On a weighted total variation minimization problem

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This article is dedicated to the memory of Thomas Lachand-Robert.

Abstract

The present article is devoted to the study of a constrained weighted total variation minimization problem, which may be viewed as a relaxation of a generalized Cheeger problem and is motivated by landslide modelling. Using the fact that the set of minimizers is invariant by a wide class of monotone transformations, we prove that level sets of minimizers are generalized Cheeger sets and obtain qualitative properties of the minimizers: they are all bounded and all achieve their essential supremum on a set of positive measure.

Keywords: *generalized Cheeger sets, total variation minimization*

1 Introduction

Given an open bounded subset of $\mathbb{R}^N$ and nonnegative functions f and g (more precise assumptions on the data Ω, f and g will be given later on), we are interested in the following

$$\mu := \inf_{u \in BV_0} \mathcal{R}(u) \tag{1}$$

where

$$BV_0 := \{u \in BV(\mathbb{R}^N),\ u \equiv 0 \text{ on } \mathbb{R}^N \setminus \overline{\Omega}\}, \tag{2}$$

and for $u \in BV_0$ such that $\int_\Omega fu \neq 0$,

$$\mathcal{R}(u) := \frac{\displaystyle\int_{\mathbb{R}^N} g(x)\, \mathrm{d}|Du(x)|}{\left|\displaystyle\int_\Omega f(x)u(x)\, \mathrm{d}x\right|}. \tag{3}$$

Whenever $\int_\Omega fu = 0$, we set $\mathcal{R}(u) = +\infty$.

This problem is motivated by a landslide model proposed by Ionescu and Lachand-Robert [8] in which f and g respectively represent the body forces and the (inhomogoneous) yield limit distribution. These functions are determined by the properties of the considered geomaterials and, roughly speaking, taking a non constant f captures the idea that the mechanical properties of the geomaterials (e.g. the way they are compacted by their own weight) vary with depth. When $g = f = 1$ (which is not always a relevant assumption in landslides modelling), it is well-known that the infimum in (1) coincides with the infimum of $\mathcal{R}$ over characteristic functions of sets of finite perimeter. In this case, (1) appears as a natural relaxation of:

$$\lambda(\Omega) := \inf_{A \subset \overline{\Omega},\ \chi_A \in BV} \frac{\|D\chi_A\|(\mathbb{R}^N)}{|A|} \tag{4}$$

where $|A|$ and $\|D\chi_A\|(\mathbb{R}^N)$ denote respectively the Lebesgue measure of A and the total variation of $D\chi_A$. Problem (4) is famous and known as Cheeger's problem [3], its value $\lambda(\Omega)$ is called the Cheeger constant of Ω and its minimizers are called Cheeger sets of Ω (see [9], [10] and the references therein). Note also that $\lambda(\Omega)$ is the first eigenvalue of the 1-Laplacian on Ω, see for instance [5], [6].

Throughout the paper, we will assume that

- Ω is a nonempty open bounded subset of $\mathbb{R}^N$ with a Lipschitz boundary,
- $f \in L^\infty(\Omega)$, $f \geq f_0$ for a positive constant f_0,
- $g \in C^1(\overline{\Omega})$, $g \geq g_0$ for a positive constant g_0.

Let us remark that the space $BV(\mathbb{R}^N)$ is the natural one to search for a minimizer of (1). Indeed the infimum is usually not achieved in a Sobolev space like $W^{1,1}(\mathbb{R}^N)$. It is also clear that one always have $\mathcal{R}(|u|) \leq \mathcal{R}(u)$ so that we can restrict the minimization problem to non-negative functions.

