Belief Dispersion and Convex Cost of Adjustment in the Stock Market and in the Real Economy*

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Abstract

We develop a continuous-time general equilibrium model with a continuum of states of the world and a continuum of agents endowed with heterogeneous beliefs. The model permits to analyze the interactions between financial markets and production. There is a single firm that faces convex adjustment costs and maximizes its terminal value. Equivalently, the firm uses a decreasing returns to scale risk-return technology. The model is tractable and matches many of the empirical regularities in aggregate output and stock prices, such as a financial volatility that is higher than the macroeconomic volatility, skewness, kurtosis, short-term momentum, and volatility risk premium during recessions. All these aspects disappear when one assumes beliefs homogeneity or constant returns to scale. In particular, the impact of beliefs heterogeneity observed in endowment economies does not pertain when introducing production unless one assumes decreasing returns to scale in the risk-return technology.


Keywords: Asset pricing, belief dispersion, production equilibrium, decreasing returns, adjustment costs, heterogeneous beliefs, excessive volatility, asset pricing puzzles.

1 Introduction

Officer (1973) relates stock market volatility to the volatility of macroeconomic variables. However, Shiller (1981a,b) argues that the level of stock market volatility is too high relative to the ex post variability of dividends:

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It has often been objected in popular discussions that stock price indexes are too "volatile", i.e., that the movements in stock price indexes could not realistically be attributed to any objective new information, since movements in the indexes are "too big" relative to actual subsequent-movements in dividends.

Financial leverage is a possible explanation, as predicted by Black (1976) and Christie (1982), although this factor explains only a small part of the variation in stock volatility.

Recent literature on heterogeneous beliefs in financial markets explores the impact of belief dispersion on stock returns (Abel, 1989, Anderson, Ghysels and Juergens, 2005, David, 2008, Banerjee and Kremer, 2010) and on stock volatility (e.g., Scheinkman and Xiong, 2003, Buraschi and Jiltsov, 2006, Dumas, Kurshev and Uppal, 2009). As shown by Jouini and Napp (2011), in such models, the economy is dominated by the pessimistic agent(s) in bad states of the world and by the optimistic agent(s) in good states of the world. Since pessimistic (optimistic) beliefs are associated to a higher (lower) risk premium, this leads to risk premium fluctuations. Therefore, even when dogmatic beliefs are assumed (no belief change at the individual belief level and no disagreement shock), there are belief fluctuations (sentiment risk) at the representative agent level that act as an additional source of risk and increase both the risk premium and the financial assets volatility (see also Basak 2000, 2005, Jouini and Napp, 2007, Bhamra and Uppal, 2014, Atmaz and Basak, 2018).

However, in almost all these papers\(^1\), the payoff/production process is considered as given while belief heterogeneity should also impact investment decisions and, doing so, the production process itself, its growth rate and the associated level of risk. As underlined by Greenberg et al. (1978), "although many interesting and useful results emerge from the analysis of exchange of fixed quantities of risk and return, an equally important set of issues arises in connection with the fact that the firm may vary the risk-return combination it offers to the market".

Hence, in a production setting, the optimistic beliefs observed in good (bad) states of the world should lead to an activity expansion (contraction). This should generate additional volatility that might either go the same way as the volatility resulting from sentiment risk and risk premium fluctuations or the other way.

In this paper, our first result is that when investment is perfectly responsive to shocks, it perfectly offsets the increase in financial market volatility observed in pure endowment models: capital and price volatility coincide, and differences in beliefs affect risk quantities but cannot explain excess volatility. In such a context, our results challenge previous findings on belief heterogeneity as a possible explanation of the excess volatility puzzle.

In fact, the mechanism is the following. It is well known that without belief heterogeneity nor disagreement, the financial volatility is equal to the production volatility. When production is exogenously given, good (bad) news and the resulting optimism (pessimism) put an upward (downward) pressure on the demand for consumption good leading to an increase (decrease) of financial

\(^1\)Panageas (2005) and Buss et al. (2016) are among the few exceptions.
assets prices. These price fluctuations lead to additional financial assets volatility while production volatility is exogenously given. Hence, the volatility ratio increases with belief heterogeneity. When production is perfectly responsive to shocks, good (bad) news and the resulting optimism (pessimism) lead to an expansion (contraction) of the production activity in order to perfectly address the increasing (decreasing) demand in consumption goods. This leads to an increase of the production volatility and to an equivalent increase of the financial volatility: the volatility ratio remains constant equal to 1. Hence excess volatility does not only result from disagreement nor from production.

Our second finding is that excessive volatility results from output rigidity to shocks. Specifically, when the transformation of output into capital is relatively more costly at high levels of capital, the expansion (contraction) of output does not fully offset the additional upward (downward) pressure on the demand for consumer goods. As a result, price fluctuations are larger than output fluctuations and the volatility ratio is greater than 1, this lower bound corresponding to the situation where output is exogenously given.

More precisely, we consider a financial markets equilibrium model that has a finite horizon and that evolves in continuous time. There is one consumption good. The economy is assumed to be large, as it is populated by a continuum of investors with standard constant relative risk aversion (CRRA) preferences and heterogeneous beliefs. Considering a continuum of investors permits a parsimonious description of beliefs dispersion as in Atmaz and Basak (2018). Furthermore, as explained therein, such a setting leads to a non-vanishing beliefs dispersion (no investor dominates the economy in relatively extreme states) and "generate(s) intuitive, simple and uniform results, which are not immediately possible in (a) two-investor economy". We assume that there is only one firm that can be seen as an aggregated representation of all firms in the economy. As in a large strand of the financial macroeconomics literature (e.g. Lucas and Prescott, 1971, Abel and Eberly, 1994), we rely on the neoclassical theory of investment to model the behavior of the firm and its production function. Along the lines initiated by Uzawa (1969) (see also Hayashi, 1982), we assume that transforming production into capital is relatively more costly at high levels of capital or, in other words the cost of adjustment function is convex. Such an assumption fits the aggregate\(^2\) data reasonably well as shown by Cooper and Haltiwanger (2006).

We show that such a model is equivalent to a model à la Greenberg et al. (1978)\(^3\) where the firm maximizes its market value by choosing from among an attainable set of risk-return combinations. That set represents the "technology" for the production of risk and return. Thus, in our model, everything works as if the capital of the firm were produced through a technology along which, return is a function of the level of exposure to economic shocks: more risk leads to a larger expected quantity of output. The convexity of the cost of adjustment function appears as equivalent to a decreasing returns to scale assumption on the risk-return production function.

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\(^2\)Whereas the cost of adjustment function might present convexities as well as nonconvexities at the individual firm level.

\(^3\)Note that while Greenberg et al. (1978) adopt a partial equilibrium approach, we adopt a general equilibrium approach.
At the firm level, our model analyzes the impact of belief dispersion and links existing micro-founded asset pricing models (Atmaz and Basak, 2018) with investment models (Van Binsbergen and Opp, 2019). Seen as a condensed form of the whole economy, it also links stock market models and macroeconomic growth models. In both cases, the introduction of a production technology permits us to capture the possibility for the firm/economy to expand during good times and to contract during bad times. Due to decreasing returns to scale, expansion is increasingly costly.

Our reduced form model allows us to use the very powerful Arrow-Debreu approach, to establish the existence of an Arrow-Debreu equilibrium and to fully characterize them. When the production process is exogenously given, the divergence of opinion among the agents can be interpreted as a dogmatic divergence of beliefs in an otherwise standard exchange equilibrium model, as in Atmaz and Basak (2018). This provides a first benchmark for our analysis. Another benchmark is provided by a model with the endogenous determination of the production process but without a divergence of opinion. In both benchmarks, the model generates a constant risk exposure while it generates a stochastic one in the general framework.

Our model appears as particularly tractable and very parsimonious: it relies on three main ingredients namely, belief heterogeneity, leverage possibilities and decreasing returns to the installation of new capital, and on a very limited number of parameters (risk aversion, two beliefs distribution parameters and two risk-return production function parameters). It allows us to derive closed-form solutions for all the equilibrium characteristics. It also allows us to explain a large number of empirical studies on both macroeconomic variables (growth rate and macroeconomic volatility) and financial market variables (stock prices, risk premia, market volatility, individual assets volatility, option prices). Almost all the empirical regularities we retrieve disappear in the homogenous beliefs and/or exchange economy settings.

A third feature of our model is that it generates a negative relationship between the aggregate stock market risk premium and aggregate stock return volatility (e.g., French et al., 1987, Campbell and Hentschel, 1992). In line with Grulon et al. (2012) explanation of this phenomenon, this is not because the fundamental relation between these variables is negative but because both variables are affected by the same underlying macroeconomic factor namely, production volatility.

Among the testable consequences of our model, we derive Sharpe ratio and volatility ratio (between financial and macroeconomic volatilities) bounds that depend only on the level of risk aversion and of belief heterogeneity; we also derive a relation between financial volatility and risk premium, macroeconomic volatility, risk aversion and the instantaneous average belief.

**Related literature.** Our paper is very close to Atmaz and Basak (2018). However, there is no production therein and, as explained above, one of our contributions is to show that the excess volatility obtained in their framework does not resist to the introduction of production unless the adjustment function is convex or, equivalently, unless if there are decreasing returns in the risk-return production function. There are few papers studying disagreement in production economies among them, Baker et al., 2016, Heyerdahl-Larsen and Walden, 2018, Li and Lowenstein, 2019. The first one is the closest to our. It considers two investors who disagree about expected output growth
in a Cox-Ingersoll-Ross economy with adjustment costs. However, they do not allow for leverage and they do not have the negative relation between the aggregate stock market risk premium and aggregate stock return volatility. Furthermore, none of these papers provides explicit formulas that, in particular, link the price of risk and the production technology parameters.

To our knowledge, our paper provides the first example of a fully calculable general equilibrium with production in a large number of states of the world and a large number of agents setting.

Organization of the paper. Section 2 introduces the model. Section 3 characterizes the Arrow-Debreu equilibrium. Section 4 introduces and analyzes the securities market. Section 5 extends our result on excess volatility to a continuous time consumption setting. Section 6 concludes. All proofs are in the Appendix. An Online Appendix considers some additional properties of the model (closed-form formula for the representative agent belief, autocorrelation properties of the capital process, results about the trading volume, impact on stock characteristics when taking into account debt constraints, leverage and embedded options). It also provides an asymptotic analysis of the model when the horizon becomes arbitrarily large. This permits to show that the model generates a long-run risk component that leads to long-run risk premia that are higher than the short-run one.

2 The model

We consider a continuous time model with one firm and a large number of consumers. The set of dates is described by $\mathbb{T} = [0, T]$. There is a single source of risk in the economy, which is represented by a Brownian motion $(W_t)_{t \in \mathbb{T}}$ and we denote the associated Brownian filtration by $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$.

There is only one output that is produced by the firm and reinvested in it. At the terminal date $T$, shareholders receive the share of the terminal stock of capital that corresponds to their initial number of shares of the firm. We assume that markets are dynamically complete and shareholders can trade a financial asset whose terminal payoff corresponds to the terminal stock of capital. Equivalently, as in Duffie and Huang (1985), we may consider that shareholders face, at $t = 0$, a complete set of contracts whose terminal payoff is contingent to date $T$ states of the world. Even though, there is no intertemporal consumption, the model does have an intertemporal dimension because information is revealed dynamically and, as developed below, the production and stock of capital processes are dynamic and depend crucially on how information is revealed over time and on the dynamic strategy adopted by the firm.

2.1 The firm

2.1.1 A stochastic AK model with adjustment costs

Jones and Manuelli (2005) introduce a stochastic analog of a standard AK model with a twist. We add adjustment costs to their model. The firm faces two types of technologies to produce consumption (alternatively, the model can be interpreted as a two-sector model with goods that are perfect substitutes). We interpret the first technology as a productive technology and the second
technology as an alternative "safe" technology. Both technologies follow a constant productivity AK model with the same productivity parameter $A$. Without loss of generality, we take $A = 1$. We denote by $K^i_t$ the total stock of capital in technology $i$, it includes physical capital as traditionally measured, but also intangible capital (such as, patents, know-how, brand value, human capital and organizational capital). The date $t$ to $t+dt$ production is then given by $K^i_t dt$, $i = 1, 2$. Since there is no intermediate consumption, production is fully reinvested, and we assume that it is reinvested in the same technology and contributes to the corresponding stock of capital. Let us denote by $\theta^i_t$ the ratio of total capital invested in the $i^{th}$ technology (i.e. $K^i_t = \theta^i_t K_t$, $i = 1, 2$, with $\theta^1_t + \theta^2_t = 1$). Because the two technologies have the same productivity coefficient and because the production through a given technology is reinvested in the same technology, $\theta^i_t$ also corresponds to the ratio of the total date–$t$ production that is reinvested in the $i^{th}$ technology.

Hence, we have $I^i_t = K^i_t$ where $I^i_t$ is the date $t$ investment in the $i^{th}$ technology. Note that the reinvestment in the same technology assumption is innocuous because we assume that capital can costlessly be reallocated across technologies: there is no exogenous constraints on the dynamics of $(\theta^i_t)$.

As underlined by Hayashi (1982), the cost of installing $I$ units of investment is likely to depend on the size of $I$ with respect to $K$. In our two-sector/technology framework, we assume that the capital increase per unit of output reinvested in the $i^{th}$ technology is a function of the investment-capital ratio, i.e. of the form $\psi^i(I^i_t/K_t)$ $I^i_t$ where $I^i_t = K^i_t$ is the invested amount and $K_t \equiv K^1_t + K^2_t$, the total stock of capital$^4$.

The stock of capital $K^i_t$ follows

$$dK^i_t = (\psi^i(\theta^i_t) - d^i)K^i_t dt + \sigma^i_K K^i_t dW_t$$

where $W_t$ is the standard Brownian motion process. The nonnegative constant $d^i$ captures depreciation effects on the capital invested in the $i^{th}$ technology. The nonnegative constant $\sigma^i_K$ reflects the impact of stochastic shocks on the capital invested in the $i^{th}$ technology (called stochastic depreciation of capital by Pyndick and Wang, 2013) while the function $\psi^i$ reflects the adjustment costs that capture costs of installing capital$^5$.

Hence, the total stock of capital evolves according to

$$dK_t = [\theta^1_t(\psi^1(\theta^1_t) - d^1) + \theta^2_t(\psi^2(\theta^2_t) - d^2)]K_t dt + [\theta^1_t\sigma^1_K + \theta^2_t\sigma^2_K]K_t dW_t.$$  

When $\psi^1$ and $\psi^2$ are constant, investment is transformed into capital without frictions at a constant rate. In a one sector model, Li and Loewenstein (2019) introduce such constant rates but with two different values for investment ($\psi^+$ for $\theta_t > 0$) and disinvestment ($\psi^-$ for $\theta_t < 0$) and they assume that the capital stock is more costly to adjust when it becomes more productive and

$^4$Baker et al. (1996) consider a similar functional form in a one sector setting. In our model, $\psi^i$ is a function of $I^i_t/K_t$ and not of $I^i_t/K^i_t$ (nor $I_t/K_t$) because due to the absence of consumption, these last ratios are, by definition, constant and equal to 1.

$^5$When $\psi(\cdot)$ is linear, the adjustment cost is equal to $I^i_t^2/K_t$ which is a quite usual form in the literature.
is less costly to adjust when it is less productive (i.e., $\psi^+ < \psi^-$).

More generally, nonincreasing $\psi(.)$ leads to decreasing returns in the conversion process which in turn leads to interesting economic trade-offs (Eberly and Wang, 2009).

**Remark 1** Our AK model can also be derived in a model where the productive technology is described by a neoclassical production function $F(K, L)$ in capital ($K$) and labor ($L$) where, as usual, $F$ is a constant-returns to scale function, increasing and concave in each variable. In particular, $F$ might be a Cobb-Douglas or a CES production function. See Online Appendix.

### 2.1.2 The firm in reduced form

In this paper, we assume that $\psi^1$ and $\psi^2$ are affine and decreasing. As will be seen, this specific choice leads to a quadratic (concave) relation between the total capital growth rate and volatility. This is a key assumption to make the model tractable and to derive closed form expressions for all equilibrium characteristics. However, all our qualitative results pertain as long as $\psi^1$ and $\psi^2$ are decreasing (and the risk-return relation remains concave).

More precisely, we assume that $\psi^1(\hat{\theta}) = \alpha_1 - \beta_1 \hat{\theta}$ and $\psi^2(\hat{\theta}) = \alpha_2 - \beta_2 \hat{\theta}$ where $\alpha_1, \beta_1, \alpha_2$ and $\beta_2$ are nonnegative constants. Because the second technology is considered a safe technology, we assume $\sigma_K^2 > \sigma^2$ and $\alpha_1 - d^1 > \alpha_2 - d^2$ (which means that the productive technology is more efficient than the safe one at $\hat{\theta}_t = 0$).

The capital dynamics is then of the form

$$dK_t = m(\theta_t)K_t dt + \theta_t K_t dW_t, \ K_0 = 1$$

where $\theta_t = (\sigma_K^2 - \sigma_{\hat{\theta}}^2) \hat{\theta}_t + \sigma_{\hat{\theta}}^2$ is a scaled version of $\hat{\theta}_t$, where the function $m$ is defined by $m(\theta) = a + b \theta - c \theta^2$, and where the expressions of $a$, $b$ and $c$ can be easily derived by identification. The function $m$ is concave which means that ensuring a given increase in returns is increasingly costly in terms of additional risk to be borne. It models the trade-off between the uncertainty measured by $\theta_t$ and the expected growth rate $m(\theta_t)$ or how risk is transformed into return. We call it risk-return production function. The parameter $b$ measures the intensity of the risk-return relationship while $c$ measures its concavity.

Hence, the model can be described as governed by a total stock of capital dynamics that results from the different exogenous shocks modeled by $(W_t)_{t \in [0,T]}$ as well as from the management decisions modeled by the random progressively measurable process $\theta$ where $\theta(t, \varrho)$ represents the degree of exposure to risk at date $t$ and in state $\varrho$. The process $\theta(t, \varrho)$ is then a control process for the firm.

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6We have $a = \frac{(\alpha_1 - d^1 - \beta_1)(\sigma_K^2)^2 + (\alpha_2 - d^2 - \beta_2)(\sigma_{\hat{\theta}}^2)^2 + \sigma_K^2(\alpha_2 - d^2 - \alpha_2)}{(\sigma_K^2 - \sigma_{\hat{\theta}}^2)^2}$, $b = \frac{\sigma_K^2(\alpha_2 - d^2 - \alpha_2)}{(\sigma_K^2 - \sigma_{\hat{\theta}}^2)^2}$, $c = \frac{(\alpha_1 - d^1 - \beta_1)(\sigma_{\hat{\theta}}^2)^2 + (\alpha_2 - d^2 - \beta_2)(\sigma_K^2)^2 + \sigma_{\hat{\theta}}^2(\alpha_1 - d^1 - \beta_1)}{(\sigma_K^2 - \sigma_{\hat{\theta}}^2)^2}$.

7This dynamics resembles the dynamics considered by Baker et al. (2016). However, in this reference, $\theta_t$ (therein denoted by $i_t$) is the investment ratio and does not directly intervene in the volatility whereas, in our model, $\theta_t$ precisely represents the date-t macroeconomic volatility and is a scaled version of the ratio of the total stock of capital invested in the productive technology.
By choosing a given \( \theta \), the firm chooses among an attainable set of risk-return combinations for which the growth rate is a given quadratic function of the level of exposure to economic shocks.

We assume that \( \theta \) is progressively measurable and such that \( \int_0^T \theta^2 \, dt < \infty \) almost surely. We denote by \( \Theta \) the set of such processes. For a given process \( \theta \in \Theta \), the date \( T \) stock of capital/production is described by the random variable \( K_T^\theta \) in which \( K_T^\theta \) is the terminal value of the solution\(^8 \) \((K_t^\theta)_t\) of (1).

### 2.1.3 Profit maximization

Since there is no intermediate consumption, prices are described by a random variable \( p \) where 
\[
p(\varrho) dP(\varrho) = \text{the date-0 price of one unit of date-}T\text{ consumption at date } T \text{ in state } \varrho.
\]
Therefore, \( p(\varrho) \) is the state price density and the terminal value of the firm is given by \( E[pK_T] \).

Even though there is only one firm, we assume a competitive behavior which means that the firm considers the prices as given (price taking behavior). Indeed and as already mentioned, the firm must be seen as an aggregated representation of the production sector in the economy\(^9 \). Hence, for a state price density \( p \), it maximizes its value \( E[pK_T^\sigma] \) over the set \( \Theta \) of admissible strategies \( \theta \).

In the next, we will assume that shareholders disagree about the risk-return production function. Note that even though the different shareholders perceive differently the risk-return trade-off, they agree on the firm objective function described above. This point will be better discussed in the next section.

Because the firm maximizes the value of its terminal stock of capital and shareholders maximize their terminal allocation of capital, we may replace the process \( K_t \) by the process \( \hat{K}_t = \exp(-at)K_t \) and this would only correspond to a change in the unit of measure of date \( T \) stock of capital. It is easy to check that \( \frac{d\hat{K}_t}{\hat{K}_t} = (b\theta - c\theta^2) \hat{K}_t \, dt + \theta \hat{K}_t \, dW_t \). Hence, we may take \( a = 0 \) without any loss of generality. Unless otherwise specified, we take \( a = 0 \) (nonzero values of \( a \) will be introduced only for calibration purposes).

### 2.2 Consumers/shareholders

There is a continuum of consumers who own the firm. We will indifferently call them consumers, shareholders or agents. The shareholders disagree about the risk-return production function \( m \). They are indexed by their type \( \delta \), where a \( \delta \)-type shareholder believes that the risk-return production function is \( m_\delta \) defined by \( m_\delta(\theta) = m(\theta) + \delta \theta \). Hence, the agents have different beliefs about the distribution of the random variable \( K_T \) and about the dynamics of the random process \((K_t)\).

\(^8\)Note that the choice of a quadratic form for \( m \) guarantees that \( m(\theta) \) is progressively measurable and such that \( \int_0^T |m(\theta)| \, dt < \infty \) almost surely which in turns guarantees the existence of \( y \) defined by (1).

