UNIVERSAL CUTOFF FOR DYSON ORNSTEIN UHLENBECK PROCESS

JEANNE BOURSIER, DJALIL CHAFAI, AND CYRIL LABBÈ

Abstract. We study the Dyson–Ornstein–Uhlenbeck diffusion process, an evolving gas of interacting particles. Its invariant law is the beta Hermite ensemble of random matrix theory, a non-product log-concave distribution. We explore the convergence to equilibrium of this process for various distances or divergences, including total variation, entropy and Wasserstein. When the number of particles is sent to infinity, we show that a cutoff phenomenon occurs: the distance to equilibrium vanishes at a critical time. A remarkable feature is that this critical time is independent of the parameter beta that controls the strength of the interaction, in particular the result is identical in the non-interacting case, which is nothing but the Ornstein–Uhlenbeck process. We also provide a complete analysis of the non-interacting case that reveals some new phenomena. Our work relies among other ingredients on convexity and functional inequalities, exact solvability, exact Gaussian formulas, coupling arguments, stochastic calculus, variational formulas and contraction properties. This work leads, beyond the specific process that we study, to questions on the high-dimensional analysis of heat kernels of curved diffusions.

Contents
1. Introduction and main results 1
2. Additional comments and open problems 9
3. Cutoff phenomenon for the OU 14
4. General exactly solvable aspects 15
5. The random matrix cases 18
6. Cutoff phenomenon for the DOU in TV and Hellinger 21
7. Cutoff phenomenon for the DOU in Wasserstein 26
Appendix A. Distances and divergences 27
Appendix B. Convexity and its dynamical consequences 30
Acknowledgements 33
References 33

1. Introduction and main results

For a Markov process \( X = (X_t)_{t \geq 0} \) with state space \( S \) and invariant law \( \mu \) we have typically

\[
\lim_{t \to \infty} \text{dist}(\text{Law}(X_t), \mu) = 0
\]

where \( \text{dist} \) is a distance or divergence on the set of probability measures on \( S \). Suppose now that \( X = X^n \) depends on a dimension, size, or complexity parameter \( n \), and let us set \( S = S^n, \mu = \mu^n \), and \( X_0 = x_0^n \in S^n \). For example \( X^n \) can be a random walk on the symmetric group of permutations of \( \{1, \ldots, n\} \), Brownian motion on the group of \( n \times n \) unitary matrices, Brownian motion on the \( n \)-dimensional sphere, etc. In many of such examples, it has been proved that when \( n \) is large enough, the supremum over some set of initial conditions \( x_0^n \) of the quantity \( \text{dist}(\text{Law}(X^n_t), \mu^n) \) collapses abruptly to 0 when \( t \) passes a critical value \( c = c_n \) which may depend on \( n \). This is often
referred to as a cutoff phenomenon. More precisely, if dist ranges from 0 to max, then, for some subset $S_0^n \subset S^n$ of initial conditions, some critical value $c = c_n$ and for all $\varepsilon \in (0, 1)$,

$$\lim_{n \to \infty} \sup_{x_0 \in S_0^n} \text{dist} (\text{Law}(X^n_t), \mu^n) = \begin{cases} \max_{t_n} & \text{if } t_n = (1 - \varepsilon)c_n \\
 = 0 & \text{if } t_n = (1 + \varepsilon)c_n. \end{cases}$$

Of course such a statement fully makes sense as soon as the function $t \mapsto \text{dist} (\text{Law}(X^n_t), \mu^n)$ is non-increasing. Such a monotonicity leads also to introduce, for an arbitrary small threshold $\eta > 0$, the quantity $\inf \{t \geq 0 : \text{dist} (\text{Law}(X^n_t), \mu^n) \leq \eta\}$ known as the mixing time in the literature.

When $S^n$ is finite, it is customary to take $S_0^n = S^n$. When $S^n$ is infinite, it may happen that the supremum over the whole set $S^n$ of the distance to equilibrium remains equal to max at all times, in which case one has to consider strict subspaces of initial conditions. For some processes, it is possible to restrict $S_0^n$ to a single state in which case one obtains a very precise description of the convergence to equilibrium starting from this initial condition. Note that the constraint over the initial condition can be made compatible with a limiting dynamics, for instance a mean-field limit when the process describes an exchangeable interacting particle system.

The cutoff phenomenon was put forward by Aldous and Diaconis at the origin for random walks on finite sets, see for instance [1, 20, 18, 43] and references therein. The analysis of the cutoff phenomenon is the subject of an important activity, still seeking for a complete theory.

The study of the cutoff phenomenon for Markov diffusion processes goes back at least to the works of Saloff-Coste [52, 53] in relation notably with Nash–Sobolev type functional inequalities, heat kernel analysis, and Diaconis–Wilson probabilistic techniques. We also refer to the more recent work [46] for the case of diffusion processes on compact groups and symmetric spaces, in relation with group invariance and representation theory, a point of view inspired by the early works of Diaconis on Markov chains and of Saloff-Coste on diffusion processes. We emphasize that apart very few works such as [38, 12], most of the available results in the literature on the cutoff phenomenon are related to compact state spaces.

Our contribution is an exploration of the cutoff phenomenon for the Dyson–Ornstein–Uhlenbeck diffusion process, for which the state space is $\mathbb{R}^n$. This process is an interacting particle system. When the interaction is turned off, we recover the Ornstein–Uhlenbeck process, a special case that has been considered previously in the literature but for which we also provide new results.

1.1. Distances. As for dist we use several standard distances or divergences between probability measures: Wasserstein, total variation (TV), Hellinger, Entropy, $L^2$ and Fisher, surveyed in Appendix A. We take the following convention for probability measures $\mu$ and $\nu$ on the same space:

$$\text{dist}(\mu, \nu) = \begin{cases} \text{Wasserstein}(\mu, \nu) & \text{when dist = Wasserstein} \\
 \|\mu - \nu\|_{\text{TV}} & \text{when dist = TV} \\
 \text{Hellinger}(\mu, \nu) & \text{when dist = Hellinger} \\
 \text{Entropy}(\mu \mid \nu) & \text{when dist = Entropy} \\
 \|\frac{\mu}{\nu} - 1\|_{L^2(\nu)} & \text{when dist = } L^2 \\
 \text{Fisher}(\mu \mid \nu) & \text{when dist = Fisher} \end{cases},$$

(1.1)

see Appendix A for precise definitions. The maximal value max taken by dist is given by

$$\max = \begin{cases} 1 & \text{if dist }\in\{\text{TV, Hellinger}\}, \\
 +\infty & \text{if dist }\in\{\text{Wasserstein, Entropy, } L^2, \text{Fisher}\}. \end{cases}$$

(1.2)

1.2. The Dyson–Ornstein–Uhlenbeck process. The Dyson–Ornstein–Uhlenbeck process (DOU) is the solution $X^n = (X^n_t)_{t \geq 0}$ on $\mathbb{R}^n$ of the stochastic differential equation

$$X^n_0 = x^n_0 \in \mathbb{R}^n, \hspace{1cm} dX^n_t = \sqrt{\frac{2}{n}} dB^n_t - V'(X^n_t) dt + \frac{\beta}{n} \sum_{j \neq i} \frac{dt}{X^n_t - X^n_j}, \hspace{1cm} 1 \leq i \leq n,$$

(1.3)

where $(B_t)_{t \geq 0}$ is a standard $n$-dimensional Brownian motion (BM), and where

- $V(x) = \frac{x^2}{2}$ is a “confinement potential” acting through the drift $-V'(x) = -x$
- $\beta \geq 0$ is a parameter tuning the interaction strength.
The notation $X^{n,i}_t$ stands for the $i$-th coordinate of the vector $X^n_t$. The process $X^n$ can be thought of as an interacting particle system of $n$ one-dimensional Brownian particles $X^{n,1},\ldots,X^{n,n}$, subject to confinement and singular pairwise repulsion when $\beta > 0$ (respectively first and second term in the drift). Note that we take an inverse temperature of order $n$ in the dynamics (1.3) in order to obtain a mean-field limit without time-changing the process, see Section 2.5. Note also that the spectral gap is 1 for all $n \geq 1$, see Section 2.6. We refer to Section 2.8 for other parametrizations or choices of inverse temperature.

The process $X^n$ was essentially discovered by Dyson in [25], in the case $\beta \in \{1,2,4\}$, because it describes the dynamics of the eigenvalues of $n \times n$ symmetric/Hermitian/symplectic random matrices with independent Ornstein–Uhlenbeck entries, see Lemma 5.1 and Lemma 5.2 for the cases $\beta = 1$ and $\beta = 2$ respectively.

- Case $\beta = 0$ (interaction turned off). The particles become $n$ independent one-dimensional Ornstein–Uhlenbeck processes, and the DOU process $X^n$ becomes exactly the $n$-dimensional Ornstein–Uhlenbeck process $Z^n$ solving (1.8). The process lives in $\mathbb{R}^n$. The particles collide but since they do not interact, this does not raise any issue.
- Case $0 < \beta < 1$. Then with positive probability the particles collide producing a blow up of the drift, see for instance [15, 17] for a discussion. Nevertheless, it is possible to define the process for all times. For simplicity, we do not consider this case in the present work.
- Case $\beta \geq 1$. If we order the coordinates by defining the convex domain

$$D_n = \{ x \in \mathbb{R}^n : x_1 < \cdots < x_n \},$$

and if $x^n_0 \in D_n$ then the equation (1.3) admits a unique strong solution that never exits $D_n$, in other words the particles never collide and the order of the initial particles is preserved at all times, see [50]. Moreover if

$$\overline{D}_n = \{ x \in \mathbb{R}^n : x_1 \leq \cdots \leq x_n \}

$$

then it is possible to start the process from the boundary $\overline{D}_n \setminus D_n$, in particular from $x^n_0$ such that $x^n_{0,1} = \cdots = x^n_{0,n}$, and despite the singularity of the drift, it can be shown that with probability one, $X^n_t \in D_n$ for all $t > 0$. We refer to [2, Th. 4.3.2] for a proof in the Dyson Brownian Motion case that can be adapted mutatis mutandis.

In the sequel, we will only consider the cases $\beta = 0$ with $x^n_0 \in \mathbb{R}^n$ and $\beta \geq 1$ with $x^n_0 \in \overline{D}_n$.

The drift in (1.3) is the gradient of a function, and (1.3) rewrites

$$X^{n}_0 = x^n_0 \in D_n, \quad dX^n_t = \sqrt{\frac{2}{n}}dB_t - \frac{1}{n} \nabla E(X^n_t)\,dt,$$

where

$$E(x_1,\ldots,x_n) = n \sum_{i=1}^{n} V(x_i) + \beta \sum_{i>j} \log \frac{1}{|x_i - x_j|},$$

(1.5)

can be interpreted as the energy of the configuration of particles $x_1,\ldots,x_n$.

- If $\beta = 0$, then the Markov process $X^n$ is an Ornstein–Uhlenbeck process, irreducible with unique invariant law $P^n_0 = \mathcal{N}(0,\frac{1}{n}I_n)$ which is reversible.
- If $\beta \geq 1$, then the Markov process $X^n$ is not irreducible, but $D_n$ is a recurrent class carrying a unique invariant law $P^n_\beta$, which is reversible and given by

$$P^n_\beta = \frac{e^{-E(x_1,\ldots,x_n)}}{C^n_\beta} 1_{(x_1,\ldots,x_n) \in \overline{D}_n} \,dx_1 \cdots dx_n,$$

(1.6)

where $C^n_\beta$ is the normalizing factor given by

$$C^n_\beta = \int_{\overline{D}_n} e^{-E(x_1,\ldots,x_n)} \,dx_1 \cdots dx_n.$$

(1.7)

In terms of geometry, it is crucial to observe that since $-\log$ is convex on $(0,\infty)$, the map

$$(x_1,\ldots,x_n) \in D_n \mapsto \text{Interaction}(x_1,\ldots,x_n) = \beta \sum_{i>j} \log \frac{1}{x_i - x_j},$$

(1.8)
is convex. Thus, since $V$ is convex on $\mathbb{R}$, it follows that $E$ is convex on $D_n$. For all $\beta \geq 0$, the law $P_n^\beta$ is log-concave with respect to the Lebesgue measure as well as with respect to $\mathcal{N}(0, \frac{1}{\beta} I_n)$.

1.3. Non-interacting case and Ornstein–Uhlenbeck benchmark. When we turn off the interaction by taking $\beta = 0$ in (1.3), the DOU process becomes an Ornstein–Uhlenbeck process (OU) $Z^n = (Z^n_t)_{t \geq 0}$ on $\mathbb{R}^n$ solution of the stochastic differential equation

$$Z^n_0 = z^n_0 \in \mathbb{R}^n, \quad dZ^n_t = \sqrt{\frac{2}{n}} dB^n_t - Z^n_t dt,$$

where $B^n_t$ is a standard $n$-dimensional BM. The invariant law of $Z^n$ is the product Gaussian law $P_n^0 = \mathcal{N}(0, \frac{1}{\beta} I_n) = \mathcal{N}(0, \frac{1}{\beta} I^n)$. The explicit Gaussian nature of $Z^n_t \sim \mathcal{N}(z^n_0 e^{-\beta t}, \frac{e^{-2\beta t}}{\beta^2} I_n)$, valid for all $t \geq 0$, allows for a fine analysis of convergence to equilibrium, as in the following theorem.

**Theorem 1.1** (Cutoff for OU: mean-field regime). Let $Z^n = (Z^n_t)_{t \geq 0}$ be the OU process (1.8) and let $P_n^0$ be its invariant law. Suppose that

$$\lim_{n \to \infty} \frac{|z_0^n|^2}{n} > 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{|z_0^n|^2}{n} < \infty,$$

where $|z| = \sqrt{z_1^2 + \cdots + z_n^2}$ is the Euclidean norm. Then for all $\varepsilon \in (0,1)$,

$$\lim_{n \to \infty} \text{dist}(\mathcal{L}(Z^n_t), P_n^0) = \begin{cases} \max \{0, \frac{1}{2} \log(n) \} & \text{if dist = Wasserstein}, \\ \max \{0, \log(n) \} & \text{if dist} \in \{L^2, TV, Hellinger, Entropy\}, \\ \max \{0, \log(n) \} & \text{if dist = Fisher}. \end{cases}$$

where $c_n = \log(n)$. Theorem 1.1 is proved in Section 3.

The TV and Hellinger cases are already covered in [38], the other cases seem to be new.

The restriction over the initial condition in Theorem 1.1 is spelled out in terms of the second moment of the empirical distribution, a natural choice suggested by the mean-field limit discussed in Section 2.5. It yields a mixing time of order $\log(n)$, just like for Brownian motion on compact Lie groups, see [53, 46]. For the OU process and more generally for overdamped Langevin processes, the non-compactness of the space is replaced by the confinement or tightness due to the drift.

Actually, Theorem 1.1 is a particular instance of the following, much more general result that reveals that, except for the Wasserstein distance, a cutoff phenomenon always occurs.

**Theorem 1.2** (General cutoff for OU). Let $Z^n = (Z^n_t)_{t \geq 0}$ be the OU process (1.8) and let $P_n^0$ be its invariant law. Let $\text{dist} \in \{TV, Hellinger, Entropy, L^2, Fisher\}$. Then, for all $\varepsilon \in (0,1)$,

$$\lim_{n \to \infty} \text{dist}(\mathcal{L}(Z^n_t), P_n^0) = \begin{cases} \max \{0, \log(n) \} & \text{if dist} \in \{TV, Hellinger, Entropy, L^2\}, \\ \log(n) \log(\sqrt{n}|z_0^n|) & \text{if dist = Fisher}. \end{cases}$$

where $c_n = \log(n)$. Regarding the Wasserstein distance, the following dichotomy occurs:

- if $\lim_{n \to \infty} |z_0^n| = +\infty$, then for all $\varepsilon \in (0,1)$, with $c_n = \log(|z_0^n|)$,

$$\lim_{n \to \infty} \text{Wasserstein}(\mathcal{L}(Z^n_t), P_n^0) = \begin{cases} +\infty & \text{if } t_n = (1-\varepsilon)c_n, \\ 0 & \text{if } t_n = (1+\varepsilon)c_n, \end{cases}$$

- if $\lim_{n \to \infty} |z_0^n| = \alpha \in [0,\infty)$ then there is no cutoff phenomenon namely for any $t > 0$

$$\lim_{n \to \infty} \text{Wasserstein}(\mathcal{L}(Z^n_t), P_n^0)^2 = 2 + (\alpha^2 - 1)e^{-2t} - 2\sqrt{1 - e^{-2t}}.$$


Theorem 1.2 is proved in Section 3.

The observation that for every distance or divergence, except for the Wasserstein distance, a cutoff phenomenon occurs generically seems to be new.

Let us make a few comments. First, in terms of convergence to equilibrium the relevant observable in Theorem 1.2 appears to be the Euclidean norm $|z_n|$ of the initial condition. This quantity differs from the eigenfunction associated to the spectral gap of the generator, which is given by $z \mapsto z_1 + \cdots + z_n$ as we will recall later on. Second, cutoff occurs at a time that is independent of the initial condition provided that its euclidean norm is small enough: this cutoff time appears as the time required to regularize the initial condition (a Dirac mass) into a sufficiently spread out absolutely continuous probability measure; in particular this cutoff phenomenon would not hold generically if we allowed for spread out (non-Dirac) initial conditions. In this respect, the Wasserstein distance is peculiar since it is much less stringent on the local behavior of the measures at stake: for instance Wasserstein($\delta_0, \delta_{1/n}$) → 0 as $n \to \infty$ while for all other distances or divergences considered here, the corresponding quantity would remain equal to max. This explains the absence of generic cutoff phenomenon for the Wasserstein distance. Third, the explicit expressions provided in our proof allow to extract the cutoff profile in each case, but we prefer not to provide them in our statement.

1.4. Exactly solvable intermezzo. When $\beta \neq 0$, the law of the DOU process is no longer Gaussian nor explicit. However several exactly solvable aspects are available. Let us recall that a Cox–Ingersoll–Ross process (CIR) of parameters $a,b,\sigma$ is a $t$-dimensional BM, then $R_t := |Z_t|^2$, is a CIR process with parameters $a = \theta^2 d, b = 2 \rho, \sigma = 2 \theta$. When $\rho = 0$ then $Z$ is a BM while $R = |Z|^2$ is a squared Bessel process.

Our next result reveals some exactly solvable aspects of the DOU process for general $\beta \geq 1$. It is based on our knowledge of eigenfunctions associated to the first spectral values of the dynamics, see (2.6), and their remarkable properties. As in (2.6), we set $\pi(x) := x_1 + \cdots + x_n$ when $x \in \mathbb{R}^n$.

**Theorem 1.3** (From DOU to OU and CIR). Let $(X^\beta_t)_{t \geq 0}$ be the DOU process (1.3), with $\beta = 0$ or $\beta \geq 1$, and let $P^\beta_n$ be its invariant law. Then:

- $(\pi(X^\beta_t))_{t \geq 0}$ is a one-dimensional OU process weak solution of (1.8) with $\theta = \sqrt{2}$, $\rho = 1$.
- Its invariant law is $N(0,1)$. It does not depend on $\beta$, and $\pi(X^\beta_t) \sim N(\pi(x^\beta_0) e^{-t}, 1 - e^{-2t})$. Furthermore $\pi(X^\beta_t)^2$ is a CIR process of parameters $a = 2, b = 2, \sigma = 2\sqrt{2}$.

- $(|X^\beta_t|^2)_{t \geq 0}$ is a CIR process, weak solution of (1.9) with $a = 2 + \beta(n-1), b = 2, \sigma = \sqrt{8/n}$.
- Its invariant law is Gamma($\frac{3n^2 + (2-\beta) n}{4}, \frac{\beta}{2} n^2$) of mean $1 + \frac{\beta}{2}(n-1)$ and variance $\beta + 2 - \beta n$. Furthermore, if $d = \frac{n^2}{2} + (1 - \frac{\beta}{2}) n$ is a positive integer, then $(|X^\beta_t|^2)_{t \geq 0}$ has the law of $(Z_t)_{t \geq 0}$ where $(Z_t)_{t \geq 0}$ is a $d$-dimensional OU process, weak solution of (1.8) with $\theta = \sqrt{2/n}$, $\rho = 1$, and $Z_0 = x^\beta_0$ for an arbitrary $x^\beta_0 \in \mathbb{R}^d$ such that $|z^\beta_0| = |x^\beta_0|$.

