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**Replicator Dynamics and Correlated Equilibrium:  
Elimination of All Strategies in the Support of Correlated  
Equilibria**

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# Replicator Dynamics and Correlated Equilibrium: Elimination of All Strategies in the Support of Correlated Equilibria<sup>1</sup> a

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**Résumé:** Nous donnons un exemple de jeu pour lequel, sous la dynamique des réplicateurs et pour un ensemble ouvert de conditions initiales, toutes les stratégies jouées en équilibre corrélé sont éliminées.

**Abstract:** We present a family of games for which, under the replicator dynamics and from an open set of initial conditions, all strategies used in correlated equilibrium are eliminated.

**Mots clés :** Dynamique des réplicateurs, équilibre corrélé

**Key Words :** Replicator dynamics, correlated equilibrium

**Classification JEL:** C73

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# 1 Introduction

We investigate the link between strategies surviving under the replicator dynamics and strategies used in correlated equilibrium. Specifically, we present a family of  $4 \times 4$  symmetric games for which, under the replicator dynamics and from a large set of initial conditions, all strategies used in correlated equilibrium are eliminated (hence only strategies that are NOT used in equilibrium remain). In a follow-up article (Viossat, 2005), we show that this occurs for an open set of games and for vast classes of dynamics, in particular, for the best-response dynamics (Gilboa & Matsui, 1991) and for every monotonic (Samuelson & Zhang, 1992) or weakly sign preserving (Ritzberger & Weibull, 1995) dynamics which depends continuously on the payoffs and in which pure strategies initially absent remain absent.

This is related to two major themes of evolutionary game theory. The first one is the relevance of traditional solution concepts from an evolutionary perspective. A number of positive results have been reached (see, e.g., Weibull, 1995). For instance, in several classes of games, e.g., potential games, dominance solvable games or games with an interior ESS, all interior solutions of the replicator dynamics converge to a Nash equilibrium. However, it is well known that in other classes of games the replicator dynamics need not converge. We show more: all strategies used in correlated equilibrium may be eliminated. This reinforces the view that evolutionary dynamics may lead to behavior drastically distinct from (Nash or even correlated) equilibrium behavior.

The second theme to which this note is connected is the identification of classes of strategies that survive (resp. are eliminated) under most evolutionary dynamics. Hofbauer & Weibull (1996), generalizing a result of Samuelson & Zhang (1992), showed that under any convex monotonic dynamics and along all interior solutions, all iteratively strictly dominated strategies are eliminated. A dual statement is that all surviving strategies are rationalizable. Our results show that, in contrast, it may be that no strategy used in correlated equilibrium survives.

The remaining of this note is organized as follows. First, we introduce the notations and basic definitions, and recall some known results on Rock-Scissors-Paper (RSP) games. In addition, we prove that these games have a unique correlated equilibrium distribution. We then introduce a family of  $4 \times 4$  symmetric games built by adding a strategy to a RSP game. We describe in details the orbits of the replicator dynamics in these games and show that, from an open set of initial conditions, all strategies used in correlated equilibrium are eliminated. We conclude by discussing a variety of related results.

## 2 Notations and basic definitions

We consider finite, two-player symmetric games played within a single population. Such a game is given by a set  $I = \{1, \dots, N\}$  of pure strategies and a payoff matrix  $\mathbf{U} = (u_{ij})_{1 \leq i, j \leq N}$ . Here  $u_{ij}$  is the payoff of a player playing strategy  $i$  against a player playing strategy  $j$ . We use bold characters for vectors and matrices and normal characters for numbers.

