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Abstract: This paper considers Dynamic Conditional Correlations (DCC) GARCH models in which the time-varying coefficients, including the conditional correlation matrix, are functions of the realizations of an exogenous stochastic process. Time series generated by this model are in general nonstationary. Necessary and sufficient conditions are given for the existence of non-explosive solutions, and for the existence of second-order moments of these solutions. Potential applications concern the modeling of the volatility of a vector of energy prices, the model coefficients depending on the weather conditions.

Keywords: Dynamic conditional correlation, Existence of nonexplosive solutions, Multivariate GARCH, Nonstationary processes, Time-varying models.

1 Introduction

Multivariate GARCH models are used heavily within the field of financial econometrics to capture the comovements of financial returns. For instance the conditional Value at Risk of a portfolio, depends on the conditional variances and covariances of the assets in the portfolio. Plenty of multivariate GARCH formulations have been proposed and studied, including the Constant Conditional Correlation (CCC) model introduced by Bollerslev (1990), and the Dynamic conditional Correlation (DCC) models proposed by Tse and Tsui (2002) and Engle (2002). Recent reviews on the literature on multivariate GARCH models are Bauwens, Laurent and Rombouts (2006), Silvennoinen and Teräsvirta (2009). See also

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the textbook by Francq and Zakoian (2010a).

While modeling volatility of multivariate stationary time series has been the main center of attention, understanding the comovements of non stationary series is of great practical importance. In particular, energy prices are subject to complex seasonalities. For such series, the classical multivariate GARCH approach is inappropriate because the only non explosive solutions of GARCH-type models are stationary processes.

The model studied in this paper is a multivariate model extending the CCC GARCH model, and also related to DCC GARCH. It combines two effects, which have been introduced separately in the GARCH literature. First, it allows for the leverage effect and, more generally, to asymmetric impacts of the past positive and negative returns on the current volatility. Recent papers dealing with such asymmetries in the multivariate framework are McAleer, Hoti and Chan (2009), McAleer, Chan, Hoti and Liebermann (2009), Francq and Zakoian (2010b). Secondly, its coefficients are functions of an observed process, allowing to capture the influence of exogenous variables on the series of interest. In this model, the coefficients of the volatilities of the components of a vector time series are driven by the observations of a discrete stochastic process. The model of this paper can be seen as a time-varying coefficients specification. Models with periodic coefficients have been proposed by Basawa and Lund (2001), among others, and in the GARCH framework by Bollerslev and Ghysels (1996). However, periodic models can be found too restrictive when the change of dynamics do not appear regularly. Time series models in which the coefficients are subordinated to an exogenous process have been recently proposed and analyzed for the conditional mean by Bibi and Francq (2003), Francq and Gautier (2004a, 2004b), and for the conditional variance by Regnard and Zakoian (2010a, 2010b).

The paper is organized as follows. In Section 2, we discuss the model assumptions and provide examples of sample paths. In Section 3, we derive conditions ensuring the existence of non explosive solutions. We also give conditions ensuring that these solutions belong to L^2. Proofs are provided in Section 4.

2 Model and examples

We consider the m-dimensional process $\{\epsilon_t = (\epsilon_{1t}, \dots, \epsilon_{mt})'\}$, solution of the model

$$\begin{cases} \epsilon_t &= H_t^{1/2}\eta_t, \\ H_t &= D_t R_0(s_t) D_t, \quad D_t = \mathrm{diag}\left(\sqrt{h_{11,t}}, \dots, \sqrt{h_{mm,t}}\right), \\ \underline{h}_t &= \underline{\omega}_0(s_t) + \sum_{i=1}^{q} \mathbf{A}_{0i,+}(s_t)\underline{\epsilon}_{t-i}^{+} + \mathbf{A}_{0i,-}(s_t)\underline{\epsilon}_{t-i}^{-} + \sum_{j=1}^{p} \mathbf{B}_{0j}(s_t)\underline{h}_{t-j}, \end{cases} \tag{2.1}$$

where where $\underline{h}_t = (h_{11,t}, \dots, h_{mm,t})'$ and (with $x^+ = \max(x,0) = (-x)^-$)

$$\underline{\epsilon}_t^{+} = \left(\left\{\epsilon_{1t}^{+}\right\}^2, \cdots, \left\{\epsilon_{mt}^{+}\right\}^2\right)', \quad \underline{\epsilon}_t^{-} = \left(\left\{\epsilon_{1t}^{-}\right\}^2, \cdots, \left\{\epsilon_{mt}^{-}\right\}^2\right)',$$

(η_t) is an iid sequence of variables on $\mathbb{R}^m$ with identity covariance matrix, (s_t) is a sequence of numbers with values in a finite set $E = \{e_1, \dots, e_d\}$, and for any $s \in E$, $R_0(s)$ is a correlation matrix, $\underline{\omega}_0(s)$ is a vector of size $m \times 1$ with strictly positive coefficients, $\mathbf{A}_{0i}^{+}(s)$, $\mathbf{A}_{0i}^{-}(s)$ and $\mathbf{B}_{0j}(s)$ are matrices of size $m \times m$ with positive coefficients.

When $d = 1$, or equivalently when the functions $R_0(\cdot)$, $\underline{\omega}_0(\cdot)$, $\mathbf{A}_{0i}(\cdot)$ and $\mathbf{B}_{0j}(\cdot)$ are constant, this model coincides with the formulation referred to as the extended CCC-GARCH(p,q) by He and Teräsvirta (2004), and recently studied by Francq and Zakoian (2010b).[1] When $m = 1$ and when no asymmetry is introduced, this model coincides with the univariate model studied by Regnard and Zakoian (2010a, 2010b).

The inclusion of positive and negative parts of the noise allows to take into account the so-called leverage effect. Indeed, many studies have documented the fact that past negative returns tend to have more impact on the current volatility than past positive ones of the same module. As is Francq and Zakoian (2010b) a cross-leverage effect is introduced, the positive and negative past values of any component being involved in the volatilities of all components. An additional effect, which is the novelty of this model, is due to the presence of the realizations (s_t). For instance, the leverage effect is allowed to change in time, depending on the sequence (s_t). Similarly, the correlations between the components of the noise (η_t) depend on the s_t's. In this sense it is also related to the family of the DCC GARCH models.