In what follows, every $u \in BV(\Omega)$ will be extended by 0 outside $\overline{\Omega}$, and thus will also be considered as an element of $BV(\mathbb{R}^N)$, still denoted u. Let us define, for every u in BV_0:

$$\mathcal{G}(u) := \int_{\mathbb{R}^N} g(x) \, \mathrm{d} \, |Du(x)| \,. \tag{5}$$

Since $\partial\Omega$ is Lipschitz, functions in $BV(\Omega)$ have a trace on $\partial\Omega$, and one can write for $u \in BV(\Omega)$:

$$\mathcal{G}(u) = \int_{\Omega} g(x) \, \mathrm{d} \, |Du(x)| + \int_{\partial\Omega} g(x) \, |u(x)| \, \mathrm{d} \, \mathcal{H}^{N-1}(x)$$

see [4] and [7] for details. Taking advantage of the homogeneity of (1), it is convenient to reformulate (1) as the convex minimization problem

$$\mu = \inf_{u \in BV_f} \mathcal{G}(u) \tag{6}$$

where

$$BV_f := \Big\{ u \in BV(\mathbb{R}^N), \ u \geq 0, \ u \equiv 0 \text{ on } \mathbb{R}^N \setminus \overline{\Omega}, \int_{\Omega} fu = 1 \Big\}. \tag{7}$$

In analogy with the case $g = f = 1$, it is natural to consider the generalized Cheeger problem:

$$\lambda := \inf_{A \in \mathcal{E}} \frac{\displaystyle\int_{\mathbb{R}^N} g(x) \, \mathrm{d}|D\chi_A(x)|}{\displaystyle\int_A f(x) \, \mathrm{d}x} = \inf_{A \in \mathcal{E}} \mathcal{R}(\chi_A) \tag{8}$$

where

$$\mathcal{E} := \{ A \subset \overline{\Omega} \quad \text{with} \quad \int_A f(x) \, \mathrm{d}x > 0 \quad \text{and} \quad \chi_A \in BV(\mathbb{R}^N) \}. \tag{9}$$

Again (1) can be interpreted as a relaxed formulation of (8) and one aim of the present paper is to study the precise links between (1) and (8). We shall first prove that if u is a nonnegative solution of (1) then so is $H(u)$ provided H is Lipschitz, nondecreasing, $H(u) \neq 0$ and $H(0) = 0$. This invariance property will enable us to deduce very simply qualitative properties of the solutions of (6) and to study the link between (1) and (8). As a first consequence of the invariance property, we shall prove that u solves (6) if and only if *all* its level sets of positive measure solve the generalized Cheeger problem (8). This is in fact a simple generalization of what is well known when $g = f = 1$. A more involved application is that the set of solutions of the generalized Cheeger problem (8) is stable by countable union. Finally, regarding qualitative properties, we will show that solutions of (6) are all bounded and that they all achieve their essential supremum on a set of positive measure (even when $g = f = 1$, to our knowledge, this result is new).

2 Existence

This paragraph is devoted to prove the following existence result.

Theorem 1 *Let Ω, f and g satisfy the previous assumptions. Then*
1) the infimum of (6) is achieved in BV_f,
2) the infimum of (8) is achieved in $\mathcal{E}$.

In order to prove this theorem we need the following lemma

Lemma 1 *Let Ω be a bounded open set in $\mathbb{R}^N$, u be in $BV(\Omega)$ and g be in $C^1(\overline{\Omega}, \mathbb{R}_+)$, then*

$$\int_\Omega g(x)\,\mathrm{d}|Du(x)| = \sup\{\int_\Omega u(x)\mathrm{div}(g(x)\varphi(x)); \varphi \in C^1_c(\Omega, \mathbb{R}^N), |\varphi(x)| \leq 1\}. \tag{10}$$

Proof. Since u belongs to $BV(\Omega)$, there exists a Radon measure μ on Ω and a μ-measurable function $\sigma : \Omega \to \mathbb{R}^N$ such that $|\sigma(x)| = 1$ μ a.e. and $Du = \sigma\mu$. Then $|Du| = \mu$, see [4]. Since

$$-\int_\Omega u(x)\mathrm{div}(g(x)\varphi(x))\,\mathrm{d}x = \int_\Omega g(x)\varphi(x)\cdot\sigma\,\mathrm{d}\mu$$

and using the fact that $|\varphi(x)| \leq 1$, we obtain

$$-\int_\Omega u(x)\mathrm{div}(g(x)\varphi(x))\,\mathrm{d}x \leq \int_\Omega g(x)\,\mathrm{d}\mu = \int_\Omega g(x)\,\mathrm{d}|Du(x)|\,.$$