\(^9\)In particular, if we have a single source of risk and \( N \) firms, \( i = 1, \ldots, N \) and if we assume that the contribution of the \( i^{th} \) firm to the aggregate production volatility is given by \( \theta_i \) and its contribution to the aggregate production drift is \( b_i \theta_i - c_i \theta_i^2 \), the aggregate risk-return production function is obtained, as usual, by solving \( \max_{\sum_i \theta_i = \theta} \sum_i (b_i \theta_i - c_i \theta_i^2) \) under the constraint \( \sum_i \theta_i = \theta \). This gives \( m(\theta) = b\theta - c\theta^2 \) where \( b \) (\( c \)) is a weighted average of the \( b_i \)'s (\( c_i \)'s).
agent δ’s point of view, the dynamics of \((K_t)\) is given by

\[ dK_t = (m(t) + \delta t) K_t dt + \theta_t K_t dW_t^\delta \]  

(2)

where \((W_t^\delta)\) is a Brownian motion. By the Girsanov Theorem, this can be modeled by assuming that a δ-type shareholder has a subjective probability measure \(P^\delta\) whose density \(\frac{dP^\delta}{dP}\) with respect to the objective probability \(P\) is denoted by \(M_T^\delta\) and given by

\[ M_T^\delta = \exp\left(\frac{1}{2} \delta^2 T + \delta W_T\right). \]  

(3)

Beliefs at intermediary dates are given by \(M_t^\delta = \exp\left(-\frac{1}{2} \delta^2 t + \delta W_t\right)\) and the process \(M_t^\delta\) satisfies \(dM_t^\delta = \delta M_t^\delta dW_t\). This way of modeling the situation takes the same form as in the heterogeneous belief literature (see e.g., Jouini and Napp, 2007), where δ measures a bias in beliefs. However, in this literature, the agents have beliefs about the likelihood of positive and/or negative shocks\(^{10}\) : they disagree about the dynamics of \((W_t)\) and therefore about the dynamics of \((K_t)\). In the current setting, the agents agree about the dynamics of \((W_t)\) that corresponds to its objective dynamics. The divergence of opinion is about the risk-return production function and therefore about the relation between the strategy \((\theta_t)\) and the capital process \((K_t)\). In terms of the initial AK model, the divergence of opinion is about the depreciation rates and the functions \(\psi^1\) and \(\psi^2\).

As a consequence, a shareholder with positive (negative) δ puts a higher (lower) probability on high capital levels than the zero δ agent by overestimating (underestimating) the efficiency of the investment into capital transformation. We say that an agent with δ = 0 (δ > 0, δ < 0) is rational (optimistic, pessimistic). We assume that optimism and pessimism are persistent at the agents’ level. This assumption echoes the largely documented persistence of forecast errors (Ma et al., 2020).

We assume that δ can take all possible real values, which means that we take into account all possible shareholder biases.

Shareholders extract utility from their terminal contingent allocation of capital and they are assumed to be expected utility maximizers. The expected utility of a δ-type agent for a contingent allocation is defined as

\[ U^\delta(.) = E\left[M_T^\delta u(.)\right], \]  

(4)

in which \(u\) is a CRRA utility function (the same for all shareholders). That is,

\[ u(x) = \frac{1}{1-\gamma} x^{1-\gamma}, \text{ if } x \geq 0, \quad u(x) = -\infty \text{ if } x < 0, \text{ for some } \gamma > 0. \]

Each shareholder is endowed with a given number of shares of the firm. It is easy to see that, for a given δ, the aggregate behavior of all δ-type shareholders corresponds to the behavior of a "δ-type representative agent" whose initial endowment is the sum of the individual endowments of

\(^{10}\) A shareholder with positive (negative) δ puts a higher (lower) probability on positive shocks, which implies that she is relatively optimistic (pessimistic) compared to an investor with objective beliefs P.
the $\delta$-type shareholders. Therefore, without any loss of generality, we may assume that there is one shareholder of each type and, for a given type $\delta$, we will call it shareholder $\delta$.

We assume that the density of the initial number of shares $\nu_\delta$ over the space of shareholders’ types, is normal with parameters $(\delta_0, \omega^2)$. Note that the total number of shares is normalized to one. There are many ways to interpret this density function. If we consider that there is actually only one agent of each type in the economy, this density corresponds to the wealth density over the agents space parametrized by the types. If we instead consider that all the agents in the economy are endowed with the same number of shares, this function corresponds to the relative frequency of agents over the type space. In both cases, this means that the average shareholder – when shareholders are weighted by their initial endowments – is of type $\delta_0$ and is optimistic (pessimistic) if $\delta_0 > 0$ ($\delta_0 < 0$). The parameter $\omega^2$ measures the dispersion of (endowment-weighted) beliefs. For $\omega = 0$, all the agents in the economy have the same type $\delta_0$ and the economy can be represented as a single agent economy where the agent is of type $\delta_0$ and is a unique shareholder of the firm.

Note that even though shareholders have different beliefs and perceive differently the risk-return trade-off, they unanimously agree on the value maximizing strategy. This result is well-known since Grossman and Stiglitz (1977). The economic intuition is that disagreement is already integrated within the state price density and all shareholders as well as the firm face the same price system. In Bianchi et al. 2021, the authors consider a model that is very close to ours but for a different purpose (namely, the determination of optimal manager’s compensation policies) and they show that unanimity holds whenever there exists at least one shareholder who prefers a given plan to any other admissible plan and that the unanimously preferred plan is the profit/value maximizing one.

3 Arrow-Debreu equilibrium

We first show the existence of an Arrow-Debreu production equilibrium defined as follows.

**Definition 1** An Arrow-Debreu production equilibrium is characterized by an admissible strategy $\tilde{\theta} \in \Theta$, a set of individual consumption random variables $(\tilde{c}_\delta)_{\delta \in \mathbb{R}}$ and by a random price $\tilde{p}$ such that

1. $K_{T}^{\tilde{\theta}} = \arg\max_{\theta \in \Theta} E \left[ \tilde{p} K_T^{\theta} \right],$

2. $\tilde{c}_\delta = \arg\max U^{\delta}(c_\delta), \ E \left[ \tilde{p} c_\delta \right] \leq \nu_\delta E \left[ \tilde{p} K_T^{\theta} \right] \text{ for all } \delta,$

3. $\int \tilde{c}_\delta d\delta = K_T^{\tilde{\theta}}.$

\footnote{Note that there is no beliefs heterogeneity nor explicit beliefs homogeneity in this last reference. Indeed, it considers heterogeneous general utility functions (instead of expected utility functions with common $u$) and possible divergence of beliefs is hidden in such utility functions. In our framework, agent $\delta$ has a utility function $U^{\delta}$ and the fact that $U^\delta$ is an expectation of some $u$ under $P^\delta$ is irrelevant in order to apply Grossman and Stiglitz (1977) results.}

\footnote{The density of such a price system varies with the probability with respect to which it is considered but the price of a given production plan or of a given bundle is the same for all agents. Therefore, profit/value maximization has the same meaning for all agents.}
As usual in general equilibrium, agents are price-takers. The first condition means that the equilibrium strategy maximizes the value of the firm. The second condition means that, for each $\delta$, agent $\delta$ maximizes her utility under the budget constraint associated with the value of her initial endowment that consists of $\nu_{\delta}$ shares of the firm and, therefore, $\nu_{\delta}$ shares of its date $T$ stock of capital. The third condition is a market clearing condition.

### 3.1 Homogeneous/exogenous benchmarks

To analyze the impact of belief heterogeneity and of the endogenous production framework, we first analyze two benchmarks: a homogeneous belief setting and an exchange economy with heterogeneous beliefs setting or, equivalently, a production economy setting where the production process is exogenously given as in Atmaz and Basak (2018).

**Proposition 1** Without divergence of opinion ($\omega = 0$), there exists a unique Arrow-Debreu production equilibrium in which capital volatility is constant $\theta_t \equiv \frac{b+\delta_0}{2c+\gamma}$. The equilibrium price $\tilde{p}_t$ is the solution of $d\tilde{p}_t = \sigma_{h,t}^p \tilde{p}_t dW_t$ where the price volatility $\sigma_{h,t}^p$ is constant and given by $\sigma_{h,t}^p = \delta_0 - \gamma \frac{b+\delta_0}{2c+\gamma}$.

As already seen, the capital volatility coincides with the relative share of capital invested in the productive technology. As expected, this share increases with the level of optimism in the economy and decreases with the level of risk aversion: optimistic agents and less risk averse ones are willing to take more risk. It also decreases with the concavity parameter $c$, i.e., when risk is less rewarding.

Let us now analyze what happens when beliefs are heterogeneous and the production process is exogenously given. This corresponds to the Atmaz and Basak (2018) exchange economy setting where the production process (which is equivalent - in our setting - to our capital process) is given and the equilibrium is defined by Conditions 2 and 3 of Definition 1. To make things comparable between the production and exchange economy settings, we consider the situation where the exogenous capital process is the same as above, namely, $K_T^\theta_t$. With such a choice and without divergence of opinion, the production and exchange economy lead - by construction - to the same equilibrium plans and to the same price system.

**Proposition 2 (Atmaz and Basak, 2018)** If the total wealth of the economy is exogenously given by $K_T^\theta = K_T^{\theta h}$, there exists a unique Arrow-Debreu equilibrium and the price volatility is given by $\sigma_{e,t}^p = \frac{W_t-k_e}{\omega_t^2+k_e^2-\omega_t^2} - \frac{\gamma \omega_t^2}{\omega_t^2+k_e^2} \frac{b+\delta_0}{2c+\gamma}$ where $(\omega_t^2, k_e)$ are given and such that $\omega_e^2 \geq \sqrt{T}$, $\lim_{\omega \to 0} \omega_e = \infty$ and $\lim_{\omega \to 0} k_e/\omega_e^2 = -\delta_0$.

By construction, the homogeneous belief economy and the economy with exogenously\(^{13}\) given capital process have the same endowment process. However, they do not have the same state price density process. While it has a constant volatility in the homogeneous beliefs setting, its volatility fluctuates with $W_t$ in the exogenous setting. As already underlined by Jouini and Napp (2007),

\(^{13}\)To retrieve Atmaz and Basak (2018) results, it suffices to calibrate our model on theirs. Taking $\omega^2 = \frac{\theta}{\sigma^2}$, $\delta_0 = \frac{\theta}{\gamma}$ and $c = \frac{1}{2} \left( \frac{\tilde{m} + \tilde{v} - \tilde{c}^2}{\sigma^2} \right)$, we have a Gaussian distribution with parameters $(\tilde{m}, \tilde{v})$ on $\delta_0$ where $\theta_\sigma = \frac{b+\delta_0}{2c+\gamma} = \sigma$. It suffices to adjust the parameter $a$ (maintained equal to 0 until now) to reach the target drift $\mu$. 

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an heterogeneous beliefs economy behaves - locally and at each date - as an homogeneous beliefs economy where the common belief is the wealth weighted average of individual beliefs. After a positive (negative) shock, the optimists (pessimists) that are relatively more (less) exposed to risk become wealthier and their weight increases: the economy as a whole becomes more optimistic (pessimistic). This optimism (pessimism) exerts an upward (downward) pressure on the demand for the productive technology. But due to the rigidity of the production process that is exogenously given, this pressure is transformed into a pressure on prices that leads to additional price fluctuations.

As expected, we have $\sigma_{e,t}^g \rightarrow \sigma_{h,t}^p$ when $\omega \rightarrow 0$: additional price fluctuations disappear when belief heterogeneity disappears.

### 3.2 The heterogeneous belief and endogenous production process framework

#### 3.2.1 The equilibrium

In our main setting, we have the following characterization of the Arrow-Debreu equilibrium.

**Theorem 3** If $(\gamma - 1) T \omega^2 < 2c + \gamma^2$, there exists a unique Arrow-Debreu production equilibrium characterized by

$$
\theta_t = \frac{W_t - b(T - t) - k + b\overline{\omega}^2}{\varphi(t)} \quad \text{where} \quad \varphi(t) = (2c + \gamma) \overline{\omega}^2 - (T - t)(2c + 1)
$$

(5)

where $\overline{\omega}^2$ is the (higher) solution of

$$
\frac{1}{\overline{\omega}^2} = \frac{\varphi(0)\overline{\omega}^2}{2Tc(1 - \gamma) + \gamma(2c + \gamma) \overline{\omega}^2} \quad \text{and} \quad k = \frac{b(1 - \gamma) + \delta_0(2c + 1)}{2c + \gamma} T - \overline{\omega}^2 \delta_0.
$$

(6)

It is easy to check that $\overline{\omega}^2$ decreases with $\omega$ from $\infty$, for $\omega = 0$, to $\frac{2c+1}{2c+\gamma}T$ (resp. $T$), for $\omega = \infty$ (resp. $\omega = \sqrt{\frac{2c+\gamma^2}{(\gamma-1)T}}$) when $\gamma < 1$ (resp. $\gamma > 1$). In particular, this means that we always have $(2c + \gamma) \overline{\omega}^2 > (2c + 1)T$ and $\frac{\partial \theta_t}{\partial W_t} > 0$. Note also that the condition $(\gamma - 1) T \omega^2 < 2c + \gamma^2$ on $\omega^2$ is automatically satisfied for $\gamma < 1$.

The Atmaz and Basak (2018) exogenous production process framework with drift $\mu$ and volatility $\sigma$ is obtained as a special case when $c \rightarrow \infty$ with $b = 2\sigma c$, $a = \mu - c\sigma^2$. For this parametrization, $m(\sigma) = \mu$ and $\lim_{c \rightarrow \infty} m(\theta) = -\infty$ for $\theta \neq \sigma$. The optimal $\theta$ is then $\sigma$ at each date and in every state of the world.

In the general setting, $\theta_t$ is no longer constant, it is stochastic, and its sensitivity to $W_t$ is positive. As previously explained, when $W_t$ is high (low), the optimists (pessimists) have more weight, the demand for the productive technology increases (decreases) and this leads to an increase (decrease) of the relative share $\theta_t$ of capital invested in the productive technology and, consequently, to a higher (lower) risk exposure.

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14 This case corresponds to the homogeneous beliefs setting and we retrieve that $\theta$ is constant and equal to $\frac{b \delta_t}{2c+\gamma}$. 

On average, on $W_t$ and for $\delta_0 = 0$, $\tilde{\theta}_t$ increases, (decreases) with $\varpi^2$ for $\gamma \leq 1$ ($\gamma \geq 1$). This means that belief heterogeneity acts as an additional source of risk and depending on the agents being prudent ($\gamma \leq 1$) or not, an increase in belief heterogeneity leads to a decrease (increase) in the willingness to invest.

Let us analyze how the growth rate $m(\tilde{\theta}_t)$ is impacted by $\omega$. Direct computations gives us that \[ \frac{1}{2} E \left[ \frac{dm(\tilde{\theta}_t)}{d\omega} \right] \] has the sign of $2c(2c + \gamma)^2 + b^2(1 - \gamma)(-T\gamma(2c - 1) + \varpi^2\gamma(2c + \gamma))$ when $t$ is near 0. This means that, for $\gamma \geq 1$ and for a small enough $c$, the average growth rate increases with agents’ disagreement while it decreases when $c$ is large. This illustrates the Oi-Hartman-Abel Effect (Oi, 1961; Hartman, 1972; Abel, 1983): because firms can expand to exploit good outcomes and contract to insure against bad outcomes, they may benefit from increased uncertainty. However, for this mechanism to work, the expansion should not be too costly. In our setting and as seen above, when agents are less prudent ($\gamma \geq 1$), a higher level of disagreement leads to more risk exposure: the firms expands. When the adjustment cost is linear ($c = 0$), this leads to higher returns. When the adjustment cost is convex, the growth rate is positively impacted by an increase of the (average) level of risk exposure but negatively impacted by its variability. When $c$ is large enough, the second effect dominates.

### 3.2.2 Stock of capital and state price density processes

In the previous section, we have determined $\tilde{\theta}_t$. The following proposition provides a closed-form expression for $K^\tilde{\theta}_t$ and for the state price density $\bar{p}$ and characterizes the state price density process $p_t = E_t[\bar{p}]$.

**Proposition 4** At equilibrium,

1. the stock of capital $K_t$ is path independent and given by
   \[
   K_t = K(t, W_t) \text{ where } K(t, w) = \left( \frac{\varphi(t)}{\varphi(0)} \right)^{-\frac{1}{2\varpi^2+1}} \exp \frac{1}{2} \left( \varphi(t)\tilde{\theta}^2(t, w) - \frac{(k - b\varpi^2 + T\bar{b})^2}{\varphi(0)} \right),
   \]

2. the growth rate $\ln K_{s+t}/K_s$ exhibits positive skewness. For a small enough $t$, it also exhibits excess kurtosis,

3. the state price density is given by $\bar{p} = \exp \frac{1}{2} \frac{(k-W_T)^2}{\varpi^2} (K_T)^{-\gamma}$,

4. the state price density process satisfies $dp_t = \sigma_t^P \tilde{p}_tdW_t$ with
   \[
   \sigma_t^P = \frac{2c(W_t - k) - b\gamma\varpi^2}{\varphi(t) + T - t}.
   \]

Leptokurticity is easy to understand. The firm expands (contracts) in good (bad) states of the word which increases the probability of very high (low) returns. However, the productive technology
Figure 1: **The effect of belief heterogeneity on stock of capital distribution.** These figures plot the distribution function of the stock of capital process at date $t = 1$ (in black). The baseline parameter values are as in 1. In red are (a) the log-normal distribution with the same mean and variance, (b) the stock of capital distribution for $\delta_0 = 0.3$ (dashed lines) and $\delta_0 = -0.1$, and (c) the stock of capital distribution when there is less disagreement ($\omega = 0.46$ instead of 0.54).

Figure 2: **The impact of decreasing returns to scale on the growth rate.** This figure plots the average growth rate as a function of the stock of capital level. The black line corresponds to $c = 32.82$ and the red line to $c = 16.41$. The horizontal axis corresponds to $c = 0$. The other parameter values are those in Example 1.

being (in average) efficient, the average risk exposure is positive. Hence, the described phenomenon is asymmetric and leads to skewness.

Obviously, both skewness and kurtosis disappear in the homogeneous setting since the risk exposure level does not fluctuate anymore in such a setting.

Figure 1 illustrates the skewness and the excess kurtosis of the stock of capital level distribution. Skewness clearly appears in (a). Disagreement (c) and optimism (b) increase the stock of capital and production variances, which is natural since disagreement acts like a source of risk, while optimism increases risk exposure. optimism (pessimism) also decreases (increases) the stock of capital and production averages, which is due to the increasing returns to scale: high and low risk exposures are not transformed linearly into growth, the highest ones are penalized. This last point is illustrated in Figure 2, where the average growth rate is lower when $c$ is higher. It is constant and equal to 0 for $c = \infty$.

Note that $K_t$ is bounded away from 0, and its minimum at $t$ is reached when $\hat{\theta}(t, w) = 0$. This is natural since taking $\theta \equiv 0$ permits a constant capital process equal to 1. Taking this as a benchmark, agents accept having lower capital levels in some states to have higher capital levels in others as far as this permits them to increase their utility. Since their marginal utility goes to $\infty$. 

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when consumption goes to 0, the optimal stock of capital process will be bounded away from 0.

In the Online Appendix, it is shown that, for $\delta_0 = 0$ and $\gamma > 1$, the stock of capital process exhibits momentum at all horizons, which is compatible with the empirical findings of Mao and Wei (2014), who states that "in comparison with prices, earnings do not display long term reversal" (in our model earnings and stock of capital are proportional).

**Example 1 (Running Example)** Throughout the paper (unless otherwise specified), for numerical illustrations, we take $\gamma = 6$, as well as $T = 30$, which is a reasonable value for the duration of a stock. We also take $b = 2.62$, $c = 32.82$, $\omega = 0.54$ (and then $\omega^2 = 35$) and $\delta_0 = 0$. With these parameters, the macroeconomic volatility at date $t = 0$ is equal to 3.66%, and as we will see later, the financial volatility at date $t = 0$ is 17% and the risk premium at date $t = 0$ is 6%, which are reasonable values close to those of Mehra and Prescott (1985). The one-standard-deviation agent, $\delta = \pm \omega$, overestimates/underestimates the stock of capital growth rate by 2%.

Let us compare our model with Pindyck and Wang (2013) one and, for simplicity, let us assume $d^1 = d^2 = \alpha_2 = \beta_2 = \sigma_K^2 = 0$ which means that the safe technology is not productive and there is no capital depreciation on both technologies. The drift of $K$ is given by $\hat{\theta} \left( \alpha_1 - \beta_1 \hat{\theta} \right) - d^1$ in our setting and by $i - \frac{1}{2} \hat{\theta} \omega^2 - d$ in their setting, where $i$ corresponds to the investment to capital ratio and where $\hat{\theta}$ is the adjustment cost convexity parameter. When all the production is invested into the productive capital, we have $\hat{\theta} = 1$ in our setting and $i = A$ in their setting. By identification, we have $\alpha_1 = A$ and $\beta_1 = \frac{1}{2} \alpha A^2$, and from there (see footnote 6) $b = \frac{A}{\sigma}$ and $c = \frac{1}{2} \hat{\theta} A^2$.

Pindyck and Wang (2013) take $A = 0.113$ which, with $b = 2.62$ and $c = 32.82$, leads to $\hat{\theta} = 9.56$ and $\sigma = 4.3\%$.

This derived convexity parameter $\hat{\theta}$ is of the same order of magnitude as that found by Pindyck and Wang ($\hat{\theta} = 12.025$) and the derived productive technology volatility ($\sigma = 4.3\%$) is more realistic than the one they obtain ($\sigma = 13.55\%$) and closer to the Barro (2009) volatility parameter ($\sigma = 2\%$).

As far as $\sigma_{t}^p$ is concerned and as expected, we have $\lim_{\omega \to \infty} \sigma_{t}^p = \sigma_{h,t}^p$ and for $b = 2e\theta_e$, we check that $\lim_{\omega \to \infty} \sigma_t^p = \sigma_{e,t}^p$.

Interestingly, when we replace $\omega^2$ with $\omega^2 (1 + \frac{2}{\omega})$ in the definition of $\sigma_{e,t}^p$ (as well as in $k$), we retrieve $\sigma_t^p$: the volatility of the state price density in the exogenous setting corresponds to the volatility of the state price density in the endogenous setting with a lower level of heterogeneity. The endogenous determination of the level of capital (or of risk exposure) then has an effect that is in opposite direction to the effect of belief heterogeneity. Capital adjustments permit to absorb a part of the additional risk (due to the waves of optimism and pessimism) generated by belief heterogeneity.

When these adjustments are frictionless ($c = 0$), we retrieve the same deterministic volatility for the state price density as in the homogenous setting ($\omega = \infty$): the fluctuations that are generated in the exogenous setting are exactly compensated by the risk exposure adjustments.

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\footnote{The estimation of the risk-aversion parameter has been the subject of numerous studies. Some of these studies are summarized by Conine et al. (2017). Estimates are mainly between 0.35 and 10.}
To summarize, risk premium fluctuations are due to the production rigidity in the exogenous setting and to the decreasing returns to scale in the endogenous one. In both cases, they result from the imperfect adjustment of the stock of capital to economic shocks.

4 Securities markets

We assumed that the agents face a complete set of Arrow-Debreu securities. Alternatively, we may consider that agents face two long-lived securities, a risky stock and a riskless bond. The stock price $S$ is posited to have dynamics

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$$

where the stock mean return $\mu_t$ and volatility $\sigma_t$ are endogenously determined in equilibrium. The stock is in positive net supply of one unit and is a claim to the payoff $K_T$, paid at $T$, and so $S_T = K_T$. The bond is in zero net supply and pays a riskless interest rate $r$, which is set to 0 without loss of generality\(^\text{16}\).