The proportion of the population playing strategy  $i$  at time  $t$  is denoted  $x_i(t)$ . Thus, the vector  $\mathbf{x}(t) = (x_1(t), \dots, x_N(t))^T$  denotes the mean strategy at time  $t$ . It belongs to the  $N - 1$  dimensional simplex over  $I$

$$S_N := \left\{ \mathbf{x} \in \mathbb{R}^I : x_i \geq 0 \forall i \in I \text{ and } \sum_{i \in I} x_i = 1 \right\}$$

(henceforth, “the simplex”) whose vertices  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N$  correspond to the pure strategies of the game. We study the evolution of the mean strategy  $\mathbf{x}$  under the single-population replicator dynamics (Taylor & Jonker, 1978):

$$\dot{x}_i(t) = x_i(t) [(\mathbf{U}\mathbf{x}(t))_i - \mathbf{x}(t) \cdot \mathbf{U}\mathbf{x}(t)] \quad (1)$$

*Remark:* for lightness sake, we usually write  $x_i$  and  $\mathbf{x}$  instead of  $x_i(t)$  and  $\mathbf{x}(t)$ .

We now define correlated equilibrium distributions. Consider a (non necessarily symmetric) bimatrix game with strategy set  $I$  (resp.  $J$ ) for player 1 (resp. 2). Let  $g_k(i, j)$  denote the payoff of player  $k$  when player 1 plays  $i$  and player 2 plays  $j$ . A correlated equilibrium distribution (Aumann, 1974) is a probability distribution  $\mu$  on the set  $I \times J$  of pure strategy profiles (i.e.  $\mu(i, j) \geq 0$  for all  $(i, j)$  in  $I \times J$  and  $\sum_{(i, j) \in I \times J} \mu(i, j) = 1$ ) which satisfies the following inequalities:

$$\sum_{j \in J} \mu(i, j) [g_1(i, j) - g_1(i', j)] \geq 0 \quad \forall i \in I, \forall i' \in I \quad (2)$$

and

$$\sum_{i \in I} \mu(i, j) [g_2(i, j) - g_2(i, j')] \geq 0 \quad \forall j \in J, \forall j' \in J \quad (3)$$

Abusively, we may write “correlated equilibrium” for “correlated equilibrium distribution”. Though the above definition applies to general bimatrix games, from now on, we only consider *symmetric* bimatrix games.

*Definition:* the pure strategy  $i$  is *used in correlated equilibrium* if there exists a correlated equilibrium distribution  $\mu$  and a pure strategy  $j$  such that  $\mu(i, j) > 0$ .<sup>1</sup>

*Definition:* the pure strategy  $i$  is *eliminated* (for some initial condition  $\mathbf{x}(0)$ ) if  $x_i(t)$  goes to zero as  $t \rightarrow +\infty$ .

### 3 A reminder on Rock-Scissors-Paper games

A RSP (Rock-Scissors-Paper) game is a  $3 \times 3$  symmetric game in which the second strategy (Rock) beats the first (Scissors), the third (Paper) beats the second, and the first beats the third. Up to normalization (i.e. putting zeros on the diagonal) the payoff matrix is of the form:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \left( \begin{array}{ccc} 0 & -a_2 & b_3 \\ b_1 & 0 & -a_3 \\ -a_1 & b_2 & 0 \end{array} \right) \end{array} \quad \text{with } a_i > 0, b_i > 0 \text{ for all } i = 1, 2, 3. \quad (4)$$

Any RSP game has a unique Nash equilibrium (Zeeman, 1980; see also Gaunersdörfer and Hofbauer, 1995, or Hofbauer and Sigmund, 1998):

$$\mathbf{p} = \frac{1}{\Sigma} (a_2 a_3 + a_3 b_2 + b_2 b_3, a_1 a_3 + a_1 b_3 + b_3 b_1, a_1 a_2 + a_2 b_1 + b_1 b_2) \quad (5)$$

with  $\Sigma > 0$  such that  $\mathbf{p} \in S_4$ . Actually,

*Notation:* for  $\mathbf{x} \in S_N$ ,  $\mathbf{x} \otimes \mathbf{x}$  denotes the probability distribution on  $S_N$  induced by  $\mathbf{x}$ .