From Theorem 1.3, explicit computations on the moments of $\pi(X^\beta_t)$ and $|X^\beta_t|^2$ can be made and they reveal an abrupt convergence to their equilibrium values. Actually, it would be possible to treat any moment of order $p \geq 1$ of $|X^\beta_t|^2$, see Remark 4.1.

**Corollary 1.4** ("Cutoffs" for moments). Let $(X^\beta_t)_{t \geq 0}$ be the DOU process solution of (1.3) with $\beta = 0$ or $\beta \geq 1$ and let $P^\beta_n$ be its invariant law, and $\beta_n = 1 + \frac{\beta}{2}(n-1)$.
If \( \lim_{n \to \infty} \frac{\pi(x_n^0)}{n} = \alpha \neq 0 \) then for \( \varepsilon \in (0, 1) \),
\[
\lim_{n \to \infty} E[\pi(X_{t_n}^n)] = \begin{cases} +\infty & \text{if } t_n = (1 - \varepsilon) \log(n) \\ 0 & \text{if } t_n = (1 + \varepsilon) \log(n) \end{cases}.
\]

If \( \lim_{n \to \infty} \frac{|x_n|^2}{n} = \beta \neq \frac{3}{2} \) then for \( \varepsilon \in (0, 1) \),
\[
\lim_{n \to \infty} |E[|X_{t_n}^n|^2] - \beta_n| = \begin{cases} +\infty & \text{if } t_n = (1 - \varepsilon) \frac{1}{2} \log(n) \\ 0 & \text{if } t_n = (1 + \varepsilon) \frac{1}{2} \log(n) \end{cases}.
\]

It is worth noting that the critical times for the transient moments (in other words the moments at some time \( t \)) of the DOU process revealed by Corollary 1.4 are universal with respect to \( \beta \).

From the first part of Theorem 1.3 and contraction properties available for some distances or divergences, see Lemma A.2, we obtain the following lower bound on the mixing time for the DOU, which is independent of \( \beta \):

**Corollary 1.5** (Lower bound on the mixing time). Let \( (X_t^n)_{t \geq 0} \) be the DOU process (1.3) with \( \beta = 0 \) or \( \beta \geq 1 \), and invariant law \( P_\beta^n \). Let \( \text{dist} \in \{ \text{Wasserstein}, \text{TV}, \text{Hellinger}, \text{Entropy} \} \). Set
\[
c_n := \begin{cases} \log \left( \frac{\pi(x_n^0)}{\sqrt{n}} \right) & \text{if dist = Wasserstein,} \\ \log(|\pi(x_n^0)|) & \text{if dist } \in \{ \text{TV}, \text{Hellinger, Entropy} \} \end{cases},
\]
and assume that \( \lim_{n \to \infty} c_n = \infty \). Then, for \( \varepsilon \in (0, 1) \), we have
\[
\lim_{n \to \infty} \text{dist}(\text{Law}(X_{t_n}^n), P_\beta^n) = \max.
\]

Theorem 1.3, Corollary 1.4, and Corollary 1.5 are proved in Section 4.

The derivation of an upper bound on the mixing time is much more delicate: once again recall that the case \( \beta = 0 \) covered by Theorem 1.2 is specific as it relies on exact Gaussian computations which are no longer available for \( \beta \geq 1 \). In the next subsection, we will obtain results for general values of \( \beta \geq 1 \) via more elaborate arguments.

In the specific cases \( \beta \in \{1, 2\} \), there are some exactly solvable aspects that one can exploit to derive, in particular, precise upper bounds on the mixing times. Indeed, for these values of \( \beta \), the DOU process is the process of eigenvalues of the matrix-valued OU process:
\[
M_0 = m_0, \quad dM_t = \sqrt{\frac{2}{n}} dB_t - M_t dt,
\]
where \( B \) is a BM on the symmetric \( n \times n \) matrices if \( \beta = 1 \) and on Hermitian \( n \times n \) matrices if \( \beta = 2 \), see (5.4) and (5.16) for more details. Based on this observation, we can deduce an upper bound on the mixing times by contraction (for some distances or divergences).

**Theorem 1.6** (Upper bound on mixing time in matrix case). Let \( (X_t^n)_{t \geq 0} \) be the DOU process (1.3) with \( \beta \in \{0, 1, 2\} \), and invariant law \( P_\beta^n \), and \( \text{dist} \in \{ \text{Wasserstein}, \text{TV}, \text{Hellinger}, \text{Entropy} \} \). Set
\[
c_n := \begin{cases} \log(|x_n^0|) & \text{if dist = Wasserstein,} \\ \log(\sqrt{n}|x_n^0|) \lor \log(\sqrt{n}) & \text{if dist } \in \{ \text{TV, Hellinger, Entropy} \} \end{cases},
\]
and assume that \( \lim_{n \to \infty} c_n = \infty \) if dist = Wasserstein. Then, for \( \varepsilon \in (0, 1) \), we have
\[
\lim_{n \to \infty} \text{dist}(\text{Law}(X_{t_n}^n), P_\beta^n) = 0.
\]

Combining this upper bound with the lower bound already obtained above, we derive a cutoff phenomenon in this particular matrix case.

**Corollary 1.7** (Cutoff for DOU in the matrix case). Let \( (X_t^n)_{t \geq 0} \) be the DOU process (1.3), with \( \beta \in \{0, 1, 2\} \), and invariant law \( P_\beta^n \). Let \( \text{dist} \in \{ \text{Wasserstein}, \text{TV}, \text{Hellinger, Entropy} \} \). Let
(a_n) be a real sequence satisfying inf_n \sqrt{n}a_n > 0, and assume further that \lim_{n \to \infty} \sqrt{n}a_n = \infty if dist = Wasserstein. Then, for all \varepsilon \in (0, 1), we have

\lim_{n \to \infty} \sup_{x_n^j \in [−a_n,a_n]^n} \text{dist}(\text{Law}(X_n^j), P_n^\beta) = \begin{cases} \max \log(\sqrt{n}a_n) & \text{if dist = Wasserstein} \\ \log(na_n) & \text{if } \varepsilon \in \{TV, Hellinger, Entropy\} \end{cases}.

Theorem 1.6 and Corollary 1.7 are proved in Section 5.

It is worth noting that d = \frac{d}{2}n^2 + (1 - \frac{d}{2})n in Theorem 1.3 is indeed an integer in the “random matrix” cases \beta \in \{1, 2\}, and corresponds then exactly to the degree of freedom of the Gaussian random matrix models GOE and GUE respectively. More precisely, if we let X_n^\infty \sim P_n^\beta then:

- If \beta = 1 then P_n^\beta is the law of the eigenvalues of S \sim GOE_n, and |X_n^\infty|^2 = \sum_{j,k=1}^n S_{jk} which is the sum of n squared Gaussians of variance v = 1/n (diagonal) plus twice the sum of \frac{n^2}{2} squared Gaussians of variance \frac{\sqrt{2}}{2} (off-diagonal) all being independent. The duplication has the effect of renormalizing the variance from \frac{\sqrt{2}}{2} to v. All in all we have the sum of d = \frac{d}{2}n^2 independent squared Gaussians of same variance v. See Section 5.

- If \beta = 2 then P_n^\beta is the law of the eigenvalues of H \sim GUE_n, and |X_n^\infty|^2 = \sum_{j,k=1}^n |H_{jk}|^2 is the sum of n squared Gaussians of variance v = 1/n (diagonal) plus twice the sum of n^2 - n squared Gaussians of variance \frac{\sqrt{2}}{2} (off-diagonal) all being independent. All in all we have the sum of d = n^2 independent squared Gaussians of same variance v. See Section 5.

Another manifestation of exact solvability lies at the level of functional inequalities. Indeed, and following [17], the optimal Poincaré constant of P_n^\beta is given by 1/n and does not depend on \beta, and the extremal functions are translations/dilations of x \mapsto \pi(x) = x_1 + \cdots + x_n, which corresponds to a spectral gap of the dynamics equal to 1 and its associated eigenfunction. Moreover, the optimal logarithmic Sobolev inequality of P_n^\beta (Lemma B.1) is given by 2/n and does not depend on \beta, and the extremal functions are of the form x \mapsto e^{c(x_1 + \cdots + x_n)}, c \in \mathbb{R}. This knowledge of the optimal constants and extremal functions and their independence with respect to \beta is truly remarkable. It plays a crucial role in the results presented in this article. More precisely, the optimal Poincaré inequality is used for the lower bound via the first eigenfunctions while the optimal logarithmic Sobolev inequality is used for the upper bound via exponential decay of the entropy.

1.5. Cutoff in the general interacting case. Our main contribution consists in deriving an upper bound on the mixing times in the general case \beta \geq 1: the proof relies on the logarithmic Sobolev inequality, some coupling arguments and a regularization procedure.

Theorem 1.8 (Upper bound on the mixing time: the general case). Let (X_t^\alpha)_{t \geq 0} be the DOU process (1.3), with \beta = 0 or \beta \geq 1 and invariant law P_n^\beta. Take dist \in \{Wasserstein, TV, Hellinger\}. Set

\epsilon_n := \begin{cases} \log(|x_0^n|) \lor \log(\sqrt{n}) & \text{if dist = Wasserstein} \\ \log(\sqrt{n}|x_0^n|) \lor \log(n) & \text{if } \varepsilon \in \{TV, Hellinger\} \end{cases}.

Then, for all \varepsilon \in (0, 1), we have

\lim_{n \to \infty} \text{dist}(\text{Law}(X_n^\alpha_{(1+\epsilon)c_n}), P_n^\beta) = 0.

Combining this upper bound with the general lower bound that we obtained in Corollary 1.5, we deduce the following cutoff phenomenon. Observe that it holds both for \beta = 0 and \beta \geq 1, and that the expression of the mixing time does not depend on \beta.

Corollary 1.9 (Cutoff for DOU in the general case). Let (X_t^\alpha)_{t \geq 0} be the DOU process (1.3) with \beta = 0 or \beta \geq 1 and invariant law P_n^\beta. Take dist \in \{Wasserstein, TV, Hellinger\}. Let (a_n) be a real sequence satisfying inf_n a_n > 0. Then, for all \varepsilon \in (0, 1), we have

\lim_{n \to \infty} \sup_{x_n^j \in [−a_n,a_n]^n} \text{dist}(\text{Law}(X_n^\alpha), P_n^\beta) = \begin{cases} \max \log(\sqrt{n}a_n) & \text{if } t_n = (1 - \varepsilon)c_n \\ \log(na_n) & \text{if } t_n = (1 + \varepsilon)c_n \end{cases}. 
where
\[ c_n := \begin{cases} 
\log(\sqrt{n}a_n) & \text{if dist = Wasserstein} \\
(na_n) & \text{if dist \in \{TV, Hellinger\}}.
\end{cases} \]

The proofs of Theorem 1.8 and Corollary 1.9 for the TV and Hellinger distances are presented in Section 6. The Wasserstein distance is treated in Section 7.

A natural, but probably quite difficult, goal would be to establish a cutoff phenomenon in the situation where the set of initial conditions is reduced to any given singleton, as in Theorem 1.2 for the case \( \beta = 0 \). Recall that in that case, the asymptotic of the mixing time is dictated by the Euclidean norm of the initial condition. In the case \( \beta \geq 1 \), this cannot be the right observable since the Euclidean norm does not measure the distance to equilibrium. Instead one should probably consider the Euclidean norm \( |x_n^0 - \rho_n| \), where \( \rho_n \) is the vector of the quantiles of order \( 1/n \) of the semi-circle law that arises in the mean-field limit equilibrium (see Subsection 2.5). More precisely
\[ \rho_{n,i} = \inf \left\{ t \in \mathbb{R} : \int_{-\infty}^{t} \frac{\sqrt{2\beta - x^2}}{\beta \pi} e^{-x \sqrt{2\beta}} dx \geq \frac{i}{n} \right\}, \quad i \in \{1, \ldots, n\}. \]  
(1.11)

Note that \( \rho_n = 0 \) when \( \beta = 0 \).

A first step in this direction is given by the following result:

**Theorem 1.10** (DOU in the general case and singleton pointwise initial condition). Let \( (X^n_t)_{t \geq 0} \) be the DOU process (1.3) with \( \beta = 0 \) or \( \beta \geq 1 \), and invariant law \( P^n_\infty \). There hold

- If \( \lim_{n \to \infty} |x_n^0 - \rho_n| = +\infty \), then, denoting \( t_n = \log(|x_n^0 - \rho_n|) \), for all \( \varepsilon \in (0,1) \),
  \[ \lim_{n \to \infty} \text{Wasserstein}(\text{Law}(X_{(1+\varepsilon)t_n}^n), P^n_\infty) = 0. \]

- If \( \lim_{n \to \infty} |x_n^0 - \rho_n| = \alpha \in [0, \infty) \), then, for all \( t > 0 \),
  \[ \lim_{n \to \infty} \text{Wasserstein}(\text{Law}(X_t), P^n_\infty)^2 \leq \alpha^2 e^{-2t}. \]

Theorem 1.10 is proved in Section 7.

### 1.6. Non-pointwise initial conditions.

It is natural to ask about the cutoff phenomenon when the initial conditions \( X_n^0 \) is not pointwise. Even if we turn off the interaction by taking \( \beta = 0 \), the law of the process at time \( t \) is then no longer Gaussian in general, which breaks the method of proof used for Theorem 1.1 and Theorem 1.2. Nevertheless, Theorem 1.11 below provides a universal answer, that is both for \( \beta = 0 \) and \( \beta \geq 1 \), at the price however of introducing several objects and notations. More precisely, for any probability measure \( \mu \) on \( \mathbb{R}^n \), we introduce

\[ S(\mu) = \left\{ \int \frac{d\mu}{dx} \log \frac{d\mu}{dx} dx = \text{"Entropy}(\mu | dx)" \quad \text{if} \quad \frac{d\mu}{dx} \log \frac{d\mu}{dx} \in L^1(dx) \right\}, \]  
(1.12)

Note that \( S \) takes its values in the whole \( (-\infty, +\infty] \), and when \( S(\mu) < +\infty \) then \( -S(\mu) \) is the Boltzmann–Shannon entropy of the law \( \mu \). For all \( x \in \mathbb{R}^n \) with \( x_i \neq x_j \) for all \( i \neq j \), we have
\[ E(x_1, \ldots, x_n) = n^2 \int \int \Phi(x, y) 1_{x \neq y} L_n(dx)L_n(dy) \]  
(1.13)

where \( L_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \) and where \( \Phi(x, y) := \frac{n}{n-1} \frac{V(x) + V(y)}{2} + \frac{\beta}{2} \log \frac{1}{|x-y|} \).

Let us define the map \( \Psi : \mathbb{R}^n \to \overline{B}_n \) by
\[ \Psi(x_1, \ldots, x_n) := (x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \]  
(1.14)

where \( \sigma \) is any permutation of \( \{1, \ldots, n\} \) that reorders the particles non-decreasingly.

**Theorem 1.11** (Cutoff for DOU with product smooth initial conditions). Let \( (X^n_t)_{t \geq 0} \) be the DOU process (1.3) with \( \beta = 0 \) or \( \beta \geq 1 \), and invariant law \( P^n_\infty \). Let \( S, \Phi, \) and \( \Psi \) be as in (1.12), (1.13), and (1.14). Let us assume that \( \text{Law}(X^n_0) \) is the image law or push forward of a product law \( \mu_1 \otimes \cdots \otimes \mu_n \) by \( \Psi \) where \( \mu_1, \ldots, \mu_n \) are laws on \( \mathbb{R} \). Then:
(1) If \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int x \mu_i(\text{d}x) \neq 0 \) then, for all \( \varepsilon \in (0, 1) \),
\[
\lim_{n \to \infty} \text{Entropy}(\text{Law}(X_{(1-\varepsilon)\log(n)}) | P^β_n) = +\infty.
\]

(2) If \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} S(\mu_i) < \infty \) and \( \lim_{n \to \infty} \frac{1}{n} \sum_{i \neq j} \int \Phi \, \text{d}\mu_i \otimes \text{d}\mu_j < \infty \), then, for all \( \varepsilon \in (0, 1) \),
\[
\lim_{n \to \infty} \text{Entropy}(\text{Law}(X_{(1+\varepsilon)\log(n)}) | P^β_n) = 0.
\]

Theorem 1.11 is proved in Section 6.3.
It is likely that Theorem 1.11 can be extended to the case \( \text{dist} \in \{ \text{Wasserstein}, \text{Hellinger}, \text{Fisher} \} \).

1.7. Structure of the paper.
- Section 2 provides additional comments and open problems.
- Section 3 focuses on the OU process (\( \beta = 0 \)) and gives the proofs of Theorems 1.1 and 1.2.
- Section 4 concerns the exact solvability of the DOU process for all \( \beta \), and provides the proofs of Theorem 1.3, and Corollaries 1.4 and 1.5.
- Section 5 is about random matrices and gives the proofs of Theorem 1.6 and Corollary 1.7.
- Section 6 deals with the DOU process for all \( \beta \) with the TV and Hellinger distances, and provides the proofs of Theorem 1.8 and Corollary 1.9.
- Section 7 gives the Wasserstein counterpart of Section 6 and the proof of Theorem 1.10.
- Appendix A provides a survey on distances and divergences, with new results.
- Appendix B gathers useful dynamical consequences of convexity.

2. Additional comments and open problems

2.1. About the results and proofs. The proofs of our results rely among other ingredients on convexity and optimal functional inequalities, exact solvability, exact Gaussian formulas, coupling arguments, stochastic calculus, variational formulas, contraction properties and regularization.

The proofs of Theorems 1.1 and 1.2 are based on the explicit Gaussian nature of the OU process, which allows to use Gaussian formulas for all the distances and divergences that we consider (the Gaussian formula for Fisher seems to be new). Our analysis of the convergence to equilibrium of the OU process seems to go much beyond what is available in the literature, see for instance \[38\].

Theorem 1.3 is a one-dimensional analogue of \[8, \text{Th. 1.2}\]. The proof exploits the explicit knowledge of eigenfunctions of the dynamics (2.6), associated with the first two non-zero spectral values, and their remarkable properties. The first one is associated to the spectral gap and the optimal Poincaré inequality. It implies Corollary 1.5, which is the provider of all our lower bounds on the mixing time for the cutoff.

The proof of Theorem 1.6 is based on a contraction property and the upper bound for matrix OU processes. It is not available beyond the matrix cases. All the other upper bounds that we establish are related to an optimal exponential decay which comes from convexity and involves sometimes coupling, the simplest instance being Theorem 1.8 about the Wasserstein distance.

The proof of Theorem 1.8 for the TV and Hellinger relies on the knowledge of the optimal exponential decay of the entropy (with respect to equilibrium) related to the optimal logarithmic Sobolev inequality. Since pointwise initial conditions have infinite entropy, the proof proceeds in three steps: first we regularize the initial condition to make its entropy finite, second we use the optimal exponential decay of the entropy of the process starting from this regularized initial condition, third we control the distance between the processes starting from the initial condition and its regularized version. This last part is inspired by a work of Lacoin \[39\] for the simple exclusion process on the segment, subsequently adapted to continuous state-spaces \[11, 12\], where one controls an area between two versions of the process.

The (optimal) exponential decay of the entropy (Lemma B.2) is equivalent to the (optimal) logarithmic Sobolev inequality (Lemma B.1). For the DOU process, the optimal logarithmic Sobolev inequality provided by Lemma B.1 achieves also the universal bound with respect to the spectral gap, just like for Gaussians. This sharpness between the best logarithmic Sobolev constant and the spectral gap also holds for instance for the random walk on the hypercube \[21\], a process for
which a cutoff phenomenon can be established with the optimal logarithmic Sobolev inequality. If we generalize the DOU process by adding an arbitrary convex function to $V$, then we will still have a logarithmic Sobolev inequality – see [17] for several proofs including the proof via the Bakry–Émery criterion – however the optimal logarithmic Sobolev constant will no longer be explicit nor sharp with respect to the spectral gap, and the spectral gap will no longer be explicit.