Theorem 5 At the Arrow-Debreu production equilibrium, $\mu_t$ and $\sigma_t$ are given by

$$\mu_t = \omega^2 (2c + \gamma) \frac{b\omega^2 \gamma + 2ck - 2cW_t}{(\varphi(t) + T - t)^2} \bar{\Theta}_t,$$

$$\sigma_t = \frac{(2c + \gamma) \omega^2}{(\varphi(t) + T - t)} \bar{\Theta}_t,$$

It is immediate that an increase in the macroeconomic volatility $\bar{\Theta}_t$ leads to an increase in financial asset volatility $\sigma_t$. Direct computations permit to show that an increase in uncertainty (under the form of disagreement) also leads to an increase in asset volatility. This is in line with our explanations above: disagreement acts as an additional source of risk, namely risk sentiment, even though individual beliefs are kept fixed.

As for the stock of capital and its volatility, financial volatility is high in good states and increases (decreases) with future good (bad) news. Hence, an increase (decrease) in the state variable $W_t$ is amplified (mitigated) at the stock price process level and the probability of high future returns is higher after high observed returns. In bad states of the world, the risk exposure is negative but also high in absolute value. We observe then the symmetric mechanism. Together, they lead to positive autocorrelation (momentum). Since this property results from the possibility for the firm to expand/contract, there is no such momentum in the exogenous and in the homogeneous settings. This result is established in the Online Appendix for $\delta_0 = 0$ (no bias) and $\gamma > 1$.

If we take as a benchmark, the homogeneous framework where all agents perceive $m$ correctly, there is overinvestment (resp. underinvestment) in the productive technology (in our heterogeneous

\(^\text{16}\)In our setting, consumption can occur only at time $T$, and the interest rate can be taken exogenously (see, e.g., Pastor and Veronesi, 2012, Atmaz and Basak, 2018).
beliefs economy) when $\bar{\theta}_t \geq \frac{b}{2c + \gamma}$ (resp. $\leq$). When $c = 0$, we have $\mu_t = b \bar{\theta}_t$ and higher investment levels go hand in hand with higher returns (and prices), in line with q-theory. For $c$ small, $\frac{d\mu_t}{d\bar{\theta}_t}$ is still positive while it is negative for $c$ large. Actually, when the convexity of adjustment costs increases, expansions/contractions become too costly and the sensitivity of risk exposition to shocks decreases. Therefore, an equal increase of macroeconomic volatility $\bar{\theta}_t$ corresponds to a larger positive economic shock when $c$ is larger. The corresponding increase of optimism leads then to a decrease of the financial return. This is not because an optimistic investor requires less return but because she is optimistic she overestimates returns and accepts low returns thinking them higher.

For all levels of adjustment costs convexity levels, $\frac{d\mu_t}{d\bar{\theta}_t}$ increases with belief dispersion: higher belief dispersion acts as an additional risk and increases the financial return. These results are in line with Panageas (2005) (if we consider decreasing returns and convex costs of adjustment as a friction).

If the agents are rational on average ($\delta_0 = 0$) or if they are not too pessimistic, belief heterogeneity increases the risk premium. Indeed, the drift and volatility in the homogeneous setting can be derived from the formulas (7) and (8) by letting $\bar{\omega}^2$ go to $\infty$, which leads to $k \sim -\bar{\omega}^2\delta_0$ and

$$\mu_{h,t} = \frac{b\gamma - 2c\delta_0}{2c + \gamma} \theta_0, \quad \sigma_{h,t} = \theta_0.$$ 

At $t = 0$, the difference $\mu_0 - \mu_{h,0}$ is equal to $2Tc_0 b \bar{\omega} \bar{\sigma} (2c + \gamma)/(\gamma + 2T + 1)$ that has the sign of $b + \delta_0$.

When $c = 0$, the observation at a given date, of $(\bar{\theta}, \mu, \sigma)$ does not permit to distinguish between the homogeneous and heterogeneous settings. Indeed, we have $\mu_{h,t} = b \theta_{h,t}$ and $\sigma_{h,t} = \theta_{h,t}$. We check from (7) and (8) that we also have $\mu_t = b \bar{\theta}_t$ and $\sigma_t = \bar{\theta}_t$ in the heterogeneous setting. Hence, the heterogeneous beliefs economy has the same characteristics – at each date and in each state of the world – as the homogeneous belief economy associated to the instantaneous belief of the representative agent (this date–t representative agent belief characterized by $(\delta_t, \omega^2_t)$ is determined explicitly in the Online Appendix). However, they do not have the same dynamics. The waves of optimism and pessimism induced by positive and negative shocks induce fluctuations for $\delta_t$ and then for $\bar{\theta}_t = \frac{b + \delta_t}{\gamma}$ while $\theta_{h,t}$ is constant.

When $c \neq 0$ (decreasing returns to scale), financial volatility and macroeconomic volatility are no longer equal, and this will be analyzed in further detail in Section 4.1.

In Figure 1 (a), we retrieve the well-established result in the asset pricing literature that aggregate returns are negatively correlated with aggregate market volatility (e.g., French et al., 1987, Campbell and Hentschel, 1992). This figure is obtained for reasonable values of the Brownian motion (namely, $W_t \in [-5, 5]$). However, if we consider extreme values (namely, around $W_t = -18.586$), the shape is inverted and $\mu_t$ and $\sigma_t$ increase together with $K_t$ and $S_t$. Note that $W_t = -18.586$ corresponds to $\bar{\theta}_t = 0$ and to the minimum level for the stock of capital process (and/or the stock price process, as shown in the Online Appendix). These values then correspond to very bad times where other phenomena are to be taken into account as the credit constraints and, more generally, the real options that are involved in firms. This aspect is analyzed in more
Figure 3: Risk premium behavior. This figure plots the risk premium $\mu$ as a function of the financial volatility $\sigma$ (a) and as a function of the stock of capital level $K$ (b). The baseline parameter values are as in Example 1, and $W_t$ ranges in $[-5, 5]$.

detail in Section 5. Figure 1 (b) represents the risk premium as a function of the stock of capital, which appears countercyclical.

4.1 Financial and macroeconomic volatilities ratio

Except for $c = 0$ (constant returns to scale) or $\omega = 0$ (homogeneous beliefs), the financial volatility is higher than the macroeconomic volatility (stock of capital volatility). More precisely, we have

**Proposition 6** The ratio $\sigma_t/\bar{\theta}_t$ between the financial volatility and the macroeconomic volatility increases with $\omega$ from 1 (for $\omega = 0$) to $\frac{1+2c}{1+2c/T}$ (for $\omega = \infty$) when $\gamma < 1$ and from 1 (for $\omega = 0$) to $\frac{\gamma+2c}{\gamma+2c/T}$ (for $\omega = \infty$) when $\gamma > 1$.

The proof is immediate using the intervals for $\omega^2$ provided by Theorem 1.

Note that financial volatility is higher than macroeconomic volatility only due to the concavity of the risk-return production function. In particular, if $c = 0$ (constant returns to scale), we have $\sigma_t = \bar{\theta}_t$, in sharp contrast to Atmaz and Basak (2018), where belief heterogeneity always generates additional volatility and where the maximal value of the ratio is infinite. In fact, when the agents may choose their risk exposure and for $c = 0$, an increase in belief heterogeneity leads to an adjustment of the risk exposure that cancels out the effect described by Atmaz and Basak (2018). When the risk-return production function is concave, this adjustment is too costly and the volatility ratio cannot be reduced to 1.

The next proposition establishes a relation that $(\mu_t, \sigma_t, \delta_t, \bar{\theta}_t)$ should satisfy at each date and in each state of the world. Since all these characteristics\footnote{Recall that $(\delta_t, \omega_t^2)$ characterizes the representative agent belief and are explicitly determined in the Online Appendix.} are measurable at each date, this relation is a testable consequence of the theory. It also provides a zone that depends only on the horizon $T-t$, the risk aversion coefficient $\gamma$ and the heterogeneity $\omega_t$, in which the volatility ratio $\sigma_t/\bar{\theta}_t$ should lie.
Proposition 7  At equilibrium, the financial asset drift and volatility, macroeconomic volatility, average belief and risk aversion are related as follows

\[
\bar{\theta}_t = \frac{1}{\sigma_t} \frac{\mu_t - \sigma_t^2 + \sigma_t \delta_t}{\gamma - 1} \quad \text{or} \quad \frac{\sigma_t}{\bar{\theta}_t} = (\gamma - 1) \frac{\sigma_t^2}{\mu_t - \sigma_t^2 + \sigma_t \delta_t}.
\]  (9)

Furthermore, the ratio \(\frac{\sigma_t}{\bar{\theta}_t}\) takes all possible values in

\[
I_- = \left[ 1, 2 + \sqrt{\left( (T-t) \omega_t^2 - \gamma \right)^2 + 4(T-t) \omega_t^2 + (T-t) \omega_t^2 - \gamma} \right] / 2, \quad \text{for } \gamma \leq 1,
\]  (10)

\[
I_+ = I_- \cap \left( \frac{(\gamma - 1) ((T-t) \omega_t^2 - \gamma)}{\gamma}, \infty \right), \quad \text{for } \gamma \geq 1,
\]

when \(b\) and \(c\) take all admissible values.

We have already seen that return and volatility are negatively related at the aggregate level and that this finding is consistent with empirical evidence. However, Duffee (1995) finds that stock returns and volatility are positively correlated at the individual stock level. To explain this difference in behavior at the aggregate and individual levels, Gruñon et al. (2012) argue that because investors tend to be more uncertain about future real output growth during economic downturns (e.g., Veronesi, 1999), periods of high stock return volatility could coincide with periods of low stock returns even if the direct effect of volatility on firm value is positive. That is, volatility may increase when stock prices decline not because the fundamental relation between these variables is negative, but because both variables are affected by the same underlying macroeconomic factors. Equation (9) provides a theoretical justification for this argument. Indeed, we have \(\mu_t = (\gamma - 1) \sigma_t \bar{\theta}_t + \sigma_t^2 - \sigma_t \delta_t\) which, with the bounds on \(\frac{\sigma_t}{\bar{\theta}_t}\) provided by Proposition 6, gives that \(\mu_t\) is positively related to \(\sigma_t\) when we control for \(\bar{\theta}_t\), for \(\delta_t = 0\) (no average bias).

Note that the relation (9) does not involve the level of heterogeneity. However, the level of heterogeneity has an impact on the dynamics of \(\delta_t\) (see Online Appendix) and then has an impact on the fluctuations of the volatility ratio \(\frac{\sigma_t}{\bar{\theta}_t}\).

Figure 3 represents the range for the volatility ratio as a function of \(\gamma\) (resp. \(\omega\)). We remark that the upper bound on the volatility decreases when \(\gamma\) increases, which might seem to contradict (9), where the volatility ratio increases with \(\gamma\). However, while (9) holds for fixed \(b\) and \(c\), the upper bound is the maximum over the set of all possible pairs \((b, c)\), taking into account the impact of \(\gamma\), \(b\) and \(c\) on the volatility ratio.

From (9) and (10) we also may represent, for each value of \(\gamma\), the sets in which the pairs \((\bar{\theta}_t, \sigma_t)\), \((\bar{\theta}_t, \frac{\mu_t}{\sigma_t})\) and \((\sigma_t, \frac{\mu_t}{\sigma_t})\) should lie respectively, when \(\omega_t\) varies.

**Running example justification.** To better illustrate formula (9), let us assume that the risk premium is given by \(\mu_t = 6\%\), the financial volatility \(\sigma_t = 17\%\), and let us take a risk aversion coefficient \(\gamma = 6\) and assume \(\delta_t = 0\) (no bias). We have, from (9), \(\bar{\theta}_t = 3.66\%\), which gives a volatility ratio of 4.65. If the growth rate of the economy is equal to 2\%, the average belief among
Figure 4: The volatility ratio $\sigma/\theta$ as a function of risk aversion/heterogeneity. The colored area in (a) corresponds to the set of possible values for the ratio between financial volatility and macroeconomic volatility when risk aversion $\gamma$ varies. The colored area in (b) corresponds to the set of possible values for this ratio when the level of belief heterogeneity $\omega$ varies. The other parameter values are as in Example 1.

The agent is also equal to 2% ($\delta_t = 0$). If the one-standard-deviation interval is equal to $[0\%, 4\%]$, we have $\omega_t\theta_t = 2\%$ and $\omega_t = 0.54$. We take $T - t = 30$, which corresponds to a reasonable value for the duration of a stock. From (28), we have $\sigma^2 = 35$, and from (7) and (8), we can retrieve $b = 2.62$, and $c = 32.82$ and the condition $(\gamma - 1)T\omega^2 < 2c + \gamma^2$ of 3 is satisfied. To summarize, these values generate an economy with a 2% growth rate with a 3.66% volatility, as in Mehra and Prescott (1985) with a 6% risk premium and a 17% volatility of financial assets.

4.2 Volatility risk premium

Let us denote by $V$ the quantity $V_t = \sigma_t^2$; simple computations permit to show that $V_t$ satisfies a stochastic differential equation of the form

$$dV_t = \left(D_0^0 + bD_1^1\sqrt{V_t} + D_2^2V_t\right)dt + D_1\sqrt{V_t}dW,$$

where $D_0^0$, $D_1^1$, $D_2^2$ and $D_1$ are given deterministic functions of time (see Appendix).

The endogenous model then leads to a stochastic volatility, and its volatility is given by $D_t$.

For $b = 0$, this corresponds to a time-dependent Heston (1993) model with a unique source of risk (or with a perfect correlation between the 2 sources of risk considered in the Heston model). Note that $b = 0$ means that return decreases with risk, and it might seem irrational to invest in such an economy. However, remember that the risk-return relationship is perceived differently by the different shareholders and, in particular, it is increasing for small levels of risk exposure for all the agents for which $\delta > 0$. Put another way, under the belief of agent $\delta = -b$, financial volatility satisfies the same behavior as in a Heston model with perfect correlation between the two sources of risk.

It is well known that a positive correlation between the two sources of risk (here, $\rho = 1$) is a source of positive skewness (as already seen above) and explains that deep out-of-the-money calls are more expensive than those produced by the Black-Scholes model (Jackwerth and Rubinstein,
The Sharpe ratio (or risk premium by unit of risk) in our model is equal to \( \frac{\mu_t}{\sigma_t} \), the drift of \( V_t \) under the risk-neutral probability is then given by \( D_0^t + bD_1^t \sqrt{V_t} + D_2^t V_t - \frac{\mu_t}{\sigma_t} D_t \sqrt{V_t}, \) and the volatility risk premium by unit of risk is given by

\[
\Lambda_t = \frac{\mu_t}{\sigma_t} D_t = \frac{2}{((2c + \gamma) \sigma^2 - 2c (T - t))} \frac{b \omega^2 \gamma + 2ck - 2cW_t}{(W_t - b (T - t) - k + b \omega^2)}.
\]

Note that in the exogenous setting (obtained by taking \( b = 2c \) and \( c \to \infty \)), we have \( dV_t = \frac{2\sigma^2 \omega^2}{(\omega^2 - T + t)^2} \sigma_t dt \), the volatility is deterministic and, in particular, \( \Lambda_t^e = 0 \) : the volatility risk premium by unit of risk \( \Lambda_t \) is equal to 0 in the homogeneous setting and in the exogenous setting. In the endogenous setting, it is stochastic and becomes very high during recessions (when \( W_t \) approaches \( b (T - t) + k - b \omega^2 \) that corresponds to \( \theta_t = 0 \) and to the minimum possible value for the date--t stock price).

Hence our model generates a volatility risk premium (Carr and Wu, 2009), with a risk premium by unit of risk that may increase to extremely high levels and quickly in bad times in line with Corradi et al., 2013, whereas it is equal to 0 in the exogenous/homogeneous settings. The effect is much more pronounced when the level of heterogeneity increases.

Stochastic volatility and volatility risk premium are direct consequences of the stochasticity of \( \theta_t \) that results from the possibility for the firm to expands/contracts after positive/negative shocks.

### 5 Extension to a continuous time consumption setting

In this section we explore to which extent our results hold when consumption occurs in continuous time. The strategy we developed in order to explicitly solve the optimal control problem in the terminal date consumption framework can not be extended in the continuous time consumption one.

However, one of the main features of our model is the possibility for the firm to adapt its risk exposure \( \theta_t \) to external shocks and we have shown that the optimal risk exposure \( \tilde{\theta}_t \) is positively related to \( W_t \) and that almost all our results are directly related to this property.

This positive relation between \( \tilde{\theta}_t \) and \( W_t \) is maintained in a continuous time consumption framework : good (bad) states of the world induce to an increase of optimism (pessimism) at the representative agent level and then an increase (decrease) of risk exposure.

In this section, we consider an exchange economy for which the production process is exogenously given and satisfies this expansion/contraction of economic activity property. To make things comparable, we take \( \theta_t \) as in Theorem 3 and we have

\[
dy_t = m(\bar{\theta}_t)y_t dt + \bar{\theta}_ty_t dW_t, \quad y_0 = 1 \text{ with } b, c > 0.
\]  

In the next we analyze the impact of such expansions/contractions on financial/macroeconomic volatility ratio.
In this framework, agent's utility is given by 

\[ U((c; t)_{t \in [0, T]}) = \int_0^T E \left[ M^\delta u(c_{\delta,t}) \right] dt \]

and the definition of an exchange equilibrium is, as usual, given by

**Definition 2** An exchange equilibrium is characterized by a set of individual consumption processes \((\bar{c}_{\delta,t})_{t \in [0, T]} \) and by a price process \((\bar{p}_t)_{t \in [0, T]} \) such that

1. \((\bar{c}_{\delta,t})_{t \in [0, T]} = \text{argmax} U^\delta((c_{\delta,t})_{t \in [0, T]}) \), \( E \left[ \int_0^T \bar{p}_t c_{\delta,t} dt \right] \leq \nu_\delta E \left[ \int_0^T \bar{p}_t y_t dt \right] \) for all \( \delta \),
2. \( \int \bar{c}_{\delta,t} d\delta = y_t \), for all \( t \).

We also introduce the following classical equilibrium with transfers definition

**Definition 3** An exchange equilibrium with transfers is characterized by a set of individual consumption processes \((\bar{c}_{\delta,t})_{t \in [0, T]} \) and by a price process \((\bar{p}_t)_{t \in [0, T]} \) such that

1. \((\bar{c}_{\delta,t})_{t \in [0, T]} = \text{argmax} U^\delta((c_{\delta,t})_{t \in [0, T]}) \), \( E \left[ \int_0^T \bar{p}_t c_{\delta,t} dt \right] \leq E \left[ \int_0^T \bar{p}_t \bar{c}_{\delta,t} dt \right] \) for all \( \delta \),
2. \( \int \bar{c}_{\delta,t} d\delta = y_t \), for all \( t \).

It is easy to check that any exchange equilibrium is an exchange equilibrium with transfers and that any exchange equilibrium with transfers is an exchange equilibrium in the economy where the initial shares are given by \( \nu_\delta = E \left[ \int_0^T \bar{p}_t \bar{c}_{\delta,t} dt \right] / E \left[ \int_0^T \bar{p}_t y_t dt \right] \).

Both equilibrium concepts are characterized by a set \((\lambda_\delta)\) of Lagrange multipliers and a set \((\nu_\delta)\) of initial shares such that

\[ M_t^\delta u'(\bar{c}_{\delta,t}) = \vartheta - \gamma \lambda_\delta \bar{p}_t, \text{ a.e., for all } t \text{ and all } \delta, \]  
(13)

\[ \int \bar{c}_{\delta,t} d\delta = y_t, \text{ for all } t, \]  
(14)

\[ E \left[ \int_0^T \bar{p}_t \bar{c}_{\delta,t} dt \right] = \nu_\delta E \left[ \int_0^T \bar{p}_t y_t dt \right] \]  
(15)

where \( \vartheta \) is a given scaling parameter.

In the exchange equilibrium (with transfers), the set \((\nu_\delta)\) (the set \((\lambda_\delta)\)) is exogenously given and the set \((\lambda_\delta)\) (the set \((\nu_\delta)\)) results from (13) to (15).

In order to make our framework as close as possible to the terminal consumption one, we assume that the set \((\lambda_\delta)\) is exogenously given by \( \lambda_\delta = \exp \left( \frac{\sigma^2 - \gamma^2}{2} \delta \right) \) \( k_\delta \) (the same as above). If there exists an equilibrium with transfers associated to this set \((\lambda_\delta)\), then it is an exchange equilibrium in the economy where the set \((\nu_\delta)\) is given by (15). We call it \((\lambda_\delta)\)–exchange equilibrium.

Finally, for such an equilibrium, we consider two different financial assets associated to \((y_t)\). The first one, \( S_{\text{final}} \), only pays \( y_T \) at the final date \( T \), it is the analogon of the financial asset considered in our main framework. The second one, \( S_{\text{cont}} \), pays a flow of dividends \((y_t)\) at date \( T \), it is more in line with the financial asset that is usually considered in the continuous time literature. We respectively denote by \( \sigma_{S_{\delta,t}}^\text{final} \) and \( \sigma_{S_{\delta,t}}^\text{cont} \) their volatilities at date \( t \).
Proposition 8 For \( \lambda_\delta = \exp \left( \frac{-2T - \delta^2}{2} \right) \exp (k\delta) \), there exists a \( (\lambda_\delta) - \) exchange equilibrium.

1. For \( \omega \neq 0 \), and \( \tilde{\theta}_t = \sigma_\text{constant}^{18} \), \( \sigma_{S_{\text{final}}}^{\tilde{\theta}_t} > 1 \) if and only if \( \gamma < 1 \).

2. For \( \tilde{\theta}_t = \frac{W_t - b(T-t) - k + b\omega^2}{(2c+\gamma)\omega^2 - (T-t)(2c+1)} \) and \( T \) sufficiently large, we have

   (a) the stock \( S_{\text{final}} \) that only pays at date \( T \), exhibits excess volatility \( (\sigma_{S_{\text{final}}}^{\tilde{\theta}_t} > 1) \),

   (b) for \( \omega = 0 \) (no belief heterogeneity) or \( \gamma = 1 \) (log utility), there is no excess volatility for the stock with continuous payment of dividends \( (\sigma_{S_{\text{cont.}}}^{\tilde{\theta}_t} = 1) \),

   (c) for \( \omega \neq 0 \) and \( \gamma \neq 1 \), the stock with continuous payment of dividends exhibits excess volatility \( (\sigma_{S_{\text{cont.}}}^{\tilde{\theta}_t} > 1) \) if and only if \( \gamma > 1 \).

Note first that the excess volatility is not only due to the risk exposure fluctuations in the production process. For instance, it does not lead to excess volatility when there is no belief heterogeneity.

Conversely, when there is no such risk exposure fluctuations (\( \tilde{\theta}_t \) constant), we might have excess volatility but the conditions on \( \gamma \) are reversed.

Let us analyze these results more in detail.