**Proposition 1** *Any RSP game has a unique correlated equilibrium distribution:  $\mathbf{p} \otimes \mathbf{p}$ .*

**Proof.** Let  $\mu$  be a correlated equilibrium of (4). For  $i = 1$  and, respectively,  $i' = 2$  and  $i' = 3$ , the incentive constraint (2) reads:

$$\mu(1, 1)(-b_1) + \mu(1, 2)(-a_2) + \mu(1, 3)(a_3 + b_3) \geq 0 \quad (6)$$

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<sup>1</sup>Note that if  $\mu$  is a correlated equilibrium distribution of a two-player symmetric game, then so is  $\mu^T$  (defined by  $\mu^T(i, j) = \mu(j, i)$ ) and  $(\mu + \mu^T)/2$ . Thus, if a strategy is used in a correlated equilibrium distribution, it is also used in a symmetric correlated equilibrium distribution.

$$\mu(1, 1)a_1 + \mu(1, 2)(-a_2 - b_2) + \mu(1, 3)b_3 \geq 0 \quad (7)$$

Add (6) multiplied by  $a_1$  to (7) multiplied by  $b_1$ . This gives

$$-\mu(1, 2)(a_1a_2 + a_2b_1 + b_1b_2) + \mu(1, 3)(a_1a_3 + a_1b_3 + b_3b_1) \geq 0$$

That is, recalling (5):

$$p_2\mu(1, 3) \geq p_3\mu(1, 2)$$

Every choice of a player and a strategy  $i$  yields a similar inequality. So we get six inequalities which together read:

$$p_2\mu(1, 3) \geq p_3\mu(1, 2) \geq p_1\mu(3, 2) \geq p_2\mu(3, 1) \geq p_3\mu(2, 1) \geq p_1\mu(2, 3) \geq p_2\mu(1, 3)$$

Therefore all the above inequalities hold as equalities. Letting  $\lambda$  be such that the common value of the above expressions is  $\lambda p_1 p_2 p_3$ , we have:  $\mu(i, j) = \lambda p_i p_j$  for every  $j \neq i$ . Together with (6) and (7), this implies that we also have  $\mu(1, 1) = \lambda p_1^2$  (and by symmetry  $\mu(i, i) = \lambda p_i^2$  for all  $i$ ). Therefore  $\lambda = 1$  and  $\mu = \mathbf{p} \otimes \mathbf{p}$ . ■

The behaviour of the replicator dynamics in RSP games has been totally analyzed by Zeeman (1980). In particular, letting  $\partial S_3 := \{\mathbf{x} \in S_3 : x_1 x_2 x_3 = 0\}$  denote the boundary of the simplex:

**Proposition 2 (Zeeman (1980))** *If  $a_1 a_2 a_3 > b_1 b_2 b_3$ , then for every initial condition  $\mathbf{x}(0) \neq \mathbf{p}$ , the solution  $\mathbf{x}(t)$  converges to  $\partial S_3$  as  $t \rightarrow +\infty$*

In the case of *cyclic symmetry* (i.e.  $a_1 = a_2 = a_3$  and  $b_1 = b_2 = b_3$ ) then the unique Nash equilibrium is  $\mathbf{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ . Furthermore, up to division of all payoffs by the common value of the  $a_i$ , the payoff matrix may be taken of the form:

$$\begin{pmatrix} 0 & -1 & \epsilon \\ \epsilon & 0 & -1 \\ -1 & \epsilon & 0 \end{pmatrix} \text{ with } \epsilon > 0 \quad (8)$$

The condition  $a_1 a_2 a_3 > b_1 b_2 b_3$  then reduces to  $\epsilon < 1$  and in this case proposition 2 may be proved as follow: for  $\epsilon < 1$ , the Nash equilibrium  $\mathbf{p}$  is globally inferior in the sense that:

$$\forall \mathbf{x} \in S_3, \mathbf{x} \neq \mathbf{p} \Rightarrow \mathbf{p} \cdot \mathbf{U}\mathbf{x} < \mathbf{x} \cdot \mathbf{U}\mathbf{x} \quad (9)$$

More precisely,

$$\mathbf{p} \cdot \mathbf{U}\mathbf{x} - \mathbf{x} \cdot \mathbf{U}\mathbf{x} = -(\mathbf{p} - \mathbf{x}) \cdot \mathbf{U}(\mathbf{p} - \mathbf{x}) = -\left(\frac{1-\epsilon}{2}\right) \sum_{1 \leq i \leq 3} (p_i - x_i)^2 \quad (10)$$

