication map $\mathbf{c} : \mathfrak{p}_\mathbb{C} \rightarrow \text{End}(S)$. Whenever X is in \mathfrak{p} , the endomorphism $\mathbf{c}(X)$ sends S^\pm to S^\mp ; in addition, the element X of \mathfrak{g} defines a left-invariant vector field on G/K , which yields a differential operator $X^\mathcal{E}$ acting (componentwise in the natural trivialization associated to the action of G on G/K) on sections of \mathcal{E} . If $(X_i)_{i=1..2q}$ is an orthonormal basis of \mathfrak{p} , then the Dirac operator reads

$$Ds = \sum_{i=1}^{2q} \mathbf{c}(X_i) X_i^\mathcal{E} s \quad (3.1)$$

for every section s of \mathcal{E} . It splits as $D = D^+ + D^-$, with D^\pm sending sections of \mathcal{E}^\pm to sections of \mathcal{E}^\mp .

Write \mathbf{H} for the space of smooth, square integrable sections of \mathcal{E} which are annihilated by D . Since D is an essentially self-adjoint elliptic operator, \mathbf{H} is a closed subspace of the Hilbert space of square-integrable sections of \mathcal{E} . Because D is G -invariant, \mathbf{H} is invariant under the natural action of G on sections of \mathcal{E} .

Theorem 3.1 (Parthasarathy, Atiyah & Schmid). *The Hilbert space \mathbf{H} carries an irreducible unitary representation of G , whose equivalence class is that of $\pi = \mathbf{V}_G(\mu)$.*

We must point out that Atiyah and Schmid's proof shows that the solutions to the Dirac equation that are gathered in \mathbf{H} do not explore the whole fibers, but are actually sections of a sub-bundle of \mathcal{E} whose fiber, a K -module, is irreducible and of class μ . Let W denote the isotypical K -submodule of $V_{\mu^b} \otimes S^+$ for the highest weight $\bar{\mu} = \mu^b + \rho_n$; the K -module W is irreducible. Let \mathcal{W} denote the equivariant bundle on G/K associated to W .

Proposition 3.2 (Atiyah & Schmid). *If a section of \mathcal{E} is a square-integrable solution of the Dirac equation, then it is in fact a section of \mathcal{W} .*

Atiyah and Schmid comment on this result as follows:

We should remark that the arguments leading up to [the fact that the cokernel of the Dirac operator is zero] are really curvature estimates, in algebraic disguise. The curvature properties of the bundles and of the manifold G/K force all square-integrable, harmonic spinors to take values in a certain sub-bundle of $[\mathcal{E}]$, namely the one that corresponds to the K -submodule of highest weight $[\mu^b + \rho_n]$ in $[V_{\mu^b} \otimes S^+]$.

3.1.2. Contraction to the lowest K -type

We are now in a position to observe how, for every real number $t \neq 0$, the above construction yields a G_t -invariant Dirac equation on \mathfrak{p} , and study the behaviour of solutions when the parameter t goes to zero.

Fix a real number t and bring in from §2.2.2 the diffeomorphism $u_t : \mathfrak{p} \rightarrow G_t/K$, the action of G_t and the metric η_t on \mathfrak{p} . Using the action of K on $V_{\mu} \otimes S$, we use the results of the previous subsection to build a G_t -equivariant spinor bundle \mathcal{E}_t over G_t/K . The vector bundle $u_t^* \mathcal{E}_t$ over \mathfrak{p} can be trivialized using the G_t -action: this yields a vector bundle isomorphism, say T_t , between $u_t^* \mathcal{E}_t$ and the trivial bundle $\mathfrak{p} \times (V_{\mu^b} \otimes S)$.

Using the metric η_t to build a G_t -equivariant Dirac operator acting on sections of $u_t^* \mathcal{E}_t$, and trivializing using T_t , we obtain a first-order differential operator D'_t acting on $\mathbf{C}^\infty(\mathfrak{p}, V_{\mu^b} \otimes S)$. Proposition 3.2 makes it tempting to build from D'_t a differential operator acting on

$$\mathbf{E} = \mathbf{C}^\infty(\mathfrak{p}, W). \quad (\text{A})$$

We equip \mathbf{E} with the usual Fréchet topology (see [38], Chapter 10: we shall spell out semi-norms defining the topology of \mathbf{E} in the proof of Lemma 3.8). We then define the differential operator

$$\Delta_t := \text{Proj}_W \circ \left[(D'_t)^2 \Big|_{\mathbf{C}^\infty(\mathfrak{p}, W)} \right]$$

where Proj_W is the (orthogonal) isotypical projection $V_{\mu^b} \otimes S \rightarrow W$.

Applying the above construction for each t (including $t = 0$), we obtain a second-order G_0 -invariant differential operator Δ_0 on the flat space (\mathfrak{p}, η_0) , as well as G_t -invariant differential operators $\Delta_t, t \neq 0$, for the negatively-curved spaces (\mathfrak{p}, η_t) . These operators do fit together:

Lemma 3.3. *The family $(\Delta_t)_{t \in [0,1]}$ of differential operators on \mathbf{E} is weakly continuous: if f lies in \mathbf{E} , then $(\Delta_t f)_{t \in [0,1]}$ is a continuous path in \mathbf{E} .*

Proof. Fix an orthonormal basis $(X_i)_{i=1..2q}$ of \mathfrak{p} as in (3.1) and use the $2q$ corresponding Cartesian coordinates on \mathfrak{p} ; given (3.1), the operator $D'_t \Big|_{\mathbf{C}^\infty(\mathfrak{p}, W)}$ reads

$$D'_t \Big|_{\mathbf{C}^\infty(\mathfrak{p}, W)} = \sum_{i=1}^{2q} A_t^i \partial_i + K_t$$

where $A_t^i, i = 1..2q$ and K_t are continuous maps from \mathfrak{p} to the space $\mathcal{L}(W, V_{\mu^b} \otimes S)$ of linear maps $W \rightarrow V_{\mu^b} \otimes S$. Given the definitions of u_t and T_t , each of the vector fields $(T_t \circ u_t)^* X_i^{\mathcal{E}_t}$ defines a continuous endomorphism of $\mathcal{C}^\infty(\mathfrak{p} \times \mathbb{R})$ when this space is equipped with its usual Fréchet topology; given (3.1) we deduce that multiplication by $(x, t) \mapsto A_t^i(x)$ and $(x, t) \mapsto K_t(x)$ define continuous endomorphisms of $\mathcal{C}^\infty(\mathfrak{p} \times \mathbb{R})$; Lemma 3.3 follows. \square

We know from the previous subsection that the \mathbf{L}^2 kernel of $\Delta_t, t > 0$, carries “the” discrete series representation of G_t with lowest K -type μ . For every $t > 0$, we set

$$\mathbf{V}_t = \{f \in \mathbf{C}^\infty(\mathfrak{p}, W) \mid \Delta_t f = 0 \text{ and } f \in \mathbf{L}^2(\eta_t, W)\}. \quad (\text{B})$$

and call in the action of G_t on $\mathbf{E} = \mathbf{C}^\infty(\mathfrak{p}, W)$ induced by the action \cdot_t on \mathfrak{p} (of §2.2.2) and the action μ of K on W : if (k, v) is an element of $K \times \mathfrak{p}$ – interpreted as an element γ of G_t – and f is an element of \mathbf{E} , we set

$$\pi_t(k, v)f : x \mapsto \mu(k) \cdot f(\gamma^{-1} \cdot_t x) \quad (3.2)$$

where the inverse for γ is taken in G_t . From the G_t -invariance of Δ_t (and the fact that D_t^i and its square have the same \mathbf{L}^2 kernel) we know that

The subspace \mathbf{V}_t of \mathbf{E} is π_t -stable; the representation (\mathbf{V}_t, π_t) of G_t is irreducible.
It is a discrete series representation of G_t with lowest K -type μ . (B')

Recall from Lemma 2.7 that the dilation

$$z_t : x \mapsto \frac{x}{t}$$

intertwines the actions of G and G_t on \mathfrak{p} . We can use z_t to transform functions on \mathfrak{p} , setting

$$\mathbf{C}_t f := x \mapsto f(t \cdot x) \tag{C}$$

for every f in \mathbf{E} . As a consequence of Lemma 2.7(ii) and the definition of the Dirac operator in (3.1), we get

$$\mathbf{C}_t^{-1} \Delta_t \mathbf{C}_t = t^4 \cdot \Delta_1.$$

Together with the fact that $\mathbf{C}_t f$ is square-integrable with respect to η_t as soon as f is square-integrable with respect to η_1 , this means that

If f lies in \mathbf{V}_1 , then for each $t > 0$, the map $\mathbf{C}_t f$ lies in \mathbf{V}_t . (Vect1)

We remark that our contraction operators $(\mathbf{C}_t)_{t>0}$ satisfy the naturality property (Nat_{RIC}) of §2.3, and are (almost) uniquely determined by that constraint:

Lemma 3.4. Fix $t > 0$.

- (i) The contraction operator \mathbf{C}_t intertwines the representations $(\mathbf{V}_1, \pi_1 \circ \varphi_1)$ and $(\mathbf{V}_t, \pi_t \circ \varphi_t)$ of G .
- (ii) Our contraction operator \mathbf{C}_t is the only map from \mathbf{V}_1 to \mathbf{V}_t that satisfies (i) and preserves the value of functions at zero, in that $(\mathbf{C}_t f)(0) = f(0)$ for every f in \mathbf{E} .

Part (i) is immediate from Lemma 2.7, and part (ii) from the irreducibility of π_1 and π_t and Schur's lemma. □

It should be mentioned that the idea of using of a change of variable as an intertwining operator for contractions of representations is not new; see for example [34].



Let us turn to the convergence of the contraction process. Fix an f in \mathbf{V}_1 , and set $f_t = \mathbf{C}_t f$ for each $t > 0$. We first make the trivial but crucial observation that in the usual Fréchet topology on \mathbf{E} ,

When t goes to zero, f_t goes to the constant function on \mathfrak{p} with value $f(0) \in W$. (Vect2)

In our current setting, more can be said. We know that \mathbf{V}_1 splits as a direct sum according to the decomposition of $(\pi_1)|_K$ into K -types, so that we can write f as a Fourier series

$$f = \sum_{\lambda \in \widehat{K}} f_\lambda$$

where, for each λ in \widehat{K} , f_λ belongs to a finite-dimensional subspace \mathbf{V}_1^λ of \mathbf{V}_1 on which $(\pi_1)|_K$ restricts as a direct sum of copies of λ . Of course a parallel decomposition holds for \mathbf{V}_t , $t \neq 0$, and f_t , too, has a Fourier series

$$f_t = \sum_{\lambda \in \widehat{K}} f_{t,\lambda}.$$

For every λ in \widehat{K} , the dimension of the λ -isotypical subspace \mathbf{V}_t^λ is independent of t , and the support of the above Fourier series does not depend on t .

Lemma 3.5. *The Fourier component $f_{t,\lambda}$ is none other than $\mathbf{C}_t f_\lambda$.*

Proof. For each $\lambda \in \widehat{K}$, we define an endomorphism P_λ of $\mathbf{C}^\infty(\mathfrak{p}, W)$, as follows:

$$\text{For all } f \text{ in } \mathbf{C}^\infty(\mathfrak{p}, W), P_\lambda(f) \text{ is the map } \left[x \mapsto \int_K \xi_\lambda^*(k) \mu(k) \cdot f(\text{Ad}(k^{-1})x) dk \right]$$

where ξ_λ is the global character of λ – a continuous function from K to \mathbb{C} – and the star is complex conjugation. Now, if f is an element of \mathbf{V}_t , we can view K as a subgroup of G_t : since the action of K on \mathfrak{p} induced from the G_t -action \cdot_t is none other than the adjoint action, the formula for $P_\lambda f$ is exactly that which defines the isotypical projection from \mathbf{V}_t to \mathbf{V}_t^λ . From Lemma 3.4, we deduce that P_λ commutes with \mathbf{C}_t . \square

Lemma 3.6. *If $\lambda \in \widehat{K}$ is different from the lowest K -type μ , then $f_\lambda(0) = 0$.*

Proof. The origin of \mathfrak{p} is a fixed point for the action of K on \mathfrak{p} ; so

$$f_\lambda(0) = (P_\lambda f)(0) = \int_K \xi_\lambda^*(k) \mu(k) \cdot f(0) dk. \quad (3.3)$$

Recall that $f(0)$ is in W , which is an irreducible K -module of class μ : now, (3.3) is the formula for the orthogonal projection of $f(0)$ onto the isotypical component of W corresponding to $\lambda \in \widehat{K}$: this is zero whenever $\lambda \neq \mu$. \square

Lemmas 3.5 and 3.6, together with property (Vect2) above, prove that each Fourier component of f goes to zero as the contraction is performed, except for that which corresponds to the lowest K -type. We can thus refine (Vect1)-(Vect2) into the following statement. Set

$$\mathbf{V}_0 = \{ \text{constant functions on } \mathfrak{p} \text{ with values in } W \}.$$

Proposition 3.7.

- (i) *For every $f \in \mathbf{E}$, there is a limit f_0 to $\mathbf{C}_t f$ as t goes to zero (the convergence is for the topology of \mathbf{E}).*
- (ii) *The limit f_0 is the element of \mathbf{V}_0 with value $f(0)$.*
- (iii) *Suppose f is in \mathbf{V}_1 ; write f_{\min} for the orthogonal projection of f onto the isotypical subspace \mathbf{V}_1^μ of \mathbf{V}_1 for the lowest K -type μ . Then $\mathbf{C}_t(f - f_{\min})$ goes to zero as t goes to zero.*
- (iv) *The correspondence $f \rightarrow f_0$ induces a K -equivariant isomorphism between \mathbf{V}_1^μ and \mathbf{V}_0 .*

We now observe the asymptotic behavior of the operators $\pi_t(k, v)$, when (k, v) is a fixed element of $K \times \mathfrak{p}$ – interpreted for each t as an element of G_t . Fix an element f of \mathbf{V}_1 and consider the associated family $(f_t)_{t>0}$. To study $\pi_t(k, v)f_t$, remark that

$$\pi_t(k, v)f_t = \pi_t(k, v)(f_t - f_0) + \pi_t(k, v)f_0, \quad (3.4)$$

and insert the following observation:

Lemma 3.8. *Fix (k, v) in $K \times \mathfrak{p}$. The topology of \mathbf{E} can be defined by a distance with respect to which each of the $\pi_t(k, v)$, $t \in [0, 1]$, is 1- Lipschitz.*

Proof. Whenever $A \subset \mathfrak{p}$ is compact, the subset $\Pi(A) = \{(k, v) \cdot_t A \mid t \in [0, 1]\}$ is compact too (see Lemma 2.8). So there is an increasing family $(A_n)_{n \in \mathbb{N}}$ of compact subsets of \mathfrak{p} such that $\Pi(A_n) \subset A_{n+1}$ for each n and $\cup_{n \in \mathbb{N}} A_n = \mathfrak{p}$. For each (n, K) in \mathbb{N}^2 and f in \mathbf{E} , set $\|f\|_{n,K} = \sup_{|\alpha|_1 \leq K} \sup_{x \in A_n} \|\partial^\alpha f(x)\|_W$, where $\alpha \in \mathbb{N}^K$ are multi-indices and $|\cdot|_1$ is the ℓ^1 height function on \mathbb{N}^K . For all f_1, f_2 in \mathbf{E} , we have $\|\pi_t f_1 - \pi_t f_2\|_{n,K} \leq \|f_1 - f_2\|_{n+1,K}$. A distance whose associated topology is that of \mathbf{E} is $d(f, f') = \sum_n \frac{\|f - f'\|_{n,K}}{2^{n+K}(1 + \|f - f'\|_{n,K})}$; the distance $d/2$ has the desired property. \square

Taking up (3.4), we can deduce that $\pi_t(k, v)(f_t - f_0)$ goes to zero as t goes to zero from the fact that $f_t - f_0$ does and $\pi_t(k, v)$ does not increase the distance of Lemma 3.8. As for the second term $\pi_t(k, v)f_0$ in (3.4), it is equal to

$\mu(k)f_0$ independently of t . In conclusion:

$$\pi_t(k, v) \cdot f_t \text{ goes to } \mu(k) \cdot f_0 \text{ as } t \text{ goes to zero.} \quad (\text{Op1})$$

Since property (Op2) of §2.3 is immediate when we set $\pi_0(k, v) = \mu(k)$, we have proved:

Theorem 3.9. *The quadruple $(\mathbf{E}, (\mathbf{V}_t)_{t>0}, (\pi_t)_{t>0}, (\mathbf{C}_t)_{t>0})$ describes a contraction from the discrete representation π to its lowest K -type μ .*



We close our discussion of the discrete series with a side remark. It may seem annoying, in view of Lemma 3.3, to find that the space \mathbf{V}_t of \mathbf{L}^2 -solutions to the G_t -invariant Dirac equation $\Delta_t f = 0$ should converge in such a natural way as indicated by Theorem 3.9 to the G_0 -module \mathbf{V}_0 , but to note that \mathbf{V}_0 is not the \mathbf{L}^2 kernel of Δ_0 , since that kernel is zero. A simple way to diminish the annoyance is to consider *extended kernels*, setting, for each t in \mathbb{R} (including $t = 0$),

$$\tilde{\mathbf{V}}_t = \{f \in \mathbf{C}^\infty(\mathfrak{p}, W) \mid \Delta_t f = 0 \text{ and there is a constant } c \in W \text{ such that } f + c \in \mathbf{L}^2(\eta_t, W)\}.$$

Because the riemannian metric η_t has infinite volume, when f is in $\tilde{\mathbf{V}}_t$, there can be only one constant c such that $f + c$ is square-integrable.

Lemma 3.10. (i) *For $t \neq 0$, the extended kernel $\tilde{\mathbf{V}}_t$ is none other than the \mathbf{L}^2 kernel \mathbf{V}_t .*

(ii) *For $t = 0$, the extended kernel \mathbf{H}_0 is the space of constant W -valued functions on \mathfrak{p} .*

Proof. Let us come back to G/K and the Dirac operator D defined in §3.1.1. Parthasarathy [31] expresses D^2 in terms of the Casimir operator Ω acting on sections of \mathcal{E} : there is a scalar σ such that

$$D^2 := D^- D^+ = -\Omega + \sigma.$$

Suppose a G -invariant trivialization of \mathcal{E} is chosen, so that D^2 is viewed as acting on functions from G/K to $V_{\mu^b} \otimes S$, and suppose $D^2 g = 0$, where g reads $f + C$ with $f \in \mathbf{L}^2(G/K, V_{\mu^b} \otimes S)$ and C a constant in $V_{\mu^b} \otimes S$. Then

$$\Omega f = \sigma f + \sigma C. \quad (3.5)$$

What we need to show is that this can only happen if C is zero. To that end, we use Helgason's Fourier transform for functions on G/K (see [17], Chapter 3; we shall meet Helgason's analogue of plane waves in §3.2.2). The Fourier transform of a smooth function with compact support on G/K is a function on $\mathfrak{a}^* \times K/M$; the transform defines a linear isometry \mathbf{F} between $\mathbf{L}^2(G/K)$ and $\mathbf{L}^2(\mathfrak{a}^* \times K/M)$ for a suitable measure on $\mathfrak{a}^* \times K/M$. There is a notion of tempered distributions on G/K and $\mathfrak{a}^* \times K/M$; when f is a smooth and square-integrable function, both f and Ωf are tempered distributions. The equality $\Omega f = \sigma f + \sigma C$ therefore is turned into an equality of tempered distributions on $\mathfrak{a}^* \times K/M$:

$$\mathbf{F}(\Omega f) - \sigma \mathbf{F}(f) = \sigma C \delta_{(0, 1M)}$$

where $\delta_{(0, 1M)}$ is the Dirac distribution at the point $(0, 1M)$. Given the transformation properties of \mathbf{F} with respect to the G -invariant differential operators, $\mathbf{F}(\Omega f)$ is actually the product of $\mathbf{F}(f)$ — an element of $\mathbf{L}^2(\mathfrak{a}^* \times K/M)$ — with a smooth function on \mathfrak{a}^* . Thus, if f is a smooth, square-integrable solution of (3.5), then $\sigma C \delta_0$ is the product of an element in $\mathbf{L}^2(\mathfrak{a}^* \times K/M)$ with a smooth function on the same space. That can only happen if C is zero. \square

3.2. On the principal series and generic spherical representations

In this section, we consider the representations associated with Mackey parameters of the form $\delta = (\chi, \mu)$ where χ is a *regular* element of \mathfrak{a}^* . The corresponding representations of G are (minimal) principal series representations, and several existing results can be understood as giving flesh to Mackey's analogy at the level of carrier spaces.