To fix ideas, suppose that (s_t) is a sequence of temperature levels, driving the volatility of energy prices. Given the importance of temperature in the demand for energy prices, it

[1] In the simplest CCC-GARCH model introduced by Bollerslev (1990) the matrices $\mathbf{A}_i$ and $\mathbf{B}_{0j}$ are diagonal.

is not surprising that the volatility of such prices be influenced by temperature. Regnard and Zakoian (2010b) found evidence of $d = 5$ regimes for the volatility of gas prices. In moderate-temperature periods, the returns volatility was found to be more influenced by the most recent past returns than in high or low-temperature periods.

Note that the standard assumption $E\eta_t = 0$ allows to interpret H_t as the volatility (*i.e.* the conditional variance) of ϵ_t, but this assumption is not required in the sequel.

Example 2.1 (Bivariate ARCH(1) model) The simplest model of this class is the bivariate model with $p = 0$ and $q = 1$, for which a more explicit form can be given. In this case, Model (2.1) writes

$$\left\{ \begin{array}{rcl} \epsilon_t &=& \begin{pmatrix} \sqrt{h_{11t}} & 0 \\ 0 & \sqrt{h_{11t}} \end{pmatrix} \begin{pmatrix} 1 & \rho(s_t) \\ \rho(s_t) & 1 \end{pmatrix} \eta_t, \\ h_{11t} &=& \omega_1(s_t) + \alpha_{11,+}(s_t)\left\{\epsilon^+_{1,t-1}\right\}^2 + \alpha_{12,+}(s_t)\left\{\epsilon^+_{2,t-1}\right\}^2 \\ && +\alpha_{11,-}(s_t)\left\{\epsilon^-_{1,t-1}\right\}^2 + \alpha_{12,-}(s_t)\left\{\epsilon^-_{2,t-1}\right\}^2 \\ h_{22t} &=& \omega_2(s_t) + \alpha_{21,+}(s_t)\left\{\epsilon^+_{1,t-1}\right\}^2 + \alpha_{22,+}(s_t)\left\{\epsilon^+_{2,t-1}\right\}^2 \\ && +\alpha_{21,-}(s_t)\left\{\epsilon^-_{1,t-1}\right\}^2 + \alpha_{22,-}(s_t)\left\{\epsilon^-_{2,t-1}\right\}^2 \end{array} \right. \tag{2.2}$$

with $\alpha_{ij,+}(\cdot), \alpha_{ij,-}(\cdot) \geq 0$ and $\omega_i(\cdot) > 0$, for $i = 1, 2$, and $\rho(\cdot) \in (-1, 1)$. For the sake of illustration, Figure 1 displays two sets of simulations of length $n = 100$ of Model (2.2), with the parameter values displayed in Table 1. The sequence (η_t) is drawn from a standard bivariate Gaussian distribution. The (s_t) of the left panel is generated from a Bernoulli distribution, with $P(s_t = 1) = 0.8 = 1 - P(s_t = 0)$. The simulations of the right panel are obtained by inverting the regimes, that is for the sequence $(s_t^*) = (1 - s_t)$, with the same realization of the noise (η_t). With this choice of parameters, the volatility of $\epsilon_{2,t}$ only depends on its past values whereas the volatility of $\epsilon_{1,t}$ depends on the past values of both components. Moreover, the leverage effect is present only when $s_t = 1$ and there is no cross leverage effect: the volatility of each component is higher when the past values of this component are negative rather than positive, but is not influenced by the sign of the past values of the other component. It can also be noted that a strong positive correlation is introduced in the regime $s_t = 1$, this correlation being small and negative in the regime $s_t = 0$. Some of these effects can be detected by inspecting the sample paths provided in this figure.

Figure 2 displays another simulation, in which (s_t) is the realization of a Poisson dis-

Table 1: Parameter values for the simulation of Model (2.2) in Figure 1

	ρ	ω_1	ω_2	$\alpha_{11,+}$	$\alpha_{12,+}$	$\alpha_{11,-}$	$\alpha_{12,-}$	$\alpha_{21,+}$	$\alpha_{22,+}$	$\alpha_{21,-}$	$\alpha_{22,-}$
$s_t = 1$	0.9	1	1	0.1	0.2	0.5	0.2	0	0.1	0	0.5
$s_t = 0$	-0.1	1	1	0.3	0.1	0.3	0.1	0	0.3	0	0.3

Table 2: Parameter values for the simulation of Model (2.2) in Figure 2

	ρ	ω_1	ω_2	$\alpha_{11,+}$	$\alpha_{12,+}$	$\alpha_{11,-}$	$\alpha_{12,-}$	$\alpha_{21,+}$	$\alpha_{22,+}$	$\alpha_{21,-}$	$\alpha_{22,-}$
$s_t = 0$	0.1	1	1	0.1	0.1	0.1	0.1	0.1	0.1	0	0.1
$s_t = 1$	0.3	1	1	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
$s_t = 2$	0.5	1	1	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
$s_t = 3$	0.9	1	1	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5
$s_t = 4$	0.95	1	1	2	2	2	2	2	2	2	2

tribution with mean 0.7. For simplicity, the volatilities of the two components were taken symmetric, that is: $\alpha_{ij,\cdot} = 0$ for $i \neq j$, and $\alpha_{ii,+}(\cdot) = \alpha_{ii,-}(\cdot) = \alpha(\cdot)$ for $i = 1, 2$. The regimes $s(t) = i$ when i increases correspond to increasing volatilities and increasing correlations between the noise components. It is seen that, despite the presence of large ARCH coefficients in the regimes $s_t = 3$ and $s_t = 4$, the process is not explosive because the frequency of occurrence of such regimes is low.

Example 2.2 (Periodic model) Suppose that the sequence (s_t) is purely periodic, that is $s_{t+m} = s_t$ for some $m > 1$ and for any t. In this case, Model (2.1) can be called a Periodic CCC model. In the univariate framework, Periodic-GARCH models were studied by Bollerslev and Ghysels (1996), Aknouche and Bibi (2009).

Figure 3 displays a simulation, in which (s_t) is a purely periodic sequence of 0 and 1. The volatilities of the two series are strongly related in the regime $s_t = 1$: the volatility coefficients are the same in the two regimes, with a strong asymmetry only related to the sign of the first component, and the noise correlation is strong. On the contrary, the volatilities are disconnected in the regime $s_t = 0$: volatilities of the two components only depend on their own lagged value, with a small (resp. large) ARCH coefficient for the first (resp. second) component. Moreover, the correlation coefficient ρ is equal to zero in this regime. This choice of coefficients entails patterns which can be easily noticed in the sample path of Figure 3.

