Arbitrage pricing and equilibrium pricing : compatibility conditions

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Abstract

The problem of fair pricing of contingent claims is well understood in the context of an arbitrage free, complete financial market, with perfect information : the so-called arbitrage approach permits to construct a unique valuation operator compatible with the observed price processes.

In the more realistic context of partial information, the equilibrium analysis permits to construct a unique valuation operator which only depends on some particular price processes as well as on the dividends process.

In this paper we present these two approaches and we explore their links and the conditions under which they are compatible. In particular, we derive from the equilibrium conditions some links between the price processes parameters and those of the dividend processes parameters.

Keywords: arbitrage, equilibrium, optimality, incomplete markets, nonredundant assets, derivatives pricing

1. Introduction

The theory of asset pricing takes its roots in the Arrow-Debreu model (see, for instance, Debreu 1959, Chap. 7), the Black and Scholes [1973] formula, and the Cox and Ross [1976] linear pricing model. This theory and its link to arbitrage has been formalized in a general framework by Harrison and Kreps [1979], Harrison and Pliska [1981, 1983], and Duffie and Huang [1986]. In these models, security markets are assumed to be frictionless: securities can be sold short in unlimited amounts, the borrowing and lending rates are equal, and there is no transaction cost. The main result is that the price process of traded securities is arbitrage free if and only if there exists some equivalent probability measure that transforms it into a martingale, when normalized by the numeraire. Contingent claims can then be priced by taking the expected value of their (normalized) payoff with respect to any equivalent martingale measure. If this value is unique, the claim is said to be priced by arbitrage and it can be perfectly hedged (i.e. duplicated) by dynamic trading. When the markets are dynamically complete, there is only one such a *martingale-probability measure* and any contingent claim is priced by arbitrage. The *weight* of each state of the world for this probability measure can be interpreted as the state price of the economy (the price of $1 tomorrow in that state of the world) as well as the marginal utilities (for consumption in that state of the world) of rational agents maximizing their expected utility.

When there are frictions, including dynamic market incompleteness, the characterization of the no-arbitrage condition is nomore equivalent to the existence of a unique equivalent martingale measure and we generally have, more than one measure characterizing the no-arbitrage condition. However arbitrage bounds can be computed, for arbitrary contingent claims, taking the expected value of their (normalized) payoff with respect to all the measures that characterize the absence of arbitrage opportunities. These bounds are the minimum amount it costs to hedge the claim and the maximum amount that can be borrowed against it using dynamic strategies. These are the tightest bounds that can be inferred on the price of a contingent claim without knowing the agent's preferences as shown by Jouini and Kallal [1995,1999] (see Jouini and Kallal [2000] for the case where we further impose that the preferences derive from VNM utility functions). The main assumption in these models is, in fact, a necessary condition for the existence of an equilibrium: the no-arbitrage condition. These preference-free theories are based on the knowledge of the price processes for the so-called marketed assets and give results of great generality without specifying the equilibrium in its full details.

Another approach consists in a complete description of the economy (agents, preferences, endowments,...) and of the equilibrium conditions. All the price processes as well as the valuation operator are then obtained endogeneously from these equilibrium conditions. Models of competitive equilibrium go back to Walras [1874] and the first complete proof for the existence of an equilibrium in an economy with finitely many commodities was given by Arrow and Debreu

[1954]. In the chapter 7 of Debreu [1959], the author explains how this model permits to take into account dynamic markets with uncertainty. Bewley [1972] studied the competitive equilibrium in an infinite-dimensional commodity space, namely L^∞ and Mas-Colell [1986] generalized Bewley's results to Hausdorff locally convex, topological vector spaces under a "uniform properness condition" on the agent's preferences. Araujo and Monteiro [1989 a and b] and Duffie and Zame [1989] proved independently the existence of an equilibrium without Mas-Colell's uniform properness condition. However, all the previous models does not take explicitely into account dynamic security trading. Models where the agents achieve equilibrium allocations by trading in securities like the capital asset pricing model (CAPM) of Merton [1971, 1973] or the consumption based capital asset pricing model (CCAPM) of Breeden [1979] can be found in the literature going back to Merton [1971], Cox, Ingersoll and Ross [1985], Duffie and Huang [1985], Huang [1987] and Karatzas, Lehoczky and Shreve [1990]. The link between these two approaches is made by Duffie and Huang [1985] where the authors explain how an Arrow-Debreu equilibrium can be implemented by trading in securities. This role of securities was, in fact, already recognized by Arrow [1952].