There are several very well-known function spaces carrying a representation of G with class $\mathbf{M}(\delta)$ – see for instance §VII.1. in [25]. We will use two of these standard realizations: in the first, the functions are defined on K/M (where $M = Z_K(\mathfrak{a})$ is the compact group of Remark 2.4), which has the same meaning in G and G_0 ; in the second, to be considered here only when μ is trivial, the functions are defined on G/K , or equivalently on \mathfrak{p} , and the geometrical setting in §2.2.2 will prove helpful.

3.2.1. A contraction of (minimal) principal series representations in the compact picture

We first assume $\delta = (\chi, \mu)$ to be an *arbitrary* Mackey parameter and recall the “compact picture” realization for the G -representation $\mathbf{M}(\delta)$ from (2.4).

To simplify the notation, we write $P = MAN$ for the cuspidal parabolic subgroup P_χ to be induced from. Suppose V_σ is the carrier space for a tempered irreducible M -module of class $\sigma = \mathbf{V}_M(\mu)$. Fix an M -invariant inner product on V_σ , and consider

$$\mathbf{H}_\sigma^{\text{comp}} = \{f \in \mathbf{L}^2(K; V_\sigma) / f(km) = \sigma(m)^{-1}f(k), \forall (k, m) \in K \times M\}. \quad (3.6)$$

To endow $\mathbf{H}_\sigma^{\text{comp}}$ with a G -action, we call in the Iwasawa projections $\kappa, \mathbf{m}, \mathbf{a}, \nu$ sending an element of G to the unique quadruple

$$\begin{aligned} (\kappa(g), \mathbf{m}(g), \mathbf{a}(g), \nu(g)) &\in K \times \exp_G(\mathfrak{m} \cap \mathfrak{p}) \times \mathfrak{a} \times N \quad \text{such that} \\ g &= \kappa(g)\mathbf{m}(g)\exp_G(\mathbf{a}(g))\nu(g). \end{aligned} \quad (3.7)$$

We recall that if P is minimal, the map \mathbf{m} is trivial. For every g in G , an endomorphism of $\mathbf{H}_\sigma^{\text{comp}}$ is defined by setting

$$\pi_{\chi, \mu}^{\text{comp}}(g) = f \mapsto \left[k \mapsto e^{\langle -i\chi - \rho, \mathbf{a}(g^{-1}k) \rangle} \sigma(\mathbf{m}(g^{-1}k))^{-1} f(\kappa(g^{-1}k)) \right] \quad (3.8)$$

where ρ is the half-sum of those roots of $(\mathfrak{g}, \mathfrak{a})$ that are positive in the ordering used to define N . The Hilbert space (3.6) does not depend on χ , but the G -action (3.8) does.



We now assume that χ is a *regular* element of \mathfrak{a}^* , so that P_χ is a *minimal* parabolic subgroup of G , and describe a Fréchet contraction of $\pi = \mathbf{M}(\chi, \mu)$ onto $\mathbf{M}_0(\chi, \mu)$.

Recall from Remark 2.4 that in the decomposition $P_\chi = M_\chi A_\chi N_\chi$, the subgroup $M_\chi = M = Z_K(\mathfrak{a})$ is contained in K , and that $A_\chi = A$.

Fix $t > 0$, λ in \mathfrak{a}^* and σ in \widehat{M} . Then we can view (λ, σ) as a Mackey parameter for G_t and run through the constructions of §2.1 for G_t : this leads to defining a parabolic subgroup $P_{\lambda, t} = M_t A_t N_t$ of G_t , with $M_t = M$, and to considering the principal series representation $\pi_{\lambda, \sigma}^{t, \text{comp}}$ of G_t acting on

$$\mathbf{H}_\sigma^{\text{comp}} \quad \text{from (3.6)}$$

via

$$\pi_{\lambda, \sigma}^{t, \text{comp}}(\gamma) : f \mapsto \left[k \mapsto \exp \langle -i\lambda - \rho_t, \mathbf{a}_t(\gamma^{-1}k) \rangle f(\kappa_t(\gamma^{-1}k)) \right] \quad (3.9)$$

where ρ_t is the half-sum of positive roots for $(\mathfrak{g}_t, \mathfrak{a})$ associated with the ordering used to define N_t , and $\kappa_t : G_t \rightarrow K$, $\mathbf{a}_t : G_t \rightarrow \mathfrak{a}$ are the Iwasawa projections.

At this point, given our Mackey parameter (χ, μ) , it may seem natural to fix an element (k, v) in $K \times \mathfrak{p}$, view it as an element of G_t for each t and try to study the convergence of $\pi_{\chi, \mu}^{t, \text{comp}}(k, v)$ from (3.9) as t goes to zero. We shall pursue this, but we must point out that this idea surreptitiously brings in the renormalization of continuous parameters from property (Nat_{gen}) in §2.3. To see why, define a representation of G as the composition

$$\pi_{\chi, \mu}^{t, \text{comp}}(\gamma) \circ \varphi_t^{-1} : G \xrightarrow{\varphi_t^{-1}} G_t \xrightarrow{\pi_{\chi, \mu}^{t, \text{comp}}} \text{End}(\mathbf{H}_\sigma^{\text{comp}}).$$

Proposition 3.11. For each $t > 0$, $\pi_{\chi, \mu}^{t, \text{comp}}(\gamma) \circ \phi_t^{-1}$ is equal to $\pi_{\chi/t, \mu}^{\text{comp}}$.

To prove Proposition 3.11, we only have to understand what happens to the half-sum of positive roots when we go from G to G_t , and to make the relationship between the Iwasawa decompositions in both groups clear. Here are statements that apply to any parabolic subgroup.

Lemma 3.12. Suppose $P = MAN$ is an arbitrary parabolic subgroup of G with A contained in $\exp_G(\mathfrak{p})$. If $\alpha \in \mathfrak{a}^*$ is a root of $(\mathfrak{g}, \mathfrak{a})$, then $t\alpha$ is a root of $(\mathfrak{g}_t, \mathfrak{a})$.

Proof. When α is a root of $(\mathfrak{g}, \mathfrak{a})$, there is a nonzero $X \in \mathfrak{g}$ such that $[X, H] = \alpha(H)X$ for each $H \in \mathfrak{a}$. Split X as $X_e + X_h$, with $X_e \in \mathfrak{k}$ and $X_h \in \mathfrak{p}$, according to the Cartan decomposition. Then

$$[X_e, H] - \alpha(H)X_h = -([X_h, H] - \alpha(H)X_e). \quad (3.10)$$

The left-hand side of (3.10) is in \mathfrak{p} and the right-hand side is in \mathfrak{k} , so both are zero.

Now, the group morphism $\phi_t : G_t \rightarrow G$ from §2.2 induces a Lie algebra isomorphism $\phi_t : \mathfrak{g}_t \rightarrow \mathfrak{g}$. The isomorphism ϕ_t^{-1} sends X to $X^t = X_e + \frac{1}{t}X_h \in \mathfrak{g}_t$, and for each $H \in \mathfrak{a}$,

$$[X^t, H]_{\mathfrak{g}_t} = [X_e, H]_{\mathfrak{g}_t} + \frac{1}{t}[X_h, H]_{\mathfrak{g}_t} = [X_e, H]_{\mathfrak{g}} + t[X_h, H]_{\mathfrak{g}} \quad (3.11)$$

(for the last equality, recall that if U, V are in \mathfrak{g} , the bracket $[U, V]_{\mathfrak{g}_t}$ is defined as $\phi_t^{-1}[\phi_t U, \phi_t V]_{\mathfrak{g}}$, so that $[U, V]_{\mathfrak{g}_t} = [U, V]_{\mathfrak{g}}$ when U lies in \mathfrak{k} and V lies in \mathfrak{p} , whereas $[U, V]_{\mathfrak{g}_t} = t^2[U, V]_{\mathfrak{g}}$ when they both lie in \mathfrak{p}).

The right-hand side of (3.11) is $\alpha(H)X_h + t \cdot \alpha(H)X_e = t \cdot \alpha(H)X^t$, so X^t is in the $(\mathfrak{g}_t, \mathfrak{a})$ root space for $t \cdot \alpha$, whence Lemma 3.12. \square

The proof shows that the root space for $t\alpha$ is the image of \mathfrak{g}_α under ϕ_t^{-1} ; we spell out an immediate consequence.

Lemma 3.13. Suppose $P = MAN$ is a parabolic subgroup of G . Then $P_t := \phi_t^{-1}(P)$ is a parabolic subgroup of G_t , with Iwasawa decomposition $M_t A_t N_t = (\phi_t^{-1}M)A(\phi_t^{-1}N)$. The Iwasawa projections for G and G_t are related as follows: for every g in G ,

$$\begin{aligned} \kappa_t(\phi_t^{-1}g) &= \kappa(g); \\ \mathfrak{m}_t(\phi_t^{-1}g) &= \mathfrak{m}(g); \\ \mathfrak{a}_t(\phi_t^{-1}g) &= \frac{\mathfrak{a}(g)}{t}. \end{aligned}$$

We now return to the proof of Proposition 3.11, assuming again that χ is regular. Inspecting the definitions of §2.1, we know from Lemma 3.13 that the parabolic subgroup $P_{\chi, t}$ of G_t used in (3.9) is $\phi_t^{-1}(P_\chi) = MA\phi_t^{-1}(N_\chi)$. From Lemma 3.12 we know that ρ_t in (3.9) is $t\rho$. We can now conclude that for every g in G ,

$$\begin{aligned} \pi_{\chi, \mu}^{t, \text{comp}}(\phi_t^{-1}(g)) &= f \mapsto \left[k \mapsto \exp \left\langle -i \frac{\chi}{t} - \rho, t \cdot \mathfrak{a}_t(\phi_t^{-1}[g^{-1}k]) \right\rangle f(\kappa_t(\phi_t^{-1}[g^{-1}k])) \right] \\ &= f \mapsto \left[k \mapsto \exp \left\langle -i \frac{\chi}{t} - \rho, \mathfrak{a}(g^{-1}k) \right\rangle f(\kappa(g^{-1}k)) \right] \quad (\text{using Lemma 3.13}) \\ &= \pi_{\chi, \mu}^{\text{comp}}(g). \end{aligned}$$

This concludes the proof of Proposition 3.11. \square

Remark 3.14. There is an implicit choice behind the interpretation of (χ, μ) as a Mackey datum for G_t which leads to Proposition 3.11. In the present context, we have $A_t = A_1$ for all $t > 0$; as one referee pointed out, our way of viewing $e^{i\chi}$ as a character of A_t amounts to using the identification between the Pontryagin duals \widehat{A}_t and \widehat{A} induced by the identity map $\mathfrak{a}_t \rightarrow \mathfrak{a}$. There are other ways to identify \widehat{A}_t and \widehat{A} ; but they do not lead to the convergence theorems to be proven below. The renormalization of continuous parameters in property (Nat_{gen}) of §2.3 reflects the identification.

With Proposition 3.11 and its proof in hand, discussing the contraction from G to G_0 becomes particularly simple. Set

$$\mathbf{E} = \mathbf{H}_\sigma^{\text{comp}} \quad \text{from (3.6)} \quad (\text{A})$$

and for every $t > 0$,

$$\mathbf{V}_t = \mathbf{E} \quad (\text{B})$$

$$\pi_t = \pi_{\chi, \mu}^{t, \text{comp}} \quad \text{from (3.9)} \quad (\text{B}')$$

$$\mathbf{C}_t = \text{id}_{\mathbf{E}}. \quad (\text{C})$$

Fix (k, v) in $K \times \mathfrak{p}$. For every $t > 0$, the endomorphism $\pi_t(k, v)$ of \mathbf{E} is an operator for a principal series representation of G with continuous parameter $\frac{\chi}{t}$; but as t goes to zero it gets closer and closer to an operator for the representation of G_0 with Mackey parameter (χ, μ) :

Proposition 3.15. *Fix (k, v) in $K \times \mathfrak{p}$ and f in \mathbf{E} . There is a limit to $\pi_t(k, v)f$ as t goes to zero, given by the map $\pi_0(k, v)f$ in (2.3). The convergence holds both in $\mathbf{L}^2(K, V_\mu)$ and in the sense of uniform convergence in $\mathbf{C}(K, V_\mu)$.*

Recall that when k is in K and v in \mathfrak{p} ,

$$\text{the element } \exp_{G_t}(v)k \text{ of } G_t \text{ is just } (k, v) \text{ in the underlying set } K \times \mathfrak{p}. \quad (3.12)$$

As a consequence, we can rewrite (3.9) as

$$\pi_{\chi, \mu}^{t, \text{comp}}(k, v) = f \mapsto [u \mapsto \exp\langle -i\chi - t\rho, \mathbf{a}_t((k^{-1} \cdot_t \exp_{G_t}(-v) \cdot_t u)) \rangle f(\kappa_t(k^{-1} \exp_{G_t}(-v)u))].$$

where the symbols \cdot_t indicate products in G_t . On the other hand, the operator $\pi_0(k, v)$ from §2.1 reads

$$\pi_0(k, v) = f \mapsto [u \mapsto \exp\langle i\chi, \text{Ad}(u^{-1})v \rangle f(k^{-1}u)].$$

To make the two look more similar, notice that

$$\begin{aligned} [\pi_{\chi, \mu}^{t, \text{comp}}(k, v)f](u) &= e^{\langle -i\chi - t\rho, \mathbf{a}_t((k^{-1}u) \cdot_t \exp_{G_t}(-\text{Ad}(u^{-1})v)) \rangle} f(\kappa_t(k^{-1}u \exp_{G_t}(-\text{Ad}(u^{-1})v))) \\ &= e^{\langle -i\chi - t\rho, \mathbf{a}_t(\exp_{G_t}(-\text{Ad}(u^{-1})v)) \rangle} f((k^{-1}u) \cdot \kappa_t(\exp_{G_t}(-\text{Ad}(u^{-1})v))). \end{aligned}$$

We just need to see how the Iwasawa projection parts behave as t goes to zero. Let us introduce, for every $t > 0$, the Iwasawa maps

$$\begin{aligned} \mathfrak{K}_t : \mathfrak{p} &\rightarrow K & \text{and} & & \mathfrak{J}_t : \mathfrak{p} &\rightarrow \mathfrak{a} \\ v &\mapsto \kappa_t(\exp_{G_t}(v)) & & & v &\mapsto \mathbf{a}_t(\exp_{G_t}(v)). \end{aligned} \quad (3.13)$$

The Iwasawa projection $\mathfrak{J}_t : \mathfrak{p} \rightarrow \mathfrak{a}$ is nonlinear, but as t goes to zero it gets closer and closer to a linear projection:

Lemma 3.16. *As t goes to zero,*

- \mathfrak{J}_t admits as a limit (in the sense of uniform convergence on compact subsets of \mathfrak{p} of every derivative) the orthogonal projection from \mathfrak{p} to \mathfrak{a} ,
- \mathfrak{K}_t tends to the constant function on \mathfrak{p} with value $\mathbf{1}_K$.

Proof. • We check that \mathfrak{J}_t is none other than $v \mapsto \frac{1}{t}\mathfrak{J}(tv)$, where $\mathfrak{J} : \mathfrak{p} \rightarrow \mathfrak{a}$ is the projection attached to G . Since φ_t is a group morphism from G_t to G , the definition of group exponentials does imply that $\exp_G(tv) = \exp_G(d\varphi_t(1)v) = \varphi_t(\exp_{G_t}v)$. Let us write $\exp_{G_t}v = ke^{\mathfrak{J}_t(v)}n_t$ with $k \in K$ and $n_t \in N_t$, then $\varphi_t(\exp_{G_t}v) = ke^{t\mathfrak{J}_t(v)}n$, where $n = \varphi_t(n_t)$ lies in N . So we know that

$$\exp_G(tv) = ke^{t\mathfrak{J}_t(v)}n \quad (3.14)$$

and thus that $\mathfrak{J}(tv) = t\mathfrak{J}_t(v)$, as announced.

But then as t goes to zero, the limit of $\mathfrak{J}_t(v)$ is the value at v of the derivative $d\mathfrak{J}(0)$. This does yield the orthogonal projection of v on \mathfrak{a} : although the Iwasawa decomposition of \mathfrak{g} is not an orthogonal direct sum because \mathfrak{k} and \mathfrak{n} are not orthogonal to each other, they are both orthogonal to \mathfrak{a} with respect to the Killing form of \mathfrak{g} , so the direct sum $\mathfrak{k} \oplus \mathfrak{n}$ is the orthogonal of \mathfrak{a} .

- As for \mathfrak{K}_t , from (3.14) we see that $\mathfrak{K}_t(v) = \kappa_t(\exp_{G_t} v) = \kappa(\exp_G(tv))$, and this does go to the identity uniformly on compact subsets as t goes to zero. \square

Both parts of Proposition 3.15 follow immediately.

Remark 3.17. Proposition 3.15 can be viewed as a reformulation of Theorem 1 in Dooley and Rice [13]. Our reason for including it here is that in our treatment of the general case in §5, we will take up the strategy of the proof and use the lemmas here proven. The interplay with Helgason's waves (in the next subsection) may also throw some light on the phenomenon, for instance on the necessity of renormalizing continuous parameters.

3.2.2. A contraction of generic spherical representations using Helgason's waves

Helgason's waves and picture for the spherical principal series. We first recall how the compact picture of the previous subsection is related to the usual "induced picture", for which the Hilbert space is

$$\mathbf{H}_\delta^{\text{ind}} = \left\{ f : G \rightarrow V_\sigma \mid \begin{array}{l} f(gme^H n) = e^{\langle -i\chi - \rho, H \rangle} \sigma(m)^{-1} f(g) \text{ for all } (g, me^H n) \in G \times P \\ \text{and } f|_K \in \mathbf{L}^2(K; V_\sigma) \end{array} \right\}, \quad (3.15)$$

the inner product is the \mathbf{L}^2 scalar product between restrictions to K , and the G -action is given by $\pi_\delta^{\text{ind}}(g)f = [x \mapsto f(g^{-1}x)]$ for (g, f) in $G \times \mathbf{H}_\delta^{\text{ind}}$. Because of the P -equivariance condition in (3.15), every element of $\mathbf{H}_\delta^{\text{ind}}$ is completely determined by its restriction to K , so restriction to K induces an isometry (say \mathcal{I}) between $\mathbf{H}_\delta^{\text{ind}}$ and $\mathbf{H}_\sigma^{\text{comp}}$. The definition of $\pi_{\chi, \mu}^{\text{comp}}$ in (3.8) is just what is needed to make \mathcal{I} an intertwining operator.

We now consider a Mackey parameter $\delta = (\chi, \mu)$ and assume that

- χ is regular and lies in \mathfrak{a}^* , so that K_χ is equal to $M = Z_K(\mathfrak{a})$;
- μ is the trivial representation of M .

There is then a distinguished element in $\mathbf{H}_\sigma^{\text{comp}}$: the constant function on K with value one. Under the isometry \mathcal{I} , it corresponds to the function

$$\bar{e}_{\chi, 1} = ke^H n \mapsto e^{\langle -i\chi - \rho, H \rangle}$$

in $\mathbf{H}_\delta^{\text{ind}}$; this in turn defines a function on G/K if we set $\tilde{e}_{\chi, 1}(gK) = \bar{e}_{\chi, 1}(g^{-1})$, and a function on \mathfrak{p} if we set $e_{\lambda, 1}(v) = \tilde{e}_{\chi, 1}(\exp_G(v)K)$. Figure 2 shows a plot of $e_{\lambda, 1}$ when G is $SL(2, \mathbb{R})$.

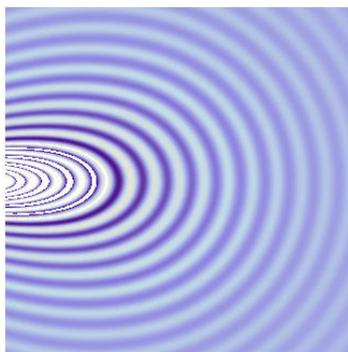


Figure 2 – Plot of the real part of the Helgason wave $e_{30, 1}$. We used the mapping from \mathbb{R}^2 to the unit disk provided by the Cartan decomposition, and the explicit formulae available on the unit disk: see [16], chapter 0. The x - and y - range is $[-1.5, 1.5]$ (this region is chosen so that the modulus varies clearly but within a displayable range, and the choice of λ is to have enough waviness in the region).