In our main framework the interest rate is exogenously given and kept fixed equal to 0. While in the continuous time framework, the date–s price of a bond paying 1 unit of date \( t \) consumption is endogenously defined and given by \( E_s [\tilde{p}_t] / \tilde{p}_s \). The price of \( S_{\text{final}} \) is given by \( \frac{E_t [\tilde{p}_{T,T}]}{\tilde{p}_t} \) (resp. \( \frac{E_t [\tilde{p}_{T,T}]}{E_t [\tilde{p}_t]} \)) in the continuous time (resp. terminal) consumption case, the ratio between these two quantities being equal to the date–t price of the bond with maturity \( T \). The volatility of \( S_{\text{final}} \) in the continuous time consumption case is then equal to the sum of its volatility in the terminal consumption case and the volatility of the bond\(^{19}\).

In the terminal consumption framework, belief heterogeneity increases the volatility ratio. In the consumption setting, this effect is still present but is combined with another effect: the impact of belief heterogeneity on the bond volatility.

When \( \tilde{\theta}_t \) is constant (no adaptation of risk exposure to external shocks), a positive shock to output increases the optimists’ wealth pushing up the average belief and leading to a higher interest rate as can be shown using the classical Ramsey rule: \( r = \delta + \gamma \mu \) where \( \delta \) is the representative agent time preference rate and \( \mu \) is her subjective growth rate. An increase in \( r \) leads to a decrease in the price of the bond. The volatility of the bond is then negative and this decreases the volatility ratio.

However, the impact of belief heterogeneity on the bond price through the belief of the representative agent is only one of three effects highlighted by Jouini and Napp (2007). The second effect is related to the representative agent’s time preference rate: it is not equal to the average time preference rate but depends on the position of \( \gamma \) with respect to 1. In our setting, the average

\(^{18}\) This case is obtained taking \( b = 2\sigma c \) and letting \( c \) go to \( \infty \).

\(^{19}\) I would like to thank an anonymous referee for helpful suggestions relative to this discussion.
time preference rate is 0 and the representative agent’s time preference rate is positive for $\gamma < 1$ and negative for $\gamma > 1$. The impact on the volatility ratio is higher for $\gamma < 1$. Since there is no excess volatility in the log setting, we obtain that the volatility ratio is higher than one if and only if $\gamma < 1$.

The third effect is related to fluctuations in $\tilde{\theta}_t$. Positive shocks increase $\tilde{\theta}_t$ and this leads to a decrease in the interest rate as can be seen in the risk-adjusted Ramsey formula $r = \delta + \gamma \mu - \frac{1}{2} \gamma (\gamma + 1) \theta^2$. This effect is in the opposite direction to the previous one and is stronger for higher $\gamma$.

Again, because there is no excess volatility in the log setting, the overall effect for $\gamma > 1$ is in the direction of excess volatility: bond volatility is not high enough (in absolute value) to cancel out the excess volatility observed in our main framework.

## 6 Conclusion

In this paper, we have attempted to link classical financial asset pricing models to the type of pricing, output, and investment models extensively studied in the economics literature. Because the firm maximizes its market value, its decisions are impacted by financial markets, and financial markets are impacted by its decisions in return. We analyzed the impact of these interactions on both macroeconomic volatility and financial volatility and derived some testable consequences.

In our model, the production technology has constant returns to scale and the decreasing returns in the risk-return production function result from the convex costs of adjustment. It may be interesting to analyze the impact of increasing/decreasing returns to scale in the production technology itself.

From the economy of the firm point of view, it may be of interest to enlarge the set of decision variables to include some parameters that we considered exogenous to the firm, such as product quality or advertising, that might have an impact on consumers’ demand or technology change.

From the financial point of view, it may be of interest to explore the possible consequences of the exhibited interactions on the structure of the firm by introducing debt as a strategic variable for the firm.

From the corporate governance point of view, it would be interesting to explore how firms’ decisions might be delegated to a manager, who has her own optimality criteria, and the impact of such a delegation.

From a macroeconomic point of view, the model might provide some useful consequences to better understand the impact of real or perceived technological changes (e.g., dot-com bubble, data economy, Uberization, etc.) on macroeconomic growth and volatility as well as on financial volatility. Indeed, such changes are likely to have an impact on both the costs of adjustment structure and on the degree of belief heterogeneity.
7 Appendix

Proof of Propositions 1 and 2. These results correspond to specific cases of Theorem 3. For the sake of completeness, direct proofs are provided in the Online Appendix.

Proof of Theorem 3. Let us take

\[ \bar{\rho} = \exp \frac{1}{2} \frac{(k - W_T)^2}{\omega^2} \left( K^\theta_T \right)^{1 - \gamma}, \]

where

\[ \vartheta = \frac{\omega}{\sqrt{2\pi\gamma}} \quad \text{and} \quad \lambda_\delta = \exp \left( \frac{\omega^2}{2} - \frac{T}{\delta^2} \right) \exp (k \delta).\]  

(16)

We check that

\[ \int \bar{c}_\delta d\delta = K^\theta_T \quad \text{and} \quad M^\delta_T u'(\bar{c}_\delta) = M^\delta_T e^{-\gamma} = \vartheta^{-\gamma} \lambda_\delta \bar{\rho} \]

which respectively corresponds to the market clearing condition and to the first order condition for utility maximization where the \( \lambda_\delta \) play the role of Lagrange multipliers.

Let us show that \( K^\theta_T \) maximizes \( E \left[ \bar{\rho} K^\theta_T \right] \) over \( \Theta \). For a given \( \theta \), let us define \( z^\theta_t = K^\theta_t \left( K^\theta_t \right)^{1 - \gamma} \), it is equivalent to show that \( \bar{\theta} \) maximizes \( E \left[ q_T z^\theta_T \right] \) where \( q_T = \exp \frac{1}{2} \frac{(k - W_T)^2}{\omega^2} \). Let us denote by \( (q_t) \) the process defined by \( q_t = E_t [q_T] \). By Ito’s Lemma, it is easy to check that \( \frac{\omega}{\sqrt{\omega^2 - T + t}} \exp \left( \frac{1}{2} \frac{(k - W_t)^2}{\omega^2 - T + t} \right) \)

is a martingale equal to \( q_T \) at date \( T \). Therefore \( q_t = \frac{\omega}{\sqrt{\omega^2 - T + t}} \exp \left( \frac{1}{2} \frac{(k - W_t)^2}{\omega^2 - T + t} \right) \) and we have

\[ dq_t = \sigma^q_t q_t dW_t \quad \text{with} \quad \sigma^q_t = \frac{W_t - k}{\omega^2 - T + t}.\]

On the other hand, we have

\[ dz^\theta_t = \left( m(\theta_t) - \gamma m(\bar{\theta}_t) + \frac{1}{2} \right. \left( \gamma + 1 \right) \theta^2_t - \gamma \bar{\theta}_t \bar{\theta}_t \right) z^\theta_t dt \quad \text{with} \quad \mu_t = \frac{1}{2} \left( \gamma + 1 \right) \theta^2_t - \gamma \bar{\theta}_t \bar{\theta}_t \]

\[ + \mu_t z^\theta_t dt + \sigma^\theta_t z^\theta_t dW_t.\]

Let us denote by \( \mathcal{V} \) the function defined by \( \mathcal{V}(t, q_t, z_t, W_t) = \max_\theta E_t [q_T z^\theta_T] \). We are facing an optimal control problem whose associated dynamic programming equation (see, for instance, Pham, 2009, Theorem 3.5.2) is given by

\[ \mathcal{V}_t + \max_\theta \left( \mu^\theta_t z^\theta_t \mathcal{V}_z + \frac{\sigma^\theta_t q_t}{2} \mathcal{V}_{qq} + \frac{\sigma^\theta_t z^\theta_t}{2} \mathcal{V}_{zz} + \mathcal{V}_{ww} + \sigma^\theta_t z^\theta_t (\sigma^\theta_t q_t \mathcal{V}_{zq} + \mathcal{V}_{zw}) + \sigma^\theta_t q_t \mathcal{V}_{qw} \right) = 0, \]

(18)

and if we find a control \( \theta^* \) and a function \( \mathcal{V} \) such that \( \theta^* \) realizes the maximum in (18) and \( \mathcal{V} \) is a solution of (18) such that \( \mathcal{V}(T, q, z, w) = q_z \), then \( \theta^* \) is the optimal control and \( \mathcal{V} \) is the value function of our problem. If we have \( \theta^* = \bar{\theta} \) then \( \bar{\theta} \) solves 1. and 3. in the definition of an Arrow-Debreu equilibrium, the budget constraint condition 2. being replaced by exogenously given Pareto weights (this situation is called equilibrium with transfers in the general equilibrium literature).
We posit that \( V \) is of the form \( V(t; q; z; w) = qzF(t; w) \). Replacing \( V \) in (18) by \( qzF(t; w) \), permits to transform the problem into finding - if they exist - solutions \( F \) and \( \theta^* \) such that \( F(T, w) = 1 \) and

\[
F_t + \mu_0^\theta F + \frac{1}{2}F_{ww} + \sigma_t^2\sigma_t^\theta F + \sigma_t^\theta F_w + \sigma_t^\theta F_w = 0, \quad (19)
\]

\[
\frac{d}{d\theta} \left( \mu_0^\theta F + \sigma_t^2\sigma_t^\theta F + \sigma_t^\theta F_w \right) \bigg|_{\theta^*} = 0. \quad (20)
\]

It suffices to check that

\[
F(t, w) = qz \sqrt{\frac{\omega^2_t+T}{\omega^2}} \exp \left( -\frac{1}{2} \frac{(T-t)(\gamma-1)}{\omega^2} \left( k - w - b\omega^2 + Tb - bt \right)^2 \right)
\]

with \( \ell = \frac{2c+1}{2c+\gamma} \), solves this equation when \( \theta^* = \tilde{\theta} \) is given as in the (5). We define \( A(\cdot), B(\cdot) \) and \( C(\cdot) \) such that \( F(t, w) = \exp(A(t)w^2 + B(t)w + C(t)) \).

Note that these functions are well defined only if \( \omega^2 - T + t \) and \( \omega^2 + \ell(t-T) \) are always positive which means \( \omega^2 > T \) and \( \omega^2 > \ell T \).

Let us check that both conditions are satisfied under our assumption \( (\gamma - 1)T\omega^2 < 2c + \gamma^2 \).

We have \( \frac{1}{\omega_T} = H(\omega^2) \) with \( H(X) = \frac{((2c+\gamma)X - T(2c+1))X}{2T(\gamma-1)\gamma(2c+\gamma)X} \). First, remark that \( H'(X) \) has the sign of \( P(X) = X^2\gamma(2c+\gamma)^2 - 4XcT(\gamma-1)(2c+\gamma) + 2c^2T^2(2c+1)(\gamma-1) \). If \( 1 - \gamma > 0 \), the discriminant of \( P \) is given by \(-8c^2T^2(\gamma-1)(2c+\gamma)^3 > 0 \) and the roots have a negative product. \( H \) is well defined for \( X > 0 \) and \( P \) has only one positive root and since \( H(0) = 0 < \lim_\infty H = \infty \), \( H \) is decreasing then increasing and \( H^{-1} \) is increasing on \( \mathbb{R}_+ \) which means that \( \omega \) decreases with \( \omega \) from \( \infty \) to \( \frac{2c+1}{2c+\gamma}T = \ell T > T \).

If \( 1 - \gamma < 0 \), \( P \) does not have real roots and is always positive. Furthermore, we have \( H(0) = 0 \), \( \lim_{X_0^+} H(X) = \infty \), \( \lim_{X_0^-} H(X) = -\infty \), \( \lim_\infty H(X) = \infty \) where \( X_0 = 2(\gamma-1)cT/(2c+\gamma) \gamma \). There are then two possible values \( \omega \) for each \( \omega \). More precisely, there exists \( G_1 \) and \( G_2 \) both decreasing with \( G_1(0) = X_0 \) and \( G_2(0) = \infty \), \( G_1(\infty) = 0 \) and \( G_2(\infty) = T(2c+1)/(2c+\gamma) \) such that both \( \omega^2 = G_1(\omega^2) \) and \( \omega^2 = G_2(\omega^2) \) solve (6). However, only the solution given by \( G_2 \) is possibly higher than \( T \) (and hence than \( \ell T \)) and if and only if \( \omega^2 < \frac{2c+1}{(2c+\gamma)\gamma}T \).

It remains to show that the budget constraints (point 2.) are respected. For this purpose, it suffices to show that \( E \left[ \bar{p} \delta \right] \) is proportional to \( \nu_\delta \).

We have

\[
\bar{p} \delta = \theta \lambda_\delta^{-\frac{1}{2}} \exp \left( -\frac{1}{2\gamma} \delta^2 T + \frac{\delta}{\gamma} W_T + \frac{1}{2} \left( 1 - \frac{1}{\gamma} \right) \frac{(k - W_T)^2}{\omega^2} \right) K_{TT}^{1-\gamma}
\]

and

\[
E \left[ \bar{p} \delta \right] = \theta \exp \left( -\frac{1}{2} \frac{\delta^2 \omega^2}{\gamma - 1} \right) E \left[ q_T^\delta z_T \right] \quad (21)
\]
with 
\[ z_T = K_T^{1-\gamma}, \quad q_T^\delta = \exp\left( \frac{1}{2} \frac{(W_T - k^\delta)^2}{\omega^2 (1 - \frac{1}{\gamma})} \right), \quad k^\delta = k + \frac{\omega^2}{1-\gamma}. \]

Let us evaluate \( E_t [q_T^\delta z_T] \) and, for this purpose, let us introduce \( q_T^{\rho,h} = \exp\left( \frac{1}{2} \frac{(W_T-h)^2}{\omega^2 (1 - \frac{1}{\gamma})} \right) \) and let us define \( q_T^{\rho,h} \) by \( q_T^{\rho,h} = E_t \left[ q_T^{\rho,h} \right] \). Using Ito’s Lemma, it is easy to check that \( q_T^\delta = q_T^{1-\gamma,k} \), that 
\[ q_T^{\rho,h} = \sqrt{\frac{\omega^2}{(t-T)\rho + \omega^2}} \exp\left( \frac{1}{2} \frac{(W_T-h)^2}{(t-T)\rho + \omega^2} \right), \]

d \( q_T^{\rho,h} dt \) \( dW_t \) \( \omega^2 \) and \( \sigma_t^{q^{\rho,h}, q_t^{\rho,h}} \) with \( \rho = \frac{W_T-h}{\omega^2} \).

As above, it is easy to check that if \( \mathcal{V}_{t}^{q,h} \) is a solution of the following partial differential equation 
\[ 0 = \mathcal{V}_{t}^{q,h} + \bar{b} \frac{\partial}{\partial z} \mathcal{V}_{z}^{q,h} + \frac{1}{2} \left( \frac{\partial^2 q^{q,h}}{\partial q \partial h} \right)^2 \mathcal{V}_{zz}^{q,h} + \frac{1}{2} \left( \frac{\partial^{\theta q,h}}{\partial q \partial h} \right)^2 \mathcal{V}_{zz}^{q,h}, \]

then \( E_t [q_T^{\rho,h} z_T] = \mathcal{V}_{t}^{1,h}(t, q, z, W_t) \).

As above, we posit that \( \mathcal{V}_{t}^{q,h} \) is of the form \( \mathcal{V}_{t}^{q,h}(t, q, z, w) = q_T^{F^{q,h}}(t, w) \).

We are interested in the solution of the previous equations when \( \rho = \left( 1 - \frac{1}{\gamma} \right) \). We start by solving the problem for \( \rho = 1 \). We can check that 
\[ F^{1,h}(t, w) = \exp(A^{1,h}(t) w^2 + B^{1,h}(t) w + C^{1,h}(t)) \]
with 
\[ A^{1,h}(t) = -\frac{\tau (\ell - 1)}{2(\tau + \omega^2)(\omega^2 + \tau \ell)} \text{ with } \tau = t - T, \]
\[ B^{1,h}(t) = \left( \frac{1}{\tau + 2ct + 2c\omega^2 + \omega^2 \gamma} \right) \left( \frac{1}{\tau + \omega^2} \right) \omega^2 (\gamma - 1) \]
\[ C^{1,h}(t) = \frac{h^2 (2c + 1) (2c + \gamma) + 2h (2c + 1) (\gamma - 1) (b\omega^2 - k)}{2\ell (\ell + \omega^2)(2c + \gamma)(2c + 1)} \]
\[ + \frac{(\gamma - 1)(2c + 1)k^2 + b^2\omega^4 (\gamma - 1)}{2\ell (\ell + \omega^2)(2c + \gamma)(2c + 1)} \]
\[ - \frac{h^2 (1 - \gamma)}{2(2c + 1)} \ell - \frac{h^2}{2(2c + 1)} \ln \left( \frac{\ell + \omega^2}{\omega^2} \right) - \frac{1}{2\ell} \ln \left( \frac{\tau + \omega^2}{\omega^2} \right) \]
\[ - \frac{h^2 (2c + 1) + 2h (2c + 1) (b\omega^2 - k) + k^2 + 2ck^2 - b^2\omega^4 + b^2\omega^4 \gamma}{2(2c + 1)^2 \omega^2} (\gamma - 1), \]

is a solution and we have, 
\[ E \left[ q_T^{1,h} z_T \right] = \sqrt{\frac{\omega^2}{\omega^2 - T}} \exp\left( \frac{1}{2} \frac{h^2}{\omega^2 - T} \right) \exp(C^{1,h}(0)). \]

For \( \rho \neq 1 \), let us define \( \mathcal{U}(\rho, h) = E \left[ q_T^{\rho,h} z_T \right] \). We have \( \mathcal{U}(\rho, h) = E \left[ \exp\left( \frac{1}{2} \rho \frac{(W_T-h)^2}{\omega^2} \right) z_T \right] \) and,
by differentiation, we check that we have
\[ \frac{\partial^2 U}{\partial h^2} = \frac{2\varrho^2}{\varpi^2} \frac{\partial U}{\partial \rho} + \frac{\rho}{\varpi^2} U \]  \tag{23}

with \( U(1, h) = E \left[ q_1^T z_T \right] \) given by (22). The solution of (23) is given by
\[ U(\rho, h) = \exp \left( R_2(\rho) h^2 + R_1(\rho) h + R_0(\rho) \right) \] with

\[ 
R_2(\rho) = \frac{\rho R_2(1)}{2\varpi^2 (1 - \rho) R_2(1) + \rho}, \quad R_1(\rho) = \frac{\rho R_1(1)}{2\varpi^2 (1 - \rho) R_2(1) + \rho}, \\
R_0(\rho) = -\frac{1}{2} \ln \left( (2R_2(1)\varpi^2 - 1)\rho - 2R_2(1)\varpi^2 \right) - \frac{1}{2} \frac{(\rho - 1) R_2^2(1)\varpi^2}{(2R_2(1)\varpi^2 - 1)\rho - 2R_2(1)\varpi^2} + R_0(1)
\]

where \( R_2(1), R_1(1) \) and \( R_0(1) \) are the coefficients of \( h^2, h \) and 1 in \( \ln E \left[ q_1^T z_T \right] \) and are given by
\[ 
R_2(1) = \frac{1}{2} \left( T \gamma - T + 2c\varpi^2 + \varpi^2 \gamma \right), \quad R_1(1) = \frac{b\varpi^2 - k}{(2c\varpi^2 - 2T c - T + \varpi^2 \gamma) \varpi^2}.
\]

From there we obtain \( U_0(0, 1 - \frac{1}{\gamma}, k^\delta) = \exp \left( R_2(1 - \frac{1}{\gamma}) (k^\delta)^2 + R_1 \left( 1 - \frac{1}{\gamma} \right) k^\delta \right) U_0(0, 1 - \frac{1}{\gamma}, 0) \) which corresponds to \( E \left[ q_1^{1-\frac{1}{\gamma}} z_T \right] = E \left[ q_T z_T \right] \). By (21), we have
\[ 
\frac{E \left[ \tilde{p} c^{\delta} \right]}{E \left[ \tilde{p} c^0 \right]} = \exp \left( -\frac{1}{2} \frac{\delta^2 \varpi^2}{\gamma - 1} \right) \exp \left( R_2(1 - \frac{1}{\gamma}) \left( (k^\delta)^2 - k^2 \right) + R_1 \left( 1 - \frac{1}{\gamma} \right) (k^\delta - k) \right)
\]

and we check that this gives
\[ 
\frac{E \left[ \tilde{p} c^{\delta} \right]}{E \left[ \tilde{p} c^0 \right]} = \frac{\nu_\delta}{\nu_0}
\]
which proves that the individual budget constraints are satisfied.

Let us end with the proof of the uniqueness of the production equilibrium. We rely on Dana (1995) and on the fact that our utility functions are homogeneous and agents have shares of aggregate endowment. However, Dana (1995) only deals with exchange equilibria with a finite number of agents. We adapt her result to our setting. Let us consider two equilibria \((\tilde{c}^\delta, K_T, \tilde{p})\) and \((\hat{c}^\delta, K_T, \hat{p})\) where without loss of generality, prices are such that \( E [\tilde{p} K_T] = E [\hat{p} K_T] = 1 \). Then \( \hat{c}^\delta \) (resp. \( \tilde{c}^\delta \)) is the optimal solution to the problem
\[ 
\max U^\delta(c^\delta) \text{ s.t. } E [\tilde{p} c^\delta] \leq \nu_\delta
\]
(resp. \(E[\hat{pc}_\delta] \leq \nu_\delta\)). From Dana (1995), Proposition 2.1, we have
\[
E\left[(\hat{\rho} - \hat{\rho}) \cdot (\hat{c}^\delta - \hat{c}^\delta)\right] < 0.
\]
Summing over \(\delta\), we obtain
\[
(\hat{\rho} - \hat{\rho}) \cdot (K_T - K_T) < 0.
\]
From the definition of a production equilibrium, we have \(E[\hat{pc}] \geq E[\hat{pK}_T]\) and \(E[\hat{pK}_T] \geq E[\hat{pK}_T]\) which leads to a contradiction hence to the uniqueness of the production equilibrium. ■

**Proof of Proposition 4.** It suffices to check that
\[
\frac{\partial K}{\partial w}(t, w) = \hat{\theta}(t, w), \quad \frac{\partial K}{\partial w}(t, w) + \frac{1}{2} \frac{\partial^2 K}{\partial w^2}(t, w) = m(\hat{\theta}(t, w)), \quad K(0, 0) = 1,
\]
and to conclude the first part by Ito’s Lemma. As far as skewness and kurtosis are concerned, let us establish the result between 0 and \(t\), the general case is similar. It is immediate that \(\ln K_t/K_0\) is proportional to \((W_t + \xi_t)^2 + \zeta_t\) where \(\xi_t\) and \(\zeta_t\) are deterministic. It suffices then to analyze the moments of \(X = (W_t + \xi)^2\) (the dependence in \(t\) of \(\xi\) is not relevant). From the moments of a normal distribution, we derive \(E[X] = t + \xi^2\), \(Var[X] = 2t(t + 2\xi^2)\), \(E[X^3] = \frac{15\xi^4 + 45\xi^2 + \xi^6 + 15\xi^3}{(2\xi^2)^{3/2}} > 0\) and \(E[X^4] / Var[X] - 3 = \frac{1}{2} \frac{420\xi^2 + 12(2\xi^4 - 6) + (28\xi^6 - 12\xi^2 + 105)}{t(t + 2\xi^2)}\) which is positive for \(t\) small enough.