The proof of Theorem 1.11 relies crucially on the tensorization property of Entropy and on the asymptotics on the normalizing constant $C_n^\beta$ at equilibrium.

2.2. Analysis and geometry of the equilibrium. The full space $\mathbb{R}^n$ is, up to a bunch of hypersurfaces, covered with $n!$ disjoint isometric copies of the convex domain $D_n$ obtained by permuting the coordinates (simplices or Weyl chambers). Following [17], for all $\beta \geq 0$ let us define the law $P_n^\beta$ on $\mathbb{R}^n$ with density proportional to $e^{-E}$, just like for $P_n^0$ in (1.6) but without the $1_{\{x_1, \ldots, x_n\} \subseteq \mathbb{R}^n}$.

If $\beta = 0$ then $P_n^0 = P_n^0 = N(0, \frac{1}{n}I_n)$ according to our definition of $P_n^0$.

If $\beta > 0$ then $P_n^\beta$ has density $(C_n^\beta)^{-1}e^{-E}$ with $C_n^\beta = n!C_n^0$ where $C_n^\beta$ is the normalization of $P_n^\beta$. Moreover $P_n^\beta$ is a mixture of $n!$ isometric copies of $P_n^\beta$, while $P_n^0$ is the image law or push forward of $P_n^0$ by the map $\Psi_n : \mathbb{R}^n \rightarrow D_n$ defined in (1.14). Furthermore for all bounded measurable $f : \mathbb{R}^n \rightarrow \mathbb{R}$, denoting $\Sigma_n$ the symmetric group of permutations of $\{1, \ldots, n\}$,

$$\int f \mathrm{d}P_n^\beta = \int f_{\text{sym}} \mathrm{d}P_n^\beta \quad \text{with} \quad f_{\text{sym}}(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Regarding log-concavity, it is important to realize that if $\beta = 0$ then $E$ is convex on $\mathbb{R}^n$, while if $\beta > 0$ then $E$ is convex on $D_n$ but is not convex on $\mathbb{R}^n$ and has $n!$ isometric local minima.

- The law $P_n^\beta$ is centered but is not log-concave when $\beta > 0$ since $E$ is not convex on $\mathbb{R}^n$.
  As $\beta \rightarrow 0^+$ the law $P_n^\beta$ tends to $P_n^0 = P_n^0 = N(0, \frac{1}{n}I_n)$ which is log-concave.
- The law $P_n^\beta$ is not centered but is log-concave for all $\beta \geq 0$.
  Its density vanishes at the boundary of $D_n$ if $\beta > 0$.
  As $\beta \rightarrow 0^+$ the law $P_n^\beta$ tends to the law of the order statistics of $n$ i.i.d. $N(0, \frac{1}{n})$.

2.3. Spectral analysis of the generator: the non-interacting case. This subsection and the next deal with analytical aspects of our dynamics. We start with the OU process ($\beta = 0$) for which everything is explicit; the next subsection deals with the DOU process ($\beta \geq 1$).

The infinitesimal generator of the OU process is given by

$$G f = \frac{1}{n} \left( \Delta - \nabla E \cdot \nabla \right) - \frac{1}{n} \sum_{i=1}^n \partial_i^2 - \sum_{i=1}^n V'(x_i) \partial_i. \quad (2.1)$$

It is a self-adjoint operator on $L^2(\mathbb{R}^n, \rho_0^n)$ that leaves globally invariant the set of polynomials. Its spectrum is the set of all non-positive integers, that is, $\lambda_0 = 0 > \lambda_1 = -1 > \lambda_2 = -2 > \cdots$. The corresponding eigenspaces $F_0, F_1, F_2, \cdots$ are finite dimensional: $F_m$ is spanned by the multivariate Hermite polynomials of degree $m$, in other words tensor products of univariate Hermite polynomials. In particular, $F_0$ is the vector space of constant functions while $F_1$ is the $n$-dimensional vector space of all linear functions.

Let us point out that $G$ can be restricted to the set of $\rho_0^n$ square integrable symmetric functions: it leaves globally invariant the set of symmetric polynomials, its spectrum is unchanged but its associated eigenspaces $E_m$ are the restrictions of the vector spaces $F_m$ to the set of symmetric functions, in other words, $E_m$ is spanned by the multivariate symmetrized Hermite polynomials of degree $m$. Note that $E_1$ is the one-dimensional space generated by $\pi(x) = x_1 + \cdots + x_n$.

The Markov semigroup $(e^{tG})_{t \geq 0}$ generated by $G$ admits $P_0^n$ as a reversible invariant law since $G$ is self-adjoint in $L^2(\mathbb{R}^n)$. Following [52], let us introduce the heat kernel $p_t(x,y)$ which is the density of $\text{Law}(X_n^t | X_n^0 = x)$ with respect to the invariant law $\rho_0^n$. The long-time behavior reads

$$\lim_{t \rightarrow \infty} p_t(x,.) = 1 \quad \text{for all} \quad x \in \mathbb{R}^n. \quad \text{Let} \quad \|f\|_p \quad \text{be the norm of} \quad L^p = L^p(\rho_0^n). \quad \text{For all} \quad 1 \leq p \leq q, \quad t \geq 0, \quad x \in \mathbb{R}^n, \quad \text{we have}$$

$$2\|\text{Law}(X_n^t | X_n^0 = x) - P_0^n\|_{TV} = \|p_t(x,\cdot) - 1\|_1 \leq \|p_t(x,\cdot) - 1\|_p \leq \|p_t(x,\cdot) - 1\|_q. \quad (2.2)$$
In the particular case $p = 2$ we can write
\[
\|p_t(x, \cdot) - 1\|^2_2 = \sum_{m=1}^{\infty} e^{-2mt} \sum_{\psi \in B_m} |\psi(x)|^2.
\]  
(2.3)
where $B_m$ is an orthonormal basis of $F_m \subset L^2(P^n_0)$, hence
\[
\|p_t(x, \cdot) - 1\|^2_2 \geq e^{-2t} \sum_{\psi \in B_1} |\psi(x)|^2,
\]  
(2.4)
which leads to a lower bound on the $L^2$ cutoff, provided one can estimate $\sum_{\psi \in B_1} |\psi(x)|^2$ which is the square of the norm of the projection of $\delta_x$ on $B_1$.

Following [52, Th. 6.2], an upper bound would follow from a Bakry–Émery curvature–dimension criterion $\text{CD}(\rho, d)$ with a finite dimension $d$, in relation with Nash–Sobolev inequalities and dimensional pointwise estimates on the heat kernel $p_t(x, \cdot)$ or ultracontractivity of the Markov semigroup. The OU process satisfies to $\text{CD}(\rho, \infty)$ but never to $\text{CD}(\rho, d)$ with $d$ finite and is not ultracontractive. Actually the OU process is a critical case, see [3, Ex. 2.7.3].

2.4. Spectral analysis of the generator: the interacting case. We now assume that $\beta \geq 1$. The infinitesimal generator of the DOU process is the operator
\[
Gf = \frac{1}{n} \left( \Delta - \nabla E \cdot \nabla \right) = \frac{1}{n} \sum_{i=1}^{\infty} \partial_i^2 - \sum_{i=1}^{\infty} V'(x_i) \partial_i + \frac{\beta}{2n} \sum_{i \neq j} \partial_i - \partial_j.
\]  
(2.5)
Despite the interaction term, the operator leaves globally invariant the set of symmetric polynomials. Following Lassalle in [40, 4], see also [17], the operator $G$ is a self-adjoint operator on the space of $P^\beta_n$ square integrable symmetric functions of $n$ variables, its spectrum does not depend on $\beta$ and matches the spectrum of the OU process case $\beta = 0$. In particular the spectral gap is 1. The eigenspaces $E_m$ are spanned by the generalized symmetrized Hermite polynomials of degree $m$. For instance, $E_1$ is the one-dimensional space generated by $\pi(x) = x_1 + \cdots + x_n$ while $E_2$ is the two-dimensional space spanned by
\[
(x_1 + \cdots + x_n)^2 - 1 = x_2^2 + \cdots + x_n^2 - 1 - \frac{\beta}{2} (n-1).
\]  
(2.6)
From the isometry between $L^2(\mathcal{D}_n, P^\beta_n)$ and $L^2_{\text{sym}}(\mathbb{R}^n, P^\beta_n)$, the above explicit spectral decomposition applies to the semigroup of the DOU on $\mathcal{D}_n$. Formally, the discussion presented at the end of the previous subsection still applies. However, in the present interacting case the integrability properties of the heat kernel are not known: in particular, we do not know whether $p_t(x, \cdot)$ lies in $L^p(P^\beta_n)$ for $t > 0$, $x \in \mathcal{D}_n$ and $p > 1$. This leads to the question, of independent interest, of pointwise upper and lower Gaussian bounds for heat kernels similar to the OU process, with explicit dependence of the constants over the dimension. We refer for example to [54, 27, 32] for some results in this direction.

2.5. Mean-field limit. The measure $P^\beta_n$ is log-concave since $E$ is convex, and its density writes
\[
x \in \mathbb{R}^n \mapsto \frac{e^{-2|x|^2}}{C_n} \prod_{i > j} (x_i - x_j)^\beta 1_{x_1 \leq \cdots \leq x_n}.
\]  
(2.7)
See [17, Sec. 2.2] for a high-dimensional analysis. The Boltzmann–Gibbs measure $P^\beta_n$ is known as the $\beta$-Hermite ensemble or $\beta HE$. When $\beta = 2$, it is better known as the Gaussian Unitary Ensemble (GUE). If $X^n \sim P^\beta_n$ then the Wigner theorem states that the empirical measure with atoms distributed according to $P^\beta_n$ converges in distribution to a semi-circle law, namely
\[
\frac{1}{n} \sum_{i=1}^{n} \delta_{X^{n,i}} \stackrel{\text{weak}}{\to}_{n \to \infty} \frac{\sqrt{2\beta - x^2}}{\beta \pi} \mathbf{1}_{x \in [-\sqrt{2\beta}, \sqrt{2\beta}]} dx,
\]  
(2.8)
and this can be deduced in this Coulomb gas context from a large deviation principle as in [6].
Let \((X^n)_{t \geq 0}\) be the process solving (1.3) with \(\beta \geq 0\) or \(\beta \geq 1\), and let
\[
\mu^n_t = \frac{1}{n} \sum_{k=1}^{n} \delta_{X^n_{t,k}},
\]  
be the empirical measure of the particles at time \(t\). Following notably [50, 7, 14, 13, 44, 23], if the sequence of initial conditions \((\mu^n_0)_{n \geq 1}\) converges weakly as \(n \to \infty\) to a probability measure \(\mu_0\), then the sequence of measure valued processes \((\mu^n_{t \geq 0})_{n \geq 1}\) converges weakly to the unique probability measure valued deterministic process \((\mu_t)_{t \geq 0}\) satisfying the evolution equation
\[
\langle \mu_t, f \rangle = \langle \mu_0, f \rangle - \int_0^t \int \left( V'(x) f'(x) \mu_s(dx) \right) ds + \frac{\beta}{2} \int_0^t \int \frac{f'(x) - f'(y)}{x - y} \mu_s(dx) \mu_s(dy) ds
\]  
for all \(t \geq 0\) and \(f \in C^1_b(\mathbb{R}, \mathbb{R})\). The equation (2.10) is a weak formulation of a McKean–Vlasov equation or free Fokker–Planck equation associated to a free OU process. Moreover, if \(\mu_0\) has all its moments finite, then for all \(t \geq 0\), we have the free Mehler formula
\[
\mu_t = \text{dil}_{-2t} \mu_0 \oplus \text{dil}_{-e^{-2t}} \mu_\infty,
\]  
where \text{dil}_t \mu\) is the law of \(\sigma X\) when \(X \sim \mu\), where "\(\oplus\)" stands for the free convolution of probability measures of Voiculescu free probability theory, and where \(\mu_\infty\) is the semi-circle law of variance \(\frac{2}{\beta}\). In particular, if \(\mu_0\) is a semi-circle law then \(\mu_t\) is a semi-circle law for all \(t \geq 0\).

The second moment \(m_t = \int x^2 \mu_t(dx)\) of \(\mu_t\) satisfies the ODE
\[
m_t = m_0 e^{-2t} + \frac{\beta}{2} \left(1 - e^{-2t}\right) \quad \text{lim} \quad t \to \infty \frac{\beta}{2},
\]  
More generally, beyond the second moment, the Cauchy–Stieltjes transform
\[
z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\} \mapsto s_t(z) = \int_{\mathbb{R}} \frac{\mu_t(dx)}{x - z},
\]  
of \(\mu_t\) is the solution of the following complex Burgers equation
\[
\partial_z s_t(z) = s_t(z) + z \partial_x s_t(z) + \beta s_t(z) \partial_x s_t(z), \quad t \geq 0, z \in \mathbb{C}_+.
\]  
The semi-circle law on \([-c, c]\) has density \(\frac{2 \sqrt{c} - x^2}{2\pi c} 1_{x \in [-c, c]}\), mean 0, second moment or variance \(\frac{2}{\beta}\), and Cauchy–Stieltjes transform \(s_t(z) = \frac{\sqrt{4z^2 - 4\beta^2}}{z^2 - 2}, t \geq 0, z \in \mathbb{C}_+\).

The cutoff phenomenon is in a sense a diagonal \((t, n)\) estimate, melting long time behavior and high dimension. When \(|z^n_0|\) is of order \(n\), cutoff occurs at a time of order \(\approx \log(n)\): this informally corresponds to taking \(t \to \infty\) in \((\mu_t)_{t \geq 0}\).

When \(\mu_0\) is centered with same second moment \(\frac{2}{\beta}\) as \(\mu_\infty\), then there is a Boltzmann H-theorem interpretation of the limiting dynamics as \(n \to \infty\): the steady-state is the Wigner semi-circle law \(\mu_\infty\), the second moment is conserved by the dynamics, the Voiculescu entropy is monotonic along the dynamics, exponentially, and is maximized by the steady-state.

2.6. \(L^p\) cutoff. Following [18], we can deduce an \(L^p\) cutoff started from \(x\) from an \(L^1\) cutoff by showing that the heat kernel \(p_t(x, \cdot)\) is in \(L^p(P^n_x)\) for some \(t > 0\). Thanks to the Mehler formula, it can be checked that this holds for the OU case, despite the lack of ultracontractivity. The heat kernel of the DOU process is less accessible.

In another exactly solvable direction, the \(L^p\) cutoff phenomenon has been studied for instance in [52, 53] for Brownian motion on compact simple Lie groups, and in [53, 46] for Brownian motion on symmetric spaces, in relation with representation theory, an idea which goes back at the origin to the works of Diaconis on random walks on groups.

2.7. Other potentials. It is natural to ask about the cutoff phenomenon for the process solving (1.3) when \(V\) is a more general \(C^2\) function. The invariant law \(P^n_\beta\) of this Markov diffusion writes
\[
\frac{e^{-n \sum_{i=1}^n V(x_i)}}{C^n_\beta} \prod_{i \neq j} (x_i - x_j) \beta 1_{(x_1, \ldots, x_n) \in \mathbb{R}^n} dx_1 \cdots dx_n.
\]
The case where $V - \frac{\rho}{2} |\cdot|^2$ is convex for some constant $\rho \geq 0$ generalizes the DOU case and has exponential convergence to equilibrium, see [17]. Three exactly solvable cases are known:

- $e^{-V(\cdot)} = e^{-\frac{\rho}{2} |\cdot|^2}$: the DOU process associated to the Gaussian law weight and the $\beta$-Hermite ensemble including HOE/HUE/HSE when $\beta \in \{1, 2, 4\}$,
- $e^{-V(\cdot)} = x^{a-1} e^{-x} 1_{x \in [0, \infty)}$: the Dyson–Laguerre process associated to the Gamma law weight and the $\beta$-Laguerre ensembles including LOE/LUE/LSE when $\beta \in \{1, 2, 4\}$,
- $e^{-V(\cdot)} = x^{a-1} (1 - x)^{b-1} 1_{x \in [0, 1]}$: the Dyson–Jacobi process associated to the Beta law weight and the $\beta$-Jacobi ensembles including JOE/JUE/JSE when $\beta \in \{1, 2, 4\}$,

up to a scaling. Following Lassalle [40, 42, 41, 4] and Bakry [5], in these three cases, the multivariate orthogonal polynomials of the invariant law $P_n^\beta$ are the eigenfunctions of the dynamics of the process. We refer to [26, 24, 45] for more information on $(H/L/J)\beta E$ random matrix models.

The contraction property or spectral projection used to pass from a matrix process to the Dyson process can be used to pass from BM on the unitary group to the Dyson circular process for which the invariant law is the Circular Unitary Ensemble (CUE). This provides an upper bound for the cutoff phenomenon. The cutoff for BM on the unitary group is known and holds at critical time $\log(n)$, see for instance [53, 52, 46].

More generally, we could ask about the cutoff phenomenon for a McKean–Vlasov type interacting particle system $(X^n_t)_{t \geq 0}$ in $(\mathbb{R}^d)^n$ solution of the stochastic differential equation of the form

$$
dX^n_t = \sigma_{n,t}(X^n_t)dB_t^n - \sum_{i=1}^n \nabla V_{n,t}(X^n_t)dt - \sum_{j \neq i}^n \nabla W_{n,t}(X^n_i - X^n_j)dt, \quad 1 \leq i \leq n, \quad (2.16)$$

for various types of confinement $V$ and interaction $W$ (convex, repulsive, attractive, repulsive-attractive, etc), and discuss the relation with the propagation of chaos. The case where $V$ and $W$ are both convex and constant in time is already very well studied from the point of view of long-time behavior and mean-field limit in relation with convexity, see for instance [14, 13, 44].

Regarding universality, it is worth noting that if $V = |\cdot|^2$ and if $W$ is convex then the proof by factorization of the optimal Poincaré and logarithmic Sobolev inequalities and their extremal functions given in [17] remains valid, paving the way to the generalization of many of our results in this spirit. On the other hand, the convexity of the limiting energy functional in the mean-field limit is of Bochner type and suggests to take for $W$ a power, in other words a Riesz type interaction.

2.8. Alternative parametrization. If $(X^n_t)_{t \geq 0}$ is the process solution of the stochastic differential equation (1.3), then for all real parameters $\alpha > 0$ and $\sigma > 0$, the space scaled and time changed stochastic process $(Y^n_t)_{t \geq 0} = (\sigma X^n_{\alpha t})_{t \geq 0}$ solves the stochastic differential equation

$$
Y^n_0 = \sigma x^n_0, \quad dY^n_t = \sqrt{\frac{2\alpha \sigma^2}{n}} dB^n_t - \alpha Y^n_{n,t} dt + \frac{\alpha \beta \sigma^2}{n} \sum_{j \neq i}^n \frac{dt}{Y^n_t - Y^n_{n,j}}, \quad 1 \leq i \leq n, \quad (2.17)
$$

where $(B_t)_{t \geq 0}$ is a standard $n$-dimensional BM. The invariant law of $(Y^n_t)_{t \geq 0}$ is

$$
e^{-\frac{n|y|^2}{C_n^\beta}} \prod_{i > j} (y_i - y_j)^{\beta} 1_{(y_1, \ldots, y_n) \in \mathcal{D}_n} dy_1 \cdots dy_n \quad (2.18)$$

where $C_n^\beta$ is the normalizing constant. This law and its normalization $C_n^\beta$ depend on the “shape parameter” $\beta$, the “scale parameter” $\sigma$, and does not depend on the “speed parameter” $\alpha$. When $\beta > 0$, taking $\sigma^2 = \beta^{-1}$, the stochastic differential equation (2.17) boils down to

$$
Y^n_0 = \frac{x^n_0}{\sqrt{\beta}}, \quad dY^n_t = \sqrt{\frac{2\alpha}{n \beta}} dB^n_t - \alpha Y^n_{n,t} dt + \frac{\alpha}{n} \sum_{j \neq i}^n \frac{dt}{Y^n_t - Y^n_{n,j}}, \quad 1 \leq i \leq n \quad (2.19)
$$

while the invariant law becomes

$$
e^{-\frac{n|y|^2}{C_n^\beta}} \prod_{i > j} (y_i - y_j)^{\beta} 1_{(y_1, \ldots, y_n) \in \mathcal{D}_n} dy_1 \cdots dy_n \quad (2.20)$$

The equation (2.19) is the one considered in [28, Eq. (12.4)] and in [37, Eq. (1.1)]. The advantage of (2.19) is that $\beta$ can be now truly interpreted as an inverse temperature and the right-hand side
in the analogue of (2.8) does not depend on $\beta$, while the drawback is that we cannot turn off the interaction by setting $\beta = 0$ and recover the OU process as in (1.3). It is worthwhile mentioning that for instance Theorem 1.8 remains the same for the process solving (2.19) in particular the cutoff threshold is at critical time $\frac{c}{n}$ and does not depend on $\beta$.