Now set $e_{\chi,b}(v) = e_{\chi,1}(b^{-1}v)$ for b in K/M and v in \mathfrak{p} (the notation comes from [17], §III.1: for \tilde{b} in K , the map $v \mapsto e_{\chi,1}(\tilde{b}^{-1}v)$ depends only on the class of \tilde{b} in K/M). Equip K/M with the measure inherited with the Haar measure of K . Then the ‘‘Poisson transform’’

$$\begin{aligned} \mathbf{L}^2(K/M) &= \mathbf{H}_\sigma^{\text{comp}} \rightarrow \mathbf{C}^\infty(\mathfrak{p}) \\ F &\mapsto \int_{K/M} e_{\chi,b} F(b) db \end{aligned}$$

intertwines $\pi_{\chi,1}^{\text{comp}}$ with the quasi-regular action of G on the Fréchet space

$$\mathbf{E} = \mathbf{C}^\infty(\mathfrak{p}) \tag{A}$$

associated with the action $(g,x) \mapsto g \cdot x$ defined below (2.8), and happens to be an injection (see [17], Chapter 3). We write

$$\mathbf{V}_{\text{Helgason}}^\chi = \left\{ \int_{K/M} e_{\chi,b} F(b) db, \quad F \in \mathbf{L}^2(K/M) \right\} \tag{3.16}$$

for the image of this map, a geometric realization for the spherical principal series representation of G with continuous parameter χ .

For each $t > 0$, we can repeat this construction to attach to every regular λ in \mathfrak{a}^* a geometric realization for the spherical principal series representation of G_t with continuous parameter λ . Using the quasi-regular action

$$\pi_t : G_t \rightarrow \text{End}(\mathbf{E}) \tag{3.17}$$

attached to the action \cdot_t of §2.2.2, and comparing the above constructions of Helgason with §3.2.1 (especially Lemma 3.12), we now define, for every λ in \mathfrak{a}^* and b in K/M , a function

$$\begin{aligned} \varepsilon_{\lambda,b}^t : \mathfrak{p} &\mapsto \mathbf{C} \\ v &\mapsto e^{\langle i\lambda + t\rho, \mathfrak{J}_t(\text{Ad}(b)\cdot v) \rangle} \end{aligned}$$

where \mathfrak{J}_t is the Iwasawa projection from (3.13). We set

$$\tilde{\mathbf{V}}_t^\lambda = \left\{ \int_{K/M} \varepsilon_{\lambda,b}^t F(b) db, \quad F \in \mathbf{L}^2(K/M) \right\}; \tag{B}$$

this is a $\pi_t(G_t)$ -stable subspace of \mathbf{E} . We note that $\tilde{\mathbf{V}}_1^\lambda$ coincides with $\mathbf{V}_{\text{Helgason}}^\lambda$ from (3.16).

Convergence of Helgason waves and contraction of spherical representations. Still assuming that χ is a regular element of \mathfrak{a}^* , suppose π is the spherical representation of G with continuous parameter χ ; then the geometric realization for π described above is strongly reminiscent of the description we gave in Remark 2.3 for the representation $\mathbf{M}_0(\chi, 1)$ of G_0 that corresponds to π in the Mackey-Higson bijection. There G_0 acted on the subspace of \mathbf{E} whose elements are combinations of plane waves whose wavevectors lie on the $\text{Ad}^*(K)$ -orbit of χ in \mathfrak{p}^* . But in view of Lemma 3.16, the two kinds of waves can be deformed onto one another (see Figure 3):

Lemma 3.18. *Fix λ in \mathfrak{a}^* and b in K/M . Then as t goes to zero, the Helgason waves $\varepsilon_{\lambda,b}^t$ converge (in the Fréchet space \mathbf{E}) to the Euclidean plane wave $v \mapsto e^{\langle i\lambda, \text{Ad}(b)\cdot v \rangle}$.*

It is then natural to use the spaces in (B) to write down a Fréchet contraction of π to the representation of G_0 on

$$\mathbf{V}_0 = \left\{ \left[v \mapsto \int_{K/M} e^{\langle i\chi, \text{Ad}(b)\cdot v \rangle} F(b) db \right], \quad F \in \mathbf{L}^2(K/M) \right\} \tag{3.18}$$

where an element (k, v) of G_0 acts on \mathbf{E} through the usual $\pi_0(k, v) : f \mapsto [x \mapsto f(k^{-1}x - k^{-1}v)]$.

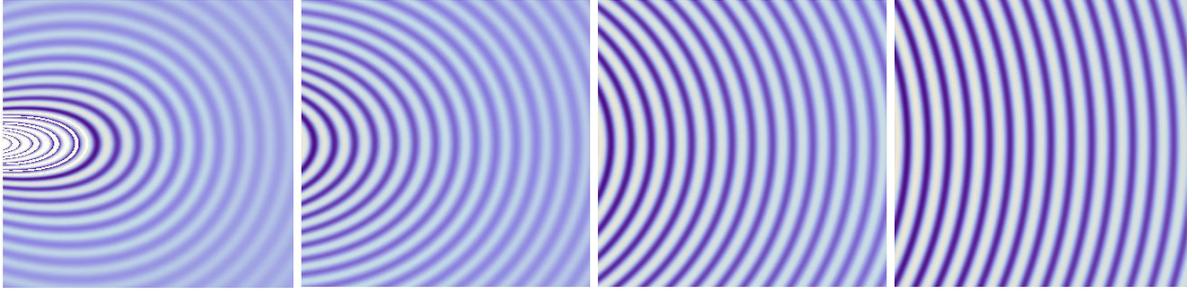


Figure 3 – Illustration of Lemma 3.18 : these are plots of $\varepsilon_{\chi,1}^{1/2^k}$, $k = 0, 1, 2, 3$, in the same domain as in Figure 2. Each of these waves is a building block for a principal series representation of G whose continuous parameter is $2^k\chi$, with $\chi = 30$ here.

The technical necessities have already been dealt with in §2.2.2 and §3.1 and involve the zooming-in operator

$$\begin{aligned} \mathbf{C}_t : \mathbf{E} &\rightarrow \mathbf{E} \\ f &\mapsto [x \mapsto f(tx)]. \end{aligned} \tag{C}$$

Lemma 3.19. *Suppose λ is an element of \mathfrak{a}^* and $t > 0$.*

- (i) *For every b in K/M , the operator \mathbf{C}_t sends $e_{\lambda,b}$ to $\varepsilon_{t\lambda,b}^t$.*
- (ii) *Fix f in \mathbf{E} . Then $\mathbf{C}_t f$ lies in $\tilde{\mathbf{V}}_t^\lambda$ if and only if f lies in $\tilde{\mathbf{V}}_1^{\lambda/t}$.*
- (iii) *The map \mathbf{C}_t intertwines the actions π and $\pi_t \circ \varphi_t^{-1}$ of G on $\tilde{\mathbf{V}}_1^{\lambda/t}$ and $\tilde{\mathbf{V}}_t^\lambda$, respectively.*
- (iv) *It is the only map satisfying (iii) which preserves the values of functions at zero.*

Proof. For (i), we use Lemma 3.13 and find that for all v in \mathfrak{p} ,

$$\varepsilon_{t\lambda,b}^t(v) = e^{\langle i\lambda + \rho, \mathfrak{I}(b \cdot (tv)) \rangle} = \varepsilon_{\lambda,b}^1(tv) = \varepsilon_{\lambda,b}^1(tv) = e_{\lambda,b}(tv).$$

Deducing (ii) is immediate. Part (ii) is from Lemma 2.7(i), and part (iv) is Schur's Lemma. \square

Remark 3.20. It may be instructive to compare (i) and (ii) of the above result with Proposition 3.11 and property (Nat_{gen}) of §2.3: we already saw that the zooming-in operator \mathbf{C}_t intertwines the G - and G_t -actions on \mathfrak{p} , so it is tempting to zoom-in on a Helgason wave for G to get a wave for G_t ; without any renormalization of the spacing between phase lines, however, they would get spread out and the zooming-in process would yield a trivial outcome.

To obtain a contraction from $\pi \simeq \mathbf{M}(\chi, \mu_{\text{triv}})$ to $\pi_0 \simeq \mathbf{M}_0(\chi, \mu_{\text{triv}})$, the above makes it natural to use, for the carrier space at time t ,

$$\mathbf{V}_t = \tilde{\mathbf{V}}_t^{\chi/t}. \tag{B, corrected}$$

Proposition 3.21. *The quadruple $(\mathbf{E}, (\mathbf{V}_t)_{t>0}, (\pi_t)_{t>0}, (\mathbf{C}_t)_{t>0})$ describes a Fréchet contraction from π to π_0 .*

The only property that still needs proof is (Op1): we take up the reasoning in Lemma 3.4, fix (k, v) in $K \times \mathfrak{p}$ and $f = \int_{K/M} \varepsilon_{\lambda,b}^1 F(b) db$ in \mathbf{V}_1 , set $f_t = \mathbf{C}_t f = \int_{K/M} \varepsilon_{\lambda,b}^t F(b) db$ for every $t > 0$, write $f_0 : v \mapsto \int_{K/M} e^{i\langle \lambda, \text{Ad}(b) \cdot v \rangle} F(b) db$ for the limit of f_t as t goes to zero, and remark that

$$\pi_t(k, v) f_t = \pi_t(k, v) (f_t - f_0) + \pi_t(k, v) f_0. \tag{3.19}$$

We now insert Lemma 3.8 (in the case of trivial fibers) to find that the first term goes to zero as t goes to zero, and Lemma 2.8 to find that the second term goes to $\pi_0(k, v) f_0$. \square

4. Real-infinitesimal-character representations

We begin our search for a Fréchet contraction in the general case. This section describes a contraction of every irreducible tempered representation of G with real infinitesimal character onto its lowest K -type, in case G is linear connected semisimple. In §5, we will reduce the general case to that one.

Fix an irreducible tempered representation π with real infinitesimal character; write μ for its lowest K -type. A body of work by Vogan, Zuckerman, Schmid, Wong and others makes it possible to build out of μ a pair (L, σ) , where

- L is a *quasi-split* Levi subgroup of G , the centralizer of an elliptic element in \mathfrak{g}^* ,
- σ is a tempered irreducible representation of L with real infinitesimal character.

One can then obtain from (L, σ) a geometric realization for π , based on the existence of a G -invariant complex structure on the elliptic coadjoint orbit G/L . Indeed, from (L, σ) we may form a holomorphic equivariant bundle \mathcal{V}^\sharp over G/L with fiber an irreducible L -module built from σ . The bundle usually has infinite rank; one of the Dolbeault cohomology spaces $H^{0,q}(G/L, \mathcal{V}^\sharp)$ yields a geometric realization for π (see §4.1.1).

In §4.1, we will prove that if there exists a Fréchet contraction of σ onto its lowest $(K \cap L)$ -type (and provided the contraction satisfies “nice” auxiliary conditions), then it is possible, using complex-analytic methods together with some of the ideas in §3.1, to find a contraction of π onto its lowest K -type. This reduces the problem of contracting π to that of finding a (“nice”) contraction of σ .

In §4.2, we will solve the latter problem. There σ may be realized as an irreducible factor in a principal series representation of L ; we will use methods of real harmonic analysis in the spirit of Helgason’s wave construction and §3.2.2 to obtain a (“nice”) contraction of σ .

4.1. Reduction to the quasi-split case

Throughout §4.1, we assume that G is a linear connected semisimple Lie group with finite center, we fix a tempered irreducible representation π of G with real infinitesimal character, and write μ for its lowest K -type. Our discussion is organized as follows:

- In §4.1.1, we give details about the realization of π by cohomological induction from σ .
- In §4.1.2, we construct, given a (“nice”) contraction of σ , a candidate for a Fréchet contraction of π onto μ , and state our main finding, Theorem 4.10.
- In §4.1.3, we prove the convergence results expressed by Theorem 4.10.
- We have gathered in §4.1.4 the proofs of several technical statements from §4.1.2.

4.1.1. Cohomological induction and real-infinitesimal-character representations

We here recall how a pair (L, σ) may be built from the lowest K -type μ of π , how to form the “cohomologically induced” module $H^{0,q}(G/L, \mathcal{V}^\sharp)$ to obtain a geometric realization for π , and how the lowest K -type μ can in turn be recovered from that geometrical construction.

Fix a maximal torus T in K . Vogan defines from μ an element $\lambda(\mu)$ in \mathfrak{t}^* (see [32], Proposition 2.3 and Corollary 2.4), as follows. Write Δ_c^+ for a system of positive roots for $\Delta(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, ρ_c for the half-sum of positive roots, $\bar{\mu}$ for the corresponding highest weight of μ . In the larger $\Delta = \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, fix a positive root system Δ^+ making $\bar{\mu} + \rho_c$ dominant; form the corresponding half-sum ρ of positive roots; finally, define $\lambda(\mu) \in \mathfrak{t}^*$ to be the projection of $\bar{\mu} + 2\rho_c - \rho$ on the convex cone of Δ^+ -dominant elements.

Consider the centralizer L of $\lambda(\mu)$ in G for the coadjoint action. This is a *quasi-split* reductive group of G , which is usually not compact; it admits $(K \cap L)$ as a maximal compact subgroup.

The representation σ of L to be used in the construction is specified, for instance, in Theorem 2.9 of [32]: it is the inverse image of π under the bijection (2.2) from there (here are some precisions : π belongs to the class denoted by $\Pi_a^{\lambda_a}(G)$ in the right-hand side of Eq. (2.2) in [32]; the inverse image σ of π is an irreducible admissible representation of L ; from Corollary 4.4 in [32] we know that σ is tempered; from the way the bijection in [32] affects infinitesimal character, we know that σ has real infinitesimal character). The unique lowest $(K \cap L)$ -type of σ is *fine*, in the sense of [39], Definition 4.3.9; as a consequence, σ appears as an irreducible constituent of a principal series representation of L – we shall say more on this in §4.2.

In the complexification $G_{\mathbb{C}}$, there exists a parabolic subgroup Q with Levi factor $L_{\mathbb{C}}$ (for an especially convenient choice, see [40], §13). Fix such a parabolic subgroup and write \mathfrak{u} for the unipotent radical of the Lie algebra \mathfrak{q} , so that $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}$; the complex dimension of \mathfrak{u} is $n = \frac{1}{2}(\dim(G) - \dim(L))$. The action of L on $\mathfrak{q}/\mathfrak{l}_{\mathbb{C}}$ induces an action on \mathfrak{u} ; thus, on the complex line $\wedge^n(\mathfrak{u})$ (top exterior power), there acts an abelian character of L : we will write $e^{2\rho(\mathfrak{u})}$ for it.

From an irreducible $(\mathfrak{l}_{\mathbb{C}}, (K \cap L))$ -module V with class σ , form the $(\mathfrak{l}_{\mathbb{C}}, (K \cap L))$ -module $V^{\sharp} = V \otimes \wedge^n(\mathfrak{u})$; the given choice of Q equips G/L with a G -invariant complex structure inherited from $G_{\mathbb{C}}/Q$; we can now form the holomorphic vector bundle $\mathcal{V}^{\sharp} = G \times_L (V^{\sharp})$ over G/L , then the Dolbeault cohomology groups $H^{(p,q)}(G/L, \mathcal{V}^{\sharp})$ (for these, see [41], §7, and of course [44], §2). Wong’s work on the closed-range property for the Dolbeault operator ([44], Theorem 2.4.(1)) equips $H^{(p,q)}(G/L, \mathcal{V}^{\sharp})$ with a Fréchet topology and a (\mathfrak{g}, K) -module structure.

The inclusion $K/(K \cap L) \hookrightarrow G/L$ is holomorphic; we write X for the (maximal) compact complex submanifold $K/(K \cap L)$ of G/L , and

$$s = \dim_{\mathbb{C}}(K/(K \cap L))$$

for the complex dimension of X .

Theorem 4.1 (from work by Vogan, Zuckerman, Knapp, and Wong). *The (\mathfrak{g}, K) -module $H^{(0,s)}(G/L, \mathcal{V}^{\sharp})$ is irreducible and its equivalence class is that of π .*

In the rest of §4.1, we will use this realization of π – we shall thus view π as a (\mathfrak{g}, K) -module and stay away from the delicate questions related to unitary globalizations.

Keeping in mind that our objective is the description of a Fréchet contraction of π onto its lowest K -type μ , we mention an analogous geometrical realization for μ .

Recall that the representation (V, σ) of L has a unique lowest $(K \cap L)$ -type: let us write μ^{\flat} for it, and W for the μ^{\flat} -isotypical subspace of V . Because μ^{\flat} occurs with multiplicity one in σ , the $(K \cap L)$ -module W is irreducible. The actions of L on V and $\wedge^n(\mathfrak{u})$ induce an action of $(K \cap L)$ on W and another action of $(K \cap L)$ on $\wedge^n(\mathfrak{u})$; we write W^{\sharp} for the $(K \cap L)$ -module $W \otimes \wedge^n(\mathfrak{u})$. We should emphasize the importance of using $\wedge^n(\mathfrak{u})$ once more: we do not switch to $\wedge^{\text{top}}(\mathfrak{u} \cap \mathfrak{k})$, although $\wedge^n(\mathfrak{u})$ does not have any meaning “internal to K ”.

We can as before form the holomorphic vector bundle $\mathcal{W}^{\sharp} = K \times_{K \cap L} (W^{\sharp})$ over $X = K/(K \cap L)$ (this one has finite rank), and obtain the Dolbeault cohomology groups $H^{(p,q)}(X, \mathcal{W}^{\sharp})$.

Proposition 4.2. *The K -module $H^{(0,s)}(X, \mathcal{W}^{\sharp})$ is irreducible and its equivalence class is μ .*

Proof. From Wong’s work ([44], Theorem 2.4) and the isomorphism theorems in [26], Chapter VIII, we know that the (\mathfrak{g}, K) -module $H^{(0,s)}(X, \mathcal{W}^{\sharp})$ defined above, the (\mathfrak{g}, K) -module $\mathcal{R}^s(V)$ defined in [26], Eq. (5.3b), and the (\mathfrak{g}, K) -module $\mathcal{L}_s(V)$ defined in [26], Eq. (5.3a), are all equivalent. To describe their common lowest K -type, we can proceed as follows: Wong’s work shows that the K -module $H^{(0,s)}(X, \mathcal{W}^{\sharp})$ is isomorphic with that written $\mathcal{L}_s^K(W)$ in [26], (5.70); because of the remark below Corollary 5.72 in [26], the latter K -module is irreducible. Finally, the bottom-layer map ([26], §V.6) induces, as recalled for instance in Theorem 2.9 in [32], a K -equivariant isomorphism between $\mathcal{L}_s^K(W)$ (viewed as a K -submodule of $\mathcal{L}_s^K(V)$) and the μ -isotypical subspace in $\mathcal{L}_s(V) \simeq \pi$. \square

4.1.2. Zooming-in operators and contraction in Dolbeault cohomology

In this section, we show that given a sufficiently pleasant Fréchet contraction of the representation σ of L onto its lowest $(K \cap L)$ -type μ^{\flat} , it is possible to build a Fréchet contraction of π onto its lowest K -type.

The construction is quite technical: bringing in the deformation $(G_t)_{t>0}$, and running through the constructions of §4.1.1 for G_t , we shall need to find a Fréchet space independent of t in which one can embed all Dolbeault cohomology spaces. Since the isotropy groups L_t , infinite-dimensional modules V_t , and Dolbeault operators that appear in the definition of $H^{(0,s)}(G_t/L_t, \mathcal{V}_t^{\sharp})$ all will depend on t in a more delicate way than the ingredients of §3, and because of the quotient in the definition of cohomology, we will need some preparation before we can state our main result. We have relegated the proof of four technical statements to §4.1.4, after the proof of Theorem 4.10.

A. On a Mostow decomposition and the structure of elliptic coadjoint orbits. We first give a meaning to the notion of “zooming-in on a neighborhood of $K/(K \cap L)$ in G/L ”. A few remarks on the structure of the coadjoint orbit G/L will help.