In this paper we explore the compatibility between the arbitrage approach and the equilibrium approach. In other words, we consider the price processes as given and we derive from there a valuation operator through the arbitrage approach and another valuation operator from the equilibrium conditions. The first operator only depends on the asset price processes and does not depend on the dividends process. The second one depends on the dividends process and only on the productive assets price processes. When these two operators coincides then our price processes are compatible with an equilibrium or, in other words, there exists utility functions and initial endowments for which our price processes are equilibrium price processes.

This problem that consists in considering specific pricing models and trying to find an economic justification for them, i.e., to find utility functions for which there is an equilibrium in the considered models has been raised by Bick [1987] for the Black and Scholes model and by Bick [1990] and He and Leland [1993] for a Markovian diffusion model in a complete market framework, where the unique risky asset is productive and available in one unit supply. They characterize the risk premium as the unique solution of a nonlinear partial differential equation, and they relate it to the shape of the representative agent utility function. In Pham-Touzi [1996], a specific stochastic volatility model, which is a particular case of a Markov setting, is considered; they apply their results to provide an economic foundation to the Hull and White model and to the minimal martingale measure.

In this paper we do consider a non markovian setting and we obtain weaker conditions on the utility functions and the dividends process than Pham and Touzi [1996]. However, we do not address the existence of an equilibrium problem. In the framework of this paper, this is done in a companion paper (Jouini and Napp [2000]).

Our framework is precisely the following: we consider a market in which there are a certain number of productive assets, referred to as the primitive stocks, whose price processes are supposed to be driven by a Brownian motion and we assume that the number of sources of uncertainty are greater than the number of primitive stocks available so that the primitive market consisting only of these primitive productive assets is incomplete. The price processes are very general diffusion processes: in particular, we don't assume that they are of Markovian type.

In addition to these productive assets, we consider purely financial assets; we assume that they complete the market and that the full market is in equilibrium. Our goal here is then to compare the arbitrage pricing approach and the equilibrium one and to explore the conditions under which they are equivalent.

For the sake of completeness, all the results about utility maximization, market completeness and equilibrium characterization are introduced step by step. The paper is then organized as follows: in section 2, we introduce a market model with only one productive asset; in section 3, we characterize the equivalent martingale measures; in section 4, we consider how economic agents actually trade in this market. In section 5, we are interested in utility maximization and optimal demand for a single agent. In section 6, we define what we call equilibrium and we use the preceding sections to obtain necessary conditions for equilibrium and provide the main results. Section 7 extends the previous results to an incomplete market framework with many productive assets.

We introduce a few notations: two probability measures P and Q, defined on the same measurable space (Ω, F) are said to be equivalent if they agree on the null sets.

All vectors are column vectors and transposition is denoted by the superscript $*$. As usual, 1_d denotes the d-dimensional vector whose every component is one. If $Z = \left(Z^1, ..., Z^n\right)$ denotes a vector in R^n, then $diagZ$ denotes the $(n \times n)$ diagonal matrix whose diagonal entries are the components of Z. We denote by $\parallel Z \parallel^2$ the nonnegative real number $\sum_{i=1}^n \left(Z^i\right)^2$.

Let (Ω, F, P) be a fixed probability space and $\mathcal{T}$ denote the interval $[0, T]$. Then $L_d^2(\mathcal{T})$ denotes the set of $(F_t)_{t\in\mathcal{T}}$-progressively measurable, R^d-valued processes $\{\Psi_t; t \in \mathcal{T}\}$ such that

$$\int_0^T \parallel \Psi_t \parallel^2 dt < \infty \quad a.s. \quad P.$$

For any R^d-valued process $\Psi = \{\Psi_t; t \in \mathcal{T}\}$ in $L_d^2(\mathcal{T})$, let the real-valued process $\mathcal{E}(\Psi) = \{\mathcal{E}_t(\Psi); t \in \mathcal{T}\}$ denote the exponential local martingale given for each t in $\mathcal{T}$ by

$$\mathcal{E}_t(\Psi) = \exp\left\{\int_0^t (\Psi_s)^* \, dW_s - 1/2 \int_0^t \| \Psi_s \|^2 \, ds\right\}.$$

For a real-valued process $u = \{u_t; t \in \mathcal{T}\}$ defined on a probability space (Ω, F, P), we denote by $u^- = \{u_t^-; t \in \mathcal{T}\}$ the process defined by $u_t^- = -\min(0, u_t)$ for all t in $\mathcal{T}$.