Other simulations not reported here show that models with large coefficients are explosive.

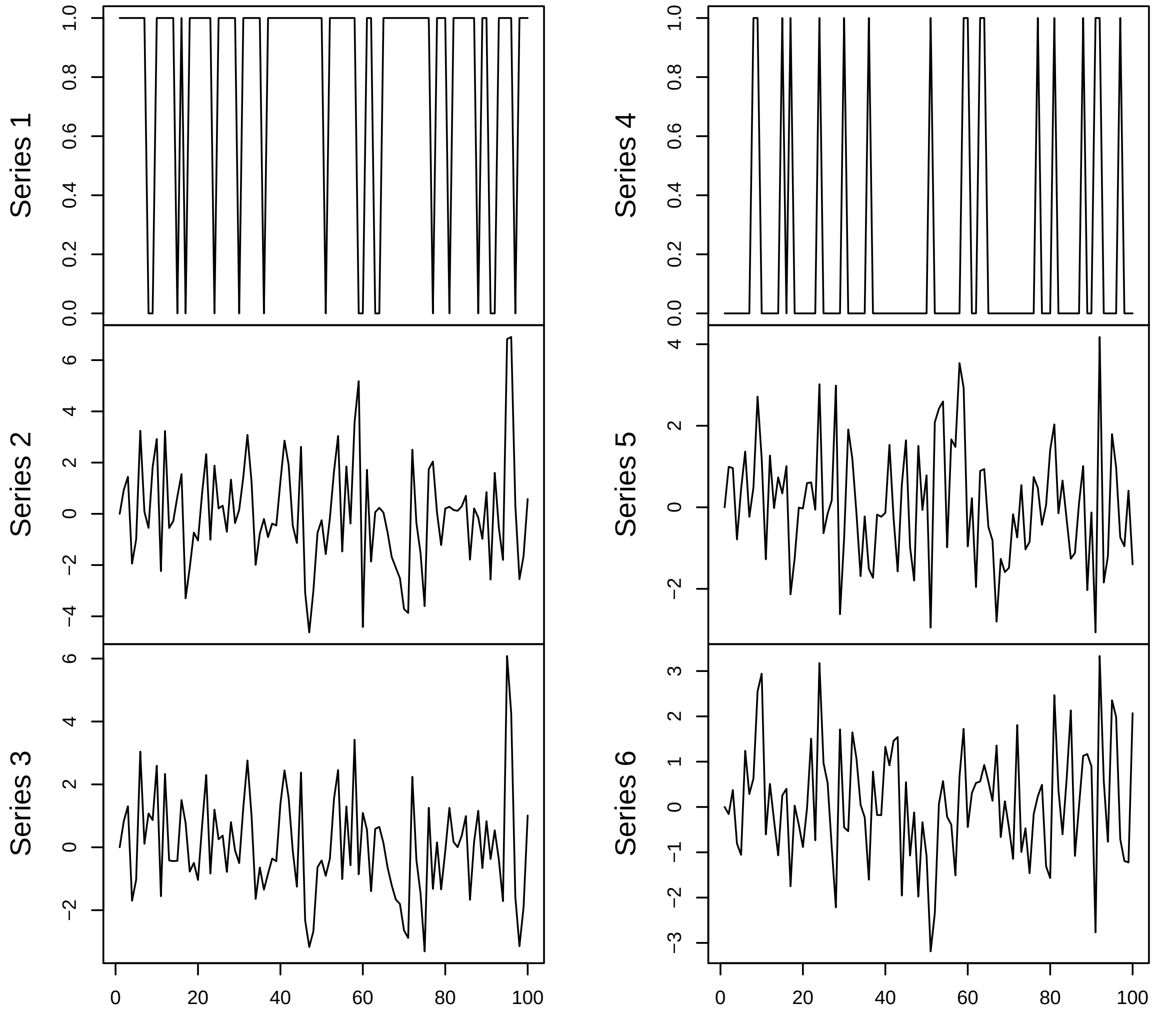

Figure 1: Simulations of Model (2.2) with the parameter values of Table 1 and $\eta_t \sim \mathcal{N}(0, I_2)$. Left panel: realization (s_t) of a $\mathcal{B}(1, 0.8)$ (top), (ϵ_{1t}) (middle) and (ϵ_{2t}) (bottom). Right panel: realization $(1 - s_t)$ (top), (ϵ_{1t}) (middle) and (ϵ_{2t}) (bottom).

Table 3: Parameter values for the simulation of Model (2.2) in Figure 3

	ρ	ω_1	ω_2	$\alpha_{11,+}$	$\alpha_{12,+}$	$\alpha_{11,-}$	$\alpha_{12,-}$	$\alpha_{21,+}$	$\alpha_{22,+}$	$\alpha_{21,-}$	$\alpha_{22,-}$
$s_t = 1$	0.8	1	1	0.1	0.1	0.6	0.1	0.1	0.1	0.6	0.1
$s_t = 0$	0	1	1	0.1	0	0.1	0	0	0.6	0	0.6