By Ito’s Lemma, we can check that
\[
q_tK_t^{-\gamma} \sqrt{\frac{(\varphi(t)^2 - T + t)(2c + \gamma)}{(2c(t - T) + \varphi^2(2c + \gamma))}} \exp\left(\frac{1}{2} \frac{(t - T)^2}{\varphi(t)^2 - T + t} \frac{(2c + \gamma)(2c + \gamma - 2c(t - T) - T + T_b + b)}{(2c + \gamma)(2c + \gamma - 2c(t - T) - T + T_b + b)}\right) \sqrt{\frac{2c + \gamma}{\varphi(t)^2 - T + t}}
\]
is a martingale whose terminal value is \(\hat{p}_T\). It is then equal to \(E_t[\hat{p}_T]\) which is of the form \(q_tK_t^{-\gamma} \exp(A^p(t)W_t^2 + B^p(t)W_t + C^p(t))\). We have then \(\sigma_t^p = (2A^p(t)W_t + B^p(t)) + \sigma_t^q - \gamma \hat{\theta}_t\). ■

**Proof of Theorem 5.** The Arrow-Debreu prices are given by \(p_T\) and since we assumed zero-interest rate, the asset price process is given by
\[
S_t = \frac{P_t}{\hat{p}_t} \quad \text{with} \quad P_t = E_t[\hat{p}_T K_T], \quad p_t = E_t[\hat{p}_T].
\]
We respectively denote by \(\mu^P\), \(\mu^p\), \(\sigma^P\) and \(\sigma^p\), the drift and volatility parameters of \(P\) and \(p\). By construction, we have \(\mu^P = \mu^p = 0\).

In the proof of Theorem 3, we have already seen that \(E_t[\hat{p}_TK_T]\) is of the form
\[
E_t[\hat{p}_TK_T] = q_tK_t^{1-\gamma} \exp(A(t)W_t^2 + B(t)W_t + C(t))
\]
which gives
\[
\sigma_t^p = (2A(t)W_t + B(t)) + \sigma_t^q + (1 - \gamma)\hat{\theta}_t
\]
and leads to
\[
\frac{1}{S_t}dS_t = \mu_t dt + \sigma_t dW_t, \quad \mu_t = \sigma_t^p (\sigma_t^p - \sigma_t^p), \quad \sigma_t = \sigma_t^p - \sigma_t^p, \text{ with}
\]
\[
\mu_t = -\sigma_t \left( (2A^p(t)W_t + B^p(t)) + \frac{W_t - k}{\omega^2 - T + t} - \gamma \tilde{\theta}_t \right),
\]
\[
\sigma_t = \tilde{\theta}_t + (2 (A(t) - A^p(t)) W_t + (B(t) - B^p(t))).
\]

From there, we have
\[
\mu_t = \omega^2 (2c + \gamma) \frac{\omega^2 \gamma + 2ck - 2c W_t}{(2c + \gamma) \omega^2 - 2c(T - t)} \tilde{\theta}_t \quad \text{and} \quad \sigma_t = \omega^2 \frac{2c + \gamma}{(2c + \gamma) \omega^2 - 2c(T - t)} \tilde{\theta}_t.
\]

**Proof of Proposition 7.** Let us rewrite the instantaneous stock return and volatility as functions of the average characteristics, we have
\[
\mu_t = \omega^2 (2c + \gamma) \frac{b \omega^2 \gamma + 2ck}{(2c + \gamma) \omega^2 - 2c(T - t)} \tilde{\theta}_t, \quad \sigma_t = \omega^2 \frac{2c + \gamma}{(2c + \gamma) \omega^2 - 2c(T - t)} \tilde{\theta}_t \tag{24}
\]

where (see Online Appendix)
\[
\delta_t = \delta_0 \frac{\varphi(0)}{\varphi(t)} + b (\gamma - 1) \frac{t}{\varphi(t)} + \frac{2c + \gamma}{\varphi(t)} W_t,
\]
\[
\omega_t^2 = \omega^2 \frac{\varphi(0)}{\varphi(t)} + (\gamma - 1) \frac{2c}{\omega^2} \frac{t}{\varphi(t)}, \tag{25}
\]
\[
\theta_t = \frac{b + \delta_t}{2c + \gamma} \quad \text{and} \quad k_t = \frac{b(1 - \gamma) + \delta_t (2c + 1)}{2c + \gamma} (T - t) - \omega^2 \delta_t.
\]

From there, we obtain
\[
b = \frac{\omega^2 \delta_t \left( \mu_t + \sigma_t \delta_t - \gamma \sigma_t^2 \right) + \sigma_t \left( -\gamma \mu_t - \sigma_t \delta_t + \gamma \sigma_t^2 \right) (T - t)}{\sigma_t^2 (1 - \gamma)(T - t) - \omega^2 \left( \mu_t + \sigma_t \delta_t - \gamma \sigma_t^2 \right)} \tag{26}
\]
\[
c = \frac{1}{2} \omega^2 \frac{\varphi(0)}{\varphi(t)} \frac{\mu_t + \sigma_t \delta_t - \gamma \sigma_t^2}{\sigma_t^2 (1 - \gamma)(T - t) - \omega^2 \left( \mu_t + \sigma_t \delta_t - \gamma \sigma_t^2 \right)} \tag{27}
\]
\[
\omega_t^2 = \gamma \left( \frac{1}{\omega_t^2} + \frac{T}{\gamma} \left( \frac{\sigma_t^2}{\varphi(0)} \right) \right) \tag{28}
\]

and plugging these expressions of $b$ and $c$ in (5), we obtain $\tilde{\theta}_t = \frac{1}{\sigma_t} \frac{\mu_t - \sigma_t^2 + \sigma_t \delta_t}{\gamma - 1}$. From (25) and (8), we derive
\[
b = 2c \tilde{\theta}_t - \delta_t + \tilde{\theta}_t \gamma \quad \text{and} \quad c = -\frac{\gamma \omega^2 (\tilde{\theta}_t - \sigma_t)}{2 (T - t) \sigma_t + \gamma \omega^2 (\tilde{\theta}_t - \sigma_t)^2}, \tag{29}
\]

and from the expression of $\omega_t^2$ (Online Appendix, Proposition 1), we derive $\omega_t^2 = \gamma \left( \frac{1}{\omega_t^2} + \frac{1}{\gamma} \frac{T \sigma_t - \sigma_t (\tilde{\theta}_t + \tilde{\theta}_t \gamma)}{\sigma_t - \sigma_t (\tilde{\theta}_t + \tilde{\theta}_t \gamma)} \right)$. 30
which permits to rewrite $c$ as

$$c = \frac{1}{2} \left( \bar{\theta}_t - \sigma_t \right) \frac{(-\bar{\theta}_t + \sigma_t + \bar{\theta}_t \gamma) \gamma + (T - t) \sigma_t \omega_t^2}{(-\bar{\theta}_t - \sigma_t)^2 + \bar{\theta}_t \gamma (\bar{\theta}_t - \sigma_t) + (T - t) \bar{\theta}_t \sigma_t \omega_t^2}.$$ 

We have now to check that the values we obtained for $c$ and $\omega^2$ are nonnegative, and that $\omega^2$ corresponds to the highest value of (7), i.e. such that $(2c + \gamma) \omega^2 - 2c(T - t) \geq 0$. Furthermore, for $\gamma \geq 1$, we have to check that $(\gamma - 1) T \omega_t^2 < 2c + \gamma^2$.

As far as the condition $(2c + \gamma) \omega^2 - 2c(T - t) \geq 0$ is concerned, it imposes and is equivalent to the fact that $\bar{\theta}_t$ and $\sigma_t$ have the same sign. Note that replacing $\bar{\theta}_t$ by $-\bar{\theta}_t$ and $\sigma_t$ by $-\sigma_t$, $c$ and $\omega^2$ are not modified and $b$ is replaced by $b'$ such that $b + \delta_t = -(b' + \delta_t)$. Hence, we can restrict our analysis to $\bar{\theta}_t \geq 0$ and $\sigma_t \geq 0$, the set of possible values for $\frac{\sigma_t}{\bar{\theta}_t}$ remaining unchanged when allowing $\bar{\theta}_t$ to be negative.

If we take $\tau = T - t$, we have the value we obtained for $\omega^2$ is nonnegative if and only if

$$-\bar{\theta}_t \gamma + \sigma_t \gamma^2 + \sigma_t \tau \omega_t^2 \geq 0.$$ 

Let us denote by $J_1$ its numerator and $J_2$ its denominator, we have

$$J_1 \geq 0 \text{ iff } \sigma_t \geq \frac{\bar{\theta}_t \gamma(1 - \gamma)}{\gamma + \tau \omega_t^2} = j_1 \text{ and } J_2 \geq 0 \text{ iff } \sigma_t \geq \bar{\theta}_t (1 - \gamma) = j_2.$$ 

Let us assume $\gamma \leq 1$, we have $j_1 \leq j_2$ and the condition on $\omega^2$ is equivalent to $0 \leq \sigma_t \leq j_1$ or $\sigma_t \geq j_2$.

As far as $c$ is concerned, we have $c = (\bar{\theta}_t - \sigma_t) \frac{J_3(\sigma_t)}{J_4(\sigma_t)}$ with $J_3(\sigma_t) = -\frac{1}{2} \left( -\bar{\theta}_t \gamma + \sigma_t \gamma^2 + \bar{\theta}_t \gamma \sigma_t \tau \omega_t^2 \right)$ and $J_4(\sigma_t) = -\bar{\theta}_t^2 - \sigma_t^2 + 2\bar{\theta}_t \sigma_t + \bar{\theta}_t^2 \gamma - \bar{\theta}_t \sigma_t \gamma + \bar{\theta}_t \sigma_t \tau \omega_t^2$. We have $J_4(\pm \infty) = -\infty$, $J_4(j_1) = 0$, $J_4(j_2) \geq 0$ and $J_4(\bar{\theta}_t) \geq 0$. If we denote by $j_4 \leq j'_4$ the roots of $J_4$, we have $j_1 \leq j_4 \leq j_2 \leq \bar{\theta}_t \leq j'_4$. Finally, we have $J_3 \geq 0$ if and only if $\sigma_t \leq j_1$. These results are summarized by $\frac{\sigma_t}{\bar{\theta}_t} \in I \overset{\text{def}}{=} \left[ 1, 1 + \frac{1}{2} \left( \sqrt{(\tau \omega_t^2 - \gamma)^2 + 4\tau \omega_t^2 + \tau \omega_t^2 - \gamma} \right) \right]$ when $\gamma \leq 1$ (in the Online Appendix, Table 1 illustrates the different conditions above as well as the conclusion).

If $\gamma \geq 1$, $j_1$ and $j_2$ are nonpositive and the condition on $\omega^2$ is then always satisfied for $\sigma_t \geq 0$. We also have that $J_3$ is always nonpositive for $\sigma_t \geq 0$ and still have $j_4 \leq \bar{\theta}_t \leq j'_4$.

Let us look at the condition $J = (\gamma - 1) \tau \omega_t^2 - 2c - \gamma^2 < 0$. It is equivalent to $J_5 \frac{\delta_t}{J_4} < 0$ with $J_5 = -\bar{\theta}_t + \sigma_t + \bar{\theta}_t \gamma - \sigma_t \gamma + \sigma_t \tau \omega_t^2$ and $J_6 = \bar{\theta}_t \gamma - \sigma_t \gamma - \bar{\theta}_t \gamma^2 - \bar{\theta}_t \tau \omega_t^2 + \bar{\theta}_t \gamma \omega_t^2$. We have that $J_5$ has the sign of $1 - \gamma + \tau \omega_t^2$ for $\sigma_t \geq j_5 \overset{\text{def}}{=} \frac{\bar{\theta}_t (1 - \gamma)}{-\gamma + \tau \omega_t^2 + 1}$ and $J_6$ is nonpositive if and only if $\sigma_t \geq j_6 \overset{\text{def}}{=} \frac{\bar{\theta}_t (1 - \gamma)}{-\gamma + \tau \omega_t^2 - \gamma}$.

We have $J_4(0) \geq 0$ and then $j_4 \leq 0$.

Furthermore, if $\gamma - \tau \omega_t^2 - 1 > 0$, we have $j_5 \geq \bar{\theta}_t$ and since $J_4(j_5) \leq 0$, this gives $j_5 \geq j'_4$. We also have $j_6 = \frac{\bar{\theta}_t (1 - \gamma)(\tau \omega_t^2 - \gamma)}{\gamma} \leq -\frac{\bar{\theta}_t (1 - \gamma)}{\gamma} \leq 0$. We have then $0 \leq \bar{\theta}_t \leq j'_4 \leq j_5$ and all these results are still summarized by $\frac{\sigma_t}{\bar{\theta}_t} \in I$ (see also Table 2 in the Online Appendix).

If $\gamma = 1 + \tau \omega_t^2$, $J_5 > 0$ and $\frac{\sigma_t}{\bar{\theta}_t}$ is still in $I$.

If $1 \leq \gamma < 1 + \tau \omega_t^2$, $j_5 \leq 0$ and $J_5 \geq 0$ for all $\sigma_t \geq 0$. Furthermore, $J_4(j_6) \geq 0$ which gives $j_6 \leq j'_4$ and $j_6 - \bar{\theta}_t = \bar{\theta}_t \left( \frac{1 + \tau \omega_t^2}{\gamma} - \frac{\tau \omega_t^2}{\gamma} \right)$ which has the sign of the polynomial in $\gamma$, $J_7(\gamma) = \left( -\gamma + \tau \omega_t^2 \right) \gamma - \tau \omega_t^2$. If $J_7 \leq 0$ then $j_6 - \bar{\theta}_t \leq 0$. This is summarized by $\frac{\sigma_t}{\bar{\theta}_t} \in I$ (see also Table 3 in
the Online Appendix).

If \( J_t \geq 0 \), we have \( \frac{\sigma_t}{\theta_t} \in I' \) where

\[
I' := \left\{ \frac{1}{2} \left( \sqrt{(\tau \omega_t^2 - \gamma)^2 + 4 \tau \omega_t^2 + \tau \omega_t^2 - \gamma} \right), 1 + \frac{1}{2} \left( \sqrt{(\tau \omega_t^2 - \gamma)^2 + 4 \tau \omega_t^2 + \tau \omega_t^2 - \gamma} \right) \right\}
\]

(see also Table 4 in the Online Appendix).

When \( 1 \leq \gamma < 1 + \tau \omega_t^2 \), we have \( I \subset I' \) and, for \( \gamma > 1 + \tau \omega_t^2 \), we have \( I' \subset I \). In both cases, we have \( \frac{\sigma_t}{\theta_t} \in I \cap I' \). ■

Proof of Equation (11) . We have \( V_t = \sigma_t^2 \) and

\[
dV_t = \left( \frac{\partial V_t}{\partial t} + \frac{1}{2} \frac{\partial^2 V_t}{\partial w^2} \right) dt + \frac{\partial V_t}{\partial w} dW_t
\]

This gives

\[
D_t = \frac{1}{\sigma_t} \frac{\partial V_t}{\partial w} = \frac{2 \omega^2 (2c + \gamma)}{((2c + \gamma) \omega^2 - 2c(T - t))((2c + \gamma) \omega^2 - (2c + 1)(T - t))}
\]

Solving for \( W_t \) as a function of \( \sigma_t \) in (8) and plugging it in \( \frac{\partial V_t}{\partial t} + \frac{1}{2} \frac{\partial^2 V_t}{\partial w^2} \), we obtain

\[
D_t^0 = \frac{\omega^4 (2c + \gamma)^2}{((2c + \gamma) \omega^2 - 2c(T - t))^2 ((2c + \gamma) \omega^2 - (2c + 1)(T - t))^2}
\]

\[
D_t^1 = \frac{2b \omega^2 (2c + \gamma)}{((2c + \gamma) \omega^2 - 2c(T - t))((2c + \gamma) \omega^2 - (2c + 1)(T - t))}
\]

\[
D_t^2 = \frac{2 - 8 c^2 \omega^2 + 8 T c^2 - 8 c^2 T - 2 c \omega^2 - \omega^2 \gamma + 4 T c - 4 c t - 4 c \omega^2 \gamma}{((2c + \gamma) \omega^2 - 2c(T - t))((2c + \gamma) \omega^2 - (2c + 1)(T - t))}
\]

(see also Table 4 in the Online Appendix).

Proof of the limit results about \( \Lambda_t \). We have \( \Lambda_t = \frac{\sigma_t}{\theta_t} \frac{D_t}{\sigma_t} \) which converges to 0 when \( \omega^2 \to \infty \): no volatility risk premium in the homogeneous setting. We also have \( \Lambda_t = \infty \) when the denominator is equal to 0. The exogenous case is obtained by taking \( b = 2 c \sigma \) and \( \sigma \to \infty \) which gives \( D_t^0 = D_t^1 = D_t^2 = D_t = \Lambda_t = 0 \). ■

Proof of proposition 8. From (13) and for the chosen \( (\lambda_\delta) \), we have

\[
c_t^\delta = \vartheta \exp(-\frac{1}{2} \delta^2 t + \frac{1}{\gamma} W_t) \exp \left( -\frac{1}{2} \frac{1}{\gamma} (\omega^2 - T) \delta^2 \right) \exp \left( -\frac{1}{\gamma} k \delta \right) (\tilde{p}_t)^{-\frac{1}{2}}
\]

and, from there, ■

\[
y_t = \vartheta (\tilde{p}_t)^{-\frac{1}{2}} \int \exp \left( -\frac{1}{2} \delta^2 \frac{1}{\gamma} (\omega^2 - T + t) - \delta \frac{1}{\gamma} (k - W_t) \right) d\delta
\]

\[
= \frac{\vartheta \sqrt{2 \pi \gamma}}{\sqrt{\omega^2 - T + t} (\tilde{p}_t)^{-\frac{1}{2}}} \exp \left( -\frac{1}{2} \left( \frac{1}{\gamma} (\omega^2 - T + t) \right) \right) \exp \left( \frac{1}{2} \left( \frac{1}{\gamma} (\omega^2 - T + t) \right)^{-1} \right).
\]
We have \( y_0 = 1 \) and if we normalize \((p_t)\) such that \( p_0 = 1 \), we have

\[
\tilde{p}_t = \left( \frac{\sqrt{\omega^2 - T}}{\sqrt{\omega^2 - T + t}} \right)^{\gamma} \exp \left( \frac{1}{2} \frac{k^2}{T - \omega^2} - \frac{1}{2} \frac{(k - W_t)^2}{T - t - \omega^2} \right) (y_t)^{-\gamma}
\]

For \( t = T \), \( \tilde{p}_t \) and \( y_t \) are as in our main framework and \( P_t = E_t [p_T|_{Y_T}] \) is such that \( \sigma^P_t = (2A(t)W_t + B(t)) + \sigma^q_t + (1 - \gamma)\tilde{\theta}_t \). By Ito’s Lemma, we have \( \sigma^P_t = \left( \frac{k - W_t}{T - t - \omega^2} - \gamma\tilde{\theta}_t \right) \) where \((p_t)\) is now the price process and not the process \( E_t [p_T] \) as before.

From there, we check that

\[
\frac{\sigma_{S_t}^{\text{final}}}{\theta_t} = \frac{\sigma^P_t - \sigma^q_t}{\theta_t} = 1 + (\gamma - 1) \frac{T - t}{T - t - \omega^2}.
\]

When \( T \) is large, we have \( \omega^2 = T^{2c+1} + \frac{2c+\gamma}{\omega^2(2c+1)} \), and \( \lim_{T \to \infty} \frac{\sigma_{S_t}^{\text{final}}}{\theta_t} = 2c + \gamma + 1 > 1 \).

When \( b = 2\sigma c \), \( \lim_{c \to \infty} \frac{\sigma_{S_t}^{\text{final}}}{\theta_t} = 1 + (1 - \gamma) \frac{\omega^2(T-t)}{t \omega^2 + 1} \).

Let us now consider a given \( t \) as well as the financial asset that only pays \( y_t \) at that specific date \( t \). Its price at date \( s \) is given by \( S^t_s = E_s [p_t y_t] / p_s \).

By Ito’s Lemma, we check that \( \exp \left( Y_2(s)w^2 + Y_1(s)w \right) (y_s)^{1-\gamma} \) has a deterministic drift \( Y_0(s) \) where

\[
Y_2(s) = \frac{1}{2} \frac{\gamma - 1}{\omega^2 (2c + \gamma) - (2c + 1)(T - s)} + \frac{1}{2} \frac{1}{s + \alpha},
\]

\[
Y_1(s) =\frac{(1 - \gamma) \left( k - b\omega^2 + 2ck + b\omega^2 \gamma \right) \left( T + \alpha + 2c\alpha - 2c\omega^2 - \omega^2 \gamma + 2Tc \right)}{(s + \alpha) (2c + 1)^2 \left( (2c + \gamma) \omega^2 - (2c + 1)(T - s) \right)}
- \frac{(2c + 1)\beta + bs(1 - \gamma)}{(2c + 1) (s + \alpha)},
\]

The process \( \exp \left( Y_2(s)w^2 + Y_1(s)w - \int_0^s Y_0(u)du \right) (y_s)^{1-\gamma} \) is then a martingale.

Furthermore, for

\[
\alpha = -t + \frac{1}{\omega^2 - T + t + \frac{1-\gamma}{\omega^2 (2c + \gamma) - (2c + 1)(T - t)}},
\]

\[
\beta = (\gamma - 1) \left( \frac{(k(2c + 1) + b\omega^2 (\gamma - 1)) \left( (2c + \gamma) \omega^2 - (2c + 1)(T + \alpha) \right)}{(2c + 1)^2 \left( (2c + \gamma) \omega^2 - (2c + 1)(T - t) \right)} - \frac{b\alpha}{(2c + 1)} \right)
+ (t + \alpha) \left( \frac{k}{\omega^2 - T + t + \frac{\gamma - 1}{2c + 1}} \right)
\]

we have \( \exp \left( Y_2(t)w^2 + Y_1(t)w - \int_0^t Y_0(u)du \right) (y_t)^{1-\gamma} \) is proportional to \( p_t y_t \). From there, we have that the volatility of \( P^t_s = E_s [p_t y_t] \) is equal to the volatility of \( \exp \left( Y_2(s)w^2 + Y_1(s)w \right) (y_s)^{1-\gamma} \) and given by \( 2Y_2(s)w + Y_1(s) + (1 - \gamma)\tilde{\theta}_s \).

The volatility of \( S^t_s = \frac{P^t_s}{P_s} \) is given by \( 2Y_2(s)w + Y_1(s) - \frac{k-w}{T-s-\omega^2} + \tilde{\theta}_s \) and the ratio finan-
cial/macroeconomic volatility is given by

$$\sigma^t_{S,s} = 1 + \frac{2Y_2(s)w + Y_1(s) - \frac{k-w}{T-s}}{\theta_s}. $$

Taking the limit when $T$ goes to $\infty$ with $\omega^2 = T \frac{2c+1}{2c+\gamma} + \frac{2c+\gamma}{\omega^2(2c+1)}$ and $k = \frac{b(1-\gamma)+\delta_0(2c+1)}{2c+\gamma}T - \omega^2\delta_0$, we obtain

$$\lim_{T \to \infty} \frac{\sigma^t_{S,s}}{\theta_s} = 1 + (\gamma - 1) \frac{\omega^2 (2c+1) (2c+\gamma) (t-s)}{(2c+\gamma)^2 + \omega^2 (2c+1) (s+2ct) + \gamma\omega^2 (t-s) (2c+1)}$$

which is higher (lower) than 1 if $\gamma > 1$ ($\gamma < 1$).