2.9. Discrete models. There are several discrete space Markov processes admitting the OU process as a scaling limit, such as for instance the random walk on the discrete hypercube, related to the Ehrenfest model, for which the cutoff has been studied in [22], and the M/M/$\infty$ queuing process, for which a discrete Mehler formula is available [16]. Certain discrete space Markov processes incorporate a singular repulsion mechanism, such as for instance the exclusion process on the segment, for which the study of the cutoff in [39] shares similarities with our proof of Theorem 1.8. It is worthwhile noting that there are discrete Coulomb gases, related to orthogonal polynomials for discrete measures, suggesting to study discrete Dyson processes. More generally, it could be natural to study the cutoff phenomenon for Markov processes on infinite discrete state spaces, under curvature condition, even if the subject is notoriously disappointing in terms of high-dimensional analysis. We refer to the recent work [51] for the finite state space case.

3. Cutoff phenomenon for the OU

In this section, we prove Theorems 1.1 and 1.2: actually we only prove the latter since it implies the former. We start by recalling a well-known fact.

Lemma 3.1 (Mehler formula). If $(Y_t)_{t \geq 0}$ is an OU process in $\mathbb{R}^d$ solution of the stochastic differential equation $Y_0 = y_0 \in \mathbb{R}^d$ and $dY_t = \sigma dB_t - \mu Y_t dt$ for parameters $\sigma > 0$ and $\mu > 0$ where $B$ is a standard $d$-dimensional Brownian motion then

$$(Y_t)_{t \geq 0} = (y_0 e^{-\mu t} + \sigma \int_0^t e^{\mu(s-t)}dB_s)_{t \geq 0} \text{ hence } Y_t \sim \mathcal{N}(y_0 e^{-\mu t}, \frac{\sigma^2}{2} e^{-2\mu t} I_d) \text{ for all } t \geq 0.$$

Moreover its coordinates are independent one-dimensional OU processes with initial condition $y_0$ and invariant law $\mathcal{N}(y_0 e^{-\mu t}, \frac{\sigma^2}{2} e^{-2\mu t} I_1), 1 \leq i \leq d.$

Proof of Theorem 1.1 and Theorem 1.2. By using Lemma 3.1, for all $n \geq 1$ and $t \geq 0$,

$$Z_t^n \sim \mathcal{N}(z_0^n e^{-t}, \frac{1}{n} I_n) = \otimes_{i=1}^n \mathcal{N}(z_0^n e^{-t}, \frac{1}{n} e^{-2t}), P_0^n = \mathcal{N}(0, I_n) = \mathcal{N}(0, \frac{1}{n}) \otimes_n.$$

$L^2$ cutoff. Denote by $p_t$ the density of $Z_t^n$ with respect to $P_0^n$. We have

$$\|p_t - 1\|_{L^2(P_0^n)}^2 = \int p_t(z)^2 P_0^n(z)dz = 1.$$

Using Lemma 3.1 for all $z \in \mathbb{R}^n$ we have

$$p_t(z)^2 P_0^n(z) = \left(\frac{1}{(1 - \frac{e^{-2t}}{\sqrt{2\pi/n}})^n}\right)^n \exp\left(-\frac{n}{2(1 - \frac{e^{-2t}}{\sqrt{2\pi/n}})}|z|^2(1 + e^{-2t}) - 4z \cdot z_0^n e^{-t} + 2|z_0^n|^2 e^{-2t}\right).$$

Then straightforward computations yield

$$\|p_t - 1\|_{L^2(P_0^n)}^2 = \frac{1}{(1 - \frac{e^{-2t}}{\sqrt{2\pi/n}})^n} \exp\left(n|z_0^n|^2 \frac{e^{-2t}}{1 + e^{-2t}}\right) = 1.$$

It follows immediately that this quantity goes to $+\infty$, respectively 0, if $t$ is much smaller, respectively much larger, than $\log(n^{1/2}|z_0^n|) \lor \log(n^{1/4})$.

Hellinger, Entropy, Fisher, and Wasserstein cutoffs. By combining Lemma A.5 and Lemma 3.1 we get, either from multivariate Gaussian formulas or univariate via tensorization,

$$\text{Hellinger}^2(\text{Law}(Z_t^n), P_0^n) = 1 - \exp\left(-\frac{n}{4} \frac{|z_0^n|^2}{2} e^{-2t} + \frac{n}{4} \log\left(\frac{4 - e^{-2t}}{2 - e^{-2t}}\right)\right),$$

$$2\text{Entropy}(\text{Law}(Z_t^n) | P_0^n) = n|z_0^n|^2 e^{-2t} - n e^{-2t} - n \log(1 - e^{-2t}),$$

$$\text{Fisher}(\text{Law}(Z_t^n) | P_0^n) = n^2 |z_0^n|^2 e^{-2t} + \frac{n^2}{2} \frac{e^{-4t}}{1 - e^{-2t}},$$

$$\text{Wasserstein}^2(\text{Law}(Z_t^n), P_0^n) = |z_0^n|^2 e^{-2t} + 2 - e^{-2t} - 2\sqrt{1 - e^{-2t}}.$$
which give the desired lower and upper bounds as before by using the hypothesis on $z_0^t$.

Total variation cutoff. By using the comparison between total variation and Hellinger distances (Lemma A.1) we deduce from (3.3) the cutoff in total variation distance at the same critical time. The upper bound for the total variation distance can alternatively be obtained by using the Entropy estimate (3.4) and the Pinsker–Csiszár–Kullback inequality (Lemma A.1). Since both distributions are tensor products, we could use alternatively the tensorization property of the total variation distance (Lemma A.4) together with the one-dimensional version of the Gaussian formula for Entropy (Lemma A.1) to obtain the result for the total variation.

\[ A_t := \text{dist}(\text{Law}(Z_t^n), \text{Law}(Z_t^n - z_0^n e^{-t})) \]

has a cutoff at time $c^A_n = \log(\sqrt{n} |z_0^n|)$, while

\[ B_t := \text{dist}(\text{Law}(Z_t^n - z_0^n e^{-t}), P_0^n) \]

admits a cutoff at time $c^B_n = \frac{1}{4} \log(n)$. The triangle inequality for dist yields

\[ |A_t - B_t| \leq \text{dist}(\text{Law}(Z_t^n), P_0^n) \leq A_t + B_t. \]

Therefore the critical time of Theorem 1.2 is dictated by either $A_t$ or $B_t$, according to whether $c^A_n > c^B_n$ or $c^A_n < c^B_n$.

Remark 3.2 (On the critical time). From the computations of the proof of Theorem 1.2, we can show that for dist $\in \{\text{TV}, \text{Hellinger}, L^2\}$

\[ A_t := \text{dist}(\text{Law}(Z_t^n), \text{Law}(Z_t^n - z_0^n e^{-t})) \]

admits a cutoff at time $c^A_n = \log(\sqrt{n} |z_0^n|)$, while

\[ B_t := \text{dist}(\text{Law}(Z_t^n - z_0^n e^{-t}), P_0^n) \]

admits a cutoff at time $c^B_n = \frac{1}{4} \log(n)$. The triangle inequality for dist yields

\[ |A_t - B_t| \leq \text{dist}(\text{Law}(Z_t^n), P_0^n) \leq A_t + B_t. \]

Therefore the critical time of Theorem 1.2 is dictated by either $A_t$ or $B_t$, according to whether $c^A_n > c^B_n$ or $c^A_n < c^B_n$.

Remark 3.3 (Total variation discriminating event for small initial conditions). Let us introduce the random variable $Z^n_\infty \sim P_0^n = \mathcal{N}(0, \frac{1}{n} I_n) = \mathcal{N}(0, \frac{1}{n}) \otimes n$, in accordance with (3.1). There holds

\[ S^n_t := \sum_{i=1}^n (Z^n_t - z_0^n e^{-t})^2 \sim \text{Gamma} \left( \frac{n}{2}, \frac{n}{2(1 - e^{-2t})} \right) \]

and

\[ |Z^n_\infty|^2 \sim \text{Gamma} \left( \frac{n}{2}, \frac{n}{2} \right). \]

We can check, using an explicit computation of Hellinger and Entropy between Gamma distributions and the comparison between total variation and Hellinger distances (Lemma A.1), that

\[ C_t := \text{dist}(\text{Law}(S^n_t), \text{Law}(|Z^n_\infty|^2)) \]

admits a cutoff at time $c^C_n = c^B_n = 4 \log(n)$. Moreover, one can exhibit a discriminating event for the TV distance. Namely, we can observe that

\[ \| \text{Gamma} \left( \frac{n}{2}, \frac{1}{2(1 - e^{-2t})} \right) - \text{Gamma} \left( \frac{n}{2}, \frac{n}{2} \right) \|_{\text{TV}} = \mathbb{P}(|Z^n_\infty|^2 \geq \alpha_t) - \mathbb{P}(S^n_t \geq \alpha_t) \]

with $\alpha_t$ the unique point where the two densities meet, which happens to be

\[ \alpha_t = -e^{2t} \log(1 - e^{-2t})(1 - e^{-2t}). \]

4. General exactly solvable aspects

In this section, we prove Theorem 1.3, Corollary 1.4 and Corollary 1.5.

The proof of Theorem 1.3 is based on the fact that the polynomial functions $\pi(x) = x_1 + \cdots + x_n$ and $|x|^2 = x_1^2 + \cdots + x_n^2$ are, up to an additive constant for the second, eigenfunctions of the dynamics associated to the spectral values $-1$ and $-2$ respectively, and that their "carré du champ" is affine. In the matrix cases $\beta \in \{1, 2\}$, these functions correspond to the dynamics of the trace, the dynamics of the squared Hilbert–Schmidt trace norm, and the dynamics of the squared trace. It is remarkable that this phenomenon survives beyond these matrix cases, yet another manifestation of the Gaussian "ghosts" concept due to Edelman.

Proof of Theorem 1.3. The process $Y_t := \pi(X^n_t)$ solves

\[ dY_t = \sum_{i=1}^n dX^{n,i}_t = \sqrt{\frac{2}{n}} \sum_{i=1}^n dB_i - \sum_{i=1}^n X^{n,i}_t dt + \frac{\beta}{n} \sum_{j \neq i} \frac{dt}{X^{n,i}_t - X^{n,j}_t}. \]
By symmetry, the double sum vanishes. Note that the process \( W_t := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} B^i_t \) is a standard one dimensional BM, so that \( dW_t = \sqrt{2} dB_t - Y_t dt \). This proves the first part of the statement. We turn to the second part. Recall that \( X_t \in D_n \) for all \( t > 0 \). By Itô’s formula
\[
\frac{d}{dt} \left( X_t^{n,i} \right)^2 = \frac{8}{n} X_t^{n,i} dB^i_t - 2 X_t^{n,i} dt + 2 \beta \frac{n}{n} \sum_{j \neq i} X_t^{n,j} \frac{d \lambda}{X_t^{n,i} - X_t^{n,j}} + \frac{2}{n} dt.
\]
Set \( W_t := \sum_{i=1}^{n} \int_0^t \frac{X_s^{n,i}}{|X_n|^2} dB^i_s \). The process \( (W_t)_{t \geq 0} \) is a BM by the Lévy characterization since
\[
\langle W \rangle_t = \int_0^t \sum_{i=1}^{n} \frac{(X_s^{n,i})^2}{|X_n|^2} ds = t.
\]
Furthermore, a simple computation shows that
\[
\sum_{i=1}^{n} X_t^{n,i} \sum_{j \neq i} \frac{1}{X_t^{n,i} - X_t^{n,j}} = \frac{n(n-1)}{2}.
\]
Consequently the process \( R_t := |X_t|^2 \) solves
\[
dR_t = \frac{8}{n} R_t dW_t + \left( 2 + \beta(n-1) - 2 R_t \right) dt,
\]
and is therefore a CIR process of parameters \( \alpha = 2 + \beta(n-1), \beta = 2, \) and \( \sigma = \sqrt{8/n} \).
When \( d = \frac{5}{2} n^2 + (1 - \frac{2}{9})n \) is a positive integer, the last property of the statement follows from the connection between OU and CIR recalled right before the statement of the theorem. □

The last proof actually relies on the following general observation. Let \( X \) be an \( n \)-dimensional continuous semi-martingale solution of
\[
dX_t = \sigma(X_t) dB_t + b(X_t) dt
\]
where \( B \) is a \( n \)-dimensional standard BM, and where
\[
x \in \mathbb{R}^n \mapsto \sigma(x) \in M_{n,n}(\mathbb{R}) \quad \text{and} \quad x \in \mathbb{R}^n \mapsto b(x) \in \mathbb{R}^n
\]
are Lipschitz. The infinitesimal generator of the Markov semigroup is given by
\[
G(f)(x) = \frac{1}{2} \sum_{i,j=1}^{n} a_{i,j}(x) \partial_{i,j} f(x) + \sum_{i=1}^{n} b_i(x) \partial_i f(x), \quad \text{where} \quad a(x) = \sigma(x)^{\top} \sigma(x),
\]
for all \( f \in C^2(\mathbb{R}^n, \mathbb{R}) \) and \( x \in \mathbb{R}^n \). Then, by Itô’s formula, the process \( M^f := (M^f_t)_{t \geq 0} \) given by
\[
M^f_t = f(X_t) - f(X_0) - \int_0^t (G f)(X_s) ds = \sum_{i,k=1}^{n} \int_0^t \partial_i f(X_s) \sigma_{i,k}(X_s) dB^k_s
\]
is a local martingale, and moreover, for all \( t \geq 0 \),
\[
\langle M^f \rangle_t = \int_0^t \Gamma(f)(X_s) ds \quad \text{where} \quad \Gamma(f)(x) = |\sigma(x)|^2 \nabla f(x) = a(x) \nabla f \cdot \nabla f.
\]
The functional quadratic form \( \Gamma \) is known as the “carré du champ” operator. If \( f \) is an eigenfunction of \( G \) associated to the spectral value \( \lambda \) in the sense that \( G f = \lambda f \) (note by the way that \( \lambda \leq 0 \) since \( G \) generates a Markov process), then we get
\[
f(X_t) = f(X_0) + \lambda \int_0^t f(X_s) ds + M^f_t, \quad \text{in other words} \quad df(X_t) = dM^f_t + \lambda f(X_t) dt.
\]
Now if \( \Gamma(f) = c \) (as in the first part of the theorem), then by the Lévy characterization of Brownian motion, the continuous local martingale \( W := \frac{1}{c} M^f \) starting from the origin is a standard BM and we recover the result of the first part of the theorem. On the other hand, if \( \Gamma(f) = cf \) (as in the second part of the theorem), then by the Lévy characterization of BM the local martingale
\[
W := \int_0^t \frac{1}{\sqrt{cf(X_s)}} dM^f_s
\]
is a standard BM and we recover the result of the second part.

At this point, we observe that the infinitesimal generator of the CIR process $R$ is the Laguerre partial differential operator
\[ L(f)(x) = \frac{4}{n} x f''(x) + (2 + \beta(n-1) - 2x)f'(x). \] (4.1)

This operator leaves invariant the set of polynomials of degree less than or equal to $k$, for all integer $k \geq 0$, a property inherited from (2.5). We will use this property in the following proof.

**Proof of Corollary 1.4.** The first item comes immediately from the formula $\mathbb{E}[\pi(X^n_t)] = \pi(x^n_0)e^{-t}$ provided by Theorem 1.3. For the second item, we first observe that by Theorem 1.3,
\[ \int |x|^2 P^n_{\beta}(dx) = \lim_{t \to \infty} \mathbb{E}[|X^n_t|^2] = 1 + \frac{\beta}{2}(n-1). \]

Alternatively, and following [17, Sec. 2.2], this formula can be deduced from the Dumitriu–Edelman tridiagonal random matrix model [24] isospectral to $\beta$-Hermite. Applying the Laguerre operator to $f(x) = x$ we deduce that the moment $m_1(t) = \mathbb{E}[R_t]$ solves the ordinary differential equation $m_1'(t) = 2 + \beta(n-1) - 2m_1(t)$ with initial condition $m(0) = |x^n_0|^2$, hence
\[ \mathbb{E}[|X^n_t|^2] = \left(1 + \frac{\beta}{2}(n-1)\right) + \left(|x^n_0|^2 - 1 - \frac{\beta}{2}(n-1)\right)e^{-2t}. \]

Provided $\frac{|x^n_0|^2}{n} \to \alpha \neq \frac{\beta}{2}$, we deduce the cutoff of $t \mapsto \mathbb{E}[|X^n_t|^2] - \beta_n$ with critical time $\frac{1}{2} \log(n)$. \qed

**Remark 4.1** (Higher-order moments). The higher-order moments can be computed. Let us show how it works for the second moment, which is some sort of fourth moment of the DOU process. We already know from Theorem 1.3 that
\[ m_1(t) = \mathbb{E}[|X^n_t|^2] = m_1(0)e^{-2t} + \beta_n(1 - e^{-2t}) \quad \text{where} \quad \beta_n = 1 + \frac{\beta}{2}(n-1). \] (4.2)

Setting now $m_2(t) = \mathbb{E}[|X^n_t|^4]$, we get by taking $f(x) = x^2$ in $Lf(x)$ the ordinary differential equation $m_2'(t) = \frac{2}{n} m_1'(t) + 4\beta_n m_1(t) - 4m_2(t)$, which gives, using the formula for $m_1(t)$,
\[ m_2(t) + 4m_2(t) = 4\left(\beta_n + \frac{2}{n}\right)\left(m_1(0)e^{-2t} + \beta_n(1 - e^{-2t})\right). \] (4.3)

The method of variation of parameter gives then
\[ m_2(t) = e^{-4t}m_2(0) + (1 - e^{-2t})\left(\beta_n + \frac{2}{n}\right)\left(2e^{-2t}m_1(0) + \beta_n(1 - e^{-2t})\right). \] (4.4)

In particular $\lim_{t \to \infty} m_2(t) = (\beta_n + \frac{2}{n})\beta_n$ and
\[ m_2(t) - \left(\beta_n + \frac{2}{n}\right)\beta_n = e^{-4t}\left(m_2(0) - \left(\beta_n + \frac{2}{n}\right)\beta_n\right) + 2e^{-2t}(1 - e^{-2t})\left(\beta_n + \frac{2}{n}\right)(m_1(0) - \beta_n). \] (4.5)

Now assume that $\lim_{n \to \infty} \frac{|x^n_0|^2}{n} = \alpha \neq \frac{\beta}{2}$. Since $m_1(0) = |x^n_0|^2$, $m_2(0) = |x^n_0|^4$ and $\lim_{n \to \infty} \frac{\beta_n}{n} = \beta$, we obtain the following “cutoff” phenomenon for $m_2$: for all $\epsilon \in (0, 1)$,
\[ \lim_{t \to \infty} \left|m_2(t)\right| = \left(\beta_n + \frac{2}{n}\right)\beta_n = \begin{cases} +\infty & \text{if } t_n = (1 - \epsilon) \log(n) \\ 0 & \text{if } t_n = (1 + \epsilon) \log(n) \end{cases} . \] (4.6)

### 4.1. Proof of Corollary 1.5.

By Theorem 1.3, $Z = \pi(X^n)$ is an OU process in $\mathbb{R}$ solution of the stochastic differential equation
\[ \dot{Z}_t = \sqrt{2d} dB_t - Z_t dt, \]
where $B$ is a standard one-dimensional BM. By Lemma 3.1, $Z_t \sim \mathcal{N}(Z_0e^{-t}, 1 - e^{-2t})$ for all $t \geq 0$ and the equilibrium distribution is $P^n_{\beta} \circ \pi^{-1} \sim \mathcal{N}(0, 1)$. Using the contraction property stated in Lemma A.2, the comparison between Hellinger and TV of Lemma A.1 and the explicit expressions for Gaussian distributions of Lemma A.5, we find
\[ \|\text{Law}(X^n_t) - P^n_{\beta}\|_{\text{TV}} \geq \|\text{Law}(Z_t) - P^n_{\beta} \circ \pi^{-1}\|_{\text{TV}} \geq \text{Hellinger}^2(\text{Law}(Z_t), P^n_{\beta} \circ \pi^{-1}) \]
\begin{align*}
= 1 - \frac{(1 - e^{-2t})^{1/4}}{(1 - 4e^{-2t})^{1/2}} \exp\left(-\frac{\pi(X_n^0)^2e^{-2t}}{4(2 - e^{-2t})}\right).
\end{align*}

Setting \( c_n := \log(\pi(X_n^0)) \) and assuming that \( \lim_{n \to \infty} c_n = \infty \), we deduce that for all \( \varepsilon \in (0,1) \)
\[
\lim_{n \to \infty} \|\text{Law}(X_n^{\varepsilon c_n}) - P_n^\beta\|_{TV} = 1.
\]

The comparison between Hellinger and TV of Lemma A.1 allows to deduce that this remains true for the Hellinger distance.