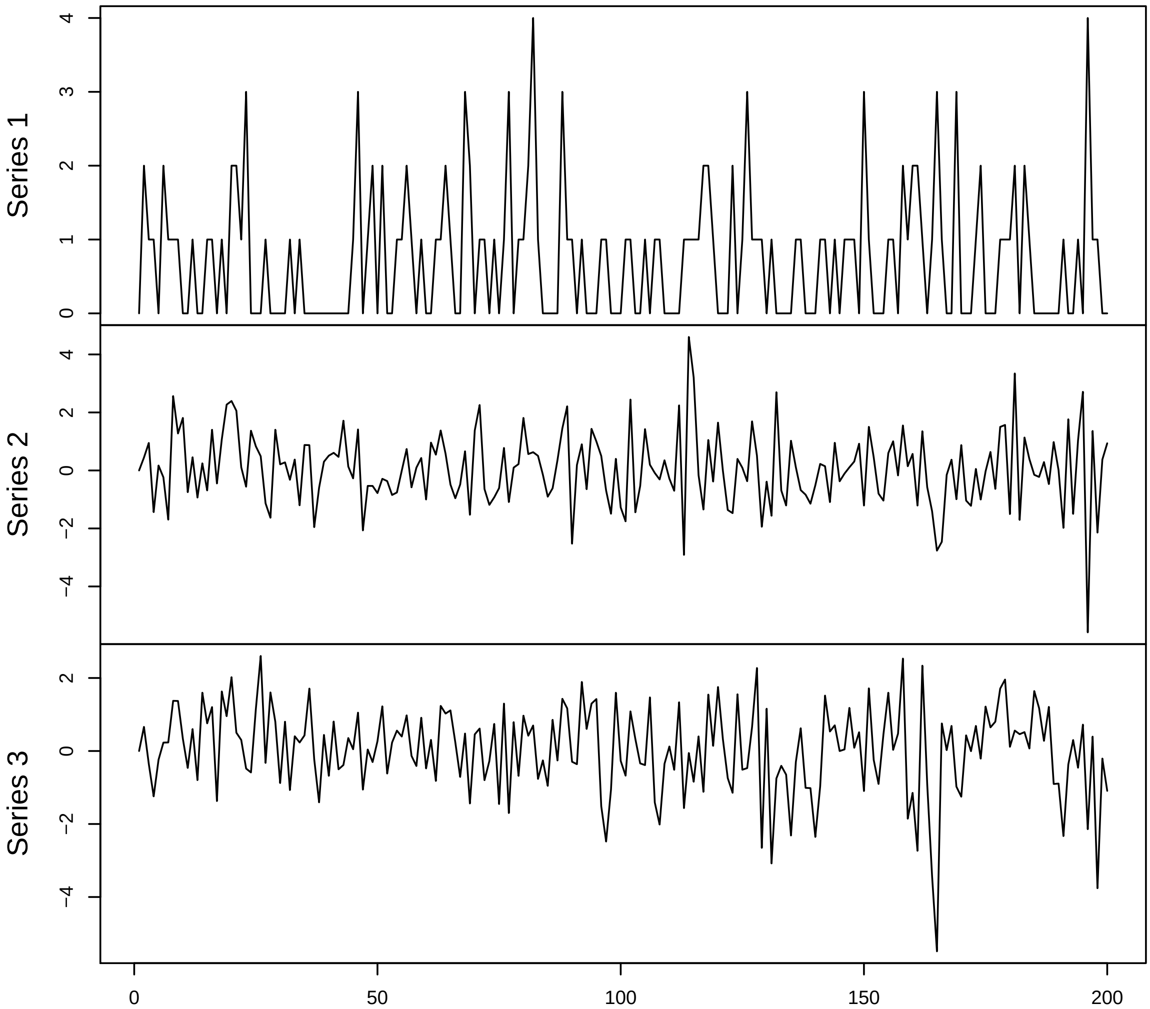

Figure 2: Simulations of Model (2.2) with the parameter values of Table 2 and $\eta_t \sim \mathcal{N}(0, I_2)$. Realization (s_t) of a $\mathcal{P}(0.7)$ distribution (top), (ϵ_{1t}) (middle) and (ϵ_{2t}) (bottom).

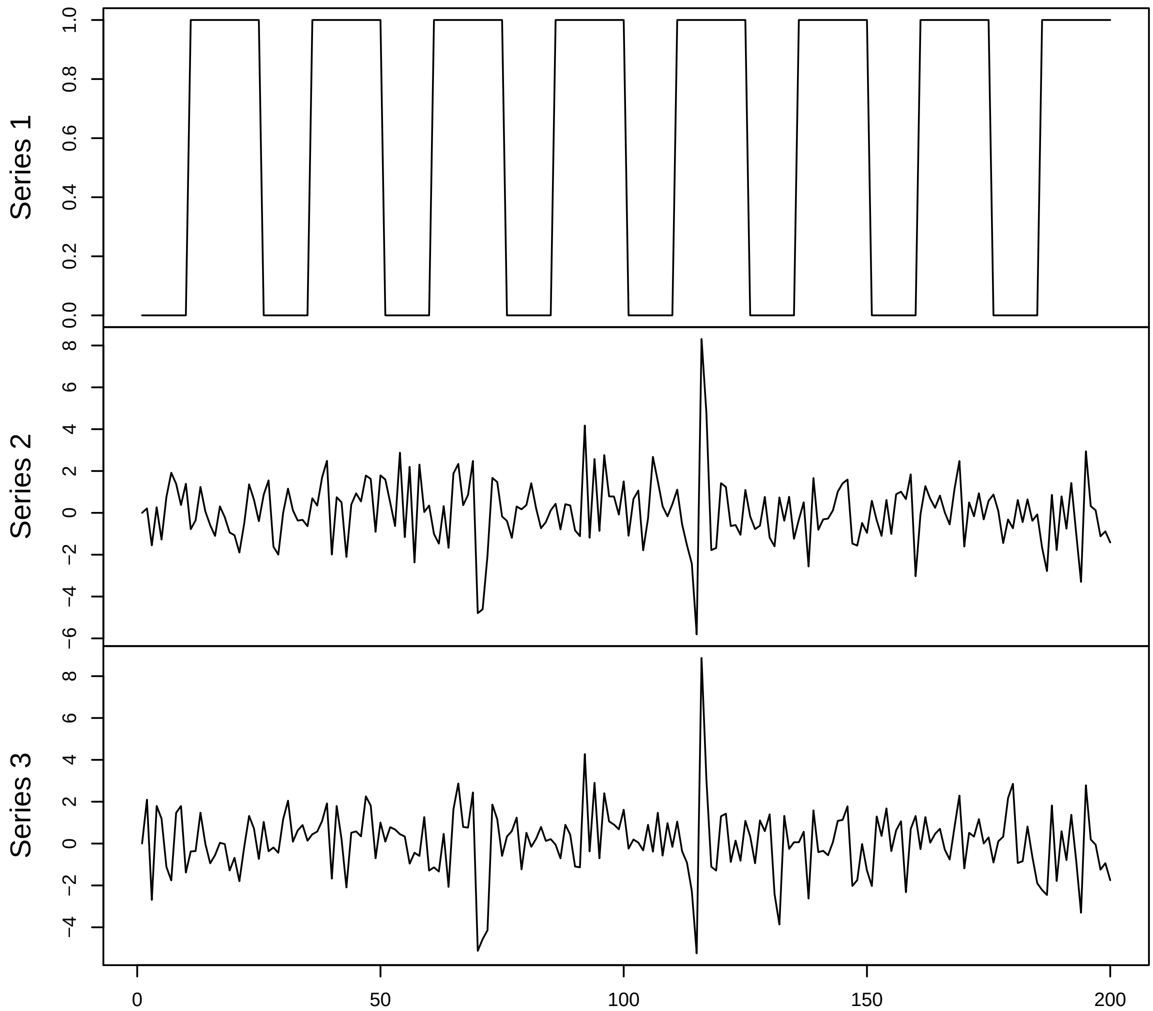

Figure 3: Simulations of Model (2.2) with the parameter values of Table 3 and $\eta_t \sim \mathcal{N}(0, I_2)$. Periodic sequence (s_t) (top), (ϵ_{1t}) (middle) and (ϵ_{2t}) (bottom).