Now, let  $\hat{V}(\mathbf{x}) := (x_1 x_2 x_3)^{1/3}$ . Note that the function  $\hat{V}$  takes its minimal value 0 on  $\partial S_3$  and its maximal value  $1/3$  at  $\mathbf{p}$ . Letting  $\hat{v}(t) := \hat{V}(\mathbf{x}(t))$  we get:

$$\dot{\hat{v}}(t) = (\mathbf{p} \cdot \mathbf{U}\mathbf{x} - \mathbf{x} \cdot \mathbf{U}\mathbf{x}) \hat{v}(t) = -\hat{v}(t) \left( \frac{1-\epsilon}{2} \right) \sum_{1 \leq i \leq 3} (p_i - x_i)^2 \quad (11)$$

The above expression is negative whenever  $\hat{v}(t) \neq 0$  and  $\mathbf{x} \neq \mathbf{p}$ . It follows that for every initial condition  $\mathbf{x}(0) \neq \mathbf{p}$ ,  $\hat{v}(t)$  decreases to zero hence  $\mathbf{x}(t)$  converges to the boundary.

## 4 A family of $4 \times 4$ games

Fix  $\epsilon$  in  $]0, 1[$ ,  $\alpha \geq 0$ , and consider the following  $4 \times 4$  symmetric game which is build by adding a strategy to a RSP game:

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \left( \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ 0 & -1 & \epsilon & & -\alpha \\ \epsilon & 0 & -1 & & -\alpha \\ -1 & \epsilon & 0 & & -\alpha \\ \frac{-1+\epsilon}{3} + \alpha & \frac{-1+\epsilon}{3} + \alpha & \frac{-1+\epsilon}{3} + \alpha & & 0 \end{array} \right) \quad (12)$$

For  $0 < \alpha < (1 - \epsilon)/3$ , the interesting case, this game is very similar to the example used by Dekel and Schotchmer (1992) to show that a discrete version of the replicator dynamics need not eliminate all strictly dominated strategies.<sup>2</sup> We now describe the main features of the above game.

Let  $\mathbf{n}_{123} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$  denote the rest-point of the replicator dynamics corresponding to the Nash equilibrium of the underlying RSP game. Let  $\mathbf{U}_\alpha$  denote the payoff matrix (12).

**The case  $\alpha = 0$ .** The strategies  $\mathbf{n}_{123}$  and  $\mathbf{e}_4$  always earn the same payoff:

$$\mathbf{n}_{123} \cdot \mathbf{U}_0 \mathbf{x} = \mathbf{e}_4 \cdot \mathbf{U}_0 \mathbf{x} \quad \forall \mathbf{x} \in S_4 \quad (13)$$

Furthermore, against  $\mathbf{e}_4$ , as against  $\mathbf{n}_{123}$ , all strategies earn the same payoff:

$$(\mathbf{x} - \mathbf{x}') \cdot \mathbf{U}_0 \mathbf{e}_4 = (\mathbf{x} - \mathbf{x}') \cdot \mathbf{U}_0 \mathbf{n}_{123} = 0 \quad \forall \mathbf{x} \in S_4, \forall \mathbf{x}' \in S_4 \quad (14)$$

<sup>2</sup>More precisely, the game obtained from (12) by multiplying all payoffs by  $-1$  belongs to the family of games *à la* Dekel & Scotchmer considered by Hofbauer and Weibull (1996). In particular, figure 1 of (Hofbauer & Weibull, 1996, p570) describes the dynamics on the boundary of the simplex in game (12), up to reversal of all the arrows and permutation of strategies 2 and 3.)