On the other hand, since

$$\int_\Omega \mathrm{d}|Du(x)| = \sup\{\int_\Omega u(x)\mathrm{div}(\varphi(x))\,\mathrm{d}x; \varphi\in C_c^1(\Omega,\mathbb{R}^N), |\varphi(x)|\leq 1\} \quad (11)$$

see [4] or [7], there exists a sequence $\varphi_k \in C_c^1(\Omega,\mathbb{R}^N)$, with $|\varphi_k(x)|\leq 1$ such that

$$-\int_\Omega u(x)\mathrm{div}(\varphi(x))\,\mathrm{d}x \to \int_\Omega \mathrm{d}\,\mu.$$

But

$$-\int_\Omega u(x)\mathrm{div}(\varphi_k(x))\,\mathrm{d}x = \int_\Omega \varphi_k(x)\cdot\sigma\,\mathrm{d}\mu$$

and then

$$\varphi_k\cdot\sigma \to 1 \quad \text{in} \quad L^1_\mu(\Omega)$$

and similarly

$$g\varphi_k\cdot\sigma \to g \quad \text{in} \quad L^1_\mu(\Omega).$$

Now, by definition

$$\sup\{\int_\Omega u(x)\mathrm{div}(g(x)\varphi(x))\,\mathrm{d}x; \varphi\in C_c^1(\Omega,\mathbb{R}^N)\} \geq \int_\Omega g(x)\varphi_k(x)\cdot\sigma\mathrm{d}\,\mu, \quad \forall k\in\mathbb{N}$$

and passing to the limit we get

$$\sup\{\int_\Omega u(x)\mathrm{div}(g(x)\varphi(x))\,\mathrm{d}x; \varphi\in C_c^1(\Omega,\mathbb{R}^N), |\varphi(x)|\leq 1\}$$
$$\geq \int_\Omega g(x)\,\mathrm{d}\mu = \int_\Omega g(x)\,\mathrm{d}|Du(x)|\,.$$

This ends the proof of the lemma. $\square$

We deduce from lemma 1 the following lower semicontinuity property.

Lemma 2 *Let Ω be a bounded open set in $\mathbb{R}^N$. The functional $F: L^1(\Omega)\to \mathbb{R}\cup\{+\infty\}$ defined by*

$$F(u) = \begin{cases} \int_\Omega g(x)\,\mathrm{d}|Du(x)| & \textit{if} \quad u\in BV(\Omega) \\ +\infty & \textit{otherwise} \end{cases}$$

is lower semicontinuous in $L^1(\Omega)$. Suppose in addition that $\partial\Omega$ is Lipschitz, then the functional $G : L^1(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$G(u) = \begin{cases} \displaystyle\int_\Omega g(x)\,\mathrm{d}\,|Du(x)| + \int_{\partial\Omega} g(x)\,|u(x)|\,\mathrm{d}\,\mathcal{H}^{N-1}(x) & \textit{if} \quad u \in BV_0(\Omega) \\ +\infty \quad \textit{otherwise} & \end{cases}$$

is lower semicontinuous in $L^1(\mathbb{R}^N)$.

From lemma 2, we deduce

Corollary 1 *1) Let $(u_n)_{n\in\mathbb{N}}$ be in $BV(\mathbb{R}^N)$ such that $u_n \to u$ in $L^1(\Omega)$. Then*

$$\int_\Omega g(x)\,\mathrm{d}|Du(x)| \le \liminf \int_\Omega g(x)\,\mathrm{d}|Du_n(x)|\,. \tag{12}$$

2) Let $(u_n)_{n\in\mathbb{N}}$ be in BV_0 such that $u_n \to u$ in $L^1(\mathbb{R}^N)$. Then

$$\begin{aligned} &\int_\Omega g(x)\,\mathrm{d}|Du(x)| + \int_{\partial\Omega} g(x)\,|u(x)|\,\mathrm{d}\mathcal{H}^{N-1}(x) \\ &\qquad \le \liminf \left(\int_\Omega g(x)\,\mathrm{d}|Du_n(x)| + \int_{\partial\Omega} g(x)\,|u_n(x)|\,\mathrm{d}\mathcal{H}^{N-1}(x) \right). \end{aligned}$$

We are now in position to prove existence.