The date-$s$ price $S^c_t$ of the stock that pays $y_t$ at each date $t$ is equal to $\int_s^T S^c_t dt$. Because the volatility of a sum is a weighted average of the volatilities of each term of the sum, the volatility of $S^c_t$ is higher (lower) than 1 if $\gamma > 1$ ($\gamma < 1$).

If instead of taking the limit when $T$ goes to $\infty$, we take $b = 2\sigma c$ and then the limit when $c$ goes to $\infty$ we obtain $1 + \omega^2 (1-\gamma) \frac{t-s}{\omega^2+1} < 1$ for $\gamma > 1$.

References


Belief Dispersion and Convex Cost of Adjustment in the Stock Market and in the Real Economy. Online Appendix.

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In this Online Appendix, we consider some additional properties of the main model as well as one extension. In Section 1, we analyze alternative production models and we show that our model might be embedded in a model with a two-factor (capital and labor) neoclassical production function. In Section 2, we provide closed-form expressions for the average belief and for the stock of capital process and we analyze the autocorrelation properties of the latter. We also provide results about the trading volume in the main model. In Section 3, we analyze option prices. We show that implicit volatility extracted from long-term options has a "smirk" shape. We also determine the value of a firm when it is modeled as a real option on its total assets. In Section 4, we analyze the impact when the horizon becomes arbitrarily large. This permits to show that the model generates a long-run risk component that leads to long-run risk premia that are higher than the short-run one (Bansal and Yaron, 2004).

All the proofs are in Section 5 where we also provide a proof of the convexity of the production set \( Y \) as well as direct proofs for Propositions 1 and 2 of the main paper, i.e. proofs that do not refer to the much more general setting of Theorem 3 and to its proof. Finally, we provide tables that illustrate the conditions and conclusions in the proof of Proposition 9. The references provided at the end are in addition to those referenced in the main paper.

1 Alternative production models

1.1 Neoclassical production function

Let us consider a firm that faces two technologies, a productive one and a riskless one whose return is equal to 0. The productive technology is described by a neoclassical production function \( F(K, L) \) in capital \((K)\) and labor \((L)\) where, as usual, \( F \) is a constant-returns to scale function (homogeneous function of degree 1), increasing and concave in each variable. In particular, \( F \) might be a Cobb-Douglas function, i.e. of the form \( F(K, L) = AK^\vartheta L^{1-\vartheta} \) where \( \vartheta \) and \((1-\vartheta)\) correspond to the
output of capital and labor respectively and are constant, or a CES production function, i.e. of the form $F(K, L) = A (v^\vartheta K + (1 - v)L^\vartheta)^{\frac{1}{\vartheta}}$ where $\vartheta$ is a constant substitution parameter and $v$ a constant share parameter. In both cases $A$ is the total factor productivity and is constant.

For given levels $K$ and $L$, the production between $t$ and $t + dt$ is given by $F(K, L)dt$. We assume that the equilibrium price of labor is equal to $w$. For a given $K$ invested in the productive technology, it is efficient to choose $L$ such that $F(K, L) - wL$, (or by the homogeneity property, $F(1, \frac{L}{K}) - w\frac{L}{K}$) is maximal. This leads to $\frac{1}{K} = l^*$ where $l^*$ is such that $\frac{\partial F}{\partial L}(1, l^*) = w$. Because $w$ is an equilibrium price, this equation admits at least one solution and this solution is unique due to the concavity of $F$. The total production is then given by $F(K, l^*K) = KF(1, l^*)$ and the net production (after wages payment) is given by $(F(1, l^*) - w)K$ of the form $AK$ as in our main model. When all shareholders have the same number of shares and labor is uniformly distributed among shareholders, they will receive the same share of the total wage $wl^*K$. Since there is no intermediate consumption, they reinvest their wages in the firm and, as a consequence, wages have no impact on the total capital $K$ nor on the distribution of shares among agents. The model is then equivalent to the main model and all the results pertain.

### 1.2 Arrow-Debreu production set

We assumed that the date—$t$ to $t + dt$ total production $y_t$ is equal to $K_t$ and fully reinvested in the production technologies. Hence, the consumers only consume the date—$T$ production $y_T = K_T$. Indexing it by $\theta$ in order to highlight its dependence in $\theta$, the date—$T$ production/consumption $y^\theta_T$ corresponds to the terminal value of the process $(y^\theta_t)$ that satisfies $y^\theta_0 = 1$ and $dy^\theta_t = m(\theta_t)y^\theta_t dt + \theta_t y^\theta_t dW_t$. The process $(y^\theta_t)$ might be seen as the arrival of news about $y^\theta_T$ as in Atmaz and Basak (2018). Hence we refer to it as the production process. We denote by $Y_0$ the set $Y_0 = \{y^\theta_T : \theta \in \Theta\}$ that corresponds to the set of all possible production plans when the process $\theta$ describes the set of all admissible random processes and by $Y = \{y : y \leq y^\theta_T \text{ for some } \theta \in \Theta\}$ the set of all production plans when free disposal is allowed. The set $Y$ describes the production possibilities and corresponds to the production set in the Arrow-Debreu description of the economy. It is shown in Section 5 that this set is convex.

Profit maximization can be rewritten as $y^\theta_T = \arg \max_Y E[\bar{p}y_T]$ which corresponds to the classical Arrow-Debreu profit maximization over the convex production set $Y$.

### 2 Additional results

#### 2.1 Average belief

At $t = 0$, the average belief is given by $\delta_0$, and the variance is given by $\omega^2$. As already explained, since all the agents that have the same belief are aggregated into a unique agent with the corresponding belief and whose wealth is their aggregated wealth, the distribution of beliefs at $t = 0$ is described

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1. When $F$ is not differentiable, it suffices to replace the derivative by a subdifferential.
by the distribution of wealth (or by the number of shares of the firm) across beliefs. We generalize this definition as follows.

**Definition 1** *The average belief at date* $t$ *is defined as the date* $t$ *wealth weighted average of beliefs. It is denoted by* $\delta_t$, *and the corresponding variance is denoted by* $\omega_t^2$.

We have the following result.

**Proposition 1** *The average belief* $\delta_t$ *and the variance* $\omega_t$ *are given by*

$$
\delta_t = \delta_0 \frac{\varphi(0)}{\varphi(t)} + b (\gamma - 1) \frac{t}{\varphi(t)} + \frac{2c + \gamma}{\varphi(t)} W_t, \\
\omega_t^2 = \omega^2 \frac{\varphi(0)}{\varphi(t)} + (\gamma - 1) \frac{2c}{\omega^2} \frac{t}{\varphi(t)} \quad \text{and we have} \theta_t = \frac{b + \delta_t}{2c + \gamma}.
$$

The evolution of $\delta_t$ and $\omega_t$ are illustrated by Figure 1 with the parameters of Example 1 (main paper). In particular, 1 (d) shows that the average belief is pessimistic in bad times and optimistic in good times.

In Section 4, it is shown that, for large $t$, $\omega_\infty = 0$ and $\delta_\infty = b \frac{\gamma - 1}{2c + 1}$, which means that in the long run, only the agent $\delta_\infty$ survives and it corresponds to the rational agent in the myopic ($\gamma = 1$) setting. The result is independent of the initial average belief. This illustrates the fact that the relative weights of agents with extreme beliefs, decreases over time.

More interestingly, 1 (c) shows that the relative disagreement about the capital growth rate $\omega_t \frac{\varphi(t)}{m(\delta_t)}$ decreases with the capital level $K_t$ and is then the highest during recessions, in line with Bloom (2014). Our model adds another mechanism to the four mechanisms identified by Bloom (2014) to explain that recessions increase uncertainty. In our framework, the result is a direct consequence of the decreasing returns to scale since the relative disagreement is constant for $c = 0$. In fact, due to the convexity of the adjustment costs (or, equivalently, to the concavity of $m$), adjustments are submitted to more frictions when the capital level is high. Hence, the equilibrium growth rate is less sensitive to beliefs heterogeneity while this sensitivity is higher in recessions.

Let us consider an economy that starts at date $t$ and ends at date $T$ with an initial average belief $\delta_t$ and an initial variance $\omega_t^2$. By the dynamic programming principle, the optimal date $s$, $s > t$, risk exposure determined in this economy that starts at date $-t$ corresponds to the optimal risk exposure in the economy that starts at date $-0$. This means that, at date $t$, all the characteristics of our economy are summarized by $(\delta_t, \omega_t)$ and all the date $-0$ results are valid at date $t$ if we replace $(\delta_0, \omega_0)$ by $(\delta_t, \omega_t)$ and $T$ by $T - t$. This is why there is no loss of generality in limiting all the numerical illustrations to $t = 0$.

**2.2 Properties of the stock of capital process**

Let us analyze the dynamic properties of $K_t$ to see if positive growth rates are followed by positive or negative ones. For this purpose, let us analyze the correlation between the short-term growth
Figure 1: Belief evolution. These figures represent the evolution of beliefs with time and with the stock of capital level. In (a), the black line corresponds to the average belief $\delta_t$ as a function of $t$ for $\delta_0 = 0$ (or $\delta_0 = -0.1$, $\delta_0 = 0.1$ in dashed lines), and the red line represents the variance $\omega_t^2$. In (b), the black line corresponds to the average belief about the growth rate as a function of the stock of capital level for $\delta_0 = 0$ (or $\delta_0 = -\omega$, $\delta_0 = \omega$ in dashed lines). In (c), the dashed lines correspond to the relative disagreement between the one-deviation agents (beliefs $\delta_0 - \omega_t$ and $\delta_0 + \omega_t$) and the average agent (belief $\delta_t$) on the growth rate as a function of the production level. In (d), the black line corresponds to the average belief $\delta_t$ as a function of the stock of capital level (where this level has been normalized to 1 for $W_t = 0$). The other parameters are those in Example 1 (main paper).
rates at date $s$ and at date $t$. It has the same sign as the covariance $C(K, s, t)$ between $\frac{K_{s+h}}{K_s}$ and $\frac{K_{t+h}}{K_t}$. A positive covariance would mean that the stock or capital process exhibits momentum, while a negative one would mean that it exhibits reversal. For a small enough $h$, we have $\frac{K_{t+h}}{K_t} \sim 1 + \left( m(\tilde{\theta}) - \frac{1}{2} \tilde{\theta}_t^2 \right) h + \tilde{\theta}_t (W_{t+h} - W_t)$. Without loss of generality, we may take $s = 0$; then, we have to evaluate

$$C(K, 0, t) = \text{cov} \left( \left( m(\tilde{\theta}_0) - \frac{1}{2} \tilde{\theta}_0^2 \right) h + \tilde{\theta}_0 W_h, \left( m(\tilde{\theta}_t) - \frac{1}{2} \tilde{\theta}_t^2 \right) h + \tilde{\theta}_t (W_{t+h} - W_t) \right).$$

**Proposition 2** The covariance $C(K, 0, t)$ has the sign of $(b + \delta_0)(b(\gamma - 1) - (2c + 1)\delta_0)$. For $\delta_0 = 0$ and $\gamma > 1$, there is a positive correlation between growth rates at date 0 and date $t$.

Note that this result is independent of the distance between 0 and $t$, which means that there is no difference between short-run and long-run behaviors. For $\delta_0 = 0$ and $\gamma > 1$, the stock of capital process exhibits momentum at all horizons, which is compatible with the empirical findings of Mao and Wei (2014), who states that "in comparison with prices, earnings do not display long term reversal". Indeed, in our model, earnings/production are proportional to the stock of capital.

The exact formula for $C(K, 0, t)$ is given in the proof of the proposition. We can check that when $b = 2c\sigma$ for some $\sigma$ and $c \to \infty$ we have $\text{cov} \left( \frac{K_h}{K_0}, \frac{K_{t+h}}{K_t} \right) \to 0$, which means that the covariance vanishes in the exogenous framework. The positive covariance is then a direct consequence of the endogenous determination of the equilibrium stock of capital process.

### 2.3 Equilibrium stock price

When the production process is exogenously given as in Atmaz and Basak (2018), the stock price $S_t$ increases with $W_t$ and it suffices to analyze the comovements of the different equilibrium characteristics with $W_t$ to know how they move with $S_t$. In our framework, this is no longer the case. Hence, we have to determine $S_t$ in order to make such an analysis. Until now, we analyzed the comovements of the different equilibrium characteristics with $K_t$. The following proposition provides a closed-form expression for the stock price process $S_t$ and establishes that $S_t$ is an increasing convex function of $K_t$.

**Proposition 3** The stock price $S_t$ is given by $S_t = H(t)K_T^{\frac{\sigma_T}{2\mu}}$ where $H$ satisfies a given linear first-order differential equation. The log-return exhibits positive skewness and excess kurtosis.

Note that this equation permits us to derive $K_t$ as a function of $S_t$. However, similar to the stock of capital process, the stock price process is non-Markovian: its characteristics (drift and volatility) do not depend only on the current value of the process.

Since the stock of capital process is bounded away from 0, the same property holds for the asset price process. This would not be the case if the asset’s terminal payoff were given by $(K_T - \kappa)^+ = \max(K_T - \kappa, 0)$, which would correspond to a framework where the company has a debt $\kappa$ that
Figure 2: **Financial volatility and the stock of capital level/stock price.** The black (red) line represents the financial volatility as a function of the stock price (stock of capital level). Both have been normalized to 1 for $W_t = 0$. The parameter values are as in Example 1 (main paper).

should be reimbursed at date $T$. Such a payoff corresponds to the payoff of an option on the date $T$ capital $K_t$. The price of such an option is analyzed in more detail in the Section 3.

We retrieve the homogeneous setting by letting $\sigma_0^2$ go to infinity, which gives $S^h_t = H^h(t)K_t$ with $H^h(t) = \exp \left( -c (t - T) \theta^h_0 \right)$. The homogeneous setting leads to a linear relation between prices and production.

In the presence of belief dispersion, the stock price is convex in cash-flow news since $\theta_1 > 1$.

As we can see in Figure 2, the volatility increases with prices as well as with the level of capital. It is more convex in the level of capital than in the stock price, which is natural since the stock price itself is convex in the level of capital.

Let us analyze the possible autocorrelation of stock returns or, in other words, the possible correlation between date $s$ returns and date $t$ returns. For this purpose, let us compute the covariance between $S_{s+h}^0$ and $S_{s+h}^0$. For a small enough $h$, we have $S_{s+h}^0 \sim 1 + (\mu_t - \frac{1}{2} \sigma_t^2) h + \sigma_t (W_{t+h} - W_t)$. Without loss of generality, we may take $s = 0$ and then have to evaluate

$$C(S, 0, t) = \lim_{h \to 0} \frac{1}{h^2} \text{cov} \left( \left( \mu^S_0 - \frac{1}{2} \sigma^2_0 \right) h + \sigma^S_0 W_h, \left( \mu_t - \frac{1}{2} \sigma_t^2 \right) h + \sigma_t (W_{t+h} - W_t) \right).$$

Since $T$ is a parameter of the model, we have to define what we mean by the short and long terms. In the next section, short-term behavior is defined as the behavior between 0 and $t$ when $t$ tends to 0, while long-term behavior is defined as the behavior between 0 and $\xi T$ for a given $\xi \in (0, 1)$.

We have the following result.

**Proposition 4**

1. If there is no initial bias ($\delta_0 = 0$), there is short-term momentum: $\lim_{t \to 0} C(S, 0, t) > 0$ for $\gamma > 1$ and $\omega$ or $T$ large enough.

2. For $\xi \in (0, 1)$, $C(S, 0, \xi T) < 0$ for $T$ or $\omega$ large enough: there is long-run reversal.

3. In the exogenous setting, we have short-term and long-run negative autocorrelations of returns.

4. There is no momentum or long-run reversal in the homogeneous framework.
Figure 3: **Autocorrelation, momentum and reversal.** This figure plots the covariance between the date 0 daily return and the date \( t \) daily return (for \( \gamma = 2, T = 30, b = 0.1464, c = 2, \) and \( \omega = 0.5 \) (in red) or \( \omega = 1 \) (in black). This last case corresponds to \( \mu = 1.3\%, \sigma = 10.26\% \) and \( \theta = 2.44\% \).

This result is illustrated in Figure 3. Price momentum and long-run reversal are among the prominent anomalies that cannot be explained by the Fama and French (1992, 1993) three-factor models. A number of explanations have been proposed to explain these two anomalies, including rational models and behavioral models. In our setting, the momentum and reversal are explained by belief heterogeneity and decreasing returns to scale in the risk-return production function. More precisely, the momentum is explained by the positive relationship between risk exposure and macroeconomic shocks. However, due to the decreasing returns to scale, a risk exposure increase has less impact when the risk level is already high. This leads to long-term reversal. Both short-term momentum and long-term reversal disappear in homogeneous settings. In an exchange economy setting à la Atmaz and Basak (2018), the autocorrelations are negative in both the short run and the long run.

### 2.4 Trading volume

The distribution of shares across agent types might have two interpretations. If we consider that there is actually only one agent of each type in the economy, this distribution corresponds to the wealth distribution over the agent space parametrized by the types. If we consider instead that all the agents in the economy are endowed with the same number of shares, this distribution corresponds to the relative frequency of agents over the type space. This section is based on this second interpretation.

Since \( \bar{c}_\delta \) is the aggregate consumption of \( \delta \)-type agents, their individual consumption is given by \( \frac{\bar{c}_\delta}{\nu_\delta} \) and the variation of their endowment is given by \( \left| \frac{\bar{c}_\delta}{\nu_\delta} - K_T \right| \) in absolute terms and by \( \left| \frac{\bar{c}_\delta}{\nu_\delta K_T} - 1 \right| \) in relative terms.

We define the trading volume as \( V = E^\delta \left[ \left| \frac{\bar{c}_\delta}{\nu_\delta K_T} - 1 \right| \right] \), where \( E^\delta \) is the expectation under the density \( \frac{1}{\sqrt{2\pi}\omega} \exp \left[ \frac{1}{2} \frac{(\delta - \delta_0)^2}{\omega^2} \right] \). We have the following result.
Figure 4: **Trading volume.** The black (red) line represents the volume in absolute (relative) terms as a function of date $T$ stock of capital. The price has been normalized to 1 for $W_T = 0$ and $W_T$ ranges between minus 2 standard deviations and 2 standard deviations (i.e., between $-2\sqrt{30}$ and $2\sqrt{30}$). All parameter values are those in Example 1 (main paper).

**Proposition 5** The trading volume is given by

$$V = \frac{\sqrt{2}}{\pi} \left( \chi \exp \left( \frac{\varphi(0)}{\varphi''(0)} \right) \frac{\left( W_T (2c+\gamma)+Tb(\gamma-1)-(2c+1)T\delta_0 \right)^2}{\gamma \varphi(0) (\chi + (2c + \gamma) T)} - 1 \right)$$

with $\chi = \left( 2Tc \left( 1 - \gamma \right) + (2c + \gamma) \varphi'' \right)$.

It converges to 0 when $\varphi$ converges to $\infty$ (no heterogeneity). Trading volumes are higher in very good and very bad states of the world. For small levels of heterogeneity, we have

$$V = \frac{1}{\varphi''} \left( \frac{W_T (2c+\gamma)+Tb(\gamma-1)-(2c+1)T\delta_0}{\gamma (2c + \gamma)^2} \right)^2 + o \left( \frac{1}{\varphi''} \right)$$

and trading volume increases with the degree of heterogeneity.

In Figure 4, we observe that the volume increases with $K_T$ and/or $S_T$ almost on their whole domain.

3 Debt, bankruptcy and (real) options

3.1 Option prices

In this Section, we show that implicit volatility extracted from long-term options has a "smirk" shape.

Let us consider a call option with a horizon $T^*$ and for which, for the ease of the presentation but without any loss of generality, the strike $\Delta$ is expressed as a proportion $\kappa$ of the median value of $S_{T^*}$, i.e. the value of $S_{T^*}$ for $W_{T^*} = 0$. The $T^*$–payoff is given by $C_{T^*}^{\kappa T^*}(w) = [S_{T^*}(w) - \kappa S_{T^*}(0)]^+$. Let us denote by $C_{T^*}^{\kappa T^*}$ its price at date $t$. Note that for $T^* = T$, the terminal payoff $[S_T - \kappa S_T(0)]^+$ coincides with $[K_T - \kappa K_T(0)]^+$ and the option is also an option on the terminal date stock of capital.
Proposition 6 The price of a call option with a maturity $T^*$ and strike $\kappa S_{T^*}(0)$ is given by

$$C^\kappa_{T^*}(w) = S(t, w) \left( \mathcal{N}(d_1) + \mathcal{N}(d'_1) \right) - \kappa S_{T^*}(0) \left( \mathcal{N}(d_2) + \mathcal{N}(d'_2) \right),$$

$$d_1 = \frac{U - P}{\sqrt{Q}}, \quad d'_1 = \frac{P - V}{\sqrt{Q}}, \quad d_2 = \frac{U - p}{\sqrt{q}}, \quad d'_2 = \frac{p - V}{\sqrt{q}},$$

$$P = Q \left( \frac{w}{T^* - t} - \frac{b \gamma \omega^2 + 2ck}{\omega^2 (2c + \gamma) - 2c(T - T^*)} \right)$$

$$- Q \left( \frac{1}{(\omega^2 (2c + \gamma) - (2c + 1)(T - T^*)) (\omega^2 (2c + \gamma) - 2c(T - T^*))} \right)$$

$$Q = (T^* - t) \frac{\omega^2 (2c + \gamma) - (2c + 1)(T - T^*)}{\omega^2 (2c + \gamma) - (2c + 1)(T - t)}$$

$$p = q \left( \frac{w}{T^* - t} - \frac{b \gamma \omega^2 + 2ck}{\omega^2 (2c + \gamma) - 2c(T - T^*)} \right),$$

$$q = (T^* - t) \frac{\omega^2 (2c + \gamma) - 2c(T - T^*)}{\omega^2 (2c + \gamma) - 2c(T - t)}$$

where $\mathcal{N}(x) = \int_{-\infty}^{x} \exp \left( -\frac{x^2}{2} \right) dx$, and where $U < V$ are the solutions of $L_2 x^2 + L_1 x = \ln \kappa$ where

$$L_2 = 1 - \frac{\omega^2 (2c + \gamma)}{((2c + \gamma) \omega^2 - 2c(T - T^*)) ((2c + \gamma) \omega^2 + (2c + 1)(T^* - T))},$$

$$L_1 = 1 - \frac{\omega^2 (2c + \gamma) (2b \omega^2 - 2k + 2b(T^* - T))}{((2c + \gamma) \omega^2 + (2c + 1)(T^* - T))}.$$
Figure 5: Implicit volatility. This figure represents the implicit volatility as a function of the strike parameter $\kappa$ for $T^* = 10$. All other parameter values are those in Example 1 (main paper).