It is therefore necessary to obtain stability conditions, indicating which parameter values are likely to be compatible with real series.

3 Stability conditions

We introduce the following assumption.

A0: (s_t) is a realization of a process (S_t) which is stationary, ergodic, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ as (η_t), and independent of (η_t).

For instance, (S_t) could be a Markov chain on E. However, even in this case, the proposed model is very different from the so-called Markov-switching models introduced by Hamilton (1989). In such models, the process (S_t) is hidden, while it is observed in our model. Moreover, under appropriate conditions, Markov-switching models have stationary solutions. On the contrary, a solution (ϵ_t) to Model (2.1), when existing, is in general non stationary because the model is conditional on (s_t).

3.1 Existence of non explosive solutions

The existence of nonexplosive solutions to Model (2.1) requires additional conditions. In this section we are interested in *nonanticipative solutions* (ϵ_t), *i.e.* such that ϵ_t is a function of the variables $\eta_{t-i}, i \geq 0$, for a given sequence (s_t).

Write

$$\epsilon_t = D_t \tilde{\eta}_t, \quad \text{where} \quad \tilde{\eta}_t = (\tilde{\eta}_{1t}, \ldots, \tilde{\eta}_{mt}) = R_0^{1/2}(s_t)\eta_t \tag{3.1}$$

and

$$\underline{\epsilon}_t^+ = \Upsilon_t^+(s_t)\underline{h}_t, \quad \underline{\epsilon}_t^- = \Upsilon_t^-(s_t)\underline{h}_t, \quad \text{where} \quad \Upsilon_t^\circ(s_t) = \text{diag}\left\{(\tilde{\eta}_{1t}^\circ)^2, \ldots, (\tilde{\eta}_{mt}^\circ)^2\right\}. \tag{3.2}$$

Introducing the $m \times pm$ matrix $\mathbf{B}_{01:p}(s_t) = (\mathbf{B}_{01}(s_t) \cdots \mathbf{B}_{0p}(s_t))$, and similar other notations, let the $(p+2q)m \times (p+2q)m$ matrix

$$C(\eta_t, s_t) = \begin{pmatrix} \Upsilon_t^+(s_t)\mathbf{A}_{01:q,+}(s_t) & \Upsilon_t^+(s_t)\mathbf{A}_{01:q,-}(s_t) & \Upsilon_t^+(s_t)\mathbf{B}_{01:p}(s_t) & \\ I_{(q-1)m} & & 0_{(q-1)m\times(p+q+1)m} & \\ \Upsilon_t^-(s_t)\mathbf{A}_{01:q,+}(s_t) & \Upsilon_t^-(s_t)\mathbf{A}_{01:q,-}(s_t) & \Upsilon_t^-(s_t)\mathbf{B}_{01:p}(s_t) & \\ 0_{(q-1)m\times qm} & I_{(q-1)m} & 0_{(q-1)m\times(p+1)m} & \\ \mathbf{A}_{01:q,+}(s_t) & \mathbf{A}_{01:q,-}(s_t) & \mathbf{B}_{01:p}(s_t) & \\ & 0_{(p-1)m\times 2qm} & I_{(p-1)m} & 0_{(p-1)m\times m} \end{pmatrix} \tag{3.3}$$

Let $\|\cdot\|$ denote any norm on the space of the $(p+2q)m \times (p+2q)m$ matrices. Note that by replacing the observation s_t by the variable S_t we obtain a matrix which is a function of the strictly stationary and ergodic process (η_t, S_t), and thus we obtain a strictly stationary and ergodic sequence of random matrices $\{C(\eta_t, S_t), t \in \mathbb{Z}\}$. Moreover $E \log^+ \|C(\eta_1, S_1)\| < \infty$. We may thus introduce $\gamma(\mathbf{C_0})$, the top Lyapunov exponent of the sequence $\mathbf{C_0} = \{C(\eta_t, S_t), t \in \mathbb{Z}\}$, defined by

$$\begin{aligned}
\gamma(\mathbf{C_0}) &= \lim_{t\to\infty} \frac{1}{t} E\left(\log \|C(\eta_t, S_t)C(\eta_{t-1}, S_{t-1}) \ldots C(\eta_1, S_1)\|\right) \\
&= \inf_{t\geq 1} \frac{1}{t} E(\log \|C(\eta_t, S_t)C(\eta_{t-1}, S_{t-1}) \ldots C(\eta_1, S_1)\|) \\
&= \lim_{t\to\infty} a.s. \frac{1}{t} \log \|C(\eta_t, S_t)C(\eta_{t-1}, S_{t-1}) \ldots C(\eta_1, S_1)\|.
\end{aligned}$$

We are now in a position to state the following result.

Theorem 3.1 *Suppose that* **A0** *holds. Then, a necessary and sufficient condition for the existence of a nonanticipative solution to Model (2.1), for almost all sequence* (s_t), *is* $\gamma(\mathbf{C_0}) < 0$. *Moreover this solution is unique and is given by*

$$\epsilon_t = \{diag(\underline{z}_{2q+1,t})\}^{1/2} R_0^{1/2}(s_t)\eta_t \tag{3.4}$$

where $\underline{z}_{2q+1,t}$ *denotes the* $(2q+1)$*-th sub-vector of size* m *of*

$$\underline{z}_t = \underline{b}(\eta_t, s_t) + \sum_{n=0}^{\infty} C(\eta_t, s_t)C(\eta_{t-1}, s_{t-1}) \cdots C(\eta_{t-n}, s_{t-n})\underline{b}(\eta_{t-n-1}, s_{t-n-1}) \tag{3.5}$$

and $\underline{b}(\eta_t, s_t) = \left(\underline{\omega}_0'(s_t)\Upsilon_t^+(s_t), 0'_{m(q-1)}, \underline{\omega}_0'(s_t)\Upsilon_t^-(s_t), 0'_{(q-1)m}, \underline{\omega}_0'(s_t), 0'_{(p-1)m}\right)'$.