Proof. 1) Taking a constant u in (5), we see that the infimum in BV_f is finite. Let $(u_n)_{n\in\mathbb{N}} \subset BV_f$ be a minimizing sequence. Since $g \ge g_0 > 0$, $(u_n)_{n\in\mathbb{N}}$ is bounded in $BV(\mathbb{R}^N)$. Therefore there exists a subsequence, still denoted (u_n) and $u \in BV(\mathbb{R}^N)$ such that $u_n \to u$ in $L^1(\mathbb{R}^N)$. Using corollary 1, we get

$$\int_{\mathbb{R}^N} g(x)\,\mathrm{d}|Du(x)| \le \liminf \int_{\mathbb{R}^N} g(x)\,\mathrm{d}|Du_n(x)| = \inf_{v\in BV_f} \mathcal{G}(v). \tag{13}$$

But from the fact that $u_n \to u$ in $L^1(\mathbb{R}^N)$, we have that

$$\int_\Omega f(x)u(x)\,\mathrm{d}x = \lim \int_\Omega f(x)u_n(x)\,\mathrm{d}x = 1$$

$$u \ge 0 \quad \text{and} \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \overline{\Omega}.$$

Thus u belongs to BV_f and the infimum is achieved.

2) Let $(A_n)_{n\in\mathbb{N}}$ be a minimizing sequence of (8) in $\mathcal{E}$. Following the proof of 1), we obtain that $(\chi_{A_n})_{n\in\mathbb{N}}$ is bounded in $BV(\mathbb{R}^N)$, and then, up to a subsequence, converging to a function in $L^1(\mathbb{R}^N)$ that is still a characteristic function of a set $A \in \mathbb{R}^N$ satisfying $A \subset \overline{\Omega}$, and from corollary 1

$$\int_{\mathbb{R}^N} g(x)\,\mathrm{d}|D\chi_A(x)| \leq \liminf \int_{\mathbb{R}^N} g(x)\,\mathrm{d}|D\chi_{A_n}(x)| = \inf_{B\in\mathcal{E}} \int_{\overline{B}} g(x)\,\mathrm{d}|D\chi_B(x)|\,.$$

Therefore the infimum is achieved. $\square$

The lower semicontinuity result of corollary 1 implies that the set of solutions of (6) is closed in L^1, this fact will be used several times later on.

3 Invariance

Proposition 1 *Let $H \in W^{1,\infty}(\mathbb{R},\mathbb{R}) \cap C^\infty(\mathbb{R},\mathbb{R})$ be such that $H(0) = 0$ and $H' > 0$ on $\mathbb{R}$. If u is a solution of (6) then so is $T_H(u)$ defined by*

$$T_H(u) := \frac{H \circ u}{\displaystyle\int_\Omega f(x)H(u(x))\,\mathrm{d}x}. \tag{14}$$

Proof. Let us denote by $X_t(.)$ the flow of the ordinary differential equation

$$\dot{v} = -H(v).$$

In other words, for all $v \in \mathbb{R}$, $X_t(v)$ is defined by:

$$\partial_t X_t(v) = -H(X_t(v)),\ X_0(v) = v. \tag{15}$$

Our assumptions guarantee that $(t, v) \mapsto X_t(v)$ is well-defined and smooth on $\mathbb{R}^2$. Moreover, setting $Y_t(v) := \partial_v X_t(v)$, differentiating (15) with respect to v, we have:

$$\partial_t Y_t(v) = -H'(X_t(v))Y_t(v),\ Y_0(v) = 1$$

hence for all $t \geq 0$

$$Y_t(v) = \exp\left(-\int_0^t H'(X_s(v))ds\right)$$

Thus, for all $v \geq 0$ and $t \geq 0$, one has the bounds:

$$0 \leq Y_t(v) \leq 1, -\|H'\|_\infty \leq \partial_t Y_t(v) \leq 0. \tag{16}$$

Since $X_t(0) = 0$ (Cauchy-Lipschitz), we deduce $X_t(v) \geq X_t(0) = 0$ for all $t \geq 0$ and $v \geq 0$.