The set of symmetric Nash equilibria is the segment  $E_0 = [\mathbf{n}_{123}, \mathbf{e}_4]$ . This shall be clear from the proof of proposition 3 below. A key property is that whenever the mean strategy  $\mathbf{x}$  does not belong to the segment of equilibria  $E_0$ , every strategy in  $E_0$  earns less than the mean payoff. Formally,

$$\forall \mathbf{x} \notin E_0, \forall \mathbf{p} \in E_0, \mathbf{p} \cdot \mathbf{U}\mathbf{x} < \mathbf{x} \cdot \mathbf{U}\mathbf{x}$$

More precisely, for  $\mathbf{x} \neq \mathbf{e}_4$ , define  $\hat{x}_i$  as the share of the population that plays  $i$  relative to the share of the population that plays 1, 2 or 3. Formally,

$$\hat{x}_i = x_i / (x_1 + x_2 + x_3) \quad (15)$$

**Lemma 4.1** For every  $\mathbf{p}$  in  $E_0$  and every  $\mathbf{x} \neq \mathbf{e}_4$ ,

$$\mathbf{p} \cdot \mathbf{U}_0\mathbf{x} - \mathbf{x} \cdot \mathbf{U}_0\mathbf{x} = -\frac{(1-\epsilon)}{2}(1-x_4)^2 \sum_{1 \leq i \leq 3} (\hat{x}_i - 1/3)^2 \quad (16)$$

**Proof.** Let  $K = \mathbf{p} \cdot \mathbf{U}_0\mathbf{x} - \mathbf{x} \cdot \mathbf{U}_0\mathbf{x} = (\mathbf{p} - \mathbf{x}) \cdot \mathbf{U}_0\mathbf{x}$ . By (13),  $\mathbf{p} \cdot \mathbf{U}_0\mathbf{x} = \mathbf{n}_{123} \cdot \mathbf{U}_0\mathbf{x}$  so that  $K = (\mathbf{n}_{123} - \mathbf{x}) \cdot \mathbf{U}_0\mathbf{x}$ . Now let  $\mathbf{y} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, 0)$ . Using (14) we get:

$$K = (\mathbf{n}_{123} - \mathbf{x}) \cdot \mathbf{U}_0 [(1-x_4)\mathbf{y} + x_4\mathbf{e}_4] = (1-x_4)(\mathbf{n}_{123} - \mathbf{x}) \cdot \mathbf{U}_0\mathbf{y}$$

Noting that  $\mathbf{n}_{123} - \mathbf{x} = (1-x_4)(\mathbf{n}_{123} - \mathbf{y}) + x_4(\mathbf{n}_{123} - \mathbf{e}_4)$  and using (13), we get:  $K = (1-x_4)^2(\mathbf{n}_{123} - \mathbf{y}) \cdot \mathbf{U}_0\mathbf{y}$ . Now apply (10). This gives (16) and concludes the proof. ■

**The case  $\alpha > 0$ .** The mixed strategy  $\mathbf{n}_{123}$  is no longer an equilibrium. Actually:

**Proposition 3** If  $\alpha > 0$ , then the game with payoffs (12) has a unique correlated equilibrium distribution:  $\mathbf{e}_4 \otimes \mathbf{e}_4$ .

**Proof.** Assume, by contradiction, that there exists a correlated equilibrium  $\mu$  different from  $\mathbf{e}_4 \otimes \mathbf{e}_4$ . Since  $\mathbf{e}_4$  is a strict Nash equilibrium, there must exist  $1 \leq i, j \leq 3$  such that  $\mu(i, j) > 0$ . Define the correlated distribution of the underlying RSP game  $\hat{G}$  by:

$$\hat{\mu}(i, j) = \frac{\mu(i, j)}{K} \quad 1 \leq i, j \leq 3$$

with  $K = \sum_{1 \leq i, j \leq 3} \mu(i, j)$ . For  $1 \leq i, i' \leq 3$ , we have  $u_{i4} = u_{i'4} (= -\alpha)$ , so that:

$$\sum_{j=1}^3 \hat{\mu}(i, j) [u_{ij} - u_{i'j}] = \sum_{j=1}^3 \frac{\mu(i, j)}{K} [u_{ij} - u_{i'j}] = \frac{1}{K} \sum_{j=1}^4 \mu(i, j) [u_{ij} - u_{i'j}] \geq 0$$



(The latter inequality holds because  $\mu$  is a correlated equilibrium)