Remark 3.1 When $d = 1$, that is when the model coefficients are constant, we retrieve the condition established by Francq and Zakoian (2010b): in this case, the solution is strictly stationary, as a function of the process (η_t). In the general case, the distribution of ϵ_t depends on the coefficients $s_t, s_{t-1}, \ldots$ and is thus time-dependent.

Remark 3.2 When $m = 1$, that is in the univariate setting, we obtain a condition which is similar to the one established by Regnard and Zakoian (2010a) for symmetric volatilities. Indeed, when $m = 1$, the coefficient $\gamma(\mathbf{C_0})$ reduces to

$$\gamma(\mathbf{C_0}) = E\left(\log |C(\eta_1, S_1)|\right) = \sum_{i=1}^{d} E\left(\log |C(\eta_1, e_j)|\right) \pi_j,$$

because the process (η_t, S_t) is stationary and ergodic.

Remark 3.3 In the ARCH case, that is when $\mathbf{B}_{0j} = 0, \quad j = 1, \dots, p$, the stability condition takes a simpler form. Introducing the $2qm \times 2qm$ matrix

$$C^*(\eta_t, s_t) = \begin{pmatrix} \Upsilon_t^+(s_t)\mathbf{A}_{01:q,+}(s_t) & \Upsilon_t^+(s_t)\mathbf{A}_{01:q,-}(s_t) \\ I_{(q-1)m} & 0_{(q-1)m\times(q+1)m} \\ \Upsilon_t^-(s_t)\mathbf{A}_{01:q,+}(s_t) & \Upsilon_t^-(s_t)\mathbf{A}_{01:q,-}(s_t) \\ 0_{(q-1)m\times qm} & I_{(q-1)m} \; 0_{(q-1)m\times m} \end{pmatrix}, \tag{3.6}$$

the stability condition takes the form $\gamma(\mathbf{C_0^*}) < 0$ where $\gamma(\mathbf{C_0^*})$ is the top Lyapunov exponent of the sequence $\{C^*(\eta_t, S_t), t \in \mathbb{Z}\}$.

The top Lyapunov exponent $\gamma(\mathbf{C_0})$ is greater than that of the matrix obtained by replacing all blocks of $C(\eta_t, S_t)$ by null matrices, except the right-lower $pm \times pm$ block. From this observation, we can deduce a necessary condition for the existence of a solution.

Corollary 3.1 *Let*

$$\mathbb{B}_0(S_t) = \begin{pmatrix} \mathbf{B}_{01:p}(S_t) \\ I_{(p-1)m} \quad 0_{(p-1)m\times m} \end{pmatrix}.$$

Then, $\gamma(\mathbf{C_0}) < 0$ *implies* $\gamma(\mathbb{B}_0) < 0$.

The following result shows that the stability condition entails the existence of moments of small order. A similar property was proven in the univariate GARCH framework by Berkes, Horváth and Kokoszka (2003).

Corollary 3.2 *Suppose* $\gamma(\mathbf{C_0}) < 0$. *Let* ϵ_t *be the strictly stationary and non anticipative solution of Model (2.1). There exists* $s > 0$ *such that* $E\|\underline{h}_t\|^s < \infty$ *and* $E\|\epsilon_t\|^{2s} < \infty$.

3.2 Existence of a solution in L^2

We now turn to the existence of second-order moments.

Theorem 3.2 *Let* $\pi_i = P(S_t = e_i)$, *for* $i = 1, \dots, d$. *Suppose that* **A0** *holds. Then if*

$$\prod_{j=1}^{d} \{E\|C(\eta_1, e_j)\|\}^{\pi_j} < 1, \tag{3.7}$$

the solution defined in Theorem 3.1 has finite second-order moments.

Note that the second-order stationarity of the GARCH regimes, that is, $E\|C(\eta_1, e_j)\| < 1$ for all j, is sufficient but non necessary for the global second-order condition. Note also that the condition (3.7) does not involve the dependence structure of the process (S_t). In general, this is not the case of the condition $\gamma(\mathbf{C_0}) < 0$ ensuring the existence of a solution.

Under (3.7), the second-order moments of the components of (ϵ_t) can be obtained using the expansion (3.5). We have, using the independence of the variables η_t,

$$E\underline{z}_t = E\underline{b}(\eta_1, s_t) + \sum_{n=0}^{\infty} EC(\eta_1, s_t)EC(\eta_1, s_{t-1}) \cdots EC(\eta_1, s_{t-n})E\underline{b}(\eta_1, s_{t-n-1}).$$

It is clear from this formula that the second-order moments of the components of ϵ_t are time-dependent, whenever (s_t) is non constant.

4 Proofs

4.1 Proof of Theorem 3.1

Following the approach developed by Bougerol and Picard (1992a, 1992b) in the case of univariate GARCH models, we introduce a Markov representation of Model (2.1). For any vector Z_t and any $h > 0$ we denote by $Z_{t:(t-h)}$ the vector $(Z'_t, \dots, Z'_{t-h})'$. Let $\underline{z}_t = \left(\underline{\epsilon}^{+'}_{t:(t-q+1)}, \underline{\epsilon}^{-'}_{t:(t-q+1)}, \underline{h}'_{t:(t-p+1)}\right)'$. In view of (3.2) and (2.1), we have

$$\underline{z}_t = \underline{b}(\eta_t, s_t) + C(\eta_t, s_t)\underline{z}_{t-1}, \tag{4.1}$$

Provided that the infinite sum converges, we thus have

$$\underline{z}_t = \underline{b}(\eta_t, s_t) + \sum_{n=0}^{\infty} C(\eta_t, s_t)C(\eta_{t-1}, s_{t-1}) \cdots C(\eta_{t-n}, s_{t-n})\underline{b}(\eta_{t-n-1}, s_{t-n-1}) \tag{4.2}$$