For $t \geq 0$, define u_t by $u_t(x) = X_t(u(x))$, it is immediate to check that $u_t \in BV_0$ and $u_t \geq 0$. Let us also define

$$h(t) := \int_{\mathbb{R}^N} g(x)\,\mathrm{d}|Du_t(x)| - \mu \int_\Omega f(x) u_t(x) dx.$$

Since u_t belongs to BV_0 and $u_t \geq 0$, we have $h(t) \geq 0$ and since $u_0 = u$ solves (6), we have $h(0) = 0$. For all $t > 0$, this yields:

$$\frac{h(t) - h(0)}{t} \geq 0. \tag{17}$$

By the chain rule for BV functions (see [1]), and since $\partial_v X_t(u(x)) \geq 0$, we can also write $h(t)$ as

$$\begin{aligned} h(t) = &\int_{\mathbb{R}^N} g(x) \partial_v X_t(u(x))\,\mathrm{d}\gamma(x) + \int_{J_u} g(x)|X_t(u^+(x)) - X_t(u^-(x))|\,\mathrm{d}\mathcal{H}^{N-1}(x) \\ &- \mu \int_\Omega f(x) u_t(x)\,\mathrm{d}x \end{aligned}$$

where J_u is the jump set of u and the nonnegative measure $d\gamma$ is the sum of the absolutely continuous part and of the Cantor part of $|Du|$ (see [1]). We may then rewrite:

$$\frac{h(t) - h(0)}{t} = I_t + J_t - \mu K_t$$

with

$$\begin{aligned} I_t &:= \frac{1}{t} \int_{\mathbb{R}^N} g(x)(Y_t(u(x)) - 1)\,\mathrm{d}\gamma(x), \\ J_t &:= \frac{1}{t} \int_{J_u} g(x) \left(|X_t(u^+(x)) - X_t(u^-(x))| - |u^+(x) - u^-(x)| \right) \mathrm{d}\mathcal{H}^{N-1}(x), \\ K_t &:= \frac{1}{t} \int_\Omega f(x)(X_t(u(x)) - u(x))\,\mathrm{d}x. \end{aligned}$$

By construction, we have pointwise convergence of $t^{-1}(X_t - \mathrm{id})$ and $t^{-1}(Y_t - 1)$ respectively to $-H$ and $-H'$. Using the monotonicity of H and of X_t, the bounds (16), and the Dominated Convergence Theorem, we thus get

$$\begin{aligned}
\lim_{t\to 0^+} I_t &= -\int_{\mathbb{R}^N} g(x)H'(u(x))\,\mathrm{d}\gamma(x),\\
\lim_{t\to 0^+} J_t &= -\int_{J_u} g(x)|H(u^+(x)) - H(u^-(x))|\,\mathrm{d}\mathcal{H}^{N-1}(x),\\
\lim_{t\to 0^+} K_t &= -\int_{\Omega} f(x)H(u(x))\,\mathrm{d}x.
\end{aligned}$$

Putting everything together and passing to the limit in (17) yields

$$\begin{aligned}
0 \geq \int_{\mathbb{R}^N} g(x)H'(u(x))\,\mathrm{d}\gamma(x) + \int_{J_u} g|H(u^+) - H(u^-)|\,\mathrm{d}\mathcal{H}^{N-1}\\
- \mu \int_{\Omega} f(x)H(u(x))\,\mathrm{d}x.
\end{aligned}$$

By the chain rule for BV functions again, the right-hand side of the previous inequality can also be rewritten as:

$$\int_{\mathbb{R}^N} g(x)\,\mathrm{d}|D(H\circ u)(x)| - \mu\int_{\Omega} f(x)H(u(x))\,\mathrm{d}x.$$

This finally proves that $H\circ u$ minimizes $\mathcal{R}$, or equivalently, that $T_H(u)$ solves (6). □

Remark 1. Taking H bounded shows the existence of bounded solutions to (6). We shall see in Theorem 4 that in fact *every* solution (6) is in fact L^∞.