Together with symmetric inequalities, this implies that  $\hat{\mu}$  is a correlated equilibrium of  $\hat{G}$ . By proposition 1, this implies that for every  $1 \leq i, j \leq 3$ , we have  $\hat{\mu}(i, j) = 1/9$  hence  $\mu(i, j) = K/9$ . It follows that for any  $1 \leq i, j \leq 3$ ,

$$\sum_{1 \leq j \leq 4} \mu(i, j) [u_{ij} - u_{4j}] \leq \sum_{1 \leq j \leq 3} \mu(i, j) [u_{ij} - u_{4j}] = -\frac{K\alpha}{3} < 0$$

This contradicts the fact that  $\mu$  is a correlated equilibrium. ■

Nevertheless, for  $\alpha < (1 - \epsilon)/3$ , the above game has a best-response cycle:  $\mathbf{e}_1 \rightarrow \mathbf{e}_2 \rightarrow \mathbf{e}_3 \rightarrow \mathbf{e}_1$ . We will show that for  $\alpha > 0$  small enough, the corresponding set

$$\Gamma := \{\mathbf{x} \in S_4, x_4 = 0 \text{ and } x_1 x_2 x_3 = 0\} \quad (17)$$

attracts all nearby orbits. We first show that the (replicator) dynamics in the interior of  $S_4$  may be decomposed in two parts: an increase or decrease in  $x_4$ , and an outward spiralling movement around the segment  $E_0 = [\mathbf{n}_{123}, \mathbf{e}_4]$ .

## 5 Decomposition of the dynamics

First, note that for every  $\mathbf{x}$  in  $E_0$ , we have:  $(\mathbf{U}\mathbf{x})_1 = (\mathbf{U}\mathbf{x})_2 = (\mathbf{U}\mathbf{x})_3$ . This implies that the segment  $E_0$  is globally invariant. Second, recall the definition (15) of  $\hat{x}_i$ . For  $\mathbf{x} \neq \mathbf{e}_4$ , let  $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ . Let  $\hat{\mathbf{U}}$  denote the payoff matrix (8) of the underlying RSP game.

**Lemma 5.1** *Let  $\mathbf{x}(\cdot)$  be a solution of the replicator dynamics (1) with  $\mathbf{x}(0) \neq \mathbf{e}_4$ . For every  $i$  in  $\{1, 2, 3\}$ ,*

$$\dot{\hat{x}}_i = (1 - x_4) \hat{x}_i [(\hat{\mathbf{U}}\hat{\mathbf{x}})_i - \hat{\mathbf{x}} \cdot \hat{\mathbf{U}}\hat{\mathbf{x}}] \quad (18)$$

**Proof.** Let  $i$  in  $\{1, 2, 3\}$ . If  $x_i = 0$ , then (18) holds trivially. Otherwise, for every  $j$  in  $\{1, 2, 3\}$  such that  $x_j$  is positive,

$$\frac{\dot{\hat{x}}_i}{\hat{x}_i} - \frac{\dot{\hat{x}}_j}{\hat{x}_j} = \frac{d}{dt} \ln \left( \frac{\hat{x}_i}{\hat{x}_j} \right) = \frac{d}{dt} \ln \left( \frac{x_i}{x_j} \right) = (\mathbf{U}\mathbf{x})_i - (\mathbf{U}\mathbf{x})_j = (1 - x_4) [(\hat{\mathbf{U}}\hat{\mathbf{x}})_i - (\hat{\mathbf{U}}\hat{\mathbf{x}})_j]$$

Multiplying the above equality by  $\hat{x}_j$  and summing over all  $j$  such that  $x_j > 0$  yield (18). ■

The lemma means that, up to a change of velocity,  $\hat{\mathbf{x}}$  follows the replicator dynamics for the game with payoff matrix  $\hat{\mathbf{U}}$  (and thus spirals towards the boundary).<sup>3</sup> Now, recall equation (11) and the definition of  $\hat{V}$ . For  $\mathbf{x} \neq \mathbf{e}_4$ , let  $V(\mathbf{x}) := \hat{V}(\hat{\mathbf{x}})$ . That is,

$$V(\mathbf{x}) = (\hat{x}_1 \hat{x}_2 \hat{x}_3)^{1/3} = \frac{(x_1 x_2 x_3)^{1/3}}{x_1 + x_2 + x_3}$$