We turn to Entropy. The contraction property stated in Lemma A.2 and the explicit expressions for Gaussian distributions of Lemma A.5 yield
\[
2\text{Entropy}(\text{Law}(X_t^n) \mid P_n^\beta) \geq 2\text{Entropy}(\text{Law}(Z_t) \mid P_n^\beta \circ \pi^{-1})
\]
\[
= z_n^2e^{-2t} - e^{-2t} - \log(1 - e^{-2t}).
\]

This is enough to deduce that
\[
\lim_{n \to \infty} \text{Entropy}(\text{Law}(X_{(1-\varepsilon)c_n}) \mid P_n^\beta) = +\infty.
\]

Regarding the Wasserstein distance, we have \( \|\pi\|_{\text{Lip}} := \sup_{x \neq y} \frac{|\pi(x) - \pi(y)|}{|x - y|} \leq \sqrt{n} \) from the Cauchy–Schwarz inequality, and by Lemma A.2, for all probability measures \( \mu \) and \( \nu \) on \( \mathbb{R}^n \),
\[
\text{Wasserstein}(\mu \circ \pi^{-1}, \nu \circ \pi^{-1}) \leq \sqrt{n}\text{Wasserstein}(\mu, \nu).
\]

Using the explicit expressions for Gaussian distributions of Lemma A.5, we thus find
\[
\text{Wasserstein}^2(\text{Law}(X_t^n), P_n^\beta) \geq \frac{1}{n} \text{Wasserstein}^2(\text{Law}(Z_t), P_n^\beta \circ \pi^{-1})
\]
\[
= \frac{1}{n} \left(\pi(X_n^0)^2e^{-2t} + 2 - e^{-2t} - 2\sqrt{1 - e^{-2t}}\right).
\]

Setting \( c_n := \log\left(\frac{\pi(X_n^0)}{\sqrt{n}}\right) \) and assuming \( c_n \to \infty \) as \( n \to \infty \), we thus deduce that for all \( \varepsilon \in (0,1) \)
\[
\lim_{n \to \infty} \text{Wasserstein}(\text{Law}(X_{(1-\varepsilon)c_n}), P_n^\beta) = +\infty.
\]

5. The random matrix cases

In this section, we prove Theorem 1.6 and Corollary 1.7 that cover the matrix cases \( \beta \in \{1,2\} \).
For these values of \( \beta \), the DOU process is the image by the spectral map of a matrix OU process, connected to the random matrix models GOE and GUE. We could consider the case \( \beta = 4 \) related to GSE. Beyond these three algebraic cases, it could be possible for an arbitrary \( \beta \geq 1 \) to use random tridiagonal matrices dynamics associated to \( \beta \) Dyson processes, see for instance [35].

The next two subsections are devoted to the proof of Theorem 1.6 in the \( \beta = 2 \) and \( \beta = 1 \) cases respectively. The third section provides the proof of Corollary 1.7.

5.1. Hermitian case \( (\beta = 2) \). Let \( \text{Her}_n \) be the set of \( n \times n \) complex Hermitian matrices, namely the set of \( h \in \mathcal{M}_{n,n}(\mathbb{C}) \) with \( h_{i,j} = \overline{h_{j,i}} \) for all \( 1 \leq i,j \leq n \). An element \( h \in \text{Her}_n \) is parametrized by the \( n^2 \) real variables \( (h_{i,i})_{1 \leq i \leq n}, (\mathbb{R}h_{i,j})_{1 \leq i < j \leq n}, (\mathbb{I}h_{i,j})_{1 \leq i < j \leq n} \). We define, for \( h \in \text{Her}_n \) and \( 1 \leq i,j \leq n \),
\[
\pi_{i,j}(h) = \begin{cases} h_{i,i} & \text{if } i = j \\ \sqrt{2} \mathbb{R}h_{i,j} & \text{if } i < j \\ \sqrt{2} \mathbb{I}h_{i,j} & \text{if } i > j \end{cases}.
\]

Note that
\[
\text{Tr}(h^2) = \sum_{i,j=1}^n |h_{i,j}|^2 = \sum_{i=1}^n h_{i,i}^2 + 2 \sum_{i<j} (\mathbb{R}h_{i,j})^2 + 2 \sum_{i<j} (\mathbb{I}h_{i,j})^2 = \sum_{i,j} \pi_{i,j}(h)^2.
\]

We thus identify \( \text{Her}_n \) with \( \mathbb{R}^n \times \mathbb{R}^{2n^2-n} = \mathbb{R}^{n^2} \), this identification is isometrical provided \( \text{Her}_n \) is endowed with the norm \( \sqrt{\text{Tr}(h^2)} \) and \( \mathbb{R}^{n^2} \) with the Euclidean norm.
The Gaussian Unitary Ensemble $\text{GUE}_n$ is the Gaussian law on $\text{Herm}_n$ with density

$$h \in \text{Herm}_n \mapsto \frac{e^{-\frac{1}{2} \text{Tr}(h^2)}}{C_n}$$

where

$$C_n := \int_{\mathbb{R}^{n^2}} e^{-\frac{1}{2} \text{Tr}(h^2)} \prod_{i=1}^{n} dh_i \prod_{i<j} d\text{Im} h_{i,j} \prod_{i<j} d\text{Re} h_{i,j}.$$  

(5.2)

If $H$ is a random $n \times n$ Hermitian matrix then $H \sim \text{GUE}_n$ if and only if the $n^2$ real random variables $\pi_{i,j}(H)$, $1 \leq i, j \leq n$, are independent Gaussian random variables with

$$\pi_{i,j}(H) \sim \mathcal{N}(0, \frac{1}{n}), \quad 1 \leq i, j \leq n.$$  

(5.3)

The law $\text{GUE}_n$ is the unique invariant law of the Hermitian matrix OU process $(H_t)_{t \geq 0}$ on $\text{Herm}_n$ solution of the stochastic differential equation

$$H_0 = h_0 \in \text{Herm}_n, \quad dH_t = \sqrt{\frac{2}{n}} dB_t - H_t dt,$$  

(5.4)

where $B = (B_t)_{t \geq 0}$ is a Brownian motion on $\text{Herm}_n$, in the sense that the stochastic processes $(\pi_{i,j}(B_t))_{t \geq 0}$, $1 \leq i \neq j \leq n$, are independent standard one-dimensional BM. The coordinates stochastic processes $(\pi_{i,j}(H_t))_{t \geq 0}$, $1 \leq i, j \leq n$, are independent real OU processes.

For any $h$ in $\text{Herm}_n$, we denote by $\Lambda(h)$ the vector of the eigenvalues of $h$ ordered in non-decreasing order. Lemma 5.1 below is an observation which dates back to the seminal work of Dyson [25], hence the name DOU for $X^n$. We refer to [28, Ch. 12] and [2, Sec. 4.3] for a mathematical approach using modern stochastic calculus.

**Lemma 5.1** (From matrix OU to DOU). The image of $\text{GUE}_n$ by the map $\Lambda$ is the Coulomb gas $P_n^3$ given by (1.6) with $\beta = 2$. Moreover the stochastic process $X^n = (X^n_t)_{t \geq 0} = (\Lambda(H_t))_{t \geq 0}$ is well-defined and solves the stochastic differential equation (1.3) with $\beta = 2$ and $x_n^0 = \Lambda(h_0)$.

Let $\beta = 2$. Let us assume from now on that the initial value $h_0 \in \text{Herm}_n$ of $(H_t)_{t \geq 0}$ has eigenvalues $x_n^0$ and that $x_n^0$ is as in Theorem 1.6. We start by proving the upper bound on the entropy stated in Theorem 1.6: it will be an adaptation of the proof of the upper bound of Theorem 1.1 applied to the Hermitian matrix OU process $(H_t)_{t \geq 0}$ combined with the contraction property of the entropy. Indeed, by Lemma 5.1 and the contraction property of Lemma A.2

$$\text{Entropy}(\text{Law}(X_t^n) \mid P_n^3) \leq \text{Entropy}(\text{Law}(H_t) \mid \text{GUE}_n).$$  

(5.5)

We claim now that the right-hand side tends to 0 as $n \to \infty$ when $t = t_n$ is well chosen. Indeed, using the identification between $\text{Herm}_n$ and $\mathbb{R}^{n^2}$ mentioned earlier, we have $\text{GUE}_n = \mathcal{N}(m_2, \Sigma_2)$ where $m_2 = 0$ and where $\Sigma_2$ is an $n^2 \times n^2$ diagonal matrix with

$$\Sigma_2(i,i,j,j) = \frac{1}{n}.$$  

(5.6)

On the other hand, the Mehler formula (Lemma 3.1) gives $\text{Law}(H_t) = \mathcal{N}(m_1, \Sigma_1)$ where $m_1 = e^{-t}h_0$ and where $\Sigma_1$ is an $n^2 \times n^2$ diagonal matrix with

$$\Sigma_1(i,i,j,j) = \frac{1 - e^{-2t}}{n}.$$  

(5.7)

Therefore, using Lemma A.5, the analogue of (3.4) reads

$$2 \text{Entropy}(\text{Law}(H_t) \mid \text{GUE}_n) = n|h_0|^2 e^{-2t} - n^2 e^{-2t} - n^2 \log(1 - e^{-2t}),$$  

(5.8)

where

$$|h_0|^2 = \sum_{1 \leq i, j \leq n} \pi_{i,j}(h_0)^2 = \sum_{1 \leq i, j \leq n} |(h_0)_{i,j}|^2 = \text{Tr}(h_0^2) = |x_n^0|^2.$$  

(5.9)

Taking now $c_n := \log(\sqrt{n} |x_n^0|) \vee \log(\sqrt{n})$, for any $\varepsilon \in (0, 1)$, we get

$$\text{Entropy}(\text{Law}(X_{(1+\varepsilon)c_n}^n) \mid P_n^3) \leq \text{Entropy}(\text{Law}(H_{(1+\varepsilon)c_n}) \mid \text{GUE}_n) \longrightarrow 0. \quad (5.10)$$

(5.10)

In the right-hand side of (5.8), the factor $n^2$ is the dimension of the $\mathbb{R}^{n^2}$ to which $\text{Herm}_n$ is identified, while the factor $n$ in the first term is due to the $1/n$ scaling in the stochastic differential equation of the process. This explains the difference with the analogue (3.4) in dimension $n$. 
From the comparison between Hellinger, TV and Entropy stated in Lemma A.1, we easily deduce that the previous convergence remains true upon replacing Entropy by TV or Hellinger.

It remains to cover the upper bound for the Wasserstein distance. This distance is more sensitive to contraction arguments: according to Lemma A.2, one needs to control the Lipschitz norm of the “contraction map” at stake. It happens that the spectral map, restricted to the set $\mathcal{H}_n$ of $n \times n$ Hermitian matrices, is 1-Lipschitz: more precisely, the Hoffman–Wielandt inequality, see [34] and [36, Th. 6.3.5], asserts that for any two such matrices $A$ and $B$, denoting $\Lambda(A) = (\lambda_i(A))_{1 \leq i \leq n}$ and $\Lambda(B) = (\lambda_i(B))_{1 \leq i \leq n}$ the ordered sequences of their eigenvalues, we have

$$\sum_{i=1}^{n} |\lambda_i(A) - \lambda_i(B)|^2 \leq \sum_{i,j} |A_{i,j} - B_{i,j}|^2.$$ 

Applying Lemma A.2, we thus deduce that

$$\text{Wasserstein}(\text{Law}(X_n^\beta), P_n^\beta) \leq \text{Wasserstein}(\text{Law}(H_t), \text{GUE}_n).$$

(5.11)

Following the Gaussian computations in the proof of Theorem 1.2, we obtain

$$\text{Wasserstein}^2(\text{Law}(H_t), \text{GUE}_n) = |\sigma_0^2| e^{-2t} + 2 - e^{-2t} - 2\sqrt{1 - e^{-2t}}.$$ 

(5.12)

Set $c_n := \log(|\sigma_0^2|)$. If $c_n \to \infty$ as $n \to \infty$ then for all $\varepsilon \in (0, 1)$ we find

$$\text{Wasserstein}(\text{Law}(X_n^{(1+\varepsilon)c_n}), P_n^\beta) \to 0 \quad n \to \infty.$$ 

This completes the proof of Theorem 1.6.

5.2. Symmetric case ($\beta = 1$). The method is similar to the case $\beta = 2$. Let us focus only on the differences. Let $\text{Sym}_n$ be the set of $n \times n$ real symmetric matrices, namely the set of $s \in \mathcal{M}_{n,n}(\mathbb{R})$ with $s_{i,j} = s_{j,i}$ for all $1 \leq i, j \leq n$. An element $s \in \text{Sym}_n$ is parametrized by the $n + \frac{n^2 - n}{2} = \frac{n(n+1)}{2}$ real variables $(s_{i,j})_{1 \leq i \leq j \leq n}$. We define, for $s \in \text{Sym}_n$ and $1 \leq i \leq j \leq n$,

$$\pi_{i,j}(s) = \begin{cases} s_{i,i} & \text{if } i = j \\ \sqrt{2}s_{i,j} & \text{if } i < j \end{cases}.$$ 

(5.13)

Note that

$$\text{Tr}(s^2) = \sum_{i,j=1}^{n} s_{i,j}^{2} = \sum_{i=1}^{n} s_{i,i}^{2} + 2\sum_{i<j} s_{i,j}^{2}.$$ 

We thus identify isometrically $\text{Sym}_n$, endowed with the norm $\sqrt{\text{Tr}(s^2)}$, with $\mathbb{R}^{n} \times \mathbb{R}^{\frac{n^2 - n}{2}}$ endowed with the Euclidean norm.

The Gaussian Orthogonal Ensemble $\text{GOE}_n$ is the Gaussian law on $\text{Sym}_n$ with density

$$s \in \text{Sym}_n \mapsto \frac{e^{-\frac{2}{n}\text{Tr}(s^2)}}{C_n} \text{ where } C_n := \int_{\mathbb{R}^\frac{n(n+1)}{2}} e^{-\frac{2}{n}\text{Tr}(s^2)} \prod_{1 \leq i \leq j \leq n} \text{ds}_{i,j}.$$ 

(5.14)

If $S$ is a random $n \times n$ real symmetric matrix then $S \sim \text{GOE}_n$ if and only if the $\frac{n(n+1)}{2}$ real random variables $\pi_{i,j}(S), 1 \leq i \leq j \leq n$, are independent Gaussian random variables with

$$\pi_{i,j}(S) \sim \mathcal{N}\left(0, \frac{1}{n}\right), \quad 1 \leq i \leq j \leq n.$$ 

(5.15)

The law $\text{GOE}_n$ is the unique invariant law of the real symmetric matrix OU process $(S_t)_{t \geq 0}$ on $\text{Sym}_n$ solution of the stochastic differential equation

$$S_0 = s_0 \in \text{Sym}_n, \quad dS_t = \sqrt{\frac{2}{n}} dB_t - S_t dt$$

(5.16)

where $B = (B_t)_{t \geq 0}$ is a Brownian motion on $\text{Sym}_n$, in the sense that the stochastic processes $(\pi_{i,j}(B_t))_{t \geq 0}, 1 \leq i \leq j \leq n$, are independent standard one-dimensional BM. The coordinates stochastic processes $(\pi_{i,j}(S_t))_{t \geq 0}, 1 \leq i \leq j \leq n$, are independent real OU processes.

For any $s$ in $\text{Sym}_n$, we denote by $\Lambda(s)$ the vector of the eigenvalues of $s$ ordered in non-decreasing order. Lemma 5.2 below is the real symmetric analogue of Lemma 5.1.
Lemma 5.2 (From matrix OU to DOU). The image of GOE\(_n\) by the map \(\Lambda\) is the Coulomb gas \(P_n^{\beta}\) given by (1.6) with \(\beta = 1\). Moreover the stochastic process \(X^n = (X^n_t)_{t \geq 0} = (\Lambda(S_t))_{t \geq 0}\) is well-defined and solves the stochastic differential equation (1.3) with \(\beta = 1\) and \(x^n_0 = \Lambda(n_0)\).

As for the case \(\beta = 2\), the idea now is that the DOU process is sandwiched between a real OU process and a matrix OU process.

By similar computations to the case \(\beta = 2\), the analogue of (5.8) becomes

\[
2\text{Ent}\left(\text{Law}(S_t) \mid \text{GOE}_n\right) = n|x_0^n|^2e^{-2t} - \frac{n(n+1)}{2}e^{-2t} - \frac{n(n+1)}{2} \log(1 - e^{-2t}).
\] (5.17)

This allows to deduce the upper bound for TV, Hellinger and Entropy. Regarding the Wasserstein distance, the analogue of (5.12) reads

\[
\text{Wasserstein}^2(\text{Law}(S_t), \text{GOE}_n) = |x_0^n|^2e^{-2t} + 2 - e^{-2t} - 2\sqrt{1 - e^{-2t}}.
\] (5.18)

If \(\lim_{n \to \infty} \log(|x_0^n|) = \infty\) then we deduce the proof of Theorem 1.6.

5.3. Proof of Corollary 1.7. Let \(\beta \in \{1, 2\}\). Recall the definitions of \(a_n\) and \(c_n\) from the statement. Take \(x_0^{a_i} = a_n\) for all \(i\), and note that \(\pi(x_0^n) = na_n\). Given our assumptions on \(a_n\), Corollary 1.5 yields for this particular choice of initial condition and for any \(\varepsilon \in (0, 1)\)

\[
\lim_{n \to \infty} \sup_{x^n \in [-a_n, a_n]^n} \text{dist}(\text{Law}(X^n_{(1-\varepsilon)c_n}), P_n^{\beta}) = \max.
\]

This remains true upon maximizing the distance over all initial conditions in \([-a_n, a_n]^n\), as it can be checked easily from the proof.