By standard approximation arguments, we obtain the natural extension of proposition 1 to more general monotone nonlinearities H:

Corollary 2 *Let u be a solution of (6) and $H \in W^{1,\infty}(\mathbb{R},\mathbb{R})$ be a nondecreasing function such that $H(0) = 0$. If $H\circ u \neq 0$ then $T_H(u)$ defined by (14) also solves (6).*

Proof. Let $(\eta_n)_n$ be a sequence of mollifiers and set for all $v\in\mathbb{R}$:

$$H_n(v) := (\eta_n \star H)(v) - (\eta_n\star H)(0) + \frac{1}{n}v.$$

Each H_n satisfies the assumptions of proposition 1, so that $T_{H_n}(u)$ solves (6). Obviously H_n converges uniformly to H on compact subsets of $\mathbb{R}$ and Lebesgue's Dominated Convergence Theorem then implies that $T_{H_n}(u)$ converges to $T_H(u)$ in $L^1(\mathbb{R}^N)$. Using the lower-semi continuity result of corollary 1, we thus get the result. $\square$

Note that corollary 2 applies in particular to $H(v) = (v-t_0)_+$ and $H(v) = \min(v, t_0)$.

4 Main results

4.1 Generalized Cheeger sets

Theorem 2 *Let u be a solution of (6) and for every $t \geq 0$, define $E_t := \{x \in \mathbb{R}^N \ : \ u(x) > t\}$. For every $t \geq 0$ such that E_t has positive Lebesgue measure $\frac{1}{\int_{E_t} f} \chi_{E_t}$ solves (6). In particular, $\frac{1}{\int_{\{u>0\}} f} \chi_{\{u>0\}}$ solves (6).*

Proof. Let us prove the claim first for the set $E_0 := \{u > 0\}$. Define for every $n \in \mathbb{N}^*$ and $v \in \mathbb{R}$:

$$H_n(v) := \begin{cases} 0 & \text{if } v \leq 0 \\ nv & \text{if } v \in [0, \frac{1}{n}] \\ 1 & \text{if } v \geq \frac{1}{n}. \end{cases}$$

For n large enough, $H_n \circ u \neq 0$ and corollary 2 implies that $T_{H_n}(u)$ solves (6). Since $T_{H_n}(u)$ converges in $L^1(\mathbb{R}^N)$ to $\frac{1}{\int_{\{u>0\}} f} \chi_{\{u>0\}}$, we conclude as in the proof of corollary 2.

Let $t \geq 0$ be such that E_t has positive Lebesgue measure. From corollary 2, $v := \frac{(u-t)_+}{\int f(u-t)_+}$ solves (6), hence so does

$$\frac{1}{\int_{\{v>0\}} f} \chi_{\{v>0\}} = \frac{1}{\int_{E_t} f} \chi_{E_t}.$$

$\square$

We also have a converse of Theorem 2 which simply reads as:

Proposition 2 *Let $u \in BV_0$, $u \geq 0$. If for every $t \geq 0$ such that $E_t := \{x \in \mathbb{R}^N \,:\, u(x) > t\}$ has positive Lebesgue measure, χ_{E_t} solves (1) then u solves (1).*

Proof. For $M > 0$ and $n \in \mathbb{N}^*$, let us define $F_k := E_{\frac{M(k-1)}{n}}$ $(k \in \{0, ..., n\})$ and:

$$u_{M,n} := \sum_{k=0}^{n-1} \frac{Mk}{n} \chi_{F_k \setminus F_{k+1}} + M\chi_{F_n} = \frac{M}{n} \sum_{k=1}^{n} \chi_{F_k}.$$

By assumption, for $k = 1, ..., n$, χ_{F_k} solves (1). Using the convexity and homogeneity properties of (1), we deduce that $u_{M,n}$ also solves (1) for all M and all n. Since $u_{M,n}$ converges in L^1 to $\min(u, M)$ as n tends to $+\infty$, we deduce that $\min(u, M)$ solves (1) and we finally get the desired result by letting M tend to $+\infty$. $\square$

4.2 Applications

As a first consequence of theorem 2, we deduce the following relaxation result:

Corollary 3 *The values of problems (6) and (8) coincide:*

$$\mu = \inf_{u \in BV_0} \mathcal{R}(u) = \lambda = \inf_{A \in \mathcal{E}} \mathcal{R}(\chi_A)$$

moreover the second infimum is actually a minimum.