**Corollary 5.2** *Let  $\mathbf{x}(\cdot)$  be a solution of (1) with  $\mathbf{x}(0) \neq \mathbf{e}_4$ . The function  $v(t) := V(\mathbf{x}(t))$  satisfies:*

$$\dot{v}(t) = -v(t)f(\mathbf{x}(t)) \text{ with } f(\mathbf{x}) = (1 - x_4) \left( \frac{1 - \epsilon}{2} \right) \sum_{1 \leq i \leq 3} (\hat{x}_i - 1/3)^2 \quad (19)$$

**Proof.** We have:  $v(t) = V(\mathbf{x}(t)) = \hat{V}(\hat{\mathbf{x}}(t))$ . Therefore  $\dot{v} = \vec{grad} \hat{V} \cdot \dot{\hat{\mathbf{x}}}$ , with  $\vec{grad} \hat{V} = (\partial \hat{V} / \partial \hat{x}_i)_{1 \leq i \leq 3}$ . Applying lemma 5.1 and equation (11) yield (19). ■

Note that  $v(t)$  is nonnegative and that the function  $f$  is negative everywhere but on the interval  $[\mathbf{n}_{123}, \mathbf{e}_4[$ , where  $V$  attains its maximal value  $1/3$ . Therefore, it follows from (19) that  $V$  decreases along all interior trajectories, except the ones starting (hence remaining) in the interval  $]\mathbf{n}_{123}, \mathbf{e}_4[$ . We now exploit this fact to build a Lyapunov function<sup>4</sup> for the set  $\Gamma$  defined in (17).

## 6 Main result

Let  $W(\mathbf{x}) = \max(x_4, 3V(\mathbf{x}))$  for  $\mathbf{x} \neq \mathbf{e}_4$  and  $W(\mathbf{e}_4) = 1$ . Note that  $W$  takes its maximal value 1 on the segment  $E_0 = [\mathbf{n}_{123}, \mathbf{e}_4]$  and its minimal value 0 on  $\Gamma$ . Now, for  $\delta \geq 0$ , let  $K_\delta$  denote the compact set:

$$K_\delta := \{\mathbf{x} \in \Delta(S), W(\mathbf{x}) \leq \delta\}$$

so that  $K_0 = \Gamma$  and  $K_1 = S_4$ .

<sup>3</sup>The fact that when the  $N - 1$  first strategies earn the same payoff against the  $N^{th}$  (and last) strategy, the dynamics may be decomposed as in lemma 5.1 was known to Josef Hofbauer (personal communication). This results from a combination of theorem 7.5.1 and of exercise 7.5.2 in (Hofbauer and Sigmund, 1998). I rediscovered it independently.

<sup>4</sup>For an introduction to Lyapunov functions, see, e.g., Bhatia & Szegö, 1970.

**Proposition 4** *Let  $0 < \delta < 1$ . There exists  $\gamma > 0$  such that for every game (12) with  $0 < \alpha < \gamma$  and for every initial condition  $\mathbf{x}(0)$  in  $K_\delta$ ,*

$$W(\mathbf{x}(t)) \leq W(\mathbf{x}(0)) \exp(-\gamma t) \quad \forall t \geq 0$$

*In particular, the set  $\Gamma$  attracts all solutions starting in  $K_\delta$ .*