Write \mathfrak{s} for the orthogonal complement of $(\mathfrak{l} \cap \mathfrak{p})$ in \mathfrak{p} with respect to the Killing form; the dimension of \mathfrak{s} is an even integer. Of course the map

$$\begin{aligned} \Psi : K \times \mathfrak{s} &\rightarrow G/L \\ (k, v) &\mapsto k \cdot \exp_G(v) \cdot L \end{aligned}$$

cannot be a global diffeomorphism, since the dimensions of the source and target spaces are different (the difference is $\dim(K \cap L)$). Still, this map will be useful to us:

Lemma 4.3. (a) *The map Ψ is surjective.*

(b) *Given (k, v) and (k', v') in $K \times \mathfrak{s}$, the condition $\Psi(k, v) = \Psi(k', v')$ is equivalent with the existence of some u in $K \cap L$ such that: $v = \text{Ad}(u) \cdot v'$ and $k = k' \cdot u$.*

Proof. Part (a) is essentially due to Mostow, who proved in 1955 ([29], Theorem 5) that the map

$$\begin{aligned} K \times \mathfrak{s} \times (\mathfrak{l} \cap \mathfrak{p}) &\rightarrow G \\ (k, v, \beta) &\mapsto k \cdot \exp_G(v) \cdot \exp_G(\beta) \end{aligned} \tag{4.1}$$

is a diffeomorphism. For (b), fix two pairs (k, v) and (k', v') in $K \times \mathfrak{s}$ and assume that $\Psi(k, v) = \Psi(k', v')$, in other words that $k'e^{v'} \in ke^vL$; then there is an element β in $(\mathfrak{l} \cap \mathfrak{p})$ and an element u in $K \cap L$ such that: $k'e^{v'} = ke^v u e^\beta = (k'u)e^{\text{Ad}(u)v'} e^\beta$. From this and Mostow’s decomposition theorem we get (b). We mention that the above is very close to the proof of Lemma 4.4 in [6]. \square

Now, equip $K \times \mathfrak{s}$ with the $K \cap L$ -action in which an element u of $K \cap L$ acts through $(k, v) \mapsto (uk, uv)$; write Y for the quotient manifold $(K \times \mathfrak{s})/(K \cap L)$ and $\text{proj}_Y : K \times \mathfrak{s} \rightarrow Y$ for the quotient map. Given Lemma 4.3, the unique map $\psi : Y \rightarrow G/L$ such that $\psi \circ \text{proj}_Y = \Psi$ is a global diffeomorphism. The inverse image of $X = K/(K \cap L)$ under the diffeomorphism ψ is a compact submanifold of Y with codimension $\dim(\mathfrak{s})$, which we will still denote by X . For every $t > 0$, the map $(k, v) \rightarrow (k, \frac{v}{t})$ (from $K \times \mathfrak{s}$ to itself) induces a map from Y to itself; this “zooming-in map” preserves the compact submanifold X .

B. Spaces of “differential forms” and models for the Dolbeault cohomologies. We now bring in the deformation $(G_t)_{t \in \mathbb{R}}$. Taking up notations from §2.3 and §4.1.1, we assume given a Fréchet contraction

$$(E, (V_t)_{t>0}, (\sigma_t)_{t>0}, (\mathfrak{c}_t)_{t>0}) \tag{4.2}$$

of σ onto its lowest $(K \cap L)$ -type. For every $t > 0$, we can consider the subgroup $L_t = \phi_t^{-1}(L)$ of G_t (where $\phi_t : G_t \rightarrow G$ is the isomorphism from §2.2); the maximal compact subgroup $K \cap L_t$ does not depend on t . Extending ϕ_t (resp. the derivative ϕ_t) to an isomorphism between the complexifications of G_t and G (resp. of \mathfrak{g}_t and \mathfrak{g}), we set $Q_t = \phi_t^{-1}(Q)$ to obtain a parabolic subgroup in the complexification of G_t ; let us write $\mathfrak{u}_t = \phi_t^* \mathfrak{u}$ for the unipotent radical of its Lie algebra. From the $(\mathfrak{l}_{\mathbb{C}}, K \cap L)$ -module V_t and the abelian character $e^{2\rho(\mathfrak{u}_t)}$ of L_t acting on $\wedge^n(\mathfrak{u}_t)$, we can form the $(\mathfrak{l}_{\mathbb{C}}, K \cap L)$ -module $V_t^\sharp = V_t \otimes \wedge^n(\mathfrak{u}_t)$ and the Dolbeault cohomologies $H^{p,q}(G_t/L_t, \mathcal{V}_t^\sharp)$ from § 4.1.1.

The results recalled in §4.1.1 show that $H^{0,s}(G_t/L_t, \mathcal{V}_t^\sharp)$ carries an irreducible tempered representation of G_t with lowest K -type μ .

The program described in §2.3 can only be realized if we embed all of the Fréchet spaces $H^{0,s}(G_t/L_t, \mathcal{V}_t^\sharp)$ into a common Fréchet space. To do so, we need a convenient definition for the notion of “differential form of type $(0, s)$ on G_t/L_t with values in \mathcal{V}_t^\sharp ”, and for the notions of “closed” and “exact” forms.

It will be helpful to assume the given contraction (4.2) to have additional “nice” properties:

Assumption 4.4.

- i. *The Fréchet space E is nuclear.*
- ii. *For every $t > 0$, the irreducible representation $\sigma_t : L_t \rightarrow \text{End}(V_t)$ is the restriction to V_t of a (very reducible) smooth representation $\sigma_t : L_t \rightarrow \text{End}(E)$; furthermore, for every u in $K \cap L_t = K \cap L$, the operator $\sigma_t(u)$ does not depend on t .*
- iii. *Suppose \mathfrak{B} is a compact subset of \mathfrak{p} . Then the Fréchet topology of E may be defined by a distance with the property that there exists some $\kappa > 0$ such that*
 - (a) *each the of $\sigma_t(k, v)$, $k \in K \cap L$, $v \in \mathfrak{B} \cap \mathfrak{l}_t \cap \mathfrak{l}$, $t \in]0, 1]$,*
 - (b) *each of the $\sigma_t(X)$, $X \in U(\mathfrak{l}_{t, \mathbb{C}})$ (enveloping algebra),*
 - (c) *and each of the \mathbf{c}_t , $t \in]0, 1]$,**is κ -Lipschitz as an endomorphism of V_t .*

Property *iii.* will be used at the end of this subsection, when we will need an analogue of Lemma 3.8 (see page 27). Property *ii.*, of the kind already encountered in (3.2) and (3.17), makes it possible for us to define for every $t > 0$ a holomorphic vector bundle \mathcal{E}_t^\sharp over G_t/L_t . In order to circumvent, in our discussion of the space of differential forms with values in \mathcal{E}_t^\sharp , some of the analytical difficulties related with the fact that \mathcal{E}_t^\sharp is an infinite-rank bundle, we will follow Wong [44] and define the Dolbeault complex directly: for each nonnegative integer q , set

$$\Omega_t^q = \left\{ \varphi \in \mathcal{C}^\infty \left(G_t, E^\sharp \otimes \wedge^q(\mathfrak{u}_t^*) \right) \mid \forall \ell \in L_t, \forall \gamma \in G_t, \varphi(\gamma \ell) = \zeta_t(\ell)^{-1} f(\gamma) \right\} \quad (4.3)$$

where ζ_t is the L_t -representation obtained from the action σ_t on E , from the action of L_t on $\wedge^n(\mathfrak{u}_t)$ through an abelian character denoted $e^{2\rho(\mathfrak{u}_t)}$, and from the adjoint-action-induced representation ξ_t of L_t on $\wedge^q(\mathfrak{u}_t^*)$, by setting $\zeta_t = \sigma_t \otimes e^{2\rho_{\mathfrak{u}_t}} \otimes \xi_t$. The notion of smooth function on G_t with values in $E_t^\sharp \otimes \wedge^q(\mathfrak{u}_t^*)$ is that from [38], Chapter 40 (see also [44], §2). In the present context, the Dolbeault operator $\bar{\partial}_{\Omega_t^q} : \Omega_t^q \rightarrow \Omega_t^{q+1}$ is defined in Wong [44], §2; we omit the details.

Using the map

$$\begin{aligned} \Psi_t : K \times \mathfrak{s} &\rightarrow G_t/L_t \\ (k, v) &\mapsto k \cdot \exp_{G_t}(v) \cdot L_t \end{aligned}$$

from Lemma 4.3, we can now exhibit a Fréchet space isomorphic with Ω_t^q , but contained in a Fréchet space independent of t . Note first that the derivative of Ψ_t at $(1_K, 0)$ induces a $K \cap L$ -equivariant isomorphism between

$$\mathfrak{r} = (\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{l})) \oplus \mathfrak{s}$$

and $\mathfrak{g}_t/\mathfrak{l}_t$, and between the complexifications $\mathfrak{r}_{\mathbb{C}}$ and $\mathfrak{g}_{t, \mathbb{C}}/\mathfrak{l}_{t, \mathbb{C}}$. Write $\iota_t : \mathfrak{r}_{\mathbb{C}} \rightarrow \mathfrak{g}_{t, \mathbb{C}}/\mathfrak{l}_{t, \mathbb{C}}$ for that map, and

$$\eta_t \subset \mathfrak{r}_{\mathbb{C}} \quad (4.4)$$

for the inverse image of $\mathfrak{q}_t/\mathfrak{l}_{t, \mathbb{C}}$ under ι_t . Identifying $\mathfrak{q}_t/\mathfrak{l}_{t, \mathbb{C}}$ with \mathfrak{u}_t , we obtain a $(K \cap L)$ -equivariant isomorphism

$$\iota_t : \eta_t \rightarrow \mathfrak{u}_t; \quad (4.5)$$

that one will make it possible to view η_t as the “antiholomorphic tangent space to the manifold Y from page 22 at $\text{proj}_Y[(1_K, 0)]$, when Y is equipped with the complex structure from G_t/L_t ”.

We now define our spaces of “differential forms of degree q on Y with values in \mathcal{E}^\sharp ” and “differential forms of

type $(0, q)$ for the G_t -invariant complex structure on Y'' , setting

$$A^q := \left\{ f \in \mathcal{C}^\infty \left(K \times \mathfrak{s}, E^\sharp \otimes \wedge^q(\mathfrak{r}_\mathbb{C}^*) \right) \mid \begin{array}{l} \forall u \in K \cap L, \forall (k, v) \in K \times \mathfrak{s}, \\ f(ku, \text{Ad}(u)v) = \lambda(u)^{-1} f(k, v) \end{array} \right\} \quad (4.6)$$

and

$$A^{(0, q)_t} := \left\{ f \in \mathcal{C}^\infty \left(K \times \mathfrak{s}, E^\sharp \otimes \wedge^q(\eta_t^*) \right) \mid \begin{array}{l} \forall u \in K \cap L, \forall (k, v) \in K \times \mathfrak{s}, \\ f(ku, \text{Ad}(u)v) = \lambda(u)^{-1} f(k, v) \end{array} \right\} \quad (4.7)$$

$$= \left\{ f \in A^q \quad / \quad f \text{ is } E^\sharp \otimes \wedge^q(\eta_t^*)\text{-valued} \right\},$$

where $\lambda : K \cap L \rightarrow \text{End}(E^\sharp \otimes \wedge^q(\mathfrak{r}_\mathbb{C}^*))$ denotes the action of $K \cap L$ on $E^\sharp \otimes \wedge^q(\mathfrak{r}_\mathbb{C}^*)$ induced by σ_t , by $e^{2\rho_{\mathfrak{u}_t}}$ and by the natural representation of $K \cap L$ on \mathfrak{r} . It should be noted that λ does not depend on t and preserves $E^\sharp \otimes \wedge^q(\eta_t^*)$: this is due to Assumption 4.4(i) and to the easily verified fact that $e^{2\rho(\mathfrak{u}_t)}|_{(K \cap L)}$ does not depend on t .

We equip A^q with the Fréchet topology from [38], Chapter 40 (see [44], page 5); then $A^{(0, q)_t}$ is closed in A^q .

Using Mostow's decomposition (4.1), we can extend every element in $A^{(0, q)_t}$ to an element of Ω_t^q , obtaining an isomorphism of Fréchet spaces

$$\begin{aligned} \mathcal{I}_t : A^{(0, q)_t} &\rightarrow \Omega_t^q \\ f &\mapsto \Phi_f^t : \end{aligned} \quad (4.8)$$

to be precise, when f is an element of $A^{(0, q)_t}$ and γ is in G_t , define $\Phi_f^t(\gamma)$ by writing $\gamma = k \exp_{G_t}^v \exp_{G_t}^\beta$ for the Mostow decomposition of γ (here $k \in K$, $v \in \mathfrak{s}$, $\beta \in \mathfrak{l}_t \cap \mathfrak{p}$), then setting $\Phi_f^t(\gamma) = \zeta_t(\exp_{G_t}(-\beta))f(k, v)$ (where we identified $\wedge^q(\eta_t^*)$ and $\wedge^q(\mathfrak{u}_t^*)$ through the isomorphism ι_t from (4.5)). The map Φ_f^t is an element of $\mathcal{C}^\infty(G_t, E^\sharp \otimes \wedge^q(\mathfrak{u}_t^*))$, and it is easily verified that it satisfies the L_t -equivariance condition from (4.3), so that $\Phi_f^t \in \Omega_t^q$. To check that \mathcal{I}_t is an isomorphism, we only have to add that every Φ in Ω_t^q is the image under \mathcal{I}_t of the map $(k, v) \mapsto \Phi(k \exp_{G_t}(v))$ (here we keep identifying $\wedge^q(\eta_t^*)$ and $\wedge^q(\mathfrak{u}_t^*)$ through ι_t).

By using \mathcal{I}_t to transfer the group action and Dolbeault operator on Ω_t^q to $A^{(0, q)_t}$, we obtain, for every $t > 0$:

- An action of G_t on $A^{(0, q)_t}$; when f is an element $A^{(0, q)_t}$, we will write

$$\mathbf{ac}_t(k, v)[f] \quad (4.9)$$

for the image of f under the action of an element (k, v) in G_t .

- A linear $\mathbf{ac}_t(G_t)$ -invariant differential operator

$$\bar{\partial}_t : A^{(0, q)_t} \rightarrow A^{(0, q+1)_t},$$

whose range is closed thanks to Wong's work [44].

We now define the “spaces of $\bar{\partial}_t$ -closed and $\bar{\partial}_t$ -exact forms”

$$F_t^q = \left\{ f \in A^q \quad / \quad f \text{ is } V_t^\sharp \otimes \wedge^q(\eta_t^*)\text{-valued} \quad \text{and} \quad \bar{\partial}_t f = 0 \right\}, \quad \text{and}$$

$$X_t^q = \left\{ f \in A^q \quad / \quad f \text{ is } V_t^\sharp \otimes \wedge^q(\eta_t^*)\text{-valued, and there exists } \omega \in A^{q-1} \text{ so that } \bar{\partial}_t \omega = f \right\}.$$

For every $t > 0$, F_t^q and X_t^q are $\mathbf{ac}_t(G_t)$ -invariant closed subspaces of $A^{(0, q)_t}$, hence of A^q . The induced action and topology on F_t^s/X_t^s equip that quotient with the structure of an admissible representation of G ; given the results recalled in §4.1.1, this representation is irreducible, tempered, and has real infinitesimal character and lowest K -type μ .

There is of course an analogous description of the (less subtle) notion of differential form on $K/(K \cap L)$ with

values in the (finite-rank) vector bundle \mathscr{W}^\sharp whose fiber is

- the subspace $W = \{\lim_{t \rightarrow 0} \mathbf{c}_t \phi, \phi \in V_1\}$ of E obtained by going through the Fréchet contraction of σ onto its lowest $(K \cap L)$ -type μ^\flat ,
- equipped with the action $\mu^\flat \otimes e^{2(\rho_u)}|_{(K \cap L)}$ of $K \cap L$.

The intersection $u_t \cap \mathfrak{k}_\mathbb{C}$ does not depend on t and can be identified with the antiholomorphic tangent space to $K/(K \cap L)$ at $1_K(K \cap L)$. The inverse image $u_t^{-1}(u_t \cap \mathfrak{k}_\mathbb{C})$ is a vector subspace of $(\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{l}))$ that does not depend on t and is contained in each of the $\eta_t, t > 0$: we will write η_0 for it. The space of differential forms of type $(0, q)$ on $K/(K \cap L)$ with values in \mathscr{W}^\sharp can thus be identified with

$$B^q = \left\{ f \in A^q \quad / \quad f \text{ is } W^\sharp \otimes \wedge^q(\eta_0)^* \text{-valued} \quad \text{and} \quad \forall (k, v) \in K \times \mathfrak{s}, f(k, v) = f(k, 0) \right\}.$$

For every $t > 0$, we remark that B^q is contained in $A^{(0, q)_t}$ and that the action \mathbf{ac}_t induces an action of K on B^q , which does not depend on t ; we will write $\mathbf{ac} : K \rightarrow \text{End}(B^q)$ for it. For every q is defined, as before, an $\mathbf{ac}(K)$ -invariant Dolbeault operator $\bar{\partial}_0 : B^q \rightarrow B^{q+1}$ with closed range (the closed-range property is from [43]); we set

$$F_0^q = \mathbf{Ker}(\bar{\partial}_0 : B^q \rightarrow B^{q+1}) \quad \text{and} \quad X_0^q = \mathbf{Im}(\bar{\partial}_0 : B^{q-1} \rightarrow B^q),$$

so that F_0^q and X_0^q are K -invariant subspaces of A^q ; the quotient F_0^q/X_0^q carries an irreducible K -module of class μ .

C. Contraction operators. We started this subsection by giving a meaning to the notion of “zooming-in on a neighborhood of $K \cap (K \cap L)$ in G/L ”. Furthermore, we assumed given a family of contraction operators $\mathbf{c}_t : E \rightarrow E$ that act on the fibers of the holomorphic vector bundles considered in this section. For every $t > 0$, the operator \mathbf{c}_t induces an endomorphism $\tilde{\mathbf{c}}_t$ of $E^\sharp \otimes \wedge^q \mathfrak{r}_\mathbb{C}^*$: using the decomposition $\mathfrak{r} = \mathfrak{k}/(\mathfrak{k} \cap \mathfrak{l}) \oplus \mathfrak{s}$ and writing $\tilde{\phi}_t : \mathfrak{r} \rightarrow \mathfrak{r}$ for the map $k + v \rightarrow k + tv$ induced by the isomorphism $\phi_t : G_t \rightarrow G$, and $\wedge^p \tilde{\phi}_t^*$ for the endomorphism of $\wedge^p \mathfrak{r}_\mathbb{C}^*$ induced by $\tilde{\phi}_t$, we set $\tilde{\mathbf{c}}_t = \mathbf{c}_t \otimes \wedge^p \tilde{\phi}_t^*$.

Combining these two operations on the base space and on the fibers, we obtain the linear automorphisms of A^q to be used for our contraction purposes,

$$\begin{aligned} \mathbf{C}_t : A^q &\rightarrow A^q & (4.10) \\ f &\mapsto \mathbf{C}_t f = \text{the map } (k, v) \mapsto \tilde{\mathbf{c}}_t \cdot f(k, tv) \quad (k \in K, v \in \mathfrak{s}). \end{aligned}$$

We now point out that \mathbf{C}_t intertwines the action of G_1 on $A^{(0, q)_1}$ and that of G_t on $A^{(0, q)_t}$, as well as the Dolbeault operators $\bar{\partial} := \bar{\partial}_1$ and $\bar{\partial}_t$:

Lemma 4.5. *Fix $t > 0$, f in $A^{(0, q)_1}$ and g in G_1 . Then $\mathbf{C}_t[\mathbf{ac}(g) \cdot f] = \mathbf{ac}_t(\phi_t^{-1}(g)) \cdot [\mathbf{C}_t f]$.*

Lemma 4.6. *For every $t > 0$, the operator $\mathbf{C}_t^{-1} \circ \bar{\partial}_t \circ \mathbf{C}_t$ is none other than $\bar{\partial}$.*

For the proofs of Lemmas 4.6 and 4.5, see §4.1.4. They have an immediate consequence:

Corollary 4.7. *If f lies in F^q , then $\mathbf{C}_t f$ lies in F_t^q ; if f lies in X^q , then $\mathbf{C}_t f$ lies in X_t^q .*

D. Representatives whose contraction is harmonic. Our aim of contracting π onto its lowest K -type μ now seems within reach: at this stage, we can associate, to every vector in the carrier space $H^{(0, s)}(G/L, \mathscr{V}^\sharp)$ for π , an element in the carrier space $H^{(0, s)}(K/(K \cap L), \mathscr{W}^\sharp)$ for μ . Indeed, if $f \in F^s := F_1^s$ is a representative of a cohomology class in $H^{(0, s)}(G/L, \mathscr{V}^\sharp)$, then as t goes to zero, $f_t := \mathbf{C}_t f$ goes (for the given topology on A^q) to an element f_0 in A^q that is constant in the \mathfrak{s} -directions and is $W^\sharp \otimes (\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{l}))$ -valued. Given the definitions of the Dolbeault operators, when f lies in F^s , the contraction f_0 lies in F_0^s ; furthermore, if f and g are two elements of F^s with the same cohomology class (meaning that $f - g$ lies in $X^s := X_1^s$), then f_0 and g_0 are two elements of F_0^s with the same cohomology class (meaning that $f_0 - g_0 \in X_0^s$): so the class of f_0 in F_0^s/X_0^s depends only of that of f in F^s/X^s , and defines an element in $H^{(0, s)}(K/(K \cap L), \mathscr{W}^\sharp)$.