Remark 2. In fact, one can obtain the relaxation result of corollary 3 as a direct consequence of the coarea and Cavalieri's formulae (see for instance [9], [8], [2] for similar level-sets approach for variational problems involving total variation minimization). Indeed, one obviously has $\mu \leq \lambda$ and if $u \in BV_0$, $u \geq 0$, setting $E_t := \{u > t\}$, the coarea and Cavalieri's formulae yield:

$$\int_{\mathbb{R}^N} g \, \mathrm{d}|Du(x)| - \lambda \int_{\mathbb{R}^N} fu = \int_0^\infty \left(\int_{\partial^* E_t} g \, \mathrm{d}\mathcal{H}^{N-1} - \lambda \int_{E_t} f(x) \, \mathrm{d}x \right) dt \geq 0$$

which proves that $\mu \geq \lambda$. From the previous argument, in fact, we see that the converse also holds: u solves (6) if and only if $E_t := \{u > t\}$ (which has finite perimeter for a.e. t) solves (8) for a.e. $t \geq 0$. Note that in Theorem 2, we have proved that E_t solves (8) for *all* t (and we have not used the coarea formula).

Of course, theorem 2 and its proof contain much more information than corollary 3. A more precise consequence of theorem 2 is the following

Corollary 4 *$A \in \mathcal{E}$ solves (8) if and only if there exists u solving (6) such that $A = \{u > 0\}$.*

Proof. We have seen that (8) and (6) have the same value. If $A \in \mathcal{E}$ solves (8) then $\dfrac{\chi_A}{\int_A f}$ obviously solves (6). Conversely, if u solves (6) then $\{u > 0\}$ solves (8) thanks to theorem 2.

$\square$

We then easily deduce the following consequence

Theorem 3 *Let $(A_n)_n$ be a sequence of solutions of (8) then $\bigcup_n A_n$ is also a solution of (8).*

Proof. For every n, the function $u_n := \dfrac{\chi_{A_n}}{\int_{A_n} f}$ solves (6). Define then:

$$\lambda_n := \frac{C 2^{-n}}{1 + \|u_n\|_{BV}}$$

where $C > 0$ is such that $\displaystyle\sum_0^\infty \lambda_n = 1$. Using the convexity properties of problem (6), we thus deduce that

$$u := \sum_{n=0}^{\infty} \lambda_n u_n$$

is a well-defined element of BV_f that solves (6). Since $\bigcup_n A_n = \{u > 0\}$, corollary 4 then implies that $\bigcup_n A_n$ solves (8). $\square$

Note that the fact that $\bigcup_n A_n$ is of finite perimeter is contained in the statement.

4.3 Qualitative properties

Adapting arguments of Serrin [11], as in Demengel [6], we obtain:

Theorem 4 *Let u be a solution of (6). Then u belongs to $L^\infty(\Omega)$.*

Proof. Let u be a solution of (6). For every $M > 0$ the truncated function

$$u_M = \frac{\min(u, M)}{\displaystyle\int_\Omega f(x) \min(u, M)(x)\,\mathrm{d}x}$$

is a solution of (6) thanks to Corollary 2. Using Proposition 1, it is still the case for

$$\frac{u_M^k}{\displaystyle\int_\Omega f(x) u_M^k(x)\,\mathrm{d}x}$$

where $k \in \mathbb{N}^*$. So we have

$$g_0 \int_{\mathbb{R}^N} \mathrm{d}\big|Du_M^k(x)\big| \le \int_{\mathbb{R}^N} g(x)\,\mathrm{d}\big|Du_M^k(x)\big| = \mu \int_\Omega f(x) u_M^k(x)\,\mathrm{d}x. \tag{18}$$

Since $f \in L^\infty(\Omega)$, and Ω is bounded, there exists some $t > 0$, such that

$$\left(\int_{\{f \ge t\}} f(x)^N\,\mathrm{d}x\right)^{\frac{1}{N}} < \frac{g_0}{2C\mu}.$$