Cauchy's root test shows that the series in (4.2) converges almost surely for all t if

$$\lim_{n\to\infty} a.s. \frac{1}{n} \log \|C(\eta_t, s_t)C(\eta_{t-1}, s_{t-1}) \dots C(\eta_{t-n}, s_{t-n})\underline{b}(\eta_{t-n-1}, s_{t-n-1})\| < 0.$$

Using a multiplicative norm, it suffices to show that

$$\lim_{n\to\infty} a.s. \frac{1}{n} \log \|C(\eta_t, s_t)C(\eta_{t-1}, s_{t-1}) \dots C(\eta_{t-n}, s_{t-n})\| < 0, \tag{4.3}$$

$$\lim_{n\to\infty} a.s. \frac{\log \|\underline{b}(\eta_{t-n-1}, s_{t-n-1})\|}{n} = 0. \tag{4.4}$$

The latter equality follows from the fact that $E\|\underline{b}(\eta_{t-n-1}, s_{t-n-1})\| < \infty$ (see for instance Francq and Zakoian (2010a), exercise 2.11). To prove (4.3), we use Lemma 5.2 in Regnard and Zakoian (2010a), which is a direct extension of Lemma 1 in Francq and Gautier (2004a): if $X_n = X_n(S_n, S_{n-1}, \dots, \eta_n, \eta_{n-1}, \dots)$ is a random variable which is measurable with respect to the $\sigma-$field generated by $\{S_t, \eta_t, t \leq n\}$ and if $X_n \to X$ a.s., where X is a random variable, then, for almost all sequence (s_t),

$$X_n(s_n, s_{n-1}, \dots, \eta_n, \eta_{n-1}, \dots) \to X, \quad a.s.$$

It follows that the limit in (4.3) is equal to $\gamma(\mathbf{C_0})$. Thus, when $\gamma(\mathbf{C_0}) < 0$, Cauchy's root test shows that the series in (4.2) converges almost surely for all t and satisfies (4.1). A real-valued solution to Model (2.1) is then obtained as $\epsilon_t = \{\mathrm{diag}(\underline{\tilde{z}}_{2q+1,t})\}^{1/2} R_0^{1/2}(s_t)\eta_t$. This solution is thus non anticipative. The necessary part is proven following the same lines as in Francq Zakoian (2010).

To prove uniqueness, let $(\underline{z}_t)$ denote a positive and solution of (4.1). For all $N \geq 0$,

$$\underline{z}_t = \underline{\tilde{z}}_t(N) + C(\eta_t, s_t) \dots C(\eta_{t-N}, s_{t-N})\underline{z}_{t-N-1},$$

where

$$\underline{\tilde{z}}_t(N) = \underline{b}_t + \sum_{n=0}^{N} C(\eta_t, s_t) \cdots C(\eta_{t-n}, s_{t-n})\underline{b}(\eta_{t-n-1}, s_{t-n-1}).$$

Then

$$\|\underline{z}_t - \underline{\tilde{z}}_t\| \leq \|\underline{\tilde{z}}_t(N) - \underline{\tilde{z}}_t\| + \|C(\eta_t, s_t) \dots C(\eta_{t-N}, s_{t-N})\| \|\underline{z}_{t-N-1}\|.$$

The first term in the right-hand side tends to 0 a.s. when $N \to \infty$. Using again Lemma 5.2 in Regnard and Zakoian (2010a), to prove that the second term tends to zero for almost all sequence (s_t), it suffices to show that $\|C(\eta_t, S_t) \dots C(\eta_{t-N}, S_{t-N})\| \|\underline{z}_{S,t-N-1}\|$ tends to zero, where $\underline{z}_{S,t}$ is obtained by replacing the sequence (s_t) by (S_t) in $(\underline{z}_t)$. Because the series defining $\underline{\tilde{z}}_{S,t}$ converges a.s., we have $\|C(\eta_t, S_t) \dots C(\eta_{t-N}, S_{t-N})\| \to 0$ with probability 1 when $n \to \infty$. Moreover the distribution of $\|\underline{z}_{S,t-N-1}\|$ is independent of N by stationarity. It follows that $\|C(\eta_t, S_t) \dots C(\eta_{t-N}, S_{t-N})\| \|\underline{z}_{S,t-N-1}\| \to 0$ in probability as $N \to \infty$. We have shown that $\underline{z}_t - \underline{\tilde{z}}_t \to 0$ in probability when $N \to \infty$. This quantity being independent of N we have, necessarily, $\underline{\tilde{z}}_t = \underline{z}_t$ for any t, a.s., for almost all sequence (s_t).

4.2 Proof of Corollary 3.2

It is sufficient to prove that the norm of $\underline{z}_t$, as defined in (4.2), admits a moment of some order r. Consider the multiplicative norm such that $\|A\| = \sum |a_{ij}|$, for any matrix $A = (a_{ij})$. By Lemma 2.2 in Francq and Zakoian (2010a), if X denotes a positive real random variable such that $E \log X < 0$ and $EX^u > 0$ for some $u > 0$, then $EX^r < 1$ for some $r > 0$. Now, because $\gamma(\mathbf{C_0}) = \inf_{t \geq 1} \frac{1}{t} E(\log \|C(\eta_t, S_t)C(\eta_{t-1}, S_{t-1}) \dots C(\eta_1, S_1)\|) < 0$, there exists k_0 such that $E(\log \|C(\eta_{k_0}, S_{k_0})C(\eta_{k_0-1}, S_{k_0-1}) \dots C(\eta_1, S_1)\|) < 0$. Moreover, we have

$$\begin{aligned}
& E\|C(\eta_{k_0}, S_{k_0})C(\eta_{k_0-1}, S_{k_0-1}) \dots C(\eta_1, S_1)\| \\
= & \sum_{e_1,\dots,e_{k_0} \in E} E\|C(\eta_{k_0}, e_{k_0})C(\eta_{k_0-1}, e_{k_0-1}) \dots C(\eta_1, e_1)\| P(S_{k_0} = e_{k_0}, \dots, S_1 = e_1) \\
\leq & \sum_{e_1,\dots,e_{k_0} \in E} E(\|C(\eta_{k_0}, e_{k_0})\| \|C(\eta_{k_0-1}, e_{k_0-1})\| \dots \|C(\eta_1, e_1)\|) P(S_{k_0} = e_{k_0}, \dots, S_1 = e_1) \\
\leq & \sum_{e_1,\dots,e_{k_0} \in E} \prod_{k=1}^{k_0} E(\|C(\eta_1, e_k)\| P(S_{k_0} = e_{k_0}, \dots, S_1 = e_1) < \infty.
\end{aligned}$$

Using the aforementioned lemma, it follows that $E\|C(\eta_{k_0}, S_{k_0}) \dots C(\eta_1, S_1)\|^r < 1$ for some $r > 0$. In what follows, we assume $k_0 = 2$ without generality loss. We thus have, for some $r > 0$,

$$E\|C(\eta_2, S_2)C(\eta_1, S_1)\|^r \quad = \sum_{j,k \in \{1,\dots,d\}} E\|C(\eta_2, e_j)C(\eta_1, e_k)\|^r \pi_{j,k} < 1 \tag{4.5}$$

where $\pi_{j,k} = P(S_2 = e_j, S_1 = e_k)$.