However, in order to achieve a full implementation of the program in §2.3, we need to embed all of the $F_t^s/X_t^s, t > 0$, inside a common Fréchet space. The need to mod out a closed subspace X_t^s that depends on t will make

things slightly acrobatic, and we will have to choose a representative in F_t^s for each cohomology class in F_t^s/X_t^s ; in other words, we will choose for every $t > 0$ a “sufficiently pleasant” algebraic complement to X_t^s within F_t^s .

Now, the complex manifold $K/(K \cap L)$ is compact: in the space B^q of differential forms with type $(0, q)$ with values in the (finite-rank) bundle \mathscr{W}^\sharp , we can call in the usual notion of *harmonic form* (see for instance [42], §II.6): a $\bar{\partial}$ -closed form of type $(0, q)$ on $K/(K \cap L)$ with values in \mathscr{W}^\sharp is harmonic when it is orthogonal, for the inner product on forms associated with the K -invariant Kähler metric on $K/(K \cap L)$ and a K -invariant inner product on W , to every $\bar{\partial}$ -exact form. We thus have a notion of harmonic form in B^s , and a harmonic form can be $\bar{\partial}_0$ -exact only if it is zero; in addition, if $f \in B^q$ is harmonic, then $\mathbf{ac}(k) \cdot f$ is harmonic too. For background on harmonic forms, see [42], Chapter III, especially around Theorems 5.22-5.24.

Write H_0^s for the subspace of B^s consisting of harmonic forms: this is an $\mathbf{ac}(K)$ -invariant subspace of F_0^s , and we have $F_0^s = H_0^s \oplus X_0^s$. Now, define

$$H^s := \{f \in F^s \mid f_0 \in H_0^s\},$$

a subset of $A^{(0,s)}$.

Lemma 4.8. *The space F^s of closed forms decomposes as $F^s = H^s + X^s$.*

For the proof, see §4.1.4. By calling in the “nuclear space” hypothesis in Assumption 4.4, we can go a step further: not only does every class in F^s/X^s admit exactly one representative in H^s , but we can obtain a *closed* subspace of H^s comprising exactly one representative for each cohomology class.

Lemma 4.9. *There exists a closed subspace Σ of H^s such that that $F^s = \Sigma \oplus X^s$.*

For the proof, see §4.1.4.

Now fix such a Σ . For every $t > 0$, the space

$$\Sigma_t = \{\mathbf{C}_t[f] \mid f \in \Sigma\} \tag{4.11}$$

is a closed complement to X_t^s in F_t^s ; since X_t^s and Σ_t are closed in F_t^s , the projection

$$q_t : \Sigma_t \rightarrow F_t^s/X_t^s \tag{4.12}$$

is a Fréchet isomorphism. Thus, we have defined a subspace of A^s such that every class in $H^{(0,s)}(G_t/L_t, \mathscr{V}_t^\sharp)$ has a unique representative in Σ_t .

In Lemma 4.9, we cannot expect Σ_t to be stable under the action $\mathbf{ac}_t(G_t)$ of (4.9). We can however use the isomorphism (4.12) to transfer to Σ_t the action $\mathbf{ac}_t^{\text{quo}}$ of G_t on F_t^s/X_t^s inherited from \mathbf{ac}_t : recall that the projection

$$p_t : F_t^s \rightarrow F_t^s/X_t^s \tag{4.13}$$

makes it possible to define for every γ in G_t a map $\mathbf{ac}_t^{\text{quo}}(\gamma)$ from F_t^s/X_t^s to itself, where

$$\text{if } \omega = p_t(F) \text{ for some } F \text{ in } F_t^s, \text{ then } \mathbf{ac}_t^{\text{quo}}(\gamma) \cdot \omega = p_t(\mathbf{ac}_t(\gamma)F). \tag{4.14}$$

We then set

$$\pi_t(k, v) \cdot F := q_t^{-1} [\mathbf{ac}_t^{\text{quo}}(k, v) \cdot q_t(F)] \tag{4.15}$$

$$= q_t^{-1} [p_t(\mathbf{ac}_t(k, v) \cdot F)] \tag{4.16}$$

for every (k, v) in $K \times \mathfrak{p}$ and every F in Σ_t : the representation $\pi_t : G_t \rightarrow \mathbf{End}(\Sigma_t)$ is irreducible tempered and its lowest K -type is μ .

E. Conclusion. We now have all the pieces for a Fréchet contraction of π onto μ . Consider

$$\mathbf{E} = A^s; \tag{A}$$

$$\mathbf{V}_t = \Sigma_t \text{ from (4.11);} \tag{B}$$

$$\pi_t \text{ from (4.15);} \tag{B'}$$

$$\mathbf{C}_t \text{ from (4.10).} \tag{C}$$

The following result is the announced reduction theorem.

Theorem 4.10. *The quadruple $(\mathbf{E}, (\mathbf{V}_t)_{t>0}, (\pi_t)_{t>0}, (\mathbf{C}_t)_{t>0})$ describes a Fréchet contraction of the real-infinitesimal character representation π onto its lowest K -type μ .*

4.1.3. Proof of Theorem 4.10, and a remark

We have already checked properties (Vect1) and (Vect2) of §2.3: see Corollary 4.7 and §4.1.2. D. Property (Nat_{RIC}) follows from Lemma 4.5 together with the fact that X_t^s is $\mathbf{ac}_t(G_t)$ -stable. We now need to prove properties (Op1) and (Op2).

Let us consider the outcome

$$\mathbf{V}_0 = \{f_0 / f \in \Sigma\}$$

of the contraction; this is a subspace of F_0^s . The contraction of every element in Σ is by definition a harmonic form, so that $\mathbf{V}_0 \subset H_0^s$. The reverse inclusion is true, and guarantees that the K -module \mathbf{V}_0 is irreducible of class μ :

Lemma 4.11. *Every harmonic form in B^s is the contraction of a form in Σ : we have $\mathbf{V}_0 = H_0^s$.*

Proof. Since the subspace Σ_0 of F_0^s is $\mathbf{ac}(K)$ -invariant, the projection $\Sigma_0 \rightarrow F_0^s/X_0^s$ is a morphism of K -modules; since every element in \mathbf{V}_0 is harmonic and no nonzero harmonic form can be exact, this morphism is injective. Now, the K -module F_0^s/X_0^s is irreducible, so that morphism must also be surjective, unless \mathbf{V}_0 be zero. But if that were the case, for every f in $F^s = \Sigma \oplus X^s$, the contraction f_0 would lie in X_0^s and be exact.

To prove that this is impossible, we call in the bottom-layer map of [26], §V.6 (see also [32], §2, and especially [1], §9). When applied to the $(K \cap L)$ -module W^\sharp , that map induces an injective morphism of K -modules from F_0^s/X_0^s to F^s/X^s , whose image is the subspace \tilde{W} of F^s/X^s carrying the lowest K -type of that G_1 -representation. Inspecting definitions reveals that the map sending an element of \tilde{W} to its inverse image by the bottom-layer map coincides with the map induced by the contraction $f \mapsto f_0$. If f_0 were exact for every f in \mathbf{V}_1 , we could deduce $F_0^s = X_0^s$, and that is not true. \square

Finally, fix an element F in \mathbf{V}_0 and an element f in Σ_1 satisfying $F = f_0$, and set $f_t = \mathbf{C}_t f$ for every $t > 0$. To complete the proof of Theorem 4.10, we need only check that $\pi_t(k, v)f_t$ goes to $\mathbf{ac}(k) \cdot f_0$ as t goes to zero.

Fix (k, v) in $K \times \mathfrak{p}$. Given Lemma 4.5 and the definition of our subspaces Σ_t in (4.11), we may write

$$\begin{aligned} \pi_t(k, v)f_t &= q_t^{-1} [p_t (\mathbf{ac}_t(k, v) \cdot \mathbf{C}_t f)] \\ &= q_t^{-1} \circ p_t \mathbf{C}_t [(\mathbf{ac}(k \exp_G(tv)) \cdot f)] \\ &= \mathbf{C}_t [q_1^{-1} \circ p_1 (\mathbf{ac}(k \exp_G(tv)) \cdot f)] \\ &= \mathbf{C}_t [q_1^{-1} \circ p_1 (\mathbf{ac}(k) \cdot f)] + (\mathbf{C}_t \circ q_1^{-1} \circ p_1) [\mathbf{ac}(k \exp_G(tv)) \cdot f - \mathbf{ac}(k) \cdot f]. \end{aligned}$$

Recall that f lies in Σ_1 , and note that Σ_1 is $\mathbf{ac}(K)$ -invariant, so that $p_1 (\mathbf{ac}(k) \cdot f) = q_1 (\mathbf{ac}(k) \cdot f)$; applying Lemma 4.5 again and inserting the fact that $\mathbf{ac}_t(k) = \mathbf{ac}(k)$ for all t , we see that

$$\pi_t(k, v)f_t = \mathbf{ac}(k) \cdot f_t + (\mathbf{C}_t \circ q_1^{-1} \circ p_1) [\mathbf{ac}(k \exp_G(tv)) \cdot f - \mathbf{ac}(k) \cdot f]. \tag{4.17}$$

When t goes to zero, the first term in (4.17) goes to $\mathbf{ac}(k) \cdot f_0$; what we need to check is that the second term goes to zero. We now insert the following two remarks (compare Lemma 3.8):

Lemma 4.12. (i) *The topology of \mathbf{E} may be defined by a distance with respect to which each of the \mathbf{C}_t , $t \in]0, 1]$, is 1-Lipschitz.*

(ii) *When \mathbf{E} is equipped with that distance, the Fréchet topology on F^s/X^s may be defined with a distance such that the projection $p_1 : F^s \rightarrow F^s/X^s$ is 1-Lipschitz and the inverse projection $q_1^{-1} : F^s/X^s \rightarrow \Sigma$ is 2-Lipschitz.*

Proof. For (i), build a countable family of semi-norms and a metric whose topology is that of \mathbf{E} , following the recipe in [38], page 412: using the notations of the first sentence after Definition 40.2 in that textbook, we use the subsets $\Omega_j = K \times B(0, j)$, $j \in \mathbb{Z}^+$, of $K \times \mathfrak{s}$ (where $B(0, j)$ is the open ball \mathfrak{s} with center $0_{\mathfrak{s}}$ and radius j), and use Assumption 4.4.iii.(c) to obtain a countable family of semi-norms on the Fréchet space $E^{\sharp} \otimes \wedge^s \mathfrak{t}^*$ with respect to which the operators \mathbf{c}_t , $t \in]0, 1]$, are all 1-Lipschitz.

For (ii), we note that given a Fréchet distance on F^s , the usual way to define a Fréchet topology on F^s/X^s is through a distance for which $p_1 : F^s \rightarrow F^s/X^s$ is 1-Lipschitz (see §12.16.9 in [12]); what needs proof is the statement on the inverse q_1^{-1} . Now, the inclusion $\Sigma_t \hookrightarrow F_t^s$ and the projection $F_t^s \rightarrow F_t^s/X_t^s$ are 1-Lipschitz, so the composition q_1 is 1-Lipschitz; the open mapping theorem for Fréchet spaces then shows that q_1^{-1} is actually 2-Lipschitz with respect to the mentioned distances on Σ and on F^s/X^s (see [12], §12.16.9 for the open mapping theorem and §12.16.8.2 for the Lipschitz statement). \square

Returning to the proof of Property (Op1) in Theorem 4.10, we take up (4.17) and let t go to zero. Then $\mathbf{ac}(k \exp_G(tv)) \cdot f - \mathbf{ac}(k)f$ goes to zero; from Lemma 4.12 we deduce that $\pi_t(k, v)f_t$ goes to $\mathbf{ac}(k) \cdot f_0$. Lemma 4.11 yields (Op2); the proof of Theorem 4.10 is now complete. \square

We close this section by mentioning, for future use in §5, that our Fréchet contraction satisfies a weakend analogue of Assumption 4.4:

Lemma 4.13. *Suppose U is a compact subset of \mathfrak{p} . Then there is a positive number \tilde{C} with the property that*

- (i) *the Fréchet topology of \mathbf{E} may be defined by a distance with respect to which each of the $\pi_t(k, v)$, $k \in K$, $v \in U$, $t \in]0, 1]$, and each of the \mathbf{C}_t , $t \in]0, 1]$, is \tilde{C} -Lipschitz as an endomorphism of \mathbf{V}_t .*
- (ii) *for every $t > 0$ and every (k, v) in $K \times U$, the operator $\pi_t(k, v)$ on \mathbf{V}_t may be extended to a \tilde{C} -Lipschitz operator acting on all of \mathbf{E} . This may be done so that the resulting map $\pi_t : G_t \rightarrow \text{End}(\mathbf{E})$ is continuous.*

Lemma 4.13 is a rather poor substitute for Assumption 4.4, because the extension mentioned in part (ii) is simple-minded: we do *not* try to define it so that $\pi_t : G_t \rightarrow \text{GL}(\mathbf{E})$ is a group morphism, and our Lipschitz estimates are crude. What will matter in §5 is the existence of an extension to all of \mathbf{E} and of a Lipschitz estimate uniform in k , v and t in the chosen domain.

Proof. For (i), we first unfold the definition of $\mathbf{ac}_t(\gamma)$ in (4.9) and find that for every F in $A^{(0,s)_t}$ and γ in G_t ,

$$\begin{aligned} \mathbf{ac}_t(\gamma) \cdot F &= \mathcal{I}_t^{-1} [x \mapsto (\mathcal{I}_t F)(\gamma^{-1}x)] \\ &= (k, v) \mapsto \mathfrak{c}_t \left(\exp_{G_t}^{-\beta_t(\gamma^{-1}k \exp_{G_t}^v)} \right) F [\kappa_t(\gamma^{-1}k \exp_{G_t}^v), X_t(\gamma^{-1}k \exp_{G_t}^v)] \end{aligned}$$

where $\kappa_t : G_t \rightarrow K$, $X_t : G_t \rightarrow \mathfrak{s}$ and $\beta_t : G_t \rightarrow \mathfrak{l}_t \cap \mathfrak{p}$ are the Mostow maps of Lemma 4.5.

We now observe, as in the proof of Lemma 3.8, that there exists an increasing family $(\Omega_n)_{n \in \mathbb{N}}$ of relatively compact open subsets of $K \times \mathfrak{s}$ such that for every γ in $K \exp_{G_t}(U)$ and every t in $]0, 1]$, the image of Ω_n under the map $(k, v) \mapsto \left(\kappa_t(\gamma^{-1}k \exp_{G_t}^v), X_t(\gamma^{-1}k \exp_{G_t}^v) \right)$ is contained in Ω_{n+1} . We can in fact choose Ω_n of the form $K \times B(0, R_n)$, where $(R_n)_{n \in \mathbb{N}}$ is an increasing sequence of positive radii and $B(0, R_n)$ is the ball with radius R_n in \mathfrak{s} . We then use those and Assumption 4.4.iii.(a)-(b) to follow the proof of Lemma 4.12.(i) and build the desired metric on \mathbf{E} .

For (ii), fix a closed complement Z to F^s in $\mathbf{E} = A^s$, and for all $t > 0$, set $Z_t = \mathbf{C}_t Z$, so that

$$\mathbf{E} = \Sigma_t \oplus X_t^s \oplus Z_t.$$

Write proj_{Σ_t} , $\text{proj}_{X_t^s}$ and proj_{Z_t} for the corresponding linear projections, and for every (k, v) in G_t , extend the operator π_t on Σ_t by having the extension act as the identity on X_t^s and Z_t , setting

$$\tilde{\pi}_t(k, v)F = \pi_t(k, v) [\text{proj}_{\Sigma_t} F] + \text{proj}_{X_t^s}(F) + \text{proj}_{Z_t}(F) \quad (4.18)$$

for all F in \mathbf{E} . Of course this extension is so simple-minded that it cannot define a representation of G_t on \mathbf{E} , but we note, taking up some arguments from the proof of Lemma 4.12(ii), that the projections proj_{Σ_t} , $\text{proj}_{X_t^s}$ and proj_{Z_t} are all 2-Lipschitz. Indeed, if A is a Fréchet space equipped with a compatible distance and B, C are closed subspaces such that $A = B \oplus C$, then the projection $A \rightarrow C$ is 2-Lipschitz: it can be obtained by first applying the projection $A \rightarrow A/B$, which is 1-Lipschitz if the usual distance on A/B is chosen, then applying the isomorphism $A/B \rightarrow C$, which is the inverse of the 1-Lipschitz isomorphism $C \rightarrow A/B$ and is as a consequence 2-Lipschitz.

Returning to (4.18), we know from Lemma 4.12 that $\pi_t(k, v)$ is 4-Lipschitz, and use the preceding remarks to deduce that $\tilde{\pi}_t(k, v)$ is 10-Lipschitz. The continuity of $(k, v) \mapsto \tilde{\pi}_t(k, v)$ is straightforward from that of π_t . \square

4.1.4. Proof of Lemmas 4.5, 4.6, 4.8 and 4.9

Our proofs of the Lemmas 4.5 and 4.6 use the notations of §4.1.2 B-C.

Proof of Lemma 4.5. In order to make the notations in the statement and proof lighter, we write $A^{(0,q)}$ for $A^{(0,q)_1}$ and $\mathbf{ac} : G \rightarrow \text{End}(A^{(0,q)})$ for the G -action obtained by composing \mathbf{ac}_1 and the isomorphism $\varphi_1 : G_1 \rightarrow G$.

When g is an element of G , we write its Mostow decomposition as $g = \kappa(g) \exp_G^{X(g)} \exp_G^{\beta(g)}$, getting Mostow maps $\kappa : G \rightarrow K$, $X : G \rightarrow \mathfrak{s}$ and $\beta : G \rightarrow \mathfrak{l} \cap \mathfrak{p}$. We write $\kappa_t : G_t \rightarrow K$, $X_t : G_t \rightarrow \mathfrak{s}$ and $\beta_t : G_t \rightarrow \mathfrak{l}_t \cap \mathfrak{p}$ for the analogous Mostow maps of G_t . We now fix g in G and $t > 0$.

- By definition, $\mathbf{ac}(g) \cdot f$ is a map from $K \times \mathfrak{s}$ to $E^\sharp \otimes \wedge^q \mathfrak{u}^*$, viz.

$$\mathbf{ac}(g) \cdot f : (k, v) \mapsto \zeta \left(e^{-\beta[g^{-1}k \exp_G(v)]} \right) f \left(\kappa [g^{-1}k \exp_G(v)], X [g^{-1}k \exp_G(v)] \right).$$

Thus $\mathbf{C}_t[\mathbf{ac}(g) \cdot f]$ is the map

$$\begin{aligned} (k, v) &\mapsto \tilde{\mathbf{c}}_t \cdot [\mathbf{ac}(g) \cdot f](k, tv) \\ &= \tilde{\mathbf{c}}_t \cdot \zeta \left(\exp_G^{-\beta[g^{-1}k \exp_G(tv)]} \right) f \left(\kappa [g^{-1}k \exp_G^{tv}], X [g^{-1}k \exp_G^{tv}] \right) \\ &= \zeta_t \left(\varphi_t^{-1}(\exp_G^{-\beta[g^{-1}k \exp_G^{tv}]}) \right) \tilde{\mathbf{c}}_t \cdot f \left(\kappa [g^{-1}k \exp_G^{tv}], X [g^{-1}k \exp_G^{tv}] \right) \\ &= \zeta_t \left(\exp_{G_t}^{-\frac{1}{t}\beta[g^{-1}k \exp_G^{tv}]} \right) \tilde{\mathbf{c}}_t \cdot f \left(\kappa [g^{-1}k \exp_G^{tv}], X [g^{-1}k \exp_G^{tv}] \right) \end{aligned}$$

(in the transition between the second and third line, we use property (Nat_{RIC}) for the contraction operator $\tilde{\mathbf{c}}_t$: it intertwines the representations σ and $\sigma_t \circ \varphi_t^{-1}$ of L).