Then using Hölder inequality, (18) implies

$$g_0 \int_{\mathbb{R}^N} \mathrm{d}\big|Du_M^k(x)\big| \le \mu(t \int_\Omega u_M^k(x)\,\mathrm{d}x + \frac{g_0}{2C\mu}\|u_M^k\|_{1^*}), \tag{19}$$

where $1^* = \frac{N}{N-1}$. On the other hand, from Poincaré's inequality (see for instance [4]), there exists some $C > 0$ such that

$$\|v\|_{1^*} \le C\|Dv\|(\mathbb{R}^N) \quad \text{for every} \quad v \in BV(\mathbb{R}^N). \tag{20}$$

Applying (20) to u_M^k and replacing in (19) leads to

$$g_0\|u_M^k\|_{1^*} \le C\mu t \int_\Omega u_M^k(x)\,\mathrm{d}x + \frac{g_0}{2}\|u_M^k\|_{1^*}, \tag{21}$$

thus

$$\|u_M^k\|_{1^*} \leq K \int_\Omega u_M^k(x)\,\mathrm{d}x, \tag{22}$$

where $K = \frac{2C\mu t}{g_0}$.

We now apply a bootstrap process: we start with $k = 1^*$ in (22), using monotone convergence, we pass to the limit in (22) when $M \to +\infty$ and we get

$$\|u\|_{(1^*)^2} \leq K^{\frac{1}{1^*}} \|u\|_{1^*}. \tag{23}$$

Taking $k = (1^*)^n$ in (22) leads to

$$\|u\|_{(1^*)^{n+1}} \leq K^{\frac{1-\frac{1}{(1^*)^n}}{1-\frac{1}{1^*}}} \|u\|_{1^*}. \tag{24}$$

Finally $u \in L^\infty$ and $\|u\|_\infty \leq K^N \|u\|_{1^*}$.

$\square$

Combining Theorem 2, Proposition 2 and Theorem 4, we deduce that every solution of (6) has a flat zone in the following sense:

Theorem 5 *Let u be a solution of (6), then the set $\{u = \|u\|_\infty\}$ has positive Lebesgue measure.*

Proof.

Set $m_\infty := \|u\|_\infty$ and let us assume that $|\{u = m_\infty\}| = 0$. Let $(m_k)_k$ be an increasing sequence of nonnegative real numbers converging to m_∞. Set $F_0 := \Omega$ and for all $k \in \mathbb{N}^*$, $F_k := \{x \in \Omega : \ u(x) > m_k\}$. Since $\sum |F_k \setminus F_{k+1}| < +\infty$, there exists an increasing sequence $(\beta_k)_k$ tending to $+\infty$ and such that $\beta_0 = 0$ and $\sum \beta_k |F_k \setminus F_{k+1}| < +\infty$.

For $n \in \mathbb{N}^*$, let us define:

$$v_n := \sum_{k=0}^{n} \beta_k \chi_{F_k \setminus F_{k+1}} + \beta_{n+1} \chi_{F_{n+1}}$$

Since for $k = 0, ..., n$, $\{v_n > \beta_k\} = F_{k+1}$ and u solves (6), we deduce from Theorem 2 and Proposition 2 that v_n solves (1). We next remark that $(v_n)_n$ is monotone with respect to n and that

$$\int_{\mathbb{R}^N} v_n = \sum_{k=0}^{n} \beta_k |F_k \setminus F_{k+1}| + \beta_{n+1} |F_{n+1}|.$$

Now, since $|\{u = m_\infty\}| = 0$ and $F_{n+1} = (\bigcup_{k\geq n+1} F_k \setminus F_{k+1}) \cup \{u = m_\infty\}$, we get

$$\int_{\mathbb{R}^N} v_n \leq \sum_{k=0}^{\infty} \beta_k |F_k \setminus F_{k+1}|.$$

The monotone convergence Theorem then implies that v_n converges in L^1 to some v which is an unbounded solution of (1). With Theorem 4, we thus obtain the desired contradiction.

$\square$

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