**Proof.** Fix  $\epsilon$  in  $]0, 1[$  and recall that  $\mathbf{U}_\alpha$  denotes the payoff matrix (12) with parameters  $\epsilon, \alpha$ . Since  $\delta < 1$ , the set  $K_\delta$  is disjoint from the segment  $E_0$ . Therefore, it follows from (16) that for every  $\mathbf{x}$  in  $K_\delta$ , the quantity  $(\mathbf{U}_0\mathbf{x})_4 - \mathbf{x} \cdot \mathbf{U}_0\mathbf{x}$  is negative. Similarly, it follows from the definition of the function  $f$  in (19) that for every  $\mathbf{x}$  in  $K_\delta$ ,  $f(\mathbf{x})$  is negative. Therefore, by compactness of  $K_\delta$ , there exists a positive constant  $\gamma$  such that

$$\min_{\mathbf{x} \in K_\delta} ((\mathbf{U}_0\mathbf{x})_4 - \mathbf{x} \cdot \mathbf{U}_0\mathbf{x}, f(\mathbf{x})) \leq -3\gamma < 0 \quad (20)$$

We now fix  $\alpha$  in  $]0, \gamma[$  and consider the replicator dynamics in the game with payoff matrix  $\mathbf{U}_\alpha$ . For every  $\mathbf{x}$  in  $S_4$  and every  $i$  in  $S$ ,  $|[(\mathbf{U}_\alpha - \mathbf{U}_0)\mathbf{x}]_i| \leq \alpha$ . Therefore, it follows from (20) that

$$\forall \mathbf{x} \in K_\delta, (\mathbf{U}_\alpha\mathbf{x})_4 - \mathbf{x} \cdot \mathbf{U}_\alpha\mathbf{x} \leq -3\gamma + 2\alpha \leq -\gamma$$

Since  $(\mathbf{U}_\alpha\mathbf{x})_4 - \mathbf{x} \cdot \mathbf{U}_\alpha\mathbf{x}$  is the growth rate of strategy 4, this implies that

$$\mathbf{x}(t) \in K_\delta \Rightarrow \dot{x}_4(t) \leq -\gamma x_4(t) \quad (21)$$

Now, recall the definition of  $v(t)$  in corollary 5.2. It follows from (19) and (20) that

$$\mathbf{x}(t) \in K_\delta \Rightarrow \dot{v}(t) \leq -3\gamma v(t) \leq -\gamma v(t) \quad (22)$$

Let  $w(t) := W(\mathbf{x}(t)) = \max(x_4(t), v(t))$ . Equations (21) and (22) imply that if  $\mathbf{x}(t)$  is in  $K_\delta$  (i.e.  $w(t) \leq \delta$ ) then  $w$  decreases weakly. This implies that  $K_\delta$  is forward invariant. Therefore, for every initial condition  $\mathbf{x}(0)$  in  $K_\delta$ , equations (21) and (22) apply for all  $t \geq 0$ . It follows that for all  $t \geq 0$ ,  $x_4(t) \leq x_4(0) \exp(-\gamma t)$  and  $v(t) \leq v(0) \exp(-\gamma t)$ . The result follows. ■

It follows from proposition 3 and proposition 4 that if  $\alpha > 0$  is small enough, then in the game (12) the unique strategy used in correlated equilibrium is strategy 4, but  $x_4(t) \rightarrow 0$  from an open set of initial conditions.

## 7 Discussion

1. The results of this note also show that the *two-population* replicator dynamics may eliminate all strategies used in correlated equilibrium along interior solutions. See the remark in (Hofbauer & Weibull, 1995, p. 571).

2. The basic idea is that if an attractor is disjoint from the set of equilibria, then it is likely that we may add a strategy in a way that strongly affects the set of equilibria but does not perturb much the dynamics in the neighborhood of the attractor.

3. As mentioned in the introduction, elimination of all strategies used in correlated equilibrium actually occurs on an open set of games and for vast classes of dynamics (Viossat, 2005). This robustness is crucial for the practical relevancy of our results. Indeed, in practical situations, we are unlikely to have an exact knowledge of the payoffs or of the dynamics followed by the agents.

4. Proposition 4 shows much more than nonconvergence to correlated equilibrium: all strategies used in correlated equilibrium are wiped out. In particular, no kind of time-average of the replicator dynamics can converge to the set of correlated equilibria. In contrast, Hofbauer (2004) shows that, in all  $n$ -player finite games and along all interior solutions, the time-average of the ( $n$ -population) replicator dynamics converges to the Hannan set.

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