As usual, a function $F : \mathcal{T} \times R \to R$ is said to be of class $C^{m,n}$ if the m-th derivative of $F(\cdot, x) : \mathcal{T} \to R$ and the n-th derivative of $F(t, \cdot) : R \to R$ exist and are continuous.

2. The market model

We fix a finite-time horizon $\mathcal{T} \triangleq [0, T]$, on which we are going to treat our problem: T corresponds to the terminal date for all economic activity under consideration. All processes that we shall encounter in this paper are defined on $[0, T]$. We consider a “primitive” market consisting of one bond and one single productive asset. We shall refer to these two assets as the primitive assets. As we have seen in the introduction, we assume that our full market consists not only of these primitive assets but also of additional “purely financial” assets “completing” the market. More precisely:

2.1. Conditions on the primitive market

The primitive market model is the same as in Karatzas [1989] taking $m = 1$, except that we consider here dividends paying assets.

We adopt a model for the primitive market consisting of one bond with price at time t denoted by S_t^0 such that

$$dS_t^0 = S_t^0 r_t dt, \quad S_0^0 = 1$$

and one stock (or one productive risky asset) with price per share at time t denoted by S_t satisfying the equation

$$dS_t = S_t \left[(b_t - \delta_t)\, dt + \sigma_t dW_t \right], \quad S_0 = 1. \tag{2.1}$$

Here, $W = \left\{ \left(W_t^1, ..., W_t^d \right)^* ; t \in \mathcal{T} \right\}$ is a d-dimensional Brownian motion on a probability space (Ω, F, P) and we let $(F_t)_{t \in \mathcal{T}}$ denote the P-augmentation of the natural filtration generated by W. We assume that the sample paths of W completely specify all the distinguishable events, which mathematically entails $F_T = F$. Since standard Brownian motions start from zero with probability one, F_0 is trivial.

It is assumed throughout that $d \geq 1$, i.e., the number of sources of uncertainty is larger than the number of stocks.

The real-valued interest rate process $\{r_t; t \in \mathcal{T}\}$, the real-valued process $\{b_t; t \in \mathcal{T}\}$, the real-valued dividend yield process paid by the stock $\{\delta_t; t \in \mathcal{T}\}$ as well as the volatility $(1 \times d)$-matrix-valued process $\left\{ \sigma_t = \left(\sigma_t^1, ..., \sigma_t^d \right) ; t \in \mathcal{T} \right\}$ are the coefficients of the model and are taken to be progressively measurable with respect to $(F_t)_{t \in \mathcal{T}}$ and bounded uniformly in (t, ω) in $\mathcal{T} \times \Omega$. Under these hypotheses that we shall denote by $\mathbf{H}_1$, we know that equation (2.1) admits a unique real-valued, $(F_t)_{t \in \mathcal{T}}$-adapted, continuous solution $\{S_t; t \in \mathcal{T}\}$, satisfying $E\left[\sup_{t \in \mathcal{T}} {S_t}^2 \right] < \infty$.

We assume that for all t in $\mathcal{T}$, the volatility matrix σ_t has full rank 1, which amounts to assuming that, for all t in $\mathcal{T}$, $P\left[\sigma_t^j = 0 \text{ for all } j \text{ in } \{1, ..., d\} \right] = 0$. We denote this hypothesis by $\mathbf{H}_2$. A d-dimensional process $\theta = \{\theta_t; t \in \mathcal{T}\}$ that we shall call in a natural way the relative risk process can then be defined by :

$$\theta_t \triangleq \left[(b_t - r_t) / \sum_{i=1}^{d} (\sigma_t^i)^2 \right] \sigma_t^* \quad a.s.\ P, \quad 0 \leq t \leq T.$$

With the above assumptions, θ is an $(F_t)_{t \in \mathcal{T}}$-progressively measurable process and we shall make in the remainder of the paper the hypothesis $\mathbf{H}_3$ that θ is uniformly bounded.