On the other hand, in the proof of Theorem 1.6 we saw that

\[
2\text{Ent}(\text{Law}(X^n_t) \mid P_n^{\beta}) \leq |x_0^n|^2ne^{-2t} - n^2e^{-2t} - n^2 \log(1 - e^{-2t}).
\]

Since \(|x_0^n| \leq \sqrt{n}\alpha_n\) for all \(x^n_0 \in [-a_n, a_n]^n\), and given the comparison between TV, Hellinger and Entropy stated in Lemma A.1 we obtain for \(\varepsilon \in (0, 1)\)

\[
\lim_{n \to \infty} \sup_{x^n \in [-a_n, a_n]^n} \text{dist}(\text{Law}(X^n_{(1-\varepsilon)c_n}), P_n^{\beta}) = 0,
\]

thus concluding the proof of Corollary 1.7 regarding theses distances.

Concerning Wasserstein, the proof of Theorem 1.6 shows that for any \(x^n_0 \in [-a_n, a_n]^n\) we have

\[
\text{Wasserstein}^2(\text{Law}(X^n_t), P_n^{\beta}) \leq |x_0^n|^2e^{-2t} + 2 - e^{-2t} - 2\sqrt{1 - e^{-2t}} \\
\leq na_n^2e^{-2t} + 2 - e^{-2t} - 2\sqrt{1 - e^{-2t}}.
\]

If \(\sqrt{n}\alpha_n \to \infty\), then for \(c_n = \log(\sqrt{n}\alpha_n)\) we deduce that for all \(\varepsilon \in (0, 1)\)

\[
\lim_{n \to \infty} \sup_{x^n \in [-a_n, a_n]^n} \text{dist}(\text{Law}(X^n_{(1+\varepsilon)c_n}), P_n^{\beta}) = 0.
\]

6. Cutoff phenomenon for the DOU in TV and Hellinger

In this section, we prove Theorem 1.8 and Corollary 1.9 for the TV and Hellinger distances. We only consider the case \(\beta \geq 1\), although the arguments could be adapted mutatis mutandis to cover the case \(\beta = 0\); note that the result of Theorem 1.8 and Corollary 1.9 for \(\beta = 0\) can be deduced from Theorem 1.2. At the end of this section, we also provide the proof of Theorem 1.11.

6.1. Proof of Theorem 1.8 in TV and Hellinger. By the comparison between TV and Hellinger stated in Lemma A.1, it suffices to prove the result for the TV distance, so we concentrate on this distance until the end of this section. Our proof is based on the exponential decay of the relative entropy at an explicit rate given by the optimal logarithmic Sobolev constant. However, this requires the relative entropy of the initial condition to be finite. Consequently, we proceed in three steps. First, given an arbitrary initial condition \(x_0^n \in D_n\), we build an absolutely continuous probability measure \(\mu_{x_0^n}\) on \(D_n\) that approximates \(\delta_{x_0^n}\) and whose relative entropy is not too large. Second, we derive a decay estimate starting from this regularized initial condition. Third, we control the total variation distance between the two processes starting respectively from \(\delta_{x_0^n}\) and \(\mu_{x_0^n}\).
6.1.1. Regularization. In order to have a finite relative entropy at time 0, we first regularize the initial condition by smearing out each particle in a ball of radius bounded below by \(n^{-(\kappa+1)}\), for some \(\kappa > 0\). Let us first introduce the regularization at scale \(\eta\) of a Dirac distribution \(\delta_z, z \in \mathbb{R}\) by
\[
\delta^{(\eta)}_z(du) = \text{Uniform}([z, z + \eta])(du) = \eta^{-1}1_{[z, z+\eta]}(du).
\]
Given \(x \in \overline{D}_n\) and \(\eta > 0\), we define a regularized version of \(\delta_x\) at scale \(n^{-\kappa}\), that we denote \(\mu_x\), by setting
\[
\mu_x = \otimes_{i=1}^n \delta_{x_i + \alpha_i}^{(\eta)}, \tag{6.1}
\]
where the parameters \(\alpha_i \geq 0, i = 1, \ldots, n\) are chosen so that
\[
\begin{align*}
x_i + \alpha_i + 2\eta < x_{i+1} + \alpha_{i+1} - \eta & \quad \text{for each } i \in \{1, \ldots, n - 1\}, \tag{6.2} \\
\alpha_i + \eta \leq n^{-\kappa} & \quad \text{for each } i \in \{1, \ldots, n\}, \tag{6.3} \\
\eta & = n^{-(\kappa+1)}. \tag{6.4}
\end{align*}
\]
Condition (6.2) ensures that the intervals \([x_i + \alpha_i, x_i + \alpha_i + \eta]\) are disjoint, ordered and at distance at least \(\eta\) from each other. Condition (6.3) guaranties that, provided \(X^n_0 = x\) and \(Y^n_0\) is distributed according to \(\mu_x\), almost surely \(X^n_0\) is close to \(Y^n_0\). Finally Condition (6.4) allows to bound from below the volume of configurations, or more precisely to bound from above the relative entropy of \(\mu_x\) with respect to the Lebesgue measure.

6.1.2. Convergence of the regularized process to equilibrium.

Lemma 6.1 (Convergence of regularized process). Let \((Y^n_t)_{t \geq 0}\) be a DOU process solution of (1.3), \(\beta \geq 1\), and let \(P^\beta_n\) be its invariant law. Assume that \(\text{Law}(Y^n_0)\) is the regularized measure \(\mu_{x_0}^n\) in (6.1) associated to some initial condition \(x^n_0 \in \overline{D}_n\). Then there exists a constant \(C > 0\), only depending on \(\kappa\), such that for all \(t \geq 0\), all \(n \geq 2\) and all \(x^n_0 \in \overline{D}_n\)
\[
\text{Entropy}(\text{Law}(Y^n_t) \mid P^\beta_n) \leq C(n|x^n_0|^2 + n^2 \log(n))e^{-2t}.
\]

Proof of Lemma 6.1. By Lemma B.2 and since \(\text{Law}(Y^n_0) = \mu_{x_0}^n\), for all \(t \geq 0\), there holds
\[
\text{Entropy}(\text{Law}(Y^n_t) \mid P^\beta_n) \leq \text{Entropy}(\mu_{x_0}^n \mid P^\beta_n)e^{-2t}. \tag{6.5}
\]
Now we have
\[
\text{Entropy}(\mu_{x_0}^n \mid P^\beta_n) = \mathbb{E}_{\mu_{x_0}^n} \left[ \log \frac{d\mu_{x_0}^n}{dP^\beta_n} \right].
\]
Recall the definition of \(S\) in (1.12). As \(P^\beta_n\) has density \(\frac{e^{-\beta S}}{C_n^\beta}\), we may re-write this as
\[
\text{Entropy}(\mu_{x_0}^n \mid P^\beta_n) = S(\mu_{x_0}^n) + \mathbb{E}_{\mu_{x_0}^n}[E] + \log C_n^\beta. \tag{6.6}
\]
Recall the partition function \(C_n^\beta = n!C_n^\beta\) from Subsection 2.2. It is proved in [6], using explicit expressions involving Gamma functions via a Selberg integral, that for some constant \(C > 0\)
\[
\log C_n^\beta \leq \log C_{rn}^\beta \leq Cn^2. \tag{6.7}
\]
Next, we claim that \(S(\mu_{x_0}^n) \leq n \log(n^{1+\kappa})\). Indeed since \(\mu_{x_0}^n\) is a product measure, the tensorization property of Entropy recalled in Lemma A.4 gives
\[
\text{Entropy}(\mu_{x_0}^n \mid dx) = \sum_{i=1}^n \text{Entropy}(\delta^{(\eta)}_0 \mid dx).
\]
Moreover an immediate computation yields \(\text{Entropy}(\delta^{(\eta)}_0 \mid dx) = \log(\eta^{-1})\) and using (6.4) we get
\[
\text{Entropy}(\mu_{x_0}^n \mid dx) = n \log(n^{\kappa+1}). \tag{6.8}
\]
We turn to the estimation of the term \(\mathbb{E}_{\mu_{x_0}^n}[E]\). The confinement term can be easily bounded:
\[
\mathbb{E}_{\mu_{x_0}^n} \left[ \frac{n}{2} \sum_{i=1}^n x_i^2 \right] \leq (n|x^n_0|^2 + n^2 \eta^2).
\]
Let us now estimate the logarithmic energy of \( \mu_{x_0^n} \). Using the fact that the logarithmic function is increasing, together with Condition (6.2), we notice that for any \( i > j \) there holds
\[
E_{\mu_{x_0^n}} [\log |x_i - x_j|] = \iint \log |x - y| \delta_{\bar{\eta}}(dy) \delta_{\bar{\eta}}(dx) \\
\geq \iint \log |x - y| \delta_{\bar{\eta}}^n(dx) \delta_{\bar{\eta}}^n(dy) \\
\geq \log \eta.
\]
It follows that the initial logarithmic energy cannot be much larger than \( n^2 \log n \):
\[
E_{\mu_{x_0^n}} \left[ \sum_{i > j} \log \frac{1}{|x_i - x_j|} \right] \leq \frac{n(n - 1)}{2} \log n^{k+1}.
\]
This implies that there exists a constant \( C > 0 \), only depending on \( \kappa \), such that for all \( n \geq 2 \)
\[
E_{\mu_{x_0^n}} [E] = E_{\mu_{x_0^n}} \left[ \frac{n}{2} \sum_{i=1}^{n} |x_i|^2 + \beta \sum_{i > j} \log \frac{1}{|x_i - x_j|} \right] \leq C \left(n|x_0^n|^2 + n^2 \log n\right). \tag{6.9}
\]
Inserting (6.7), (6.8) and (6.9) into (6.6) we obtain (for a different constant \( C > 0 \))
\[
\text{Entropy}(\mu_{x_0^n}|P_\beta^n) \leq C \left(n|x_0^n|^2 + n^2 \log n\right).
\]
This bound, combined with (6.5), concludes the proof of Lemma 6.1. \( \square \)

6.1.3. Convergence to the regularized process in total variation distance. Let \( (X^n_t)_{t \geq 0} \) and \( (Y^n_t)_{t \geq 0} \) be two DOU processes with \( X^n_0 = x_0^n \) and \( \text{Law}(Y^n_0) = \mu_{x_0^n} \), where the measure \( \mu_{x_0^n} \) is defined in (6.1). Below we prove that, as soon as the parameter \( \kappa \) is small enough, the total variation distance between \( \text{Law}(X^n_t) \) and \( \text{Law}(Y^n_t) \) tends to 0, for any fixed \( t > 0 \).

Note that at time 0, almost surely, there holds \( X^n_{0,i} \leq Y^n_{0,i} \), for every \( i \in \{1, \ldots, n\} \). We now introduce a coupling of the processes \((X^n_t)_{t \geq 0}\) and \((Y^n_t)_{t \geq 0}\) that preserves this ordering at all times. Consider two independent standard BM \( B^n \) and \( W^n \) in \( \mathbb{R}^n \). Let \( X^n \) be the solution of (1.3) driven by \( B^n \), and let \( Y^n \) be the solution of
\[
dY^n_{t,i} = \sqrt{\frac{2}{n}} \left( 1_{\{Y^n_{t,i} \neq X^n_{t,i}\}}dW^n_{i,t} + 1_{\{Y^n_{t,i} = X^n_{t,i}\}}dB^n_{i,t}\right) - Y^n_{t,i} dt + \frac{\beta}{n} \sum_{j \neq i} \frac{dt}{Y^n_{t,j} - Y^n_{t,i}}, \quad 1 \leq i \leq n.
\]
We denote by \( \mathbb{P} \) the probability measure under which these two processes are coupled. It is elementary to check that the ordering is preserved at all times under \( \mathbb{P} \), and that if \( X^n_s = Y^n_s \) for some \( s \geq 0 \), then it remains true at all times \( t \geq s \). This is a consequence of the monotonicity result stated in Lemma B.4.

As in (A.7), the total variation distance between the laws of \( X^n_t \) and \( Y^n_t \) may be bounded by
\[
\|\text{Law}(Y^n_t) - \text{Law}(X^n_t)\|_{TV} \leq \mathbb{P}(X^n_t \neq Y^n_t),
\]
for all \( t \geq 0 \). We wish to establish that for any given \( t > 0 \),
\[
\lim_{n \to \infty} \mathbb{P}(X^n_t \neq Y^n_t) = 0.
\]
To do so, we work with the area between the two processes \( X^n \) and \( Y^n \), defined by
\[
A^n_t := \sum_{i=1}^{n} (Y^n_{t,i} - X^n_{t,i}) = \pi(Y^n_t) - \pi(X^n_t), \quad t \geq 0.
\]
As the two processes are ordered at any time, this is nothing but the geometric area between the two discrete interfaces \( i \mapsto X^n_{t,i} \) and \( i \mapsto Y^n_{t,i} \) associated to the configurations \( X^n_t \) and \( Y^n_t \). We deduce that the merging time of the two processes coincide with the hitting time of 0 by this area, that we denote by \( \tau = \inf \{t \geq 0 : A^n_t = 0\} \).

The process \( A^n \) has a very simple structure: it is a semimartingale that behaves like an OU process with a randomly varying quadratic variation. Let \( N_t \) be the number of coordinates that do not coincide at time \( t \), that is
\[
N_t := \# \{ i \in \{1, \ldots, n\} : X^n_{t,i} \neq Y^n_{t,i} \}.
\]
Then $A^n$ satisfies
\[ dA^n_t = -A^n_t \, dt + dM_t, \]
where $M$ is a centered martingale with quadratic variation
\[ d\langle M \rangle_t = \frac{2}{n} N_t \, dt. \]  
(6.10)

Note that whenever $t < \tau$ we have
\[ d\langle M \rangle_t \geq \frac{2}{n}. \]
This \textit{a priori} lower bound on the quadratic variation of $M$, combined with the Dubins–Schwarz theorem, allows to check that $\tau < \infty$ almost surely. Note that in view of the coupling between $X^n_t$ and $Y^n_t$, we have $X^n_t = Y^n_t$ for all $t \geq \tau$.

Recall the following informal fact: with large probability, a Brownian motion starting from $a$ hits $b$ by a time of order $(a - b)^2$. For a continuous martingale, this becomes: with large probability, a continuous martingale starting from $a$ accumulates a quadratic variation of order $(a - b)^2$ up to its first hitting time of $b$. Our next lemma states such a bound on the supermartingale $A^n$.

**Lemma 6.2.** Let $a > b \geq 0$. Let $\tau_n = \inf\{t > 0 : A_t = b\} < \infty$ almost surely. Then, for all $u \geq 1$,
\[ P(\langle A \rangle_{\tau_n} \geq (a - b)^2 u \mid A_0 = a) \leq 4u^{-1/2}. \]

**Proof.** Without loss of generality one can assume that $A_0 = a$ almost surely.

By Itô’s formula, for all $\lambda > 0$, the process
\[ S_t = \exp\left(-\lambda A_t - \frac{\lambda^2}{2}\langle A \rangle_t\right), \]
defines a submartingale (taking its values in $[0, 1]$). Doob’s stopping theorem yields
\[ E[e^{-\frac{\lambda^2}{2}\langle A \rangle_{\tau_n}}] = e^{\lambda b} E[S_{\tau_n}] \geq e^{\lambda b} E[S_0] = e^{-\lambda(a - b)}. \]

On the other hand, for $\lambda = 2(a - b)^{-1} u^{-1/2}$, there holds
\[
E[e^{-\frac{\lambda^2}{2}\langle A \rangle_{\tau_n}}] \leq P(\langle A \rangle_{\tau_n} < (a - b)^2 u) + e^{-\frac{\lambda^2}{2}(a - b)^2 u} P(\langle A \rangle_{\tau_n} \geq (a - b)^2 u) \\
\leq 1 - (1 - e^{-\frac{\lambda^2}{2}(a - b)^2 u}) P\left(\langle A \rangle_{\tau_n} \geq (a - b)^2 u\right) \\
\leq 1 - \frac{1}{2} P\left(\langle A \rangle_{\tau_n} \geq (a - b)^2 u\right).
\]

Consequently one deduces that
\[ P(\langle A \rangle_{\tau_n} \geq (a - b)^2 u) \leq 2(1 - e^{-\lambda(a - b)}) \leq 4u^{-1/2}. \]

\[ \square \]

We are now ready to prove the following lemma:

**Lemma 6.3.** If $\kappa > \frac{3}{2}$, then for every sequence of times $(t_n)_{\kappa}$ with $\lim_{n \to \infty} t_n > 0$, we have
\[ \lim_{n \to \infty} \sup_{\alpha \in \mathcal{D}_n} \|\text{Law}(Y^n_{t_n}) - \text{Law}(X^n_{t_n})\|_{TV} = 0. \]

**Proof of Lemma 6.3.** Let $(t_n)_{\kappa}$ be a sequence of times such that $\lim_{n \to \infty} t_n > 0$. In view of the assumptions on the parameters $\eta$ and $\alpha_i$’s, the initial area satisfies almost surely
\[ A^n_0 \leq n^{1 - \kappa}. \]

According to Lemma 6.2, with a probability that goes to 1, one has
\[ \langle A^n \rangle_{\tau} - \langle A^n \rangle_0 < n^{2 - 2\kappa} \log n. \]

On the other hand, by (6.10), we have the following control on the quadratic variation:
\[ \langle A \rangle_{\tau} - \langle A \rangle_0 \geq \frac{2}{n}. \]
One deduces that, with a probability that goes to 1, 
\[ \tau \leq \frac{1}{2} n^{3 - 2\kappa} \log n, \]
and this quantity goes to 0 as \( n \to \infty \), whenever \( \kappa > \frac{3}{2} \). Therefore for \( \kappa > \frac{3}{2} \), there holds
\[ \lim_{n \to \infty} \sup_{x_0^n \in D_n} \mathbb{P}(X^n_{t_n} \neq Y^n_{t_n}) = 0, \]
thus concluding the proof of Lemma 6.3.

**Proof of Theorem 1.8 in \( TV \) and Hellinger.** Let \( \kappa > \frac{3}{2} \) and fix some initial condition \( x_0^n \in D_n \). By the triangle inequality for \( TV \), there holds
\[ \|\text{Law}(X^n_t) - P^n_{\beta t}\|_{TV} \leq \|\text{Law}(Y^n_t) - P^n_{\beta t}\|_{TV} + \|\text{Law}(X^n_t) - \text{Law}(Y^n_t)\|_{TV}. \]
Taking \( t = t_n(1 + \varepsilon) \) with \( t_n = \log(\sqrt{n}|x_0^n|) \vee \log(n) \), one deduces from Lemma 6.1 and the Pinsker inequality stated in Lemma A.1 that the first term in the right-hand side of (6.11) vanishes as \( n \) tends to infinity. Meanwhile Lemma 6.3 guarantees that the second term tends to 0 as \( n \) tends to infinity. We also conclude using the comparison between \( TV \) and Hellinger (see Lemma A.1) that
\[ \lim_{n \to \infty} \text{Hellinger}(\text{Law}(X^n_{t_n}), P^n_{\beta t_n}) = 0. \]

**6.2. Proof of Corollary 1.9 in \( TV \) and Hellinger.**

**Proof of Corollary 1.9 in \( TV \) and Hellinger.** By Lemma A.1 and the triangle inequality for \( TV \), we have
\[ \sup_{x_0^n \in [-a_n, a_n]|n} \|\text{Law}(X^n_t) - P^n_{\beta t}\|_{TV} \leq \sup_{x_0^n \in [-a_n, a_n]|n} \|\text{Law}(Y^n_t) - \text{Law}(X^n_t)\|_{TV} \]
\[ + \sup_{x_0^n \in [-a_n, a_n]|n} \sqrt{2 \text{Entropy}(\text{Law}(Y^n_t) \mid P^n_{\beta t})}. \]
Take \( t = (1 + \varepsilon)c_n \) with \( c_n = \log(na_n) \). Lemmas 6.1 and 6.3, combined with the assumption made on \( (a_n) \), show that the two terms on the right-hand side vanish as \( n \to \infty \). Using Lemma A.1, the same result holds for Hellinger.