- We need to compare this with $\mathbf{ac}_t(\varphi_t^{-1}(g)) \cdot [\mathbf{C}_t f]$, which is the map

$$\begin{aligned} (k, v) &\mapsto \zeta_t \left(\exp_{G_t}^{\beta_t(\varphi_t^{-1}(g)^{-1}k \exp_{G_t}(v))} \right) \cdot [\mathbf{C}_t f] \left(\kappa_t(\varphi_t^{-1}(g)^{-1}k \exp_{G_t}^v), X_t(\varphi_t^{-1}(g)^{-1}k \exp_{G_t}^v) \right) \\ &= \zeta_t \left(\exp_{G_t}^{\beta_t(\varphi_t^{-1}(g)^{-1}k \exp_{G_t}(v))} \right) \cdot \tilde{\mathbf{c}}_t \cdot f \left(\kappa_t(\varphi_t^{-1}(g)^{-1}k \exp_{G_t}^v), t \cdot X_t(\varphi_t^{-1}(g)^{-1}k \exp_{G_t}^v) \right). \end{aligned}$$

If $\gamma = ke^v e^\beta$ is an element of G , then $\varphi_t^{-1}(\gamma) = k \exp_{G_t}^{X/t} \exp_{G_t}^{\beta/t}$, so that $\kappa(\gamma) = \kappa_t(\varphi_t^{-1}(\gamma))$, $X(\gamma) = t \cdot X_t(\varphi_t^{-1}(\gamma))$ and $\beta(\gamma) = t \cdot \beta_t(\varphi_t^{-1}(\gamma))$. Using this to compare the last lines of our descriptions of $\mathbf{C}_t[\mathbf{ac}(g) \cdot f]$ and $\mathbf{ac}_t(\varphi_t^{-1}(g)) \cdot [\mathbf{C}_t f]$, we see that they are equal. \square

Proof of Lemma 4.6. Recall that our Dolbeault operator $\bar{\partial}$ on $A^{(0,q)}$ is defined by transferring to $A^{(0,q)}$ the Dolbeault operator acting on Ω_1^q (defined in [44] §2), through the Fréchet isomorphism $\mathcal{A} : A^{(0,q)} \rightarrow \Omega_1^q$ from (4.8).

For every $t > 0$, the operator $\bar{\partial}_t$ acting on $A^{(0,q)_t}$ similarly comes from Wong's Dolbeault operator on Ω_t^q , transferred to $A^{(0,q)_t}$ via the Fréchet isomorphism $\mathcal{S}_t : A^{(0,q)_t} \rightarrow \Omega_t^q$.

We now remark that the linear automorphism \mathbf{C}_t of A^q induces a linear isomorphism between Ω_1^q and Ω_t^q , the map $\mathcal{S}_t \circ \mathbf{C}_t \circ \mathcal{S}_1^{-1}$. Inspecting definitions, we notice that for every $F : G_1 \rightarrow E^\sharp \otimes \wedge^q \mathfrak{u}^*$ in Ω_1^q , the map $\mathcal{S}_t \circ \mathbf{C}_t \circ \mathcal{S}_1^{-1}(F)$ is none other than

$$\begin{aligned} \mathcal{C}_t(F) : G_t &\rightarrow E^\sharp \otimes \wedge^q \mathfrak{u}^* \\ (k, v) &\mapsto (\mathbf{c}_t \otimes \phi_t^*) \cdot F(\phi_t(k, v)) \end{aligned}$$

where $\phi_t^* : \wedge^q \mathfrak{u}^* \rightarrow \wedge^q \mathfrak{u}_t^*$ is the linear isomorphism induced by the derivative of ϕ_t^{-1} at the identity.

This means that $\mathcal{C}_t(F)$ is the element Ω_t^q induced by the group isomorphism $\phi_1 \circ \phi_t^{-1}$ between G_1 and G_t . That isomorphism induces an isomorphism of holomorphic vector bundles between the bundle \mathcal{V}_1^\sharp over G_1/L_1 and the bundle \mathcal{V}_t^\sharp over G_t/L_t ; by naturality of the Dolbeault operator, that bundle map must intertwine the Dolbeault operators acting on Ω_1^q and Ω_t^q . So

$$\mathcal{C}_t^{-1} \bar{\partial}_{\Omega_t^q} \mathcal{C}_t = \bar{\partial}_{\Omega_1^q},$$

which proves Lemma 4.6. □



Our proofs of Lemmas 4.8 and 4.9 use the notations of §4.1.2 D.

Proof of Lemma 4.8. Fix f in F^s ; the contraction f_0 of f is $\bar{\partial}_0$ -closed, so we can write f_0 as $a + b$, where $a \in H_0^s$ is a harmonic form and $b \in X_0^s$ is a $\bar{\partial}_0$ -exact form. Write $b = \bar{\partial}_0 \beta$, where β lies in B^{s-1} ; we can extend β trivially to $K \times \mathfrak{s}$, obtaining an element γ of A^{s-1} that is constant in the \mathfrak{s} -directions. Set $\omega = \bar{\partial} \gamma$; then the form ω lies in X^s , and $f - \omega$ lies in H^s (because $f_0 - \omega_0 = a$ is harmonic). We have proved that f can be written as a sum of a form in H^s and another in X^s ; that is the lemma. □

Proof of Lemma 4.9. We need only find an algebraic complement Σ for $H^s \cap X^s$ in H^s that is closed for the topology of H^s . The fact that a *closed* Σ may be chosen follows from the theory of nuclear spaces (see Trèves [38], chapters 50 and 51, and Grothendieck [14]): because of Assumption 4.4, the Fréchet space A^s is nuclear (see [38], Proposition 50.1 and §51.8), so the closed subspaces H^s , X^s and $H^s \cap X^s$ are both nuclear; because $X^s \cap H^s$ is closed in H^s , the identity $H^s \cap X^s \rightarrow H^s \cap X^s$ is a nuclear linear map ([14], corollaire 2, p. 101); it admits ([14], théorème 3, p. 103) a nuclear extension $\mathcal{F} : H^s \rightarrow H^s \cap X^s$. We define Σ as the kernel of \mathcal{F} : it is a closed complement to $H^s \cap X^s$ in H^s , and the lemma follows. □

4.2. The case of quasi-split groups and fine K -types

In this section, we consider a pair (G, π) where

- G is a *quasi-split* linear group with abelian Cartan subgroups,
 - π is a tempered irreducible representation of G with real infinitesimal character and a *fine* lowest K -type μ ,
- and we set out to build a Fréchet contraction of π onto μ .

In the present context, the representation π can be realized as an irreducible factor in a (usually nonspherical) *principal series* representation of G , as in Chapter 4 in [39]. Suppose $P = MAN$ is a *minimal* parabolic subgroup in G , then in our quasi-split case:

- M is abelian (and compact, but not connected),
- and there exists a character σ of M such that μ is one of the lowest K -types in

$$\Pi = \text{Ind}_{MAN}^G(\sigma \otimes \mathbf{1}).$$

The representation Π is a finite direct sum of irreducible tempered subrepresentations, each of which has real infinitesimal character and a unique lowest K -type; the irreducible factor that contains μ is equivalent with π as a G -representation.

Recall that in the compact picture from §3.2.1, the representation Π acts on

$$\mathbf{L}_\sigma^2(K) := \{ \Phi : K \rightarrow \mathbb{C} / \forall (u, m) \in K \times M, \Phi(um) = \sigma(m)^{-1} \Phi(u) \}$$

through the action obtained by choosing $\lambda = 0$ in (3.9). The subspace

$$\mathbf{H} = \text{irreducible subspace of } \mathbf{L}_\sigma^2(K) \text{ containing the } K\text{-type } \mu \quad (4.19)$$

thus carries a (not very explicit) realization of π on a space of complex-valued functions on K .

Because the induced representation Π is reducible and has continuous parameter zero, the results of §3.2.1 cannot be applied here; yet some of the interplay between §3.2.1 and §3.2.2 will be helpful.

4.2.1. A Helgason-like realization

We now describe another realization, in which G acts on a space of sections of an equivariant vector bundle over G/K ; it mimics the realization of spherical principal series representations in §3.2.2 using Helgason waves.

For the rest of §4.2.1, we fix an irreducible K -module V^μ of class μ , write \mathcal{E} for the equivariant bundle $G/K \times_K V^\mu$ over G/K and $\Gamma(\mathcal{E})$ for the space of smooth sections of \mathcal{E} .

We first describe the special sections of \mathcal{E} which will play in our construction the part Helgason waves did in §3.2.2.

From Frobenius reciprocity and the fact that the K -type μ occurs with multiplicity one in \mathbf{H} , we know that the restriction $\mu|_M$ contains σ with multiplicity one. We fix a vector $v \in V^\mu$ belonging to the one-dimensional σ -isotypical subspace and introduce

$$\begin{aligned} \tilde{\gamma} : G &\rightarrow V^\mu \\ ke^{Hn} &\mapsto e^{-\langle \rho, H \rangle} \mu(k)^{-1} v. \end{aligned}$$

Lemma 4.14. *For all (g, m, H, n) in $G \times M \times \mathfrak{a} \times N$, we have $\tilde{\gamma}(gme^{Hn}) = e^{-\langle \rho, H \rangle} \sigma(m)^{-1} \gamma(g)$.*

Proof. We use the Iwasawa decomposition (3.13) to find

$$\begin{aligned} \tilde{\gamma}(gman) &= \tilde{\gamma}(\kappa(g)e^{H(g)}v(g)man) \\ &= \tilde{\gamma}(\kappa(g)me^{H+H'}\tilde{n}) \text{ for some } \tilde{n} \in N \quad (\text{recall that } N \text{ normalizes } MA \text{ and } M \text{ centralizes } A) \\ &= e^{-\langle \rho, H+H' \rangle} \mu(\kappa(g)m)^{-1} v \\ &= \sigma(m)^{-1} e^{-\langle \rho, H \rangle} \left[e^{-\langle \rho, H(g) \rangle} \mu(\kappa(g))^{-1} v \right] \quad (\text{since } \mu(m)^{-1} v = \sigma(m)^{-1} v \text{ and } \sigma(m) \text{ is scalar}) \\ &= e^{-\langle \rho, H \rangle} \sigma(m)^{-1} \gamma(g). \end{aligned} \quad \square$$

Now set $\tilde{\gamma} : g \mapsto \tilde{\gamma}(g^{-1})$. A simple calculation shows that $\tilde{\gamma}(gk) = \mu(k^{-1})\gamma(g)$ for every k in K and g in G . As a consequence, $\tilde{\gamma}$ induces a section γ of the vector bundle \mathcal{E} over G/K . We will take up γ as an analogue in $\Gamma(\mathcal{E})$ of the Helgason “wave” $e_{\lambda, 1_K}$ with (not very wavy) frequency $\lambda = 0$.

For every b in K , the map $\tilde{\gamma}_b : g \mapsto \tilde{\gamma}(b^{-1}g)$ induces another element γ_b in $\Gamma(\mathcal{E})$, which we will take up as an analogue of the Helgason “wave” $e_{0, b}$. From Lemma 4.14, we know that $\gamma_{bm} = \sigma(m)^{-1} \gamma_b$ for every $(b, m) \in K \times M$.

Using these special sections, we build an invariant subspace of $\Gamma(\mathcal{E})$ (for the usual action of G on $\Gamma(\mathcal{E})$). We associate a section of \mathcal{E} with each of the functions on K belonging to the Hilbert space \mathbf{H} from (4.19): set

$$\begin{aligned} \mathcal{T} : \mathbf{H} &\rightarrow \Gamma(\mathcal{E}) \\ \Phi &\mapsto \int_K \gamma_b \Phi(b) db; \end{aligned}$$

note from the previous M -equivariance properties of Φ and $b \mapsto \gamma_b$ that the integrand is invariant under the change of variables $b \leftarrow b \cdot m$.

Lemma 4.15. *The map \mathcal{T} is G -equivariant and injective.*

Proof. Fix x in G and write $x = \kappa(x)e^{\mathbf{a}(x)}\mathbf{n}(x)$ for its Iwasawa decomposition. To prove that \mathcal{T} is G -equivariant, we must prove that for every Φ in \mathbf{H} , the sections

$$\begin{aligned}\mathcal{T}(\pi(x)\phi) &= \mathcal{T}\left[b \mapsto e^{-\langle \rho, \mathbf{a}(x^{-1}b) \rangle} \cdot \Phi(\kappa(x^{-1}b))\right] \\ &= \left(gK \mapsto \int_K e^{-\langle \rho, \mathbf{a}(x^{-1}b) \rangle} \gamma_b(gK) \cdot \Phi(\kappa(x^{-1}b)) db\right) \\ &= \left(gK \mapsto \int_K e^{-\langle \rho, \mathbf{a}(x^{-1}b) \rangle} e^{-\langle \rho, \mathbf{a}(g^{-1}b) \rangle} \cdot \Phi(\kappa(x^{-1}b)) (\mu(\kappa(g^{-1}b)) \cdot v) db\right)\end{aligned}\quad (4.20)$$

and

$$\begin{aligned}\lambda(x)\mathcal{T}(\phi) &= \left(g \mapsto \int_K \gamma_b(x^{-1}gK) \cdot \Phi(b) db\right) \\ &= \left(g \mapsto \int_K e^{-\langle \rho, \mathbf{a}(g^{-1}xb) \rangle} \cdot \Phi(b) (\mu(\kappa(g^{-1}xb)) \cdot v) db\right)\end{aligned}\quad (4.21)$$

are equal. Going from (4.20) to (4.21) by substituting $b \leftarrow \kappa(x^{-1}b)$ in (4.21) is made possible by the fact that the Iwasawa maps $\kappa, \mathbf{a}, \mathbf{n}$ satisfy cocycle relations. The necessary steps are identical with those taken by Camporesi for his proof of Proposition 3.3 in [10], so we omit the details.

The injectivity of \mathcal{T} comes from the irreducibility of π : from the equivariance we know that $\mathcal{T}^{-1}(\{0\})$ is a closed G -invariant subspace of \mathbf{H} . It is either $\{0\}$ or \mathbf{H} . If it were equal to \mathbf{H} , the map \mathcal{T} would be identically zero. Fixing a K -invariant inner product on V_σ , we notice that the matrix element $\xi_v : k \mapsto \langle v, \mu(k^{-1})v \rangle_{V_\sigma}$ (attached to the special v above) lies in $L^2_\sigma(K)$; by the Peter-Weyl theorem it transforms under $\Pi|_K$ according to μ , so ξ_v lies in \mathbf{H} . The value of $\mathcal{T}(\xi_v)$ at the origin of \mathfrak{p} is the vector $\int_K (\mu(k^{-1})v) \cdot \xi_v(k) dk$ in V^μ , whose scalar product with v is nonzero because of Schur's relations. This proves that $\mathcal{T}(\xi_v)$ is not zero, so \mathcal{T} must be injective. \square

We have obtained a geometric realization for π in which G acts on the subspace

$$\left\{ \int_K \gamma_b \cdot \Phi(b) db, \Phi \in \mathbf{H} \right\}$$

of $\Gamma(G/K, V^\mu)$, through the usual action of G on $\Gamma(G/K, V^\mu)$.

4.2.2. Contraction to the lowest K -type

We can now describe a Fréchet contraction of π onto μ using the same zooming-in operators as we did for the discrete series in §3.1 and for the Helgason-wave picture for spherical principal series in §3.2.2. Set

$$\mathbf{E} = \mathcal{C}^\infty(\mathfrak{p}, V^\mu). \quad (\text{A})$$

For every $t > 0$, we transfer to functions on \mathfrak{p} the sections of $\Gamma(G_t/K, V^\mu)$ which the construction in §4.2.1 yields for G_t . Calling in the Iwasawa maps $\mathfrak{J}_t : \mathfrak{p} \rightarrow \mathfrak{a}$ and $\kappa_t : \mathfrak{p} \rightarrow K$ from (3.13), we define

$$\begin{aligned}\Gamma_b^t &: \mathfrak{p} \rightarrow V^\mu \\ u &\mapsto e^{\langle t\rho, \mathfrak{J}_t(\text{Ad}(b)\cdot v) \rangle} \mu^{-1}(\kappa_t(u))v.\end{aligned}$$

In view of the previous subsection, we define

$$\mathbf{V}_t = \left\{ \int_K \Gamma_b^t \cdot \Phi(b) db, \Phi \in \mathbf{H}_\pi \right\}. \quad (\text{B})$$

The action of G_t on $\Gamma(G_t/K, V^\mu)$ yields a G_t -action on \mathbf{E} , viz.

$$\begin{aligned} G_t \times \mathbf{E} &\rightarrow \mathbf{E} \\ ((k, v), F) &\mapsto \pi_t(k, v)F = (x \mapsto \mu(k) \cdot F[(k, v) \cdot_t x]) \end{aligned} \quad (\text{B}')$$

where \cdot_t is the G_t -action on \mathfrak{p} defined in §2.2.2. The subspace \mathbf{V}_t of \mathbf{E} is then π_t -stable.

Bringing in the zooming-in map $z_t : u \mapsto \frac{u}{t}$ from \mathfrak{p} to itself, we recall from Lemma 3.4 that

$$\mathbf{C}_t : f \mapsto f \circ z_t^{-1} \quad (\text{C})$$

intertwines the G_1 - and G_t -actions on \mathbf{E} ; from Lemma 3.19 we deduce that \mathbf{C}_t sends Γ_b^1 to Γ_b^t (the main difference with the situation in §3.2.2 is that we need no renormalization of spatial frequencies here, because we are dealing with representations with continuous parameter zero). Thus,

$$\mathbf{C}_t \text{ sends } \mathbf{V}_1 \text{ to } \mathbf{V}_t. \quad (\text{Vect1})$$

Now if $f = (\int_K \Gamma_b^1 \Phi(b) db)$ is any element in \mathbf{V}_1 , then as t goes to zero, $\mathbf{C}_t f$ converges in \mathbf{E} to

$$f_0 = \text{the constant function with value } \int_K (\mu(b) \cdot v) \Phi(b) db. \quad (\text{Vect2})$$

If (k, v) is a fixed element in $K \times \mathfrak{p}$ and f is an element in \mathbf{V}_1 , we proved in §3.1 (see (3.4) and Lemma 3.8) that

$$\text{As } t \text{ goes to zero, } \pi_t(k, v) f_t \text{ goes to } \mu(k) \cdot (\text{the value of } f_0). \quad (\text{Op1})$$

From the Schur orthogonality relations and formula (Vect2), we know that

- if Φ lies in a K -isotypical subspace of \mathbf{H} for a K -type other than μ , then $f_0 = 0$;
- but the map $f \mapsto f_0$ induces a K -equivariant isomorphism between the μ -isotypical subspace of \mathbf{H} (which is spanned, if we fix an orthogonal basis (v_j) for V^μ , by the matrix elements $k \mapsto \langle v_j, \mu(k)^{-1} v \rangle$) and the fiber V^μ .

We obtain the following conclusion:

$$\begin{aligned} \text{The outcome space } \mathbf{V}_0 = \{f_0, f \in \mathbf{H}_1\} &\text{ identifies, as a } K\text{-module, with } V^\mu; \\ \text{for every } (k, v) \text{ in } K \times \mathfrak{p}, &\text{ the limit of } \pi_t(k, v) f_t \text{ then identifies with } \mu(k) \cdot f_0. \end{aligned} \quad (\text{Op2})$$

This concludes our search for a contraction of π in the present case:

Theorem 4.16. *The quadruple $(\mathbf{E}, (\mathbf{V}_t)_{t>0}, (\pi_t)_{t>0}, (\mathbf{C}_t)_{t>0})$ describes a Fréchet contraction of π onto μ . Assumption 4.4 is satisfied by this Fréchet contraction.*

(Concerning the statement about Assumption 4.4, parts *i.* and *ii.* of Assumption 4.4 are clearly satisfied here; for part *iii.*, take up the proof of Lemma 3.8 using star-shaped subsets of \mathfrak{p} in the construction of seminorms.)

5. Contraction of an arbitrary tempered representation

5.1. Real-infinitesimal-character representations for disconnected groups

Our analysis of real-infinitesimal-character representations in §4 crucially used Wong's work [44] on the closed-range property for the Dolbeault operator. The setting in [44] is that of a linear connected semisimple group: that was our reason for assuming G to be linear connected semisimple in §4.

Now suppose G is a linear connected reductive group. When (χ, μ) is an arbitrary Mackey parameter, the representation of G built from (χ, μ) in §2.1 is induced from a parabolic subgroup $P_\chi = M_\chi A_\chi N_\chi$, using the real-infinitesimal-character representation σ of M_χ with lowest K_χ -type μ .