(proportional to $dt$) and, as a consequence, the stock of capital process was positive. There was no room for bankruptcy. This is natural since our firm is an aggregate representation of the production sector.

We have also seen that our single firm can be interpreted as the aggregation of a large number of firms, each of which is very small relative to the economy as a whole. In this section, we propose to price one such small firm by assuming that it has a total asset process that is proportional to the total capital stock $K_t$ (let us denote it by $\varepsilon K_t$) and that it also has a debt $\varepsilon \Delta$ whose maturity is given by some $T^* \leq T$.

Such a firm can be modeled as a "real option" (Bernanke 1983, Brennan and Schwartz, 1985, McDonald and Siegel, 1986) where the value of the firm itself is viewed as an option on its total assets process. Up to the scaling parameter $\varepsilon$, the terminal value of the assets is given by $K_T$ and their date–$t$ value is given by $S_t$. Therefore, the value of the firm at $T^*$ is given by $[S_{T^*} - \Delta]^+ = \max(0, S_{T^*} - \Delta)$: the firm is able to pay its debt by giving a given share of its total assets only if the value at date $T^*$ of these is higher than $\Delta$.

Until now and with these notations, we considered only the case $\Delta = 0$. It is likely that $\Delta > 0$ would have an impact on the equilibrium stock’s characteristics. Considering such a setting would allow us to analyze the impact of beliefs heterogeneity in presence of long-term debt and leverage and possibility of going bankrupt.

The literature mentioned above has identified decreasing returns to scale (as in our framework) as one of the two alternative requirements (the other one being imperfect competition) for real options to have an impact on investment decisions. In this section, we show that the embedded options have a direct impact on stock characteristics in our framework.

In this section, the process $S(t)$ defined above no longer corresponds to the value of the firm but instead corresponds to the date–$t$ total value of firm’s assets while $\Delta$ corresponds to the face value of outstanding debt which is assumed to be in the form of a single zero-coupon whose maturity is $T^*$. Then, the value of the firm (or of the stock) corresponds to the value of the call option on $S$ with strike $\Delta$ and maturity $T^*$ as characterized in Proposition 6.

From this formula, we can derive $\sigma_C = \frac{1}{\Phi} \frac{\partial \Phi}{\partial \mu}$ (see the proof of Proposition 6), and from (9), we have $\mu_C$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Implicit volatility. This figure represents the implicit volatility as a function of the strike parameter $\kappa$ for $T^* = 10$.}
\end{figure}
Figure 6: **Volatility and ratio of volatilities.** In (a), (resp. (b)), the black line represents the financial volatility as a function of the stock price (resp. stock of capital level) for $T^* = 10$ and $\kappa = 0.5$. In (c), the risk premium is shown as a function of financial volatility. In (d), the ratio between the option and macroeconomic volatilities is given as a function of the normalized option price. The stock price and stock of capital level have been normalized to 1 for $W_t = 0$. All the other parameter values are those of Example 1 (main paper), except for the red line in (a), which represents the financial volatility as a function of the stock price for $\omega_t = 0.0245$.

As seen in Figure 8, volatility (as well as the risk premium) rises steeply in recessions in accordance with the empirical evidence in the literature (Schwert, 1989, for the volatility, Fama and French, 1989, Ferson and Harvey, 1991, for the risk premium). It also shows that the ratio between financial and macroeconomic volatility can be even higher when taking into account the embedded options when evaluating financial assets.

### 4 Large horizon

The previous analysis can be extended to an infinite horizon taking the limit when $T$ goes to $\infty$. When $T$ is large, we have $\sigma^2 \sim T \frac{2c+1}{2c+\gamma} + \frac{2\gamma}{\theta^2(2c+1)} + O \left( \frac{1}{T} \right)$, and replacing it in the different expressions above, we obtain the following result.
Proposition 7 Asymptotically (when $T$ is very large), we have

$$
\omega_t^2 = \omega^2 \left( \frac{(2c+\gamma)^2}{(2c+\gamma)^2 + t\omega^2(2c+1)^2} \right),
$$

$$
\delta_t = \delta_0 + \frac{2c+1}{2c+\gamma} \omega_t^2 W_t - \frac{b(1-\gamma) + \delta_0(2c+1)}{(2c+\gamma)^2} (2c+1) t\omega_t^2,
$$

$$
\bar{\theta}_t = \frac{b+\delta_t}{2c+\gamma}, \quad \mu_t = b \left( 2c+1 \right) \bar{\theta}_t, \quad \sigma_t = \left( 2c+1 \right) \bar{\theta}_t,
$$

$$
S(t, w) = \frac{\exp \left( \left( \omega^2 (2c+1) w^2 + 2w(2c+\gamma)(b+\delta_0) b\omega^2 (2c+1) + t \left( b\omega^2 (2c+1) + (b+\delta_0)(2c+2\gamma-1)b-(2c+1)\delta_0) \right) \right) }{\sqrt{1 + t\omega^2 \left( \frac{2c+1}{2c+\gamma} \right)^2}}.
$$

The parameter $b$ determines the Sharpe ratio, while the parameter $c$ determines the volatility ratio.

Figure 9 illustrates the evolution of $\delta_t$ and $\omega_t$. For large $t$, we have $\omega_\infty = 0$ and $\delta_\infty = \frac{\gamma - 1}{2c+1}$, which means that in the long run, only the agent $\delta_\infty$ survives with $\delta_\infty = \frac{b}{(2c+1)} (\gamma - 1)$, which corresponds to the rational agent in the myopic ($\gamma = 1$) setting. The result is independent of the initial average belief. In general, this does not correspond to the rational agent: the surviving agent is optimistic (pessimistic) for $\gamma > 1$ ($\gamma < 1$). Note that this result might seem to contradict the survival literature introduced by Yan (2008), where the surviving agent is the rational agent. However, those papers consider infinite horizon frameworks with continuous time consumption, while we consider finite horizon frameworks with consumption at the final date only, and we let the horizon go to $\infty$.

To retrieve the analogous results in the exogenously given production process setting, it suffices to take $b = 2c\sigma$ (where $\sigma$ is the macroeconomic volatility) and to let $c$ go to $\infty$. We obtain $\omega_\infty = 0$, which means that, once again, only the agent $\delta_\infty$ survives, but now, we have $\delta_\infty = (\gamma - 1) \sigma$. By definition, we have $\theta_\infty = \sigma$, but we have $\sigma_\infty \to \pm \infty$ and $\mu_\infty = \pm \infty$ depending on the sign of $(b+\delta_0)$. The ratio between financial volatility and macroeconomic volatility converges to $\infty$.

When there is no heterogeneity ($\omega = 0$), all the equilibrium characteristics are constant, and we have $\theta = \frac{b+\delta_0}{2c+\gamma}$, $\mu = b \left( 2c+1 \right) \frac{2c+1}{2c+\gamma}$ and $\sigma = \left( 2c+1 \right) \frac{2c+1}{2c+\gamma}$.

The heterogeneous setting then corresponds to the homogeneous setting where at each date $\delta_0$ is replaced by the current average belief $\bar{\theta}_t$. The results do not depend on $\omega_t^2$. In the large horizon setting, belief heterogeneity does not intervene directly in the equilibrium characteristics; it only governs the fluctuations of the average belief.

Since $T = \infty$, we may analyze the term structure of the drift, volatility and Sharpe ratio. We define the average volatility $\sigma_{[0,t]}$ over $[0,t]$ by $\sigma_{[0,t]} = \sqrt{\frac{1}{t} \text{VAR} \left[ \ln S_t \right]}$, the average return $\mu_{[0,t]}$ over $[0,t]$ by $\mu_{[0,t]} = \frac{1}{t} \text{E} \left[ \ln S_t \right] + \frac{1}{2} \sigma_{[0,t]}^2$, and the Sharpe ratio over $[0,t]$ by $\text{SHARPE}_{[0,t]} = \frac{\mu_{[0,t]}}{\sigma_{[0,t]}}$. We also define the average instantaneous volatility by $\bar{\sigma}_t = \text{E} \left[ \sigma_t \right]$, the average instantaneous return as

\[\text{If we denote by } \delta_t^\infty (\text{resp. } \delta_\infty) \text{ the date } t-\text{average belief in the } T-\text{horizon (resp. } \infty-\text{horizon) model, our } \delta_\infty \text{ corresponds to } \lim_{T \to \infty} \lim_{t \to \infty} \delta_t^T. \text{ We can check that it also corresponds to } \lim_{T \to \infty} \delta_t^T. \text{ Yan’s surviving belief corresponds to } \lim_{t \to \infty} \delta_t^\infty. \text{ However, while we can define } \lim_{T \to \infty} \delta_t^T \text{ in our framework, } \delta_t^\infty \text{ has no meaning.}\]
Figure 7: **Large horizon and the evolution of beliefs.** From bottom to top, the black lines represent the average belief $\delta_t$ as a function of $t$, respectively, for $\delta_0 = -0.1$, 0 and 0.1. The red line represents the standard deviation in beliefs $\omega_t$. The values of $b$, $c$, and $\omega$ are as in Example 1 (main paper).

$\bar{\mu}_t = \mathbb{E}[\mu_t]$ and the instantaneous Sharpe ratio by $\text{Sharpe}_t = \frac{\bar{\mu}_t}{\sigma_t}$. In our large horizon setting, we have $\text{Sharpe}_t = \frac{\bar{\mu}_t}{\sigma_t} = \frac{\bar{\mu}_t}{\sigma_t} = b$ independent of $t$.

When drift and volatility are constant equal to $\mu$ and $\sigma$, we have $\mu_{[0,t]} = \bar{\mu}_t = \mu$, $\sigma_{[0,t]} = \bar{\sigma}_t = \sigma$ and $\text{Sharpe}_{[0,t]} = \text{Sharpe}_t = \frac{\mu}{\sigma}$ for all $t$. This is no longer the case in our setting where a long-run risk component emerges with an associated long-run risk premium that is higher than the short-run one (Bansal and Yaron, 2004). Figure 10 shows that average volatility (risk premium) over $[0,t]$ increases with $t$ and is always above (below) the average instantaneous volatility (risk premium), while the Sharpe ratio over $[0,t]$ decreases then increases and is always below the instantaneous Sharpe ratio. The long-run risk (return) is then higher (lower) on average than the short-run risk (return), and the long-run risk premium by unit of risk is lower on average than its short-run counterpart.

5 **Proofs**

**Convexity of the production set** Let us consider $y^1$ and $y^2$ in $Y$, there exists $\theta^1$ and $\theta^2$ such that $y^1 \leq z^1_T$ and $y^2 \leq z^2_T$ where $z^1$ and $z^2$ are the solutions of

\[
\begin{align*}
    dz^1_t &= m(\theta^1_t)z^1_t dt + \theta^1_t z^1_t dW_t, \ z^1_0 = 1, \\
    dz^2_t &= m(\theta^2_t)z^2_t dt + \theta^2_t z^2_t dW_t, \ z^2_0 = 1.
\end{align*}
\]
Let us define $\theta$, $z$ and $\bar{z}$ such that

\[
\theta = \frac{\theta_{1} z_{1} + \theta_{2} z_{2}}{z_{1} + z_{2}},
\]

\[
dz_t = m(\theta_t) z_t dt + \theta_t z_t dW_t, \quad z_0 = 1,
\]

\[
\bar{z} = \frac{z_{1} + z_{2}}{2}.
\]

We have $d \bar{z}_t = \frac{m(\theta_{1}) z_{1} + m(\theta_{2}) z_{2}}{z_{1} + z_{2}} \bar{z}_t dt + \frac{\theta_{1} z_{1} + \theta_{2} z_{2}}{z_{1} + z_{2}} \bar{z}_t dW_t$ with $\bar{z}_0$. Furthermore, each time we have $z_t = \bar{z}_t$, their respective volatilities are equal and, by concavity of $m$, $z_t$ has a higher drift. Therefore, by a classical comparison argument, we have $\bar{z} \leq z \in Y_0$ and $\bar{z} \in Y$.

**Proof of Proposition 1 (main paper)** Without divergence of opinion, everything works like if we had one agent that believes that the risk-return production function is $m_0(\theta) = m(\theta) + \delta_0 \theta$. Let us take

\[
\tilde{c}_h = \exp \left( \int_0^T \left( m(\theta_t) - \frac{1}{2} \theta_{0}^2 \right) dt + \int_0^T \theta_0 dW_t \right), \quad \theta_0 = \frac{b + \delta_0}{2c + \gamma}, \quad \tilde{p}_h = M_T^{\delta_0} K_{h,T}^{-\gamma}.
\]

where $K_{h,t} = \exp \left( \int_0^t \left( m(\theta_t) - \frac{1}{2} \theta_{0}^2 \right) dt + \int_0^t \theta_0 dW_t \right)$.

The first order condition for utility maximization is satisfied as well as the market clearing one. Let us show that $\theta_0$ maximizes $E \left[ \tilde{p}_h K_{T}^{\vartheta} \right]$ for $\vartheta \in \Theta$. For a given $\vartheta$, let us define $z_t^\vartheta = z_t^\vartheta = K_t^{\vartheta} K_{h,t}^{-\gamma}$, it is equivalent to show that $\theta_0$ maximizes $E \left[ M_T^{\delta_0} z_{T}^{\vartheta} \right]$ where $dM_t^{\delta_0} = \delta_0 M_t^{\delta_0} dW_t$ and

\[
dz_t^\vartheta = \left( m(\vartheta_t) - \gamma m(\theta_0) + \frac{1}{2} \gamma (\gamma + 1) \theta_0^2 - \gamma \theta_t \theta_0 \right) z_t^\vartheta dt + (\vartheta_t - \gamma \theta_0) z_t^\vartheta dW_t
\]

\[
= \mu_t^\vartheta z_t^\vartheta dt + \sigma_t^\vartheta z_t^\vartheta dW_t.
\]
Let us denote by $V$ the function defined by $V(t;M^0_\theta, z_t, W_t) = \max_{\vartheta} E_t \left[ M^0_\theta z_T^2 \right]$. We are facing an optimal control problem whose associated dynamic programming equation is given by

$$
V_t + \max_{\vartheta} \left( \mu_{\theta} z_t \vartheta V_z + \left( \delta_0 M^0_{\theta} \right)^2 B_{\vartheta}^2 + \frac{1}{2} \sigma_{\theta}^2 \vartheta^2 B_{\vartheta}^2 + \delta_0 M^0_{\theta} \sigma_{\theta}^2 z_t \vartheta B_{\vartheta} \right) = 0,
$$

and if we find a control $\vartheta^*$ and a function $V$ such that $\vartheta^*$ realizes the maximum in (3), and $V$ is a solution of (3) such that $V(T, M^0_\theta, z, w) = M^0_\theta z$, then $\vartheta^*$ is the optimal control and $V$ is the value function of (3).

We posit that $V$ is of the form $V(t, M^0_\theta, z, w) = M^0_\theta z F(t, w)$ and solving the problem consists in finding - if they exist - solutions $F$ and $\vartheta^*$ of

$$
F_t + \mu_{\theta} F + \frac{1}{2} F_{ww} + \delta_0 \sigma_{\theta}^2 F + \sigma_{\theta}^2 \vartheta F + \delta_0 F_{\vartheta} = 0, \quad F(T, w) = 1.
$$

Note that $\frac{d}{d\vartheta} \left( \mu_{\theta} F + \delta_0 \sigma_{\theta}^2 F + \sigma_{\theta}^2 \vartheta F \right) \bigg|_{\vartheta^*} = 0$ is equivalent to $\vartheta^* = \arg \max \mu_{\theta} F + \delta_0 \sigma_{\theta}^2 F + \sigma_{\theta}^2 \vartheta F$ because this last function is concave in $\vartheta$.

Since our aim is to show that $\theta_0$ maximizes $E \left[ \hat{\vartheta}_T z_T^{\theta} \right]$, it suffice to take $\vartheta^* = \theta_0$ and to see that

$$
F(t, w) = \exp \left( \left( 1 - \gamma \right) m(\theta_0) + \frac{1}{2} \gamma (\gamma - 1) \theta_0 - \frac{1}{2} (1 - \gamma) \theta_0 \delta_0 \right) (T-t) \right)
$$

solves the equations above with the terminal condition $F(T, w) = 1$.

From there we also derive $\sigma^0_\theta = \delta_0 - \gamma \theta_0$.

**Proof of Proposition 2 (main paper)** By construction, $\bar{c}^\delta_\theta$ satisfies the first order conditions for utility maximization. Furthermore, we have

$$
\bar{p}_e = \exp \left( \frac{1}{2} \frac{(W_T - \bar{k}_e)^2}{\bar{\omega}^2} \right) \left( K_e \right)^{-\gamma}, \quad \bar{c}_\theta = \vartheta_c \lambda^\delta - \frac{1}{2} \left( M^0_\theta \right)^\frac{1}{2} \bar{p}_e^{\frac{1}{2}}
$$

where $K_e = \exp \left( \int_0^t \left( m(\vartheta_c) - \frac{1}{2} \bar{\omega}^2_\vartheta \right) dt + \int_0^t \vartheta_c dW_t \right), \quad \bar{\omega}^2_e \geq T$ and

$$
\frac{1}{\omega^2} = \bar{\omega}^2_e \frac{T}{(1 - \gamma) + \bar{\omega}^2_e \gamma}, \quad \delta_0 = \frac{T b (1 - \gamma) - k_e (2c + \gamma)}{(2c + \gamma) \bar{\omega}^2_e - T (2c + 1)}, \quad \lambda^\delta = \exp \left( \frac{1}{2} \left( \bar{\omega}^2_e - T \right) \delta^2 \right) \exp (k\delta), \quad \vartheta_c = \frac{\bar{\omega}}{\sqrt{2\pi} \gamma}.
$$

Note that $\frac{\bar{\omega}^2_e}{\bar{\omega}^2_e - T} \frac{T}{(1 - \gamma) + \bar{\omega}^2_e \gamma}$ is equal to 0 when $\bar{\omega}_e = \sqrt{T}$ and to $\infty$ when $\bar{\omega}_e = \infty$. Furthermore, it is easy to check that it increases with $\bar{\omega}_e$ for $\bar{\omega}_e \geq \sqrt{T}$. There is then one and only one $\bar{\omega}_e \geq \sqrt{T}$.
satisfying Equation (7) above.

We check that

\[ Z_c = \#p_1 \]

Finally, we have

\[ E \left[ \tilde{p}_c c^\delta \right] = \vartheta_c \exp \left( \frac{\delta^2 (T - \omega_c^2) - k_e^2 (1 - \gamma) - T \delta^2}{2 \gamma} \right) \]

\[ E \left[ \exp \left( -\frac{1}{2 \omega_e^2} W_T^2 \left( \frac{1}{\gamma} - 1 \right) \right) \right] \]

and using the fact that

\[ E \left[ \exp (\alpha W_T^2 + \beta W_T) \right] = \frac{1}{\sqrt{1 - 2 \alpha T}} \exp \left( \frac{1}{2} \frac{\beta^2}{1 - 2 \alpha T} \right) \]

for \( \alpha < \frac{1}{2 T} \), we have

\[ E \left[ \tilde{p}_c c^\delta \right] = \kappa \exp \left( -\frac{1}{2 \omega^2} + \frac{\delta}{\omega^2} \right) \]

for some \( \kappa > 0 \) which gives

\[ E \left[ \tilde{p}_c c^\delta \right] = \nu_b E \left[ \tilde{p}_c K_{e,T} \right] \]

By Ito’s Lemma, it is easy to check that

\[ \frac{\varphi_t}{\omega_t^2 - T + t} \]

is a martingale equal to \( p_T \) for \( t = T \). It is then equal to \( p_t \) at all dates and, by Ito’s Lemma, \( \sigma_{e,t} = \frac{W_t - k - \theta_0 (T - t)}{\omega_t} - \frac{b + \delta_0}{2 e + \gamma} \).