On the other hand, take \( x_0^n = a_n \) for all \( i \) and note that \( \pi(x_0^n) = na_n \) goes to \(+\infty\) as \( n \to \infty \). By Corollary 1.5 we find
\[ \lim_{n \to \infty} \sup_{x_0^n \in [-a_n, a_n]|n} \text{dist}(\text{Law}(X^n_{(1-\varepsilon)c_n}), P^n_{\beta t}) = 1 \]
whenever \( \text{dist} \in \{TV, \text{Hellinger}\} \).

**6.3. Proof of Theorem 1.11.**

**Proof of Theorem 1.11. Lower bound.** The contraction property provided by Lemma A.2 gives
\[ \text{Entropy}(\text{Law}(X^n_t) \mid P^n_{\beta t}) \geq \text{Entropy}(\text{Law}(\pi(X^n_t)) \mid P^n_{\beta t} \circ \pi^{-1}). \]
By Theorem 1.3 \( P_n \circ \pi^{-1} = \mathcal{N}(0, 1) \) and \( Y = \pi(X^n) \) is an OU process weak solution of \( Y_0 = \pi(X^n_0) \) and \( dY_t = \sqrt{2d}B_t - Y_t dt \). In particular for all \( t \geq 0 \), \( \text{Law}(Y_t) \) is a mixture of Gaussian laws in the sense that for any measurable test function \( g \) with polynomial growth,
\[ E_{\text{Law}(Y_t)}[g] = E[g(Y_t)] = E[G_t(Y_0)] \quad \text{where} \quad G_t(y) = E_{\mathcal{N}(y^2 - t, 1, e^{-2t})}[g]. \]
Now we use (again) the variational formula used in the proof of Lemma A.2 to get
\[ \text{Entropy}(\text{Law}(\pi(X^n_t)) \mid P^n_{\beta t} \circ \pi^{-1}) = \sup_g \{E_{\text{Law}(\pi(X^n_t))[g] - \log E_{\mathcal{N}(0, 1)}[e^g]} \}, \]
and taking for \( g \) the linear function defined by \( g(x) = \lambda x \) for all \( x \in \mathbb{R} \) and for some \( \lambda \neq 0 \) yields
\[ \text{Entropy}(\text{Law}(\pi(X^n_t)) \mid P^n_{\beta t} \circ \pi^{-1}) \geq \lambda e^{-1} \sum_{i=1}^{n} \int x \mu_i(dx) - \frac{\lambda^2}{2}. \]
Finally, by using the assumption on first moment and taking λ small enough we get, for all ε ∈ (0, 1),
\[
\lim_{n \to \infty} \text{Entropy}(\text{Law}(\pi(X_{n}^{(1-\epsilon)\log(n)}) | \mathcal{P}_{n}^{\beta} \circ \pi^{-1}) = +\infty,
\]

**Upper bound.** From Lemma B.2 we have, for all t ≥ 0,
\[
\text{Entropy}(\text{Law}(X_{t}^{n}) | \mathcal{P}_{n}^{\beta}) \leq \text{Entropy}(\text{Law}(X_{0}^{n}) | \mathcal{P}_{n}^{\beta}) e^{-2t}.
\]

Arguing like in the proof of Lemma 6.1 and using the contraction property of Entropy provided by Lemma A.2 for the map Ψ defined in (1.14), we can write the following decomposition
\[
\text{Entropy}(\text{Law}(X_{0}^{n}) | \mathcal{P}_{n}^{\beta}) \leq \sum_{i=1}^{n} S(\mu_{i}) + \sum_{i \neq j} \int \Phi d\mu_{i} \otimes d\mu_{j} + Cn^{2}.
\]
Combining (6.7) with the assumptions on the µi’s yields for some constant C > 0
\[
\text{Entropy}(\text{Law}(X_{0}^{n}) | \mathcal{P}_{n}^{\beta}) \leq Cn^{2}
\]
and it follows finally that for all ε ∈ (0, 1),
\[
\lim_{n \to \infty} \text{Entropy}(\text{Law}(X_{(1+\epsilon)\log(n)}) | \mathcal{P}_{n}^{\beta}) = 0.
\]

\[\square\]

7. CUTOFF PHENOMENON FOR THE DOU IN WASSERSTEIN

7.1. **Proofs of Theorem 1.8 and Corollary 1.9 in Wasserstein.** Let \( (X_{t})_{t \geq 0} \) be the DOU process. By Lemma B.2, for all t ≥ 0 and all initial conditions \( X_{0} \in \mathcal{D}_{n} \),
\[
\text{Wasserstein}^{2}(\text{Law}(X_{t}), \mathcal{P}_{n}^{\beta}) \leq e^{-2t} \text{Wasserstein}^{2}(\text{Law}(X_{0}), \mathcal{P}_{n}^{\beta}).
\]
Suppose now that \( \text{Law}(X_{n}^{0}) = \delta_{x_{0}^{n}} \). Then the triangle inequality for the Wasserstein distance gives
\[
\text{Wasserstein}^{2}(\delta_{x_{0}^{n}}, \mathcal{P}_{n}^{\beta}) = \int |x_{0}^{n} - x|^{2} \mathcal{P}_{n}^{\beta}(dx) \leq 2|x_{0}^{n}|^{2} + 2 \int |x|^{2} \mathcal{P}_{n}^{\beta}(dx).
\]
By Theorem 1.3, the mean at equilibrium of \( |X_{t}^{n}|^{2} \) equals \( 1 + \frac{\beta}{2}(n - 1) \) and therefore
\[
\int |x|^{2} \mathcal{P}_{n}^{\beta}(dx) = 1 + \frac{\beta}{2}(n - 1).
\]
We thus get
\[
\text{Wasserstein}^{2}(\text{Law}(X_{t}^{n}), \mathcal{P}_{n}^{\beta}) \leq 2(|x_{0}^{n}|^{2} + 1 + \frac{\beta}{2}(n - 1)) e^{-2t}.
\]
Set \( c_{n} := \log(|x_{0}^{n}|) \lor \log(\sqrt{n}) \). For any ε ∈ (0, 1), we have
\[
\lim_{n \to \infty} \text{Wasserstein}(\text{Law}(X_{(1+\epsilon)c_{n}}^{n}), \mathcal{P}_{n}^{\beta}) = 0
\]
and this concludes the proof of Theorem 1.8 in the Wasserstein distance.

Regarding the proof of Corollary 1.9, if \( x_{0}^{n} \in [-a_{n}, a_{n}]^{n} \) then \(|x_{0}^{n}| \leq \sqrt{n}a_{n} \). Therefore if \( \inf_{n} a_{n} > 0 \), setting \( c_{n} = \log(\sqrt{n}a_{n}) \) we find, as required,
\[
\lim_{n \to \infty} \sup_{x_{0}^{n} \in [-a_{n}, a_{n}]^{n}} \text{Wasserstein}(\text{Law}(X_{(1+\epsilon)c_{n}}^{n}), \mathcal{P}_{n}^{\beta}) = 0.
\]
7.2. Proof of Theorem 1.10. This is an adaptation of the previous proof. We compute
\[
\text{Wasserstein}^2(\delta_{x^n}, P_n^\beta) = \int |x_n^\alpha - x|^2 P_n^\beta(\,dx) \\
\leq 2|x_n^\alpha - \rho_n|^2 + 2 \int |\rho_n - x|^2 P_n^\beta(\,dx),
\]
where \(\rho_n \in D_n\) is the vector of the quantiles of order \(1/n\) of the semi-circle law as in (1.11). The rigidity estimates established in [10, Th. 2.4] justify that
\[
\lim_{n \to \infty} \int |\rho_n - x|^2 P_n^\beta(\,dx) = 0.
\]
If \(|x_n^\alpha - \rho_n|\) diverges with \(n\), we deduce that for all \(\varepsilon \in (0, 1)\), with \(t_n = \log(|x_n^\alpha - \rho_n|)\),
\[
\lim_{n \to \infty} \text{Wasserstein}(\text{Law}(X_n^{\alpha+\varepsilon t_n}), P_n^\beta) = 0.
\]
On the other hand, if \(|x_n^\alpha - \rho_n|\) converges to some limit \(\alpha\) then we easily get, for any \(t \geq 0\),
\[
\lim_{n \to \infty} \text{Wasserstein}^2(\text{Law}(X_n^\alpha), P_n^\beta) \leq \alpha^2 e^{-2t}.
\]

Remark 7.1 (High-dimensional phenomena). With \(X_n \sim P_n^\beta\), in the bias-variance decomposition
\[
\int |\rho_n - x|^2 P_n^\beta(\,dx) = |EX_n - \rho_n|^2 + E(||X_n - EX_n||^2),
\]
the second term of the right hand side is a variance term that measures the concentration of the log-concave random vector \(X_n\) around its mean \(EX_n\), while the first term in the right hand side is a bias term that measures the distance of the mean \(EX_n\) to the mean-field limit \(\rho_n\). Note also that \(E(||X_n - EX_n||^2) = E(||X_n||^2) - ||EX_n||^2 = 1 + \frac{\beta}{2}(n-1) - ||EX_n||^2\), reducing the problem to the mean. We refer to [33] for a fine asymptotic analysis in the determinantal case \(\beta = 2\).

Appendix A. Distances and divergences

We use the following standard distances and divergences to quantify the trend to equilibrium of Markov processes and to formulate the cutoff phenomena.

The Wasserstein–Kantorovich–Monge transportation distance of order 2 and with respect to the underlying Euclidean distance is defined for all probability measures \(\mu\) and \(\nu\) on \(\mathbb{R}^n\) by
\[
\text{Wasserstein}(\mu, \nu) = \left(\inf_{(X,Y)} E[|X - Y|^2] \right)^{1/2} \in [0, +\infty]
\]
where \(|x| = \sqrt{x_1^2 + \cdots + x_n^2}\) and where the inf runs over all couples \((X, Y)\) with \(X \sim \mu\) and \(Y \sim \nu\).

The total variation distance between probability measures \(\mu\) and \(\nu\) on the same space is
\[
||\mu - \nu||_{TV} = \sup_A |\mu(A) - \nu(A)| \in [0, 1]
\]
where the supremum runs over Borel subsets. If \(\mu\) and \(\nu\) are absolutely continuous with respect to a reference measure \(\lambda\) with densities \(f_\mu\) and \(f_\nu\) then \(||\mu - \nu||_{TV} = \frac{1}{2} \int |f_\mu - f_\nu| \,d\lambda = \frac{1}{2} ||f_\mu - f_\nu||_{L^1(\lambda)}\).

The Hellinger distance between probability measures \(\mu\) and \(\nu\) with densities \(f_\mu\) and \(f_\nu\) with respect to the same reference measure \(\lambda\) is
\[
\text{Hellinger}(\mu, \nu) = \left(\int \frac{1}{2} (\sqrt{f_\mu} - \sqrt{f_\nu})^2 \,d\lambda \right)^{1/2} = \left(1 - \int \sqrt{f_\mu f_\nu} \,d\lambda \right)^{1/2} \in [0, 1].
\]
This quantity does not depend on the choice of \(\lambda\). We have \(\text{Hellinger}(\mu, \nu) = \frac{1}{\sqrt{2}} ||\sqrt{f_\mu} - \sqrt{f_\nu}||_{L^2(\lambda)}\). Note that an alternative normalization is sometimes considered in the literature, making the maximal value of the Hellinger distance equal \(\frac{1}{\sqrt{2}}\).

The Kullback–Leibler divergence or relative entropy is defined by
\[
\text{Entropy}(\nu \mid \mu) = \int \log \frac{d\nu}{d\mu} \,d\nu = \int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} \,d\mu \in [0, +\infty]
\]
if \(\nu\) is absolutely continuous with respect to \(\mu\), and \(\text{Entropy}(\nu \mid \mu) = +\infty\) otherwise.
The $L^2$ distance is given by
\begin{equation}
\left| \frac{\text{d}v}{\text{d}\mu} - 1 \right|_{L^2(\mu)} = \left( \int \left| \frac{\text{d}v}{\text{d}\mu} - 1 \right|^2 \text{d}\mu \right)^{1/2} \in [0, +\infty] \tag{A.5}
\end{equation}
and this quantity is set to $+\infty$ if $\nu$ is not absolutely continuous with respect to $\mu$. It is related to the chi-square divergence as follows: $\chi^2(\nu \mid \mu) = \text{Var}_\mu(\frac{\text{d}v}{\text{d}\mu}) = \left| \frac{\text{d}v}{\text{d}\mu} - 1 \right|_{L^2(\mu)}^2 = \left| \frac{\text{d}v}{\text{d}\mu} \right|_{L^2(\mu)} - 1$.

The Fisher information or divergence is defined by
\begin{equation}
\text{Fisher}(\nu \mid \mu) = \int \nabla \log \frac{\text{d}v}{\text{d}\mu}^2 \text{d}v = \int \left| \nabla \frac{\text{d}v}{\text{d}\mu} \right|^2 \text{d}\mu = 4 \int \left| \nabla \sqrt{\frac{\text{d}v}{\text{d}\mu}} \right|^2 \text{d}\mu \in [0, +\infty] \tag{A.6}
\end{equation}
if $\nu$ is absolutely continuous with respect to $\mu$, and $\text{Fisher}(\nu \mid \mu) = +\infty$ otherwise.

Each of these distances or divergences has its advantages and drawbacks. In some sense, the most sensitive is Fisher due to its Sobolev nature, then $L^2$, then Entropy which can be seen as a sort of $L^{1+} = L \log L$ norm, then TV and Hellinger, which are comparable, then Wasserstein, but this rough hierarchy misses some subtleties related to some scales and nature of the arguments.

Some of these distances or divergences can generically be compared as the following result shows.

**Lemma A.1 (Inequalities).** For any probability measures $\mu$ and $\nu$ on the same space,
\begin{align*}
\|\mu - \nu\|_{TV}^2 & \leq 2 \text{Entropy}(\nu \mid \mu) \\
2 \text{Hellinger}^2(\mu, \nu) & \leq \text{Entropy}(\nu \mid \mu) \\
\text{Entropy}(\nu \mid \mu) & \leq 2 \left\| \frac{\text{d}v}{\text{d}\mu} - 1 \right\|_{L^2(\mu)} + 2 \left\| \frac{\text{d}v}{\text{d}\mu} - 1 \right\|_{L^2(\mu)}^2 \\
\text{Hellinger}^2(\mu, \nu) & \leq \|\mu - \nu\|_{TV} \leq \text{Hellinger}(\mu, \nu)(2 - \text{Hellinger}(\mu, \nu)^2)^{1/2}.
\end{align*}

We refer to [48, p. 61-62] for a proof. The inequality between the total variation distance and the relative Entropy is known as the Pinsker or Csiszár–Kullback inequality, while the inequalities between the total variation distance and the Hellinger distance are due to Kraft. There are many other metrics between probability measures, see for instance [49, 30] for a discussion.

The total variation distance can also be seen as a special Wasserstein distance of order 1 with respect to the atomic distance, namely
\begin{equation}
\|\mu - \nu\|_{TV} = \inf_{(X,Y)} \mathbb{P}(X \neq Y) = \inf_{(X,Y)} \mathbb{E}[\mathbb{1}_{X \neq Y}] \in [0, 1] \tag{A.7}
\end{equation}
where the infimum runs over all couplings $X \sim \mu$ and $Y \sim \nu$. This explains in particular why TV is more sensitive than Wasserstein at short scales but less sensitive at large scales, a consequence of the sensitivity difference between the underlying atomic and Euclidean distances. The probabilistic representations of TV and Wasserstein make them comparable with techniques of coupling, which play an important role in the literature on convergence to equilibrium of Markov processes.

We gather now useful results on distances and divergences.

**Lemma A.2 (Contraction properties).** Let $\mu$ and $\nu$ be probability measures on the same space $S$.

- **If** $f : S \to T$ is a measurable function for some space $T$, then\[
\|\mu \circ f^{-1} - \nu \circ f^{-1}\|_{TV} \leq \|\mu - \nu\|_{TV}.
\]
- **If** $S = \mathbb{R}^n$, $T = \mathbb{R}^k$ and $f : S \to T$ is measurable then, denoting $\|f\|_{\text{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$,\[
\text{Wasserstein}(\mu \circ f^{-1}, \nu \circ f^{-1}) \leq \|f\|_{\text{Lip}} \text{Wasserstein}(\mu, \nu).
\]
- **If** $f : S \to T$ is a measurable function for some space $T$, then\[
\text{Entropy}(\nu \circ f^{-1} \mid \mu \circ f^{-1}) \leq \text{Entropy}(\nu \mid \mu).
\]

The notation $f^{-1}$ stands for the reciprocal map $f^{-1}(A) = \{y \in S : f(x) \in A\}$ and $\mu \circ f^{-1}$ is the image measure or push-forward of $\mu$ by the map $f$, defined by $(\mu \circ f^{-1})(A) = \mu(f^{-1}(A))$. In terms of random variables we have $Y \sim \mu \circ f^{-1}$ if and only $Y = f(X)$ where $X \sim \mu$. 

The proof of the contraction properties of Lemma A.2 are all based on variational formulas. Note that following [55, Ex. 22.20 p. 588], there is a variational formula for Fisher that comes from its dual representation as an inverse Sobolev norm. We do not develop this idea in this work.

**Proof.** All these contraction properties are consequences of variational formulas (supremum or infimum). The first property comes from the definitions of $\| \cdot \|_{TV}$ and of measurability. The second comes from the fact that every coupling of $\mu$ and $\nu$ produces a coupling for $\mu \circ f^{-1}$ and $\nu \circ f^{-1}$.

The third property can be proved using the following well known variational formula:

$$\text{Entropy}(\nu \mid \mu) = \sup_g \{ \mathbb{E}_\nu[g] - \log \mathbb{E}_\mu[e^g] \}$$

where the supremum runs over all $g \in L^1(\nu)$, or by approximation when the supremum runs over all bounded measurable $g$. This variational formula can be derived for instance by applying Jensen’s inequality to $-\log \mathbb{E}_\nu[e^g \frac{d\nu}{d\mu}]$. Equality is achieved for $g = \log(d\nu/d\mu)$. Now, taking $g = h \circ f$ gives

$$\text{Entropy}(\nu \mid \mu) \geq \text{Entropy}(\nu \circ f^{-1} \mid \mu \circ f^{-1})$$

The variational formula for $\text{Entropy}(- \mid \mu)$ is a manifestation of its convexity, it expresses this functional as the envelope of its tangents, its Fenchel–Legendre transform or dual is the log-Laplace transform. This variational formula is equivalent to tensorization, see for instance [16, Th. 4.4]. □

**Lemma A.3** (Scale invariance versus homogeneity). The total variation distance is scale invariant while the Wasserstein distance is homogeneous just like a norm, namely for all probability measures $\mu$ and $\nu$ on $\mathbb{R}^n$ and all scaling factor $\sigma \in (0, \infty)$, denoting $\mu_\sigma = \text{Law}(\sigma X)$ where $X \sim \mu$, we have

$$\| \mu_\sigma - \nu_\sigma \|_{TV} = \| \mu - \nu \|_{TV} \quad \text{while} \quad \text{Wasserstein}(\mu_\sigma, \nu_\sigma) = \sigma \text{Wasserstein}(\mu, \nu).$$

**Lemma A.4** (Tensorization). For all probability measures $\mu_1, \ldots, \mu_n$ and $\nu_1, \ldots, \nu_n$ on $\mathbb{R}$, we have

$$\text{Hellinger}^2\left(\otimes_{i=1}^n \mu_i, \otimes_{i=1}^n \nu_i \right) = 1 - \prod_{i=1}^n \left(1 - \text{Hellinger}^2(\mu_i, \nu_i)\right),$$

$$\text{Entropy}(\otimes_{i=1}^n \nu_i \mid \otimes_{i=1}^n \mu_i) = \sum_{i=1}^n \text{Entropy}(\nu_i \mid \mu_i),$$

$$\text{Fisher}(\otimes_{i=1}^n \nu_i \mid \otimes_{i=1}^n \mu_i) = \sum_{i=1}^n \text{Fisher}(\nu_i \mid \mu_i),$$

$$\text{Wasserstein}^2\left(\otimes_{i=1}^n \mu_i, \otimes_{i=1}^n \nu_i \right) = \sum_{i=1}^n \text{Wasserstein}^2(\mu_i, \nu_i),$$

$$\max_{1 \leq i \leq n} \| \mu_i - \nu_i \|_{TV} \leq \| \otimes_{i=1}^n \mu_i - \otimes_{i=1}^n \nu_i \|_{TV} \leq \sum_{i=1}^n \| \mu_i - \nu_i \|_{TV}.$$

The equality for the Wasserstein distance comes by taking the product of optimal couplings.