Of course M_χ is usually disconnected, although the disconnectedness is limited in that M_χ satisfies the axioms

in §1 of [22]. It is possible that the theorem of Wong mentioned in 4.1.1 is true for linear (disconnected) reductive groups such as M_χ , but I have not been able to find a reference; if we are to use the results of §4 in a future discussion of parabolic induction from P_χ , we must check that they extend to reasonably disconnected groups.

For the rest of §5.1, we thus consider a reductive group G satisfying the axioms of section 1 in [22]: G is a matrix group with a finite number of connected components, the identity component G_e of G has compact center and a reductive Lie algebra, and $G \subset G^\mathbb{C} \cdot \text{Cent}_{\text{GL}}(G)$, where $\text{Cent}_{\text{GL}}(G)$ is the centralizer of G in the total general linear group of matrices and $G^\mathbb{C}$ is the analytic linear group with Lie algebra $\mathfrak{g}_\mathbb{C}$.

Consider the identity component G_e . It is a connected Lie group and can be decomposed as $G_e = G_{ss}(Z_G)_e$, with G_{ss} a connected semisimple Lie group with finite center. The abelian group $(Z_G)_e$ is compact and central in G_e ([25], §V.5).

Suppose we start with an irreducible tempered representation of G_e with real infinitesimal character. Then the elements in $(Z_G)_e$ will act as scalars, defining an abelian character of $(Z_G)_e$. The restriction to G_{ss} of our representation will be irreducible and have real infinitesimal character.

A representation π_e of G_e in the class $(\widetilde{G_e})_{\text{RIC}}$ is thus uniquely specified by a π_{ss} in the class $(\widetilde{G_{ss}})_{\text{RIC}}$ and an abelian character ξ of $(Z_G)_e$ whose restriction to $G_{ss} \cap (Z_G)_e$ coincides with $(\pi_{ss})|_{G_{ss} \cap (Z_G)_e}$. Given π_{ss} and ξ , any carrier space for π_{ss} furnishes a carrier space for π_e , with $g = g_{ss}g_{(Z_G)_e}$ acting through $\xi(g_{(Z_G)_e})\pi_{ss}(g_{ss})$.

Now that we know how to describe the class $(\widetilde{G_e})_{\text{RIC}}$, let us write G^\sharp for the subgroup $G_e Z_G$ of G ; because of [25], Lemma 12.30, G_e has finite index in G^\sharp ; in fact there is a finite, abelian subgroup F of K (it is the subgroup called $F(B^-)$ in [25]), lying in the center of G (hence of G^\sharp), such that

$$G^\sharp = G_e F.$$

The arguments we used for G_e go through here: a representation π^\sharp in the class $(\widetilde{G^\sharp})_{\text{RIC}}$ is uniquely specified by a representation π_e in $(\widetilde{G_e})_{\text{RIC}}$ and an abelian character χ of F whose restriction to $G_e \cap F$ coincides with $(\pi_e)|_{G_e \cap F}$. Given π_e and χ , any carrier space for π_e furnishes a carrier space for π^\sharp , with $g = g_e f$ acting through $\chi(f)\pi_e(g_e)$.

To obtain a tempered representation of G , we can start from a representation π^\sharp in the class $(\widetilde{G^\sharp})_{\text{RIC}}$ and set

$$\pi = \text{Ind}_{G^\sharp}^G(\pi^\sharp).$$

It turns out that π is irreducible, has real infinitesimal character, and that $\pi^\sharp \mapsto \pi$ maps the class $(\widetilde{G^\sharp})_{\text{RIC}}$ bijectively onto $\widetilde{G}_{\text{RIC}}$: this comes from Knapp and Zuckerman's work on the discrete series (see e.g. [25], Proposition 12.32), and from the fact that every representation in $\widetilde{G}_{\text{RIC}}$ occurs in one induced from one in the discrete series of a Levi subgroup). In addition, the restriction of π to G^\sharp decomposes as

$$\pi|_{G^\sharp} = \sum_{\omega \in G/G^\sharp} \omega \pi^\sharp$$

where $\omega \pi^\sharp$ is $g \mapsto \pi^\sharp(\omega^{-1}g\omega)$. Since G^\sharp has finite index in G (see (12.74) in [25]), the sum is finite here.

With the above description in hand, it is a simple matter to reduce the contraction problem for representations in the class $\widetilde{G}_{\text{RIC}}$ to the already solved contraction problem for representations in the class $(\widetilde{G_{ss}})_{\text{RIC}}$.

Fix a representation π in $\widetilde{G}_{\text{RIC}}$; the above discussion says how π may be constructed from a triple (χ, ξ, σ) , where σ is a representation of G_{ss} in the class $(\widetilde{G_{ss}})_{\text{RIC}}$ and χ, ξ are characters of $(Z_G)_e$ and F .

Write μ^b for the lowest K_{ss} -type of σ and assume given a Fréchet contraction

$$(E, (V_t)_{t>0}, (\sigma_t)_{t>0}, (\mathbf{c}_t)_{t>0}) \tag{5.1}$$

of σ onto μ^b . Remark that $(Z_G)_e$ and F are contained in K : there is in fact a direct product decomposition

$$K = (Z_G)_e \cdot F \cdot K_{ss}, \quad (5.2)$$

thus for each $t > 0$ the group $G_t^\sharp = \varphi_t^{-1}(G^\sharp)$ decomposes as

$$G_t^\sharp = (Z_G)_e \cdot F \cdot \varphi_t^{-1}(G_{ss}).$$

For every $t > 0$, a representation of G_t may be obtained by following the above procedure: write σ_t^\sharp for the irreducible tempered representation of G_t^\sharp obtained from the representation σ_t of $\varphi_t^{-1}(G_{ss})$, so that every element $\gamma_{(Z_G)_e} \gamma_F \gamma_{ss}$ of G_t^\sharp acts on V_t through the operator $\xi(\gamma_{(Z_G)_e}) \chi(\gamma_F) \sigma_t(\gamma_{ss})$. For each $t > 0$, the representation

$$\pi_t := \text{Ind}_{G_t^\sharp}^{G_t} \left(\sigma_t^\sharp \right) \quad (5.3)$$

is irreducible tempered, has real infinitesimal character; its lowest K -type is the lowest K -type μ of π .

Turning to K -types, there is (applying the above remarks to the reductive group K) an analogous description of μ as induced from a representation of K^\sharp : from the representation μ^b of K_{ss} and the given characters ξ, χ , form the representation $\mu^\sharp := (\mu^b)^\sharp$ of K^\sharp where any $k^\sharp = k_{(Z_G)_e} k_F k_{ss}$ acts through $\xi(k_{(Z_G)_e}) \chi(k_F) \sigma_t(k_{ss})$, then induce to K : the result is an irreducible K -module, and we do have

$$\mu \simeq \text{Ind}_{K^\sharp}^K \left(\mu^\sharp \right). \quad (5.4)$$

We now construct a Fréchet contraction of π onto μ . The representation space for π_t is a finite direct sum of copies of V_t , indexed by G_t/G_t^\sharp . Remarking that the inclusion of K in G_t induces a bijection

$$K/K^\sharp \xrightarrow{\sim} G_t/G_t^\sharp, \quad (5.5)$$

we will thus write the direct sum of copies of V_t as

$$\mathbf{V}_t = \sum_{\omega \in K/K^\sharp} V_{t,\omega}. \quad (\text{B})$$

The action of G_t on \mathbf{V}_t may be described by choosing a section of the projection $K \rightarrow K/K^\sharp$, that is, a finite collection $(\kappa_\omega)_{\omega \in K/K^\sharp}$ of elements of K : for every γ in G_t , there is a collection $(\gamma_a^\sharp)_{a \in K/K^\sharp}$ of elements of G_t^\sharp satisfying

$$\forall \omega \in K/K^\sharp, \quad \gamma \kappa_\omega = \kappa_{[\gamma\omega]_t} \gamma_{[\gamma\omega]_t}^\sharp, \quad (5.6)$$

where the action $\omega \rightarrow [\gamma\omega]_t$ of γ on K/K^\sharp is that induced from the action of γ on G_t/G_t^\sharp through the bijection (5.5). One can then set

$$\pi_t(\gamma) = \text{the operator } \sum_{\omega} x_\omega \mapsto \sum_{\omega} \sigma_t^\sharp(\gamma_{[\gamma\omega]_t}^\sharp) x_{[\gamma\omega]} \quad (\text{B}')$$

to obtain a realization for the representation (5.3) of G_t .

Now, each of the \mathbf{V}_t , $t > 0$, is contained in a finite direct sum of copies of the Fréchet space E from (5.1); we will denote that direct sum by

$$\mathbf{E} = \sum_{\omega \in K/K^\sharp} E_\omega, \quad (\text{A})$$

and equip it with the direct product topology. For each $t > 0$, the contraction operator \mathbf{c}_t acts on every summand;

if we write \mathbf{c}_t^ω for the action on the summand with index ω , we obtain an operator

$$\mathbf{C}_t : \mathbf{V}_1 \rightarrow \mathbf{V}_t \tag{C}$$

$$\left(\sum_{\omega \in K/K^\sharp} f_\omega \right) \mapsto \left(\sum_{\omega \in K/K^\sharp} \mathbf{c}_t^\omega f_\omega \right).$$

Proposition 5.1. *The quadruple $(\mathbf{E}, (\mathbf{V}_t)_{t>0}, (\pi_t)_{t>0}, (\mathbf{C}_t)_{t>0})$ describes a Fréchet contraction of π onto μ .*

We start with property (Nat_{RIC}):

Lemma 5.2. *The map \mathbf{C}_t intertwines $\pi_t \circ \varphi_1^{-1}$ and $\pi_1 \circ \varphi_1^{-1}$.*

Proof. Fix g in G , write γ_1 for the element $\varphi_1^{-1}(g)$ of G_1 and γ_t for the element $\varphi_t^{-1}(g)$ of G_t . Suppose $f = \sum_{\omega} f_\omega$ is an element of \mathbf{V}_1 . Taking up (B'), we use the intertwining relation for \mathbf{c}_t to find

$$\mathbf{C}_t(\pi_1(\gamma_1)f) = \mathbf{C}_t \left(\sum_{\omega} \sigma_t^\sharp(\gamma_{[\gamma_1\omega]}^\sharp) f_{[\gamma_1\omega]} \right) = \sum_{\omega} \mathbf{c}_t^\omega \left[\sigma_1^\sharp(\gamma_{[\gamma_1\omega]}^\sharp) f_{[\gamma_1\omega]} \right] = \sum_{\omega} \sigma_1^\sharp(\varphi_t^{-1}\varphi_1(\gamma_{[\gamma_1\omega]}^\sharp)) \cdot [\mathbf{c}_t^\omega f_{[\gamma_1\omega]}]. \tag{5.7}$$

For every ω in K/K^\sharp , the isomorphism $\varphi_t \circ \varphi_1^{-1}$ sends the class of $\gamma_1 \kappa_\omega$ in G_1/G_1^\sharp to the class of $\gamma_t \kappa_\omega$ in G_t/G_t^\sharp . As a consequence, $[\gamma_1\omega]_1$ and $[\gamma_t\omega]_t$ coincide. Together with (5.6), this means that

$$\varphi_t \varphi_1^{-1} \left(\gamma_{[\gamma_1\omega]}^\sharp \right) = \varphi_t \varphi_1^{-1} \left(\gamma_1 \kappa_\omega \kappa_{[\gamma_1\omega]}^{-1} \right) = \varphi_t \varphi_1^{-1} \gamma_1 \left(\kappa_\omega \kappa_{[\gamma_1\omega]}^{-1} \right) = \gamma_t \kappa_\omega \kappa_{[\gamma_t\omega]}^{-1} = \gamma_{[\gamma_t\omega]}^\sharp.$$

Inserting this into (5.7), we find $\mathbf{C}_t(\pi_1(\gamma_1)f) = \sum_{\omega} \sigma_t^\sharp(\gamma_{[\gamma_t\omega]}^\sharp) \cdot [\mathbf{c}_t^\omega f_{[\gamma_t\omega]}] = \pi_t(\gamma_t) \cdot [\mathbf{C}_t f]$, as desired. \square

For each vector f in \mathbf{V}_1 , the contraction $f_t := \mathbf{C}_t f$ does of course admit a limit as t goes to zero. The set of possible limits is

$$\mathbf{V}_0 := \{f_0 \mid f \in \mathbf{E}\} = \sum_{\omega \in K/K^\sharp} V_{0,\omega},$$

where $V_{0,\omega} := \{\lim(\mathbf{c}_t f) \mid f \in V_{1,\omega}\}$ carries an irreducible representation of K^\sharp with class μ^\sharp .

Turning to the convergence of operators, fix $F = \sum_{\omega} f_{0,\omega}$ in \mathbf{V}_0 and $f = \sum_{\omega} f_\omega$ in \mathbf{V}_1 such that $F = f_0$. Fix (k, v) in $K \times \mathfrak{p}$. To observe $\pi_t(k, v)f_t$, we need to insert $\gamma = \exp_{G_t}(v)k$ in (B'); for every ω in K/K^\sharp , we remark that $\gamma_{[\gamma\omega]}^\sharp$ from (5.6) is equal to $\exp_{G_t}(v)k_{k\omega}^\sharp$, where $k_{k\omega}^\sharp$ is the element of K^\sharp satisfying $k\kappa_\omega = \kappa_{k\omega}k_{k\omega}^\sharp$ (compare (5.6); here $k\omega$ is just the class of $k\kappa_\omega$ in K/K^\sharp). This indicates that

$$\pi_t(k, v)f_t = \sum_{\omega \in K/K^\sharp} \pi_t^\sharp(k_{k\omega}^\sharp, v)f_{t,\omega}.$$

As t goes to zero, the fact that we started with a contraction from σ onto μ^\flat means that for every ω , $\pi_t^\sharp(k_{k\omega}^\sharp, v)f_{t,\omega}$ goes to $\mu^\flat(k_{k\omega}^\sharp)f_{0,\omega}$. Thus,

$$\pi_t(k, v)f_t \text{ goes to } \pi_0(k, v)F := \sum_{\omega \in K/K^\sharp} \mu^\flat(k_{k\omega}^\sharp)f_{0,\omega}.$$

In the last formula, we recognize the K -action in $\text{Ind}_{K^\sharp}^K(\mu^\flat)$; from (5.4) we deduce that the G_0 -representation (\mathbf{V}_0, π_0) is the extension to G_0 of an irreducible representation of K equivalent with μ , proving Proposition 5.1. \square

We note that if the contraction (5.1) from σ onto μ^\flat satisfies Lemma 4.13, then the contraction from π onto μ we built in Proposition 5.1 obviously satisfies Lemma 4.13, too.

5.2. Parabolic induction and reduction to real infinitesimal character

We return to our linear connected reductive group G . Consider an arbitrary Mackey parameter (χ, μ) and call in the parabolic subgroup $P_\chi = M_\chi A_\chi N_\chi$ from (2.4), together with the irreducible tempered representation $\sigma = V_{M_\chi}(\mu)$ with real infinitesimal character and lowest K_χ -type μ (recall that $M_\chi \cap K = K_\chi$). The representation of G that we attached to (χ, μ) in §2.1 is

$$\pi = \text{Ind}_{M_\chi A_\chi N_\chi} (\sigma \otimes e^{i\chi}), \quad (5.8)$$

and the representation of G_0 paired with π in the Mackey-Higson bijection is

$$\pi_0 = \text{Ind}_{K_\chi \times \mathfrak{p}} (\mu \otimes e^{i\chi}). \quad (5.9)$$

We will start from a Fréchet contraction

$$(E, (V_t)_{t>0}, (\sigma_t)_{t>0}, (\mathbf{c}_t)_{t>0}) \quad (5.10)$$

of σ onto μ , assume that it has the properties in Lemma 4.13, and build a Fréchet contraction of π onto π_0 .

For every $t > 0$, the space

$$\mathbf{V}_t = \left\{ f : K \xrightarrow{\text{continuous}} V_t \mid \forall u \in K_\chi, \forall k \in K, f(ku) = \sigma_t(u^{-1})f(k) \right\} \quad (5.11)$$

can be equipped with a family of irreducible representations of G_t : recall from Lemma 3.13 that the inverse image of $M_\chi A_\chi N_\chi$ under the isomorphism $\phi_t : G_t \rightarrow G$ is a parabolic subgroup $M_{t,\chi} A_\chi N_{t,\chi}$ of G_t ; for every λ in \mathfrak{a}_χ^* , we use the Iwasawa maps

$$\kappa_t : G_t \rightarrow K, \quad \mathbf{m}_t : G_t \rightarrow M_{t,\chi} \cap \exp_{G_t}(\mathfrak{p}), \quad \mathbf{a}_t : G_t \rightarrow \mathfrak{a}_\chi,$$

and the half-sum ρ_t of positive roots in the ordering used to define $N_{t,\chi}$, to define an endomorphism of \mathbf{V}_t : we set

$$\pi_{\lambda,\mu}^{t,\text{comp}}(\gamma) := f \mapsto [k \mapsto \exp \langle -i\lambda - \rho_t, \mathbf{a}_t(\gamma^{-1}k) \rangle \sigma(\mathbf{m}_t(\gamma^{-1}k)) f(\kappa_t(\gamma^{-1}k))] \quad (5.12)$$

for every f in \mathbf{V}_t and every γ in G_t . See the discussion of the compact picture in §3.2.1.

Embedding these spaces into a fixed Fréchet space is easy: they are all vector subspaces of

$$\mathbf{E} = \{ \text{continuous functions from } K \text{ to } E \}, \quad (\text{A})$$

a Fréchet space when equipped with the topology of uniform convergence. Now for $t > 0$, consider

$$\mathbf{V}_t \text{ from (5.11)} \quad (\text{B})$$

$$\pi_t = \pi_{\chi,\mu}^{t,\text{comp}} \text{ from (5.12)} \quad (\text{B}')$$

$$\mathbf{C}_t = \text{pointwise composition with the map } \mathbf{c}_t \text{ from (5.10)}. \quad (\text{C})$$

We are ready to complete the task we set ourselves.

Theorem 5.3. *Suppose G is a linear connected reductive group, π is an irreducible tempered representation of G , and π_0 is the representation of G_0 that corresponds to π in the Mackey-Higson bijection of Theorem 2.5. The quadruple $(\mathbf{E}, (\mathbf{V}_t)_{t>0}, (\pi_t)_{t>0}, (\mathbf{C}_t)_{t>0})$ above describes a Fréchet contraction of π onto π_0 .*

It is perhaps simplest to start by establishing the naturality condition (Nat_{gen}) of §2.3: we need only mimic the case of principal series representations and use the facts encountered in the proof of Proposition 3.11 concerning the normalization of continuous parameters.