As we deal with homogeneous utility functions and since agents have shares of aggregate endowment, the uniqueness of the exchange equilibrium is derived from Dana (1995).³

Proof of Proposition 1. Let us consider an economy that starts at date \( t \) and ends at date \( T \) with an initial average belief \( \delta_t \) and an initial variance \( \omega_t^2 \). It suffices to adapt the results of Theorem 3 (main paper) to obtain that the optimal date—s risk exposure in this economy is given by

\[ \tilde{\theta}^t (s, W_s) = W_s - W_t - b (T - s) - k_t + b \omega_t^2 \]

where \( \varphi_t (u) = (2 c + \gamma) \omega_t^2 - (1 + 2 c) ((T - t) - u) \)

and where \( \omega_t \) and \( k_t \) are defined by

\[ \frac{1}{\omega_t^2} = \frac{\varphi_t (0) \omega_t^2}{2 (T - t) c (1 - \gamma) + \gamma (2 c + \gamma) \omega_t^2}, \]

\[ k_t = \frac{b (1 - \gamma) + \delta_t (2 c + 1)}{2 c + \gamma} (T - t) - \omega_t^2 \delta_t. \]

By the dynamic programming principle, we have \( \tilde{\theta}^t (s, W_s) = \tilde{\theta} (s, W_s) \) for all \( t \) and all \( s \). It is easy to check that these equations lead to \( \omega_t^2 = \omega^2 \), \( k_t = k - W_t \) as well as to the expressions for \( \delta_t \) and \( \omega_t^2 \) provided in the Proposition. ■

³Dana (1995) considers a setting with a finite number of agents. The proof can be adapted easily. In the proof of Theorem 5 (main paper), we provide such an adaptation to a large number of agents in a production setting.
Proof of Proposition 2. For \( s = 0 \), \( (m(\theta_0) - \frac{1}{2} \bar{\theta}_0^2) \) is deterministic and we have to evaluate, for \( h \) small,

\[
\begin{align*}
\operatorname{cov} \left( \frac{K_h}{K_0}, \frac{K_{t+h}}{K_t} \right) &= \tilde{\theta}_0 \operatorname{cov} \left( W_h, \left( m(\bar{\theta}_t) - \frac{1}{2} \bar{\theta}_t^2 \right) h + \bar{\theta}_t (W_{t+h} - W_t) \right) \\
&= \tilde{\theta}_0 E \left[ W_h \left( \left( m(\bar{\theta}_t) - \frac{1}{2} \bar{\theta}_t^2 \right) h + \bar{\theta}_t (W_{t+h} - W_t) \right) \right] \\
&- \tilde{\theta}_0 E \left[ W_h \right] E \left[ \left( m(\bar{\theta}_t) - \frac{1}{2} \bar{\theta}_t^2 \right) h + \bar{\theta}_t (W_{t+h} - W_t) \right] \\
&= \tilde{\theta}_0 E \left[ W_h E_t \left[ \left( m(\bar{\theta}_t) - \frac{1}{2} \bar{\theta}_t^2 \right) h + \bar{\theta}_t (W_{t+h} - W_t) \right] \right] = h \tilde{\theta}_0 E \left[ W_h E_h \left[ m(\bar{\theta}_t) - \frac{1}{2} \bar{\theta}_t^2 \right] \right].
\end{align*}
\]

We have

\[
\begin{align*}
E_h \left[ b \bar{\theta}_t - c \bar{\theta}_t^2 - \frac{1}{2} \bar{\theta}_t^2 \right] &= E_h \left[ b \frac{(W_t - b(T-t) - k + b \bar{x}^2)}{(2c + \gamma) \bar{x}^2 - (1 + 2c)(T-t)} - \left( c + \frac{1}{2} \right) \left( \frac{(W_t - b(T-t) - k + b \bar{x}^2)}{(2c + \gamma) \bar{x}^2 - (1 + 2c)(T-t)} \right)^2 \right] \\
&= \left( \frac{W_h^2}{(2c + \gamma) - (2c + 1)(T-t)^2} \right)^2 \left( c + \frac{1}{2} \right) \\
&+ W_h \left( T - t - 2c \bar{x}^2 - \bar{x}^2 \gamma + 2Tc - 2ct \right)^2 \\
&+ \frac{1}{2} (k - b \bar{x}^2 + Tb - bt) \left( -k + b \bar{x}^2 + Tb - 2ck - bt - 2bc \bar{x}^2 - 2b \bar{x}^2 \gamma + 2Tbc - 2bct \right) \\
&= \frac{(T - t - 2c \bar{x}^2 - \bar{x}^2 \gamma + 2Tc - 2ct)^2}{(T - t - 2c \bar{x}^2 - \bar{x}^2 \gamma + 2Tc - 2ct)^2}
\end{align*}
\]

and

\[
\begin{align*}
\operatorname{cov} \left( \frac{K_{t+h}}{K_0}, \frac{K_{t+h}}{K_t} \right) &= h^2 \left( -2c \bar{x}^2 \gamma (2c + \gamma) \frac{k - b \bar{x}^2 + 2ck + b \bar{x}^2 \gamma}{(2c + \gamma) \bar{x}^2 - (2c + 1)(T-t)} \right)^2 \\
&= \left( \frac{(2c + \gamma) \bar{x}^2 - (2c + 1)(T-t)}{2(2c + \gamma) \bar{x}^2 - (2c + 1)(T-t)} \right)^2 \\
&= \left( \frac{(2c + \gamma) \bar{x}^2 - (2c + 1)(T-t)}{2(2c + \gamma) \bar{x}^2 - (2c + 1)(T-t)} \right)^2 \\
&= \frac{(2c + \gamma) \bar{x}^2 - (2c + 1)(T-t)}{(2c + \gamma) \bar{x}^2 - (2c + 1)(T-t)}
\end{align*}
\]

and since we know that \((2c + \gamma) \bar{x}^2 - (2c + 1)(T-t) > 0\), we have \(C(K, 0, t)\) has the sign of \((b(\gamma - 1) - (2c + 1) \delta_0) (b + \delta_0)\).

\[\blacksquare\]

**Proof of Proposition 3.** Let \( H \) be such that \( H(T) = 1 \) and

\[
\frac{dH}{dt} = -c (2c + \gamma) \bar{x}^2 \left( \frac{1}{2c+1} \ln \left( \frac{(2c+\gamma) \bar{x}^2 - (1+2c)(T-t)}{(2c+\gamma) \bar{x}^2 - (1+2c)(T-t)} \right) + \frac{(k-b \bar{x}^2 + Tb)^2}{(2c+\gamma) \bar{x}^2 - (1+2c)(T-t)} \right) H,
\]

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and let $\mathcal{G}$ be defined by $\mathcal{G}(t, K_t) = H(t)K_t^{\frac{\sigma_t^2}{2}}$. We have $\mathcal{G}(T, K_t) = K_t$ and

$$
\frac{\partial \mathcal{G}}{\partial w}(t, K_t) = \sigma_t \mathcal{G}(t, K_t),
$$

and then $S_t = \mathcal{G}(t, K_t)$. The results on skewness and kurtosis are derived as in Proposition 4 (main paper).

**Proof of Proposition 4.** Without loss of generality, we may take $s = 0$ and we have then to evaluate

$$
\text{cov} \left( \left( \mu_0^S - \frac{1}{2} (\sigma_0^S)^2 \right) h + \sigma_0^S W_h, \left( \mu_t - \frac{1}{2} \sigma_t^2 \right) h + \sigma_t (W_{t+h} - W_t) \right),
$$

$$
= \sigma_0^S \text{cov} \left( W_h, \left( \mu_t - \frac{1}{2} \sigma_t^2 \right) h + \sigma_t (W_{t+h} - W_t) \right),
$$

$$
= \sigma_0^S E \left[ W_h \left( \left( \mu_t - \frac{1}{2} \sigma_t^2 \right) h + \sigma_t (W_{t+h} - W_t) \right) \right],
$$

$$
= \sigma_0^S E \left[ W_h E_t \left( \left( \mu_t - \frac{1}{2} \sigma_t^2 \right) h + \sigma_t (W_{t+h} - W_t) \right) \right],
$$

$$
= \sigma_0^S E \left[ W_h \left( \mu_t - \frac{1}{2} \sigma_t^2 \right) \right].
$$

We have $\mu_t - \frac{1}{2} \sigma_t^2$ of the form $\alpha_{0,t} + \alpha_{1,t} W_t + \alpha_{2,t} W_t^2$ and

$$
E \left[ W_h \left( \mu_t - \frac{1}{2} \sigma_t^2 \right) \right] = E \left[ W_h E_h \left[ \alpha_{0,t} + \alpha_{1,t} W_t + \alpha_{2,t} W_t^2 \right] \right] = h \alpha_{1,t}.
$$

We also have

$$
\text{Var} \left( \left( \mu_0^S - \frac{1}{2} (\sigma_0^S)^2 \right) h + \sigma_0^S W_h \right) = \left( \alpha_{1,t}^2 h^2 + 3 \alpha_{2,t}^2 h^2 \right) + \left( \sigma_t^2 (W_{t+h} - W_t)^2 \right) - \alpha_{2,t}^2 h^2.
$$

We have $\sigma_t^2$ of the form $\beta_{0,t} + \beta_{1,t} W_t + \beta_{2,t} W_t^2$ and

$$
E \left[ \sigma_t^2 (W_{t+h} - W_t)^2 \right] = \left( \beta_{0,t} + \beta_{2,t} h \right),
$$

and

$$
\text{cov} \left( \left( \mu_0^S - \frac{1}{2} (\sigma_0^S)^2 \right) h + \sigma_0^S W_h, \left( \mu_t - \frac{1}{2} \sigma_t^2 \right) h + \sigma_t (W_{t+h} - W_t) \right)
$$

$$
= \frac{\sqrt{\left( \alpha_{1,t}^2 h^2 + 2 \alpha_{2,t}^2 h^2 \right) + \left( \beta_{0,t} + \beta_{2,t} h \right)}}{h \alpha_{1,t}},
$$

that has the sign of $\alpha_{1,t}$. For $t$ near to 0, no bias ($\delta_0 = 0$) and $\omega$ or $T$ sufficiently large, we have
\[ \omega^2 \sim \frac{2c+1}{2c+\gamma} T + \frac{2c+\gamma}{2c+1} \frac{1}{\omega} \] and
\[ \alpha_{1,t} \sim \left( \omega^2 T (2c+1) (\gamma-1) + (\gamma-2c-1)(2c+\gamma)^2 \right) \left( (2c+\gamma)^2 + \omega^2 T (2c+1)^2 \right) (2c+1) \omega^2 b \\
(4c\gamma + 2c^2 + \gamma^2 + T \omega^2 + 2Tc\omega^2)^2 (2c+\gamma)^3 \]

which is positive for \( \gamma > 1 \) and \( \omega \) or \( T \) sufficiently large.

For \( t = \xi T \) for some \( 0 < \xi < 1 \), we have \( \alpha_{1,t} = -T^2 2bc \xi^2 \omega^4 (2c+1)^3 + o(T^2) = -T^2 2bc \xi^2 \omega^4 (2c+1)^3 + o(\omega^2) \) and is then negative for \( T \) or \( \omega \) large enough.

If we take \( b = 2\sigma c \) and we let \( c \) go to \( \infty \), we have \( \text{cov} \left( \frac{S_T}{S_0}, \frac{S_T+b}{S_T} \right) \rightarrow -h^2 \sigma^2 \frac{\omega^4}{(T-\omega^2)^2(T+\omega^2)} < 0 \).

In the case \( t = \xi T \) with \( \xi > 0 \), we have \( \text{cov} \left( \frac{S_T}{S_0}, \frac{S_T+b}{S_T} \right) \rightarrow -h^2 \sigma^2 \frac{\omega^4}{(T+T\xi+\omega^2)^2(T+\omega^2)} < 0 \). When \( \omega^2 \rightarrow \infty \), all auto-correlations vanish. ■

**Proof of Proposition 5.** Let us first compute the quadratic variation \( V_2 = E_\delta \left[ \left( \frac{\bar{e}_t}{\bar{v}_t K_T} - 1 \right)^2 \right] \).

We have
\[
E_\delta \left[ \left( \frac{\bar{e}_t}{\bar{v}_t K_T} - 1 \right)^2 \right] = \sqrt{2\pi \omega t^2} \int \exp \left( -\frac{t}{2} (\omega^2 - T) \delta^2 \right) \exp \left( \frac{-2h \delta}{\gamma} \right) \left( \exp(\frac{-\delta^2}{T} + 2\frac{\delta}{T} W_T) \right) \exp \left( \frac{(\delta-\delta_0)^2}{2\omega^2} \right) \exp \left( \frac{1}{\gamma} \left( - k - W_T \right)^2 \right) \right) \\
= \frac{(2Tc (1-\gamma) + \gamma (2c+\gamma) \omega^2) \exp \left( \frac{((2c+\gamma)\omega^2 - T(2c+1))(W_T(2c+\gamma) + Tb(\gamma-1) - (2c+1)T\delta_0)^2}{\omega^2(2c+\gamma)^2(2Tc(1-\gamma) + (2c+\gamma)(\gamma\omega^2 + T))} \right) \right)}{\sqrt{\gamma ((2c + \gamma) \omega^2 - T(2c+1))(2Tc (1-\gamma) + (2c+\gamma)(\gamma \omega^2 + T))} - 1}
\]

and since \( E_\delta \left[ \frac{\bar{e}_t}{\bar{v}_t K_T} \right] = 1 \), we have
\[
V_2 = \frac{(2Tc (1-\gamma) + \gamma (2c+\gamma) \omega^2) \exp \left( \frac{((2c+\gamma)\omega^2 - T(2c+1))(W_T(2c+\gamma) + Tb(\gamma-1) - (2c+1)T\delta_0)^2}{\omega^2(2c+\gamma)^2(2Tc(1-\gamma) + (2c+\gamma)(\gamma\omega^2 + T))} \right) \right)}{\sqrt{\gamma ((2c + \gamma) \omega^2 - T(2c+1))(2Tc (1-\gamma) + (2c+\gamma)(\gamma \omega^2 + T))} - 1}
\]

Since we deal with normal distributions, we have \( E_\delta \left[ \left| \frac{\bar{e}_t}{\bar{v}_t K_T} - 1 \right| \right] = \frac{\sqrt{2}}{\pi} E_\delta \left[ \frac{\bar{e}_t}{\bar{v}_t K_T} - 1 \right] \).

**Proof of Proposition 6.** Since markets are dynamically complete, we have \( dC_t = \mu_t C_t dt + \sigma_t C_t dW_t \) and \( \mu_t^C \) and \( \sigma_t^C \) are such that the Sharpe ratio is the same for the option and for the stock, i.e.
\[
\frac{\mu_t^C}{\sigma_t^C} = \frac{\mu_t}{\sigma_t} = \frac{-2cW_t + 2ck + b\omega^2 \gamma}{2c(t - T) + 2c\omega^2 + \omega^2 \gamma}.
\] (9)

Let us denote by \( C_t = \Phi(t, W_t) \) the price of the option at any date as a function of \( W_t \). By Ito’s Lemma, the option drift is \( \frac{\partial \Phi}{\partial t} + \frac{1}{2} \frac{\partial^2 \Phi}{\partial W^2} \) and the option volatility is \( \frac{\partial \Phi}{\partial W} \). Since markets are complete,
all assets have the same Sharpe ratio and we have to solve for

\[
\left( \frac{\partial \Phi}{\partial t} + \frac{1}{2} \frac{\partial^2 \Phi}{\partial w^2} \right) = -2cW_t + 2ck + b\omega^2 \gamma \frac{\partial \Phi}{\partial w},
\]

\[
\Phi(T, w) = \left[ S_{T^*} - \kappa S_{T^*}(0) \right]^+.
\]

First, we check that we have

\[
S(T^*) = N(T^*) \exp(L_2 w^2 + L_1 w) \] where \(N(T^*)\) is deterministic and

\[
L_2 = \frac{1}{2} \left( \frac{\omega^2 (2c + \gamma)}{\omega^2 (2c + \gamma - 2c(T - T^*))} \right)
\]

\[
L_1 = \frac{1}{2} \left( \frac{\omega^2 (2c + \gamma) (2b\omega^2 - 2k + 2b(T^* - T))}{\omega^2 (2c + \gamma - 2c(T - T^*))} \right).\]

Let us write \(\Phi(t, w)\) under the form

\[
\Phi(t, w) = \sqrt{\left( \frac{2c + \gamma}{\omega^2} + 2c(t - T) \right)} \exp \left( -\frac{1}{4} \frac{\left(2ck - 2cw + b\omega^2 \gamma\right)^2}{\left(2c + \gamma\right) \omega^2 + 2c(t - T)} \right) \Gamma(t, w).
\]

From (10), we have to solve

\[
\frac{\partial \Gamma}{\partial t} + \frac{1}{2} \frac{\partial^2 \Gamma}{\partial w^2} = 0,
\]

\[
\left[ S(T^*, w) - \kappa S(T^*, 0) \right]^+ \frac{\exp \left( \frac{1}{4} \frac{(2ck - 2cw + b\omega^2 \gamma)^2}{(2c + \gamma) \omega^2 + 2c(T^* - T)} \right)}{\sqrt{(2c + \gamma) \omega^2 + 2c(T^* - T)}} = \Gamma(T^*, w).
\]

If we define \(\Psi(t, w) = \Gamma(T - t, \frac{1}{\sqrt{2}}w)\) we have \(\frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial w^2}\) and \(\Psi(0, w) = \Gamma(T^*, w)\).

This is the classical one-dimensional heat equation whose solution is given by

\[
\Psi(t, x) = \int_{-\infty}^{\infty} \mathcal{H}(t, x - \xi) \Gamma(T^*, \frac{1}{\sqrt{2}}\xi) d\xi
\]

where \(\mathcal{H}(t, x) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right)\) is the heat kernel. From there, we have

\[
\Gamma(t, w) = \int_{-\infty}^{\infty} \frac{\exp \left( -\frac{(w - x)^2}{2(t - T)} \right)}{\sqrt{2\pi(t - T)}} \Phi(T^*, x) \exp \left( \frac{1}{4} \frac{(2ck - 2cx + b\omega^2 \gamma)^2}{(2c + \gamma) \omega^2 + 2c(t - T)} \right) dx,
\]

\[
\Phi(t, w) = N(T^*) \sqrt{\left( \frac{(2c + \gamma) \omega^2 + 2c(t - T)}{(2c + \gamma) \omega^2 + 2c(T^* - T)} \right)} \exp \left( -\frac{1}{4} \frac{(2ck - 2cw + b\omega^2 \gamma)^2}{(2c + \gamma) \omega^2 + 2c(t - T)} \right) \left[ \exp(L_2 w^2 + L_1 w) - \kappa \right]^+ dx.
\]
If we denote by $U$ and $V$ the solutions of $\exp(L_2x^2 + L_1x) = 0$, for any function $Z(x)$ we have

$$\int_{-\infty}^{\infty} Z(x) \left[ \exp(L_2x^2 + L_1x) - \kappa \right]^+ dx$$

$$= \int_{-\infty}^{U} Z(x) \exp(L_2x^2 + L_1x)dx + \int_{V}^{\infty} Z(x) \exp(L_2x^2 + L_1x)dx$$

$$- \kappa \int_{-\infty}^{U} Z(x)dx - \kappa \int_{V}^{\infty} Z(x)dx.$$

Applying this formula to $\Phi(t, w)$ we obtain 4 terms, the first one being

$$\int_{-\infty}^{U} \exp - \frac{(w - x)^2}{2(T^* - t)} \exp \left( \frac{1}{4c} \frac{(2ck - 2cx + b\omega^2\gamma)^2}{((2c + \gamma)\omega^2 + 2c(T^* - T))} \right) \exp(L_2x^2 + L_1x)dx$$

$$= \sqrt{2\pi Q} \exp \left(-\frac{w^2}{2T^* - 2t} + \frac{1}{4c(\omega^2(2c + \gamma) - 2c(T - T^*))} \left( b\gamma\omega^2 + 2ck \right)^2 + \frac{1}{2} \frac{p^2}{Q} \right) \mathcal{N}(d_1),$$

where $\mathcal{N}$, $d_1$, $P$ and $Q$ are as in the Proposition. Similarly, we have

$$\int_{-\infty}^{U} \exp - \frac{(w - x)^2}{2(T^* - t)} \exp \left( \frac{1}{4c} \frac{(2ck - 2cx + b\omega^2\gamma)^2}{((2c + \gamma)\omega^2 + 2c(T^* - T))} \right) dx$$

$$= \exp \left(-\frac{1}{4c} \frac{(2ck - 2cw + b\omega^2\gamma)^2}{(-2c\omega^2 - \omega^2\gamma + 2Tc - 2ct)} \right) \exp \left( \frac{1}{2} \frac{p^2}{q} \right) \sqrt{2\pi q} \mathcal{N}(d_2),$$

where $d_2$, $p$ and $q$ are as in the proposition. The integrals between $V$ and $\infty$ are treated similarly. Putting all together we obtain $\Phi(t, w)$ under the form

$$\Phi(t, w) = Z(t, w) \left( \mathcal{N}(d_1) + \mathcal{N}(d'_1) \right) - \kappa S_{T^*}(0) \left( \mathcal{N}(d_2) + \mathcal{N}(d'_2) \right).$$

If we take $\kappa = 0$, we should obtain $\Phi(t, w) = S(t, w)$ which gives us $Z(t, w) = S(t, w)$ and, from there

$$\Phi(t, w) = S(t, w) \left( \mathcal{N}(d_1) + \mathcal{N}(d'_1) \right) - \kappa S_{T^*}(0) \left( \mathcal{N}(d_2) + \mathcal{N}(d'_2) \right)$$

which is also equal to

$$\Phi(t, w) = \tilde{\mathcal{N}}(t, T^*) \exp \left( \frac{1}{2} \frac{w^2}{((2c + \gamma)\omega^2 + 2c(t - T))} \right) \exp \left( \frac{w(b\gamma\omega^2 + 2ck)}{\omega^2(2c + \gamma) - 2c(T - t)} \right)$$

$$\left( \sqrt{Q} \exp \left( \frac{p^2}{(2Q)} \right) \left( \mathcal{N}(d_1) + \mathcal{N}(d'_1) \right) - \kappa \exp \left( \frac{p^2}{(2q)} \right) \sqrt{q} \left( \mathcal{N}(d_2) + \mathcal{N}(d'_2) \right) \right)$$

for some deterministic function $\tilde{\mathcal{N}}(t, T^*)$. This alternative way to write it makes it easier to compute.
the volatility of the option. It is given by

\[
\frac{(2c + \gamma) \sigma^2 + 2c(T^* - T)}{(2c + \gamma) \sigma^2 + 2c(t - T)} + \frac{b \gamma \sigma^2 + 2ck}{\sigma^2 (2c + \gamma) - 2c(T - t)} + \frac{\frac{P}{Q} \frac{dP}{dw} - p \frac{dp}{dw} + \frac{1}{\sqrt{Q}} \frac{dP}{dw} (\exp (-d_1^2 - d_1^2)) - \frac{\kappa}{\sqrt{Q}} \frac{dp}{dw} (\exp (-d_2^2 - d_2^2))}{\exp \left( \frac{1}{2} \frac{p^2}{Q} \right) \sqrt{Q} (N(d_1) + N(d_1')) - \kappa \exp \left( \frac{1}{2} \frac{p^2}{Q} \right) \sqrt{Q} (N(d_2) + N(d_1'))}. 
\]

Tables illustrating the conditions and conclusions in the proof of Proposition 9
\[ \gamma \leq 1 \quad (0, j_1) \quad (j_1, j_4) \quad (j_4, j_2) \quad (j_2, \theta_t) \quad (\theta_t, j'_4) > j'_4 \]

| \( J_4 \) | - | - | + | + | + | - |
| \( J_3 \) | + | - | - | - | - | - |
| \( \tilde{\theta}_t - \sigma_t \) | + | + | + | + | - | - |
| \( J_1 \) | - | + | + | + | + | + |
| \( J_2 \) | - | - | - | + | + | + |

\( \omega^2 \geq 0 & c \geq 0 \quad NO \quad NO \quad NO \quad NO \quad YES \quad NO \)

Table 1

\[ \gamma > 1 + (t - T)\omega_t^2 \quad (0, \theta_t) \quad (\theta_t, j'_4) \quad (j'_4, j_5) > j_5 \]

| \( J_4 \) | + | + | + | - |
| \( J_3 \) | - | - | - | - |
| \( \tilde{\theta}_t - \sigma_t \) | + | + | - | - |
| \( J_5 \) | + | + | + | + |
| \( J_6 \) | - | - | - | - |

\( \omega^2 \geq 0 & c \geq 0 & J \leq 0 \quad NO \quad YES \quad NO \quad NO \)

Table 2

\[ 1 \leq \gamma < 1 + (t - T)\omega_t^2, \quad J_T \leq 0 \quad (0, j_6) \quad (j_6, \theta_t) \quad (\theta_t, j'_4) > j'_4 \]

| \( J_4 \) | + | + | + | - |
| \( J_3 \) | - | - | - | - |
| \( \tilde{\theta}_t - \sigma_t \) | + | + | - | - |
| \( J_5 \) | + | + | + | + |
| \( J_6 \) | + | - | - | - |

\( \omega^2 \geq 0 & c \geq 0 & J \leq 0 \quad NO \quad NO \quad YES \quad NO \)

Table 3

\[ 1 \leq \gamma < 1 + (t - T)\omega_t^2, \quad J_T \geq 0 \quad (0, \theta_t) \quad (\theta_t, j_6, j'_4) \quad (j_6, j'_4) > j'_4 \]

| \( J_4 \) | + | + | + | - |
| \( J_3 \) | - | - | - | - |
| \( \tilde{\theta}_t - \sigma_t \) | + | + | - | - |
| \( J_5 \) | + | + | + | + |
| \( J_6 \) | + | + | - | - |

\( \omega^2 \geq 0 & c \geq 0 & J \leq 0 \quad NO \quad NO \quad YES \quad NO \)

Table 4

References