The first inequality for the total variation distance comes from its contraction property (Lemma A.2), while the second comes from $| (a_1 \cdots a_n) - (b_1 \cdots b_n) | \leq \sum_{i=1}^n | a_i - b_i | (a_1 \cdots a_{i-1} b_{i+1} \cdots b_n)$, $a_1, \ldots, a_n, b_1, \ldots, b_n \in [0, +\infty)$, which comes itself from the triangle inequality on the telescoping sum $\sum_{i=1}^n (c_i - c_{i-1})$ where $c_i = (a_1 \cdots a_i) (b_{i+1} \cdots b_n)$ via $c_i - c_{i-1} = (a_i - b_i) (a_1 \cdots a_{i-1} b_{i+1} \cdots b_n)$.

**Lemma A.5** (Explicit formulas for Gaussian distributions). For all $n \geq 1$, $m_1, m_2 \in \mathbb{R}^n$, and all $n \times n$ covariance matrices $\Sigma_1, \Sigma_2$, denoting $\Gamma_1 = N(\mu_1, \Sigma_1)$ and $\Gamma_2 = N(\mu_2, \Sigma_2)$, we have

$$\text{Hellinger}^2(\Gamma_1, \Gamma_2) = 1 - \frac{\det(\Sigma_1, \Sigma_2)^{1/4}}{\det(\Sigma_1^{1/2} \Sigma_2^{1/2})^{1/2}} \exp\left(-\frac{1}{4} (\Sigma_1 + \Sigma_2)^{-1} (m_2 - m_1) \cdot (m_2 - m_1)^\top\right),$$

$$2\text{Entropy}(\Gamma_1 \mid \Gamma_2) = \Sigma_2^{-1} (m_1 - m_2) \cdot (m_1 - m_2) + \text{Tr}(\Sigma_2^{1/2} \Sigma_1^{-1} \text{Id}) + \log \det(\Sigma_2 \Sigma_1^{-1}),$$

$$\text{Fisher}(\Gamma_1 \mid \Gamma_2) = |\Sigma_2^{-1} (m_1 - m_2)|^2 + \text{Tr}(\Sigma_2^{-2} \Sigma_1 - 2 \Sigma_2^{-1} + \Sigma_1^{-1})$$
Wasserstein\(^2(\Gamma_1, \Gamma_2) = |m_1 - m_2|^2 + \text{Tr}(\Sigma_1 + \Sigma_2 - 2\sqrt{\Sigma_1 \Sigma_2} \sqrt{\Sigma_1}).\)

and the formulas for Fisher and Wasserstein rewrite, if \(\Sigma_1\) and \(\Sigma_2\) commute, \(\Sigma_1 \Sigma_2 = \Sigma_2 \Sigma_1\), to

\[
\text{Fisher}(\Gamma_1 \mid \Gamma_2) = |\Sigma_1^{-1}(m_1 - m_2)|^2 + \text{Tr}(\Sigma_2^{-2}(\Sigma_2 - \Sigma_1)^2 \Sigma_1^{-1})
\]

\[
\text{Wasserstein}^2(\Gamma_1, \Gamma_2) = |m_1 - m_2|^2 + \text{Tr}((\sqrt{\Sigma_1} - \sqrt{\Sigma_2})^2).
\]

Regarding the total variation distance, there is no general simple formula for Gaussian laws, but we can use for instance the comparisons with Entropy and Hellinger (Lemma A.1), see [19] for a discussion.

**Proof of Lemma A.5.** We refer to [47, p. 47 and p. 51] for Entropy and Hellinger, and to [31] for Wasserstein, a far more subtle case. We have not found in the literature a formula for Fisher. Let us give it here for the sake of completeness. Using \(\mathbb{E}[X, X_j] = \Sigma_{ij} + m_i m_j\) when \(X \sim \mathcal{N}(\mu, \Sigma)\) we get, for all \(n \times n\) symmetric matrices \(A\) and \(B\)

\[
\mathbb{E}[AX \cdot BX] = \sum_{i,j,k=1}^{n} A_{ij} B_{jk} \mathbb{E}[X_j X_k] = \sum_{i,j,k=1}^{n} A_{ij} B_{jk} (\Sigma_{jk} + m_j m_k) = \text{Trace}(\Sigma^2) + Am \cdot Bm
\]

and thus for all \(n\)-dimensional vectors \(a\) and \(b\),

\[
\mathbb{E}[A(X - a) \cdot B(X - b)] = \mathbb{E}[AX \cdot BX] + A(m - a) \cdot B(m - b) - Am \cdot Bm
\]

\[
= \text{Trace}(\Sigma^2) + A(m - a) \cdot B(m - b).
\]

Now, using the notation \(q_i(x) = \Sigma_i^{-1} - (x - m_i) \cdot (x - m_i)\) and \(|\Sigma_i| = \det(\Sigma_i)\),

\[
\text{Fisher}(\Gamma_1 \mid \Gamma_2) = 4 \frac{\sqrt{\Sigma_2}}{\sqrt{|\Sigma_1|}} \left[ \int e^{-\frac{q_1(x) + q_2(x)}{2}} \frac{e^{-\frac{q_i(x)}{2}}}{\sqrt{2\pi}|\Sigma_i|} dx \right]
\]

\[
= \int |\Sigma_1^{-1}(x - m_2) - \Sigma_1^{-1}(x - m_1)|^2 \frac{e^{-\frac{q_i(x)}{2}}}{\sqrt{2\pi}|\Sigma_i|} dx
\]

\[
= \int (|\Sigma_2^{-1}(x - m_2)|^2 - 2\Sigma_2^{-1}(x - m_2) \cdot \Sigma_1^{-1}(x - m_1) + |\Sigma_1^{-1}(x - m_1)|^2) \frac{e^{-\frac{q_i(x)}{2}}}{\sqrt{2\pi}|\Sigma_i|} dx
\]

\[
= \text{Trace}(\Sigma_2^{-2} \Sigma_1 \Sigma_2^{-1}) + |\Sigma_1^{-1}(m_1 - m_2)|^2 - 2\text{Trace}(\Sigma_2^{-1}) + \text{Trace}(\Sigma_1^{-1})
\]

\[
= \text{Trace}(\Sigma_2^{-2} \Sigma_1 - 2\Sigma_2^{-1} + \Sigma_1^{-1}) + |\Sigma_2^{-1}(m_1 - m_2)|^2.
\]

The formula when \(\Sigma_1 \Sigma_2 = \Sigma_2 \Sigma_1\) follows immediately. \(\square\)

**Appendix B. Convexity and its dynamical consequences.**

We gather useful dynamical consequences of convexity. We start with functional inequalities.

**Lemma B.1** (Logarithmic Sobolev inequality). Let \(P_n^\beta\) be the invariant law of the DOU process solving (1.3). Then, for all law \(\nu\) on \(\mathbb{R}^n\), we have

\[
\text{Entropy}(\nu \mid P_n^\beta) \leq \frac{1}{2n} \text{Fisher}(\nu \mid P_n^\beta).
\]

Moreover the constant \(\frac{1}{2n}\) is optimal.

Furthermore, finite equality is achieved if and only if \(d\nu / dP_n^\beta\) is of the form \(e^{\lambda(x_1 + \ldots + x_n)}\), \(\lambda \in \mathbb{R}\).

We refer to [28, 17] for a proof of Lemma B.1. This logarithmic Sobolev inequality is a consequence of the log-concavity of \(P_n^\beta\) with respect to \(\mathcal{N}(0, I_n)\). A slightly delicate aspect lies in the presence of the restriction to \(D_n\), which can be circumvented by using a regularization procedure.

There are many other functional inequalities which are a consequence of this log-concavity, for instance the Talagrand transportation inequality that states that when \(\nu\) has finite second moment,

\[
\text{Wasserstein}^2(\nu, P_n^\beta) \leq \frac{1}{n} \text{Entropy}(\nu \mid P_n^\beta).
\]
and the HWI inequality\(^1\) that states that when \(\nu\) has finite second moment,
\[
\text{Entropy}(\nu \mid P^n_\beta) \leq \text{Wasserstein}(\nu, P^n_\beta)\sqrt{\text{Fisher}(\nu \mid P^n_\beta)} - \frac{n}{2}\text{Wasserstein}^2(\nu \mid P^n_\beta),
\]
and we refer to [55] for this couple of functional inequalities, that we do not use here.

**Lemma B.2** (Sub-exponential convergence to equilibrium). Let \((X^n_t)_{t \geq 0}\) be the DOU process solution of (1.3) with \(\beta = 0\) or \(\beta \geq 1\), and let \(P^n_\beta\) be its invariant law. Then for all \(t \geq 0\), we have the sub-exponential convergences
\[
\text{Entropy}(\text{Law}(X^n_t) \mid P^n_\beta) \leq e^{-2t}\text{Entropy}(\text{Law}(X^n_0) \mid P^n_\beta),
\]
\[
\text{Fisher}(\text{Law}(X^n_t) \mid P^n_\beta) \leq e^{-2t}\text{Fisher}(\text{Law}(X^n_0) \mid P^n_\beta),
\]
\[
\text{Wasserstein}^2(\text{Law}(X^n_t), P^n_\beta) \leq e^{-2t}\text{Wasserstein}^2(\text{Law}(X^n_0), P^n_\beta).
\]

Recall that when \(\beta > 0\) the initial condition \(X^n_0\) is always taken in \(D_n\).

For each inequality, if the right-hand side is infinite then the inequality is trivially satisfied. This is in particular the case for Entropy and Fisher when \(\text{Law}(X^n_0)\) is not absolutely continuous with respect to the Lebesgue measure, and for Wasserstein when \(\text{Law}(X^n_0)\) has infinite second moment.

**Elements of proof of Lemma B.2.** It is a rather standard piece of probabilistic functional analysis, a consequence of the log-concavity of \(P^n_\beta\) again. We recall the crucial steps for the reader convenience. Let us set \(\mu_t = \text{Law}(X^n_t)\) and \(\mu = P^n_\beta\). For \(t > 0\) the density \(p_t = d\mu_t/d\mu\) exists and solves the evolution equation \(\partial_t p_t = G p_t\) where \(G\) is as in (2.5). We also have the integration by parts
\[
\int f G g d\mu = \int g F f d\mu = \frac{1}{n} \int \nabla f \cdot \nabla g d\mu.
\]

For Entropy, we find using these tools, for all \(t > 0\),
\[
\partial_t \text{Entropy}(\mu_t \mid \mu) = -\frac{1}{n}\text{Fisher}(\mu_t \mid \mu) \leq -2\text{Entropy}(\mu_t \mid \mu), \quad \text{(B.1)}
\]
where the inequality comes from the logarithmic Sobolev inequality of Lemma B.1. It remains to use the Grönwall lemma to get the exponential decay of Entropy.

The derivation of the exponential decay of the Fisher divergence follows the same lines by differentiating again with respect to time. Indeed, after a sequence of differential computations and integration by parts, we find, using again the log-concavity of \(P^n_\beta\), for all \(t > 0\),
\[
\partial_t \text{Fisher}(\mu_t \mid \mu) \leq -2\text{Fisher}(\mu_t \mid \mu).
\]

This differential approach goes back at least to Stam and was extensively developed independently by Bakry, Ledoux, Villani and their followers. We refer to [3, Ch. 5] and [55] for more details. This can be used to prove the log-Sobolev inequality.

For the Wasserstein distance, we proceed by coupling. Indeed, since the diffusion coefficient is constant in space, we can simply use a parallel coupling. Namely, let \((X^n_t)_{t \geq 0}\) be the process started from another possibly random initial condition \(X^n_0\), and satisfying to the same stochastic differential equation, with the same BM. We get
\[
d(X_t - X'_t) = -\frac{1}{n} (\nabla E(X_t) - \nabla E(X'_t)) dt,
\]
hence
\[
d(X_t - X'_t) \cdot (X_t - X'_t) = -\frac{2}{n} (\nabla E(X_t) - \nabla E(X'_t)) \cdot (X_t - X'_t) dt.
\]
Now since \(E\) is uniformly convex with \(\nabla^2 E \geq nI_n\), we get, for all \(x, y \in \mathbb{R}^n\),
\[
(\nabla E(x) - \nabla E(y)) \cdot (x - y) \geq n|x - y|^2,
\]
which gives
\[
d|X_t - X'_t|^2 \leq -2|X_t - X'_t|^2 dt
\]
and by the Grönwall lemma,
\[
|X_t - X'_t|^2 \leq e^{-2t}|X_0 - X'_0|^2.
\]

\(^1\)Here “H” is the capital \(\eta\) used by Boltzmann for entropy, “W” is for Wasserstein, “I” is for Fisher information.
It follows that
\[ \text{Wasserstein}^2(\text{Law}(X_t), \text{Law}(X'_t)) \leq e^{-2t}\mathbb{E}[||X_0 - X'_0||^2]. \]
By taking the infimum over all couplings of \( X_0 \) and \( X'_0 \) we get
\[ \text{Wasserstein}^2(\text{Law}(X_t), \text{Law}(X'_t)) \leq e^{-2t}\text{Wasserstein}^2(\text{Law}(X_0), \text{Law}(X'_0)). \]
Taking \( X'_0 \sim P_n^\beta \) we get, by invariance, for all \( t \geq 0 \),
\[ \text{Wasserstein}^2(\text{Law}(X_t), P_n^\beta) \leq e^{-2t}\text{Wasserstein}^2(\text{Law}(X_0), P_n^\beta). \]
\[ \square \]

**Remark B.3** (Monotonicity). The formula for the derivative of \( t \mapsto \text{Entropy}((X_t) \mid P_n^\beta) \) in (B.1), sometimes attributed to de Bruijn, reveals a monotonicity, which is a general property of Markov processes also referred to as “entropy dissipation”. Such a monotonicity holds also for the other distances and divergences, and for some of them it relies on the convexity of \( E \), see [9].

The convexity of the interaction – log as well as the constant nature of the diffusion coefficient in the evolution equation (1.3) allows to use simple “maximum principle” type arguments to prove that the dynamic exhibits a monotonous behavior and an exponential decay.

**Lemma B.4** (Monotonicity and exponential decay). Let \((X^n_t)\) and \((Y^n_t)\) be a pair of DOU processes solving (1.3), \( \beta \geq 1 \), driven by the same Brownian motion \((B_t)_{t \geq 0}\) on \( \mathbb{R}^n \) and with respective initial conditions \( X^n_0 \in \mathcal{D}_n \) and \( Y^n_0 \in \mathcal{D}_n \). If for all \( t \in \{1, \ldots, n\} \)
\[ X^{n,i}_t \leq Y^{n,i}_t \]
then the following properties hold true:

- (Monotonicity property) for all \( t \geq 0 \) and \( i \in \{1, \ldots, n\} \),
  \[ X^{n,i}_t \leq Y^{n,i}_t, \]
- (Decay estimate) for all \( t \geq 0 \),
  \[ \max_{i \in \{1, \ldots, n\}} (Y^{n,i}_t - X^{n,i}_t) \leq \max_{i \in \{1, \ldots, n\}} (Y^{n,i}_0 - X^{n,i}_0)e^{-t}. \]

**Proof of Lemma B.4.** The difference of \( Y^n_t - X^n_t \) satisfies
\[ \partial_t(Y^{n,i}_t - X^{n,i}_t) = \frac{\beta}{n} \sum_{j \not= i} \frac{(Y^{n,j}_t - X^{n,j}_t) - (Y^{n,i}_t - X^{n,i}_t)}{(Y^{n,j}_t - Y^{n,i}_t)(X^{n,j}_t - X^{n,i}_t)} - (Y^{n,i}_t - X^{n,i}_t). \] (B.2)

Since there are almost surely no collisions between the coordinates of \( X^n \), resp. of \( Y^n \), the right-hand side is almost surely finite for all \( t > 0 \) and every process \( Y^{n,i}_t - X^{n,i}_t \) is \( \mathcal{C}^1 \) on \((0, \infty)\). Note that at time 0 some derivatives may blow up as two coordinates of \( X^n \) or \( Y^n \) may coincide.

Let us define
\[ M(t) = \max_{i \in \{1, \ldots, N\}} (Y^{n,i}_t - X^{n,i}_t) \land m(t) = \min_{i \in \{1, \ldots, N\}} (Y^{n,i}_t - X^{n,i}_t). \]

Elementary considerations imply that \( M \) and \( m \) are themselves \( \mathcal{C}^1 \) on \((0, \infty)\) and that at all times \( t > 0 \), there exist \( i, j \) such that
\[ \partial_t M(t) = \partial_t(Y^{n,i}_t - X^{n,i}_t) \land \partial_t m(t) = \partial_t(Y^{n,j}_t - X^{n,j}_t). \]

This would not be true if there were infinitely many processes of course. Now observe that if at time \( t > 0 \) we have \( Y^{n,i}_t - X^{n,i}_t = M(t) \), then
\[ \partial_t(Y^{n,i}_t - X^{n,i}_t) \leq -(Y^{n,i}_t - X^{n,i}_t). \]

This implies that \( \partial_t M(t) \leq -M(t) \). Similarly, we can deduce that \( \partial_t m(t) \geq -m(t) \). Integrating these differential equations, we get for all \( t \geq t_0 > 0 \)
\[ M(t) \leq e^{-(t-t_0)}M(t_0), \quad m(t) \geq e^{-(t-t_0)}m(t_0). \]

Since all processes are continuous on \([0, \infty)\), we can pass to the limit \( t_0 \downarrow 0 \) and get for all \( t \geq 0 \),
\[ \min_{i \in \{1, \ldots, N\}} (Y^{n,i}_t - X^{n,i}_t) \geq 0, \quad \max_{i \in \{1, \ldots, N\}} (Y^{n,i}_t - X^{n,i}_t) \leq e^{-t} \max_{i \in \{1, \ldots, N\}} (Y^{n,i}_0 - X^{n,i}_0). \]
\[ \square \]
Remark B.5 (Beyond the DOU dynamics). The monotonicity property of Lemma B.4 relies on the convexity of the interaction \(-\log\), and has nothing to do with the long-time behavior and the strength of \(V\). In particular, this monotonicity property remains valid for the process solving (1.3) with an arbitrary \(V\) provided that it is \(C^1\) and there is no explosion, even in the situation where \(V\) is not strong enough to ensure that the process has an invariant law. If \(V\) is \(C^2\) then the decay estimate of Lemma B.4 survives in the following decay or growth form:

\[
\max_{i \in \{1, \ldots, n\}} \left( Y_t^{\ast n,i} - X_t^{\ast n,i} \right) \leq \max_{i \in \{1, \ldots, n\}} \left( Y_0^{\ast n,i} - X_0^{\ast n,i} \right) e^{-\left( -\inf_{\nu} \langle V'' \rangle \right)} , \quad t \geq 0.
\]

Acknowledgements

J.B. is supported by a grant from the “Fondation CFM pour la Recherche”.

C.L. is supported by the project SINGULAR ANR-16-CE40-0020-01.

References

On the law of large numbers for the empirical measure process

44. Songzi Li, Xiang-Dong Li, and Yong-Xiao Xie, Gaussian fluctuations of eigenvalues in the GUE


34. Alan J. Hoffman and Helmut W. Wielandt, The variation of the spectrum of a normal matrix


45. Ross A. Lippert, A matrix model for the Two singular diffusion problems,


52. Laurent Saloff-Coste, Precise estimates on the rate at which certain diffusions tend to equilibrium, Mathematische Zeitschrift 217 (1994), no. 1, 641–677. 2, 10, 11, 12, 13