Now we have, in view of in (4.2),

$$\begin{aligned}
\|\underline{z}_t\| \quad \leq & \quad \|\underline{b}(\eta_t, s_t)\| + \sum_{n=0}^{\infty} \|C(\eta_t, s_t)C(\eta_{t-1}, s_{t-1}) \cdots C(\eta_{t-n}, s_{t-n})\underline{b}(\eta_{t-n-1}, s_{t-n-1})\| \\
\leq & \quad \|\underline{b}(\eta_t, s_t)\| + \sum_{n=0}^{\infty} \|C(\eta_t, s_t)C(\eta_{t-1}, s_{t-1})\| \cdots \|C(\eta_{t-n+1}, s_{t-n+1})C(\eta_{t-n}, s_{t-n})\| \\
& \quad \times \|\underline{b}(\eta_{t-n-1}, s_{t-n-1})\|.
\end{aligned} \tag{4.6}$$

For $r \in (0, 1]$ we have the elementary inequality $(\sum_i u_i)^r \leq \sum_i u_i^r$ for any sequence of positive numbers u_i. Hence, since the sequence (η_t) is iid,

$$\begin{aligned}
E\|\underline{z}_t\|^r \quad \leq & \quad E\|\underline{b}(\eta_t, s_t)\|^r + \sum_{n=0}^{\infty} E\|C(\eta_2, s_t)C(\eta_1, s_{t-1})\|^r \cdots E\|C(\eta_2, s_{t-n+1})C(\eta_1, s_{t-n})\|^r \\
& \quad \times E\|\underline{b}(\eta_1, s_{t-n-1})\|^r.
\end{aligned}$$

For $j = 1, \dots d$ let $\mathcal{T}(t, j, k, n) = \{\tau \in \{1, \dots, n\} \mid s_{t-\tau+1} = e_j, s_{t-\tau} = e_k\}$. It follows that

$$\begin{aligned}
E\|\underline{z}_t\|^r \quad \leq & \quad E\|\underline{b}(\eta_1, s_t)\|^r + \sum_{n=0}^{\infty} \prod_{j,k=1}^{d} \{E\|C(\eta_2, e_j)C(\eta_1, e_k)\|^r\}^{|\mathcal{T}(t,j,k,n)|} \\
& \quad E\|\underline{b}(\eta_1, s_{t-n-1})\|^r.
\end{aligned} \tag{4.7}$$

Let $u_n = \prod_{j,k=1}^d \{E\|C(\eta_2, e_j)C(\eta_1, e_k)\|^r\}^{|\mathcal{T}(t,j,k,n)|} E\|\underline{b}(\eta_1, s_{t-n-1})\|^r$. We have

$$
\begin{aligned}
u_n^{1/n} &= \prod_{j,k=1}^d \{E\|C(\eta_2, e_j)C(\eta_1, e_k)\|^r\}^{|\mathcal{T}(t,j,k,n)|/n} \{E\|\underline{b}(\eta_1, s_{t-n-1})\|^r\}^{1/n} \\
&\to \prod_{j,k=1}^d \{E\|C(\eta_2, e_j)C(\eta_1, e_k)\|^r\}^{\pi_{j,k}}, \quad a.s.
\end{aligned}
$$

because (s_t) is a realization of the ergodic process (S_t). By (4.5) and the concavity of the logarithm function, $\lim u_n^{1/n} < 1$. It follows, by the Cauchy rule, that the infinite sum in (4.7) converges, which completes the proof.

□

4.3 Proof of Theorem 3.2

It is sufficient to prove that $\underline{z}_t$, as defined in (4.2), belongs to L^1. In view of (4.6), since the sequence (η_t) is iid,

$$
\begin{aligned}
& E\|\underline{z}_t\| \\
\leq\ & E\|\underline{b}(\eta_t, s_t)\| + \sum_{n=0}^{\infty} E\|C(\eta_1, s_t)\| E\|C(\eta_1, s_{t-1})\| \cdots E\|C(\eta_1, s_{t-n})\| E\|\underline{b}(\eta_1, s_{t-n-1})\|.
\end{aligned}
$$

For $j = 1, \dots d$ let $\mathcal{T}(t,j,n) = \{\tau \in \{0, \dots, n\} \mid s_{t-\tau} = e_j\}$. It follows that

$$
E\|\underline{z}_t\| \leq E\|\underline{b}(\eta_1, s_t)\| + \sum_{n=0}^{\infty} \prod_{j=1}^d \{E\|C(\eta_1, e_j)\|\}^{|\mathcal{T}(t,j,n)|} E\|\underline{b}(\eta_1, s_{t-n-1})\|. \tag{4.8}
$$

Let $u_n = \prod_{j=1}^d \{E\|C(\eta_1, e_j)\|\}^{|\mathcal{T}(t,j,n)|} E\|\underline{b}(\eta_1, s_{t-n-1})\|$. We have

$$
\begin{aligned}
u_n^{1/n} &= \prod_{j=1}^d \{E\|C(\eta_1, e_j)\|\}^{|\mathcal{T}(t,j,n)|/n} \{E\|\underline{b}(\eta_1, s_{t-n-1})\|\}^{1/n} \\
&\to \prod_{j=1}^d \{E\|C(\eta_1, e_j)\|\}^{\pi_j}, \quad a.s.
\end{aligned}
$$

because (s_t) is a realization of the ergodic process (S_t). By the Cauchy rule the infinite sum in (4.8) converges and the property is proven.

□

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