For every λ in \mathfrak{a}_χ^* , use $\varphi_1 : G_1 \rightarrow G$ to define a representation $\tilde{\pi}_\lambda$ of G acting on \mathbf{V}_1 , the composition

$$\tilde{\pi}_\lambda : G \xrightarrow{\varphi_1^{-1}} G_1 \xrightarrow{\pi_{\lambda, \mu}^{1, \text{comp}}} \text{End}(\mathbf{V}_1).$$

Consider the representation of G transferred from π_t through the isomorphism $\varphi_t : G_t \rightarrow G$,

$$\pi_t \circ \varphi_t^{-1} : G \xrightarrow{\varphi_t^{-1}} G_t \xrightarrow{\pi_{\lambda, \mu}^{t, \text{comp}}} \text{End}(\mathbf{V}_t).$$

Lemma 5.4. *The map \mathbf{C}_t intertwines $\pi_t \circ \varphi_t^{-1}$ and $\tilde{\pi}_{\chi/t}$.*

Proof. Fix f in \mathbf{V}_1 and g in G . We need to compare $(\pi_t \circ \varphi_t^{-1})(g) \cdot [\mathbf{C}_t f]$ with $\mathbf{C}_t [\tilde{\pi}_{\chi/t}(g)f]$. Lemma 3.12 proves that ρ_t is equal with $t\rho_1$, so given the definition of \mathbf{C}_t ,

$$(\pi_t \circ \varphi_t^{-1})(g) \cdot [\mathbf{C}_t f] \text{ is the map } k \mapsto e^{\langle -i\chi - t\rho, \mathbf{a}_t([\varphi_t^{-1}g]^{-1}k) \rangle} \sigma_t(\mathbf{m}_t([\varphi_t^{-1}g]^{-1}k)) [\mathbf{c}_t f] \left(\kappa_t([\varphi_t^{-1}g]^{-1}k) \right)$$

for every g in G . Lemma 3.13 identifies the maps $\kappa_t \circ \varphi_t^{-1}$, $\mathbf{m}_t \circ \varphi_t^{-1}$ and $\mathbf{a}_t \circ \varphi_t^{-1}$; inserting, from property (Nat_{RIC}) of the Fréchet contraction from σ onto μ , the fact that \mathbf{c}_t intertwines σ_1 and $\sigma_t \circ \varphi_t^{-1}$, some rearranging leads to

$$\begin{aligned} \pi_{\lambda, \sigma}^{t, \text{comp}}(\varphi_t^{-1}(g)) [\mathbf{C}_t f] &= \left[k \mapsto e^{\langle -i\frac{\lambda}{t} - \rho, t\mathbf{a}_t(\varphi_t^{-1}[g^{-1}k]) \rangle} \sigma_t(\mathbf{m}_t([\varphi_t^{-1}g]^{-1}k)) (\mathbf{c}_t f) (\kappa_t(\varphi_t^{-1}[g^{-1}k])) \right] \\ &= \left[k \mapsto e^{\langle -i\frac{\lambda}{t} - \rho, \mathbf{a}(g^{-1}k) \rangle} \{ \mathbf{c}_t \sigma(g^{-1}k) \} f(\kappa(g^{-1}k)) \right] \\ &= \mathbf{C}_t [\tilde{\pi}_{\chi/t}(g)f], \text{ as desired.} \end{aligned} \quad \square$$

Among the properties of §2.3, (Vect1) and (Vect2) are once again obvious from the fact that the quadruple $(E, (V_t)_{t>0}, (\sigma_t)_{t>0}, (\mathbf{c}_t)_{t>0})$ contracts σ onto μ : if f lies in \mathbf{V}_1 and we set $f_t = \mathbf{C}_t f$ for every $t > 0$, then f_t converges to an element of

$$\mathbf{V}_0 = \left\{ f : K \xrightarrow{\text{continuous}} V_0 \mid \forall u \in K_\chi, \forall k \in K, f(ku) = \sigma_0(u^{-1})f(k) \right\} \quad (5.13)$$

where V_0 is the outcome of the Fréchet contraction $(E, (V_t)_{t>0}, (\sigma_t)_{t>0}, (\mathbf{c}_t)_{t>0})$, an irreducible K_χ -module carrying an irreducible representation $k \mapsto \sigma_0(k)$ of class μ . Every F in \mathbf{V}_0 does arise as the limit of $\mathbf{C}_t f$ for some f in \mathbf{V}_1 .

The space \mathbf{V}_0 comes already equipped with an irreducible G_0 -representation equivalent with π_0 , that described in §2.1 and Eq. (2.3): for every F in \mathbf{V}_0 and (k, v) in G_0 , we set

$$\pi_0(k, v) \cdot F = \left[u \mapsto e^{i\langle \text{Ad}^*(u)\chi, v \rangle} F(k^{-1}u) \right]. \quad (5.14)$$

The only thing that still needs proof is the convergence of operators.

Proposition 5.5. *Fix F in \mathbf{V}_0 , fix f in \mathbf{V}_1 such that $F = \lim_{t \rightarrow 0} f_t$, and fix (k, v) in $K \times \mathfrak{p}$. Then $\pi_t(k, v) \cdot f_t$ goes to $\pi_0(k, v) \cdot F$ as t goes to zero.*

Proof. We first rearrange $\pi_t(k, v)f_t := \pi_t(\exp_{G_t}(v)k)f_t$ as

$$u \mapsto e^{\langle -i\lambda - t\rho, \mathbf{a}_t \left[k^{-1}u \exp_{G_t}^{-\text{Ad}(u^{-1})v} \right] \rangle} \sigma_t \left(\mathbf{m}_t \left[k^{-1}u \exp_{G_t}^{-\text{Ad}(u^{-1})v} \right] \right)^{-1} f_t \left(\kappa_t(k^{-1}u \exp_{G_t}^{-\text{Ad}(u^{-1})v}) \right).$$

Calling in the Iwasawa maps $\mathcal{J}_t : \mathfrak{p} \rightarrow \mathfrak{a}_\chi$ and $\mathcal{R}_t : \mathfrak{p} \rightarrow K$ from (3.13), notice that

$$\kappa_t \left(k^{-1}u \exp_{G_t}^{-\text{Ad}(u^{-1})v} \right) = k^{-1}u \mathcal{R}_t \left[-\text{Ad}(u^{-1})v \right],$$

which makes it possible to rewrite $\pi_t(k, v)f_t$ as

$$u \mapsto e^{\langle -i\lambda - t\rho, \mathfrak{J}_t(-Ad(u^{-1})v) \rangle} \sigma_t \left(\mathbf{m}_t \left[k^{-1}u \exp_{G_t}^{-Ad(u^{-1})v} \right] \right) f_t(k^{-1}u \mathfrak{R}_t[-Ad(u^{-1})v]).$$

Now recall that for all u in K ,

$$\mathbf{m}_t \left[k^{-1}u \exp_{G_t}^{-Ad(u^{-1})v} \right] = \mathbf{m}_t \left[\exp_{G_t}^{-Ad(u^{-1})v} \right]. \quad (5.15)$$

Set $\beta(u, v) = -Ad(u^{-1})v$, an element of the compact orbit $\text{Ad}(K_\chi)v$ in \mathfrak{p} . Then we just saw that

$$\pi^t(k, v)f_t = u \mapsto \exp \langle -i\chi - t\rho, \mathfrak{J}_t(\beta(u, v)) \rangle \quad (5.16)$$

$$\times \sigma_t \left(\mathbf{m}_t \left[\exp_{G_t}^{\beta(u, v)} \right] \right) f_t(k^{-1}u \mathfrak{R}_t[\beta(u, v)]). \quad (5.17)$$

When t goes to zero, we already know from Lemma 3.18 that (5.16) goes to $e^{i\langle \text{Ad}^*(u)\chi, v \rangle}$, uniformly in u because of Lemma 3.16. To complete the proof of Proposition 5.5, it is enough to establish that (5.17) goes to $F(k^{-1}u)$ as t goes to zero. For this, we call in Lemma 4.13 and use the extension of σ_t to all of E to rearrange (5.17). To simplify the notations, set $\Sigma_t(u, v) = \sigma_t \left(\mathbf{m}_t \left[\exp_{G_t}^{\beta(u, v)} \right] \right)$; then

$$\sigma_t \left(\mathbf{m}_t \left[\exp_{G_t}^{\beta(u, v)} \right] \right) f_t(k^{-1}u \mathfrak{R}_t[\beta(u, v)]) = f_t(k^{-1}u) \quad (\text{Term 1})$$

$$+ [\Sigma_t(u, v) - \text{Id}_E] [f_t(k^{-1}u) - f_0(k^{-1}u)] \quad (\text{Term 2})$$

$$+ \Sigma_t(u, v) [f_t(k^{-1}u \mathfrak{R}_t[\beta(u, v)]) - f_t(k^{-1}u)] \quad (\text{Term 3})$$

$$+ [\Sigma_t(u, v) - \text{Id}_E] [f_0(k^{-1}u)]. \quad (\text{Term 4})$$

As t goes to zero,

- *Term 1* goes to $f_0(k^{-1}u) = F(k^{-1}u)$.
- *Term 2* goes to zero because of the Lipschitz estimate in Lemma 4.13: since $\beta(u, v)$ lies on a compact orbit in \mathfrak{p} , there is a number C , independent of t and u , such that $\Sigma_t(u, v)$ is C -Lipschitz for all t, u (recall that v is fixed here). Since $f_t - f_0$ goes to zero as t goes to zero, so does Term 2.
- *Term 3* also goes to zero: it can be rewritten as $(\Sigma_t(u, v) \circ \mathbf{c}_t) [f(k^{-1}u \mathfrak{R}_t[\beta]) - (k^{-1}u)]$; from Lemma 3.16 and the compactness of $\{\beta(u, v), u \in K\}$, we know that $u \mapsto f(k^{-1}u \mathfrak{R}_t[\beta(u, v)]) - (k^{-1}u)$ goes to zero as t goes to zero, and we use the uniform Lipschitz estimate of Lemma 4.13 to conclude as we did for Term 2.
- To prove that *Term 4* goes to zero, we need to prove that $\Sigma_t(u, v)f_0(k^{-1}u)$ goes to $f_0(k^{-1}u)$. Now, f_0 is by definition a \mathbf{c}_t -invariant element of E , so for all u in K ,

$$\begin{aligned} \Sigma_t(u, v)f_0(k^{-1}u) &= \Sigma_t(u, v) \cdot \mathbf{c}_t \cdot f_0(k^{-1}u) \\ &= \sigma_t \left(\mathbf{m}_t \left[\exp_{G_t}^{\beta(u, v)} \right] \right) \cdot \mathbf{c}_t \cdot f_0(k^{-1}u) \\ &= \sigma_1 \left(\varphi_1^{-1} \circ \varphi_t \left\{ \mathbf{m}_t \left[\exp_{G_t}^{\beta(u, v)} \right] \right\} \right) f_0(k^{-1}u) \\ &= \sigma_1 \left(\mathbf{m}_1 \left[\exp_{G_1}^{t\beta(u, v)} \right] \right) f_0(k^{-1}u). \end{aligned}$$

The transition between the second line and the third uses the intertwining relation (Nat_{RIC}) for \mathbf{c}_t , and that between the third line and the fourth uses the fact that $\beta(u, v)$ lies in \mathfrak{p} .

As t goes to zero, the element $\mathbf{m}_1 \left[\exp_{G_1}^{t\beta(u, v)} \right]$ of G_1 goes to the identity of G_1 ; from the continuity of σ_1 (see Lemma 4.13) we deduce that $\Sigma_t(u, v)f_0(k^{-1}u)$ goes to $f_0(k^{-1}u)$, as desired.

All convergences are uniform in u ; Proposition 5.5 and Theorem 5.3 follow. \square

References

- [1] J. ADAMS, M. VAN LEEUWEN, P. TRAPA, and D. A. VOGAN, JR., Unitary representations of real reductive groups, arXiv:1212.2192v5 (version 5), 2017.
- [2] A. AFGOUSTIDIS, *Représentations de groupes de Lie et fonctionnement géométrique du cerveau*, Ph.D. thesis, Université Paris-7, July 2016.
- [3] A. AFGOUSTIDIS, *On the analogy between reductive Lie groups and Cartan motion groups: the Mackey-Higson bijection*, Preprint, arXiv:1510.02650v3 (version 3), 2018.
- [4] M. ATIYAH and W. SCHMID, A geometric construction of the discrete series for semisimple Lie groups, *Invent. Math.* **42** (1977), 1–62. MR 0463358. Link: <https://doi.org/10.1007/BF01389783>.
- [5] D. BARBASCH, N. HIGSON, and E. M. SUBAG, Algebraic families of groups and commuting involutions, *Internat. J. Math.* **29** no. 4 (2018), 1850030, 18. MR 3797197. <https://doi.org/10.1142/S0129167X18500301>.
- [6] L. BARCHINI and R. ZIERAU, Square integrable harmonic forms and representation theory, *Duke Math. J.* **92** no. 3 (1998), 645–664. MR 1620530. Link: <https://doi.org/10.1215/S0012-7094-98-09220-1>.
- [7] P. BAUM, A. CONNES, and N. HIGSON, Classifying space for proper actions and K -theory of group C^* -algebras, in *C^* -algebras: 1943–1993 (San Antonio, TX, 1993)*, *Contemp. Math.* **167**, Amer. Math. Soc., Providence, RI, 1994, pp. 240–291. MR 1292018. Link: <https://doi.org/10.1090/conm/167/1292018>.
- [8] J. BERNSTEIN, N. HIGSON, and E. M. SUBAG, Algebraic families of harish-chandra pairs, *International Mathematics Research Notices* (2018), rny147. <https://doi.org/10.1093/imrn/rny147>.
- [9] J. BERNSTEIN, N. HIGSON, and E. M. SUBAG, Contractions of representations and algebraic families of harish-chandra modules, *International Mathematics Research Notices* (2018), rny146. <https://doi.org/10.1093/imrn/rny146>.
- [10] R. CAMPORESI, The Helgason Fourier transform for homogeneous vector bundles over Riemannian symmetric spaces, *Pacific J. Math.* **179** no. 2 (1997), 263–300. MR 1452535. Link: <https://doi.org/10.2140/pjm.1997.179.263>.
- [11] A. CONNES, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994. MR 1303779.
- [12] J. DIEUDONNÉ, *Éléments d'analyse. Tome II: Chapitres XII à XV, Cahiers Scientifiques, Fasc. XXXI*, Gauthier-Villars, Éditeur, Paris, 1968. MR 0235946.
- [13] A. H. DOOLEY and J. W. RICE, On contractions of semisimple Lie groups, *Trans. Amer. Math. Soc.* **289** no. 1 (1985), 185–202. MR 779059. Link: <https://doi.org/10.2307/1999695>.
- [14] A. GROTHENDIECK, Résumé des résultats essentiels dans la théorie des produits tensoriels topologiques et des espaces nucléaires, *Ann. Inst. Fourier Grenoble* **4** (1952), 73–112 (1954). MR 0061754. Link: http://www.numdam.org/item?id=AIF_1952__4__73_0.
- [15] S. HELGASON, A duality for symmetric spaces with applications to group representations. III. Tangent space analysis, *Adv. in Math.* **36** no. 3 (1980), 297–323. MR 577307. [https://doi.org/10.1016/0001-8708\(80\)90019-5](https://doi.org/10.1016/0001-8708(80)90019-5).
- [16] S. HELGASON, *Groups and geometric analysis, Mathematical Surveys and Monographs* **83**, American Mathematical Society, Providence, RI, 2000, Integral geometry, invariant differential operators, and spherical functions, Corrected reprint of the 1984 original. MR 1790156. Link: <https://doi.org/10.1090/surv/083>.
- [17] S. HELGASON, *Geometric analysis on symmetric spaces*, second ed., *Mathematical Surveys and Monographs* **39**, American Mathematical Society, Providence, RI, 2008. MR 2463854. Link: <https://doi.org/10.1090/surv/039>.
- [18] R. HERMANN, The Gell-Mann formula for representations of semisimple groups, *Comm. Math. Phys.* **2** (1966), 155–164. MR 0206151. Link: <https://projecteuclid.org/euclid.cmp/1103815046>.

- [19] N. HIGSON, The Mackey analogy and K -theory, in *Group representations, ergodic theory, and mathematical physics: a tribute to George W. Mackey*, *Contemp. Math.* **449**, Amer. Math. Soc., Providence, RI, 2008, pp. 149–172. MR 2391803. Link: <https://doi.org/10.1090/conm/449/08711>.
- [20] E. İNÖNÜ and E. P. WIGNER, Representations of the Galilei group, *Nuovo Cimento (9)* **9** (1952), 705–718. MR 0050594.
- [21] E. İNÖNÜ and E. P. WIGNER, On the contraction of groups and their representations, *Proc. Nat. Acad. Sci. U. S. A.* **39** (1953), 510–524. MR 0055352.
- [22] A. W. KNAPP, Commutativity of intertwining operators for semisimple groups, *Compositio Math.* **46** no. 1 (1982), 33–84. MR 660154. Link: http://www.numdam.org/item?id=CM_1982__46_1_33_0.
- [23] A. W. KNAPP and G. J. ZUCKERMAN, Classification of irreducible tempered representations of semisimple groups, *Ann. of Math. (2)* **116** no. 2 (1982), 389–455. MR 672840. Link: <https://doi.org/10.2307/2007066>.
- [24] A. W. KNAPP and G. J. ZUCKERMAN, Classification of irreducible tempered representations of semisimple groups. II, *Ann. of Math. (2)* **116** no. 3 (1982), 457–501. MR 678478. Link: <https://doi.org/10.2307/2007019>.
- [25] A. W. KNAPP, *Representation theory of semisimple groups*, *Princeton Landmarks in Mathematics*, Princeton University Press, Princeton, NJ, 2001, Reprint of the 1986 original. MR 1880691.
- [26] A. W. KNAPP and D. A. VOGAN, JR., *Cohomological induction and unitary representations*, *Princeton Mathematical Series* **45**, Princeton University Press, Princeton, NJ, 1995. MR 1330919. Link: <https://doi.org/10.1515/9781400883936>.
- [27] G. W. MACKEY, Imprimitivity for representations of locally compact groups. I, *Proc. Nat. Acad. Sci. U. S. A.* **35** (1949), 537–545. MR 0031489.
- [28] J. MICKELSSON and J. NIEDERLE, Contractions of representations of de Sitter groups, *Comm. Math. Phys.* **27** (1972), 167–180. MR 0309502. Link: <https://projecteuclid.org.proxy/euclid.cmp/1103858248>.
- [29] G. D. MOSTOW, Some new decomposition theorems for semi-simple groups, *Mem. Amer. Math. Soc.* **No. 14** (1955), 31–54. MR 0069829.
- [30] N. MUKUNDA, Expansions of Lie groups and representations of $SL(3, C)$, *J. Mathematical Phys.* **10** (1969), 897–911. MR 0246583. <https://doi.org/10.1063/1.1664919>.
- [31] R. PARTHASARATHY, Dirac operator and the discrete series, *Ann. of Math. (2)* **96** (1972), 1–30. MR 0318398. Link: <https://doi.org/10.2307/1970892>.
- [32] S. A. SALAMANCA-RIBA and D. A. VOGAN, JR., On the classification of unitary representations of reductive Lie groups, *Ann. of Math. (2)* **148** no. 3 (1998), 1067–1133. MR 1670073. Link: <https://doi.org/10.2307/121036>.
- [33] I. E. SEGAL, A class of operator algebras which are determined by groups, *Duke Math. J.* **18** (1951), 221–265. MR 0045133. Link: <http://projecteuclid.org/euclid.dmj/1077476400>.
- [34] E. M. SUBAG, E. M. BARUCH, J. L. BIRMAN, and A. MANN, Strong contraction of the representations of the three-dimensional Lie algebras, *J. Phys. A* **45** no. 26 (2012), 265206, 25. MR 2942592. <https://doi.org/10.1088/1751-8113/45/26/265206>.
- [35] E. M. SUBAG, Symmetries of the hydrogen atom and algebraic families, *J. Math. Phys.* **59** no. 7 (2018), 071702, 20. MR 3827131. <https://doi.org/10.1063/1.5018061>.
- [36] E. M. SUBAG, The algebraic Mackey-Higson bijections, *J. Lie Theory* **29** no. 2 (2019), 473–492. MR 3942563.
- [37] Q. TAN, Y.-J. YAO, and S. YU, Mackey analogy via \mathcal{D} -modules for $SL(2, \mathbb{R})$, *Internat. J. Math.* **28** no. 7 (2017), 1750055, 20. MR 3667899. <https://doi.org/10.1142/S0129167X17500550>.

- [38] F. TRÈVES, *Topological vector spaces, distributions and kernels*, Academic Press, New York-London, 1967. [MR 0225131](#).
- [39] D. A. VOGAN, JR., *Representations of real reductive Lie groups*, *Progress in Mathematics* **15**, Birkhäuser, Boston, Mass., 1981. [MR 632407](#).
- [40] D. A. VOGAN, JR., Branching to a maximal compact subgroup, in *Harmonic analysis, group representations, automorphic forms and invariant theory*, *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.* **12**, World Sci. Publ., Hackensack, NJ, 2007, pp. 321–401. [MR 2401817](#). Link: https://doi.org/10.1142/9789812770790_0010.
- [41] D. A. VOGAN, JR., Unitary representations and complex analysis, in *Representation theory and complex analysis*, *Lecture Notes in Math.* **1931**, Springer, Berlin, 2008, pp. 259–344. [MR 2409701](#). Link: https://doi.org/10.1007/978-3-540-76892-0_5.
- [42] C. VOISIN, *Hodge theory and complex algebraic geometry. II*, *Cambridge Studies in Advanced Mathematics* **77**, Cambridge University Press, Cambridge, 2003, Translated from the French by Leila Schneps. [MR 1997577](#). Link: <https://doi.org/10.1017/CB09780511615177>.
- [43] H.-W. WONG, Dolbeault cohomological realization of Zuckerman modules associated with finite rank representations, *J. Funct. Anal.* **129** no. 2 (1995), 428–454. [MR 1327186](#). Link: <https://doi.org/10.1006/jfan.1995.1058>.
- [44] H.-W. WONG, Cohomological induction in various categories and the maximal globalization conjecture, *Duke Math. J.* **96** no. 1 (1999), 1–27. [MR 1663911](#). Link: <https://doi.org/10.1215/S0012-7094-99-09601-1>.
- [45] S. YU, *Mackey analogy as deformation of \mathcal{D} -modules*, Preprint, arXiv:1707.00240, 2017.