Let $\beta = \{\beta_t; t \in \mathcal{T}\}$ denote the process given by $\beta_t \triangleq 1/S_t^0 = \exp - \int_0^t r_s ds$. We shall also have the occasion to use the discounted price process $\tilde{S} = \left\{ \tilde{S}_t; t \in \mathcal{T} \right\}$ defined by $\tilde{S}_t \triangleq S_t \exp \int_0^t (\delta_s - r_s)\, ds$ for all t in $\mathcal{T}$. Using Itô's lemma, we easily get that $\tilde{S}$ is the unique solution of the following stochastic differential equation:

$$d\tilde{S}_t = \tilde{S}_t \left[(b_t - r_t)\, dt + \sigma_t dW_t \right] = \tilde{S}_t \sigma_t \left[\theta_t dt + dW_t \right], \quad \tilde{S}_0 = 1.$$

2.2. Conditions on the purely financial assets

We consider at least $(d-1)$ additional purely financial assets (or contingent claims), i.e., assets which are in zero net supply. Their prices C_t^i for i in $\{1, ..., d-1\}$ are driven by the d-dimensional Brownian motion W and we assume that they "complete" the market. More precisely, we assume that the prices C_t^i are governed by

$$dC_t^i = C_t^i \left[a_t^i dt + \mu_t^i dW_t \right] \qquad i = 1, ..., (d-1).$$

where the process $\left\{ \mu_t = \left[\left(\mu_j^i \right)_t \right]_{\substack{1 \leq i \leq d-1 \\ 1 \leq j \leq d}} ; t \in \mathcal{T} \right\}$ is an $(F_t)_{t \in \mathcal{T}}$-progressively measurable, uniformly bounded, $(d-1) \times d$ matrix-valued process such that for all t in $\mathcal{T}$, the $(d \times d)$-augmented volatility matrix $\bar{\sigma}_t \triangleq \begin{bmatrix} \sigma_t \\ \mu_t \end{bmatrix}$ admits an inverse: for example a matrix-valued process $\{\mu_t; t \in \mathcal{T}\}$ such that for all t, the rows of μ_t, thought of as vectors in R^d are orthonormal and in the kernel of σ_t, i.e., $\sigma_t \mu_t^* = 0$, like in Karatzas et al. [1991]. Moreover, we assume that the norms of $(\bar{\sigma}_t)^{-1}$ and of $(\bar{\sigma}_t^*)^{-1}$ are uniformly bounded: we could for instance impose for all t in $\mathcal{T}$ a nondegeneracy condition on the matrix $\bar{\sigma}_t \bar{\sigma}_t^*$. The process $\left\{a_t = \left(a_t^i\right)_{1 \le i \le d-1}; t \in \mathcal{T}\right\}$ is an $(F_t)_{t \in \mathcal{T}}$- progressively measurable, uniformly bounded $(d-1)$-dimensional vector process. We shall denote the preceding regularity assumptions by $\mathbf{H}_4$. Let $\bar{b} \triangleq \begin{bmatrix} b \\ a \end{bmatrix}$ denote the d-dimensional augmented stock appreciation vector. A d-dimensional process $\bar{\theta} = \left\{\bar{\theta}_t; t \in \mathcal{T}\right\}$ can then be defined by :

$$\bar{\theta}_t \triangleq (\bar{\sigma}_t)^{-1} \left[\left(\bar{b}_t - r_t 1_d\right)\right] \quad a.s. \quad P, \quad 0 \le t \le T.$$

With the above assumptions, $\bar{\theta}$ is $(F_t)_{t \in \mathcal{T}}$-progressively measurable and uniformly bounded.

So our full market consists of $(d+1)$ assets: the bond, the primitive stock and $(d-1)$ additional purely financial assets. We shall denote by Z the d-dimensional risky assets price process given by $Z \triangleq \left(S, C^1, ..., C^{d-1}\right)$ and by $\tilde{Z}$ the discounted price process $\tilde{Z} \triangleq \left(\tilde{S}, C^1/S^0, ..., C^{d-1}/S^0\right)$. Notice that assets prices can fluctuate in an almost arbitrary not necessarily Markovian fashion.

3. Equivalent martingale measures

Now that we have described both our primitive and our full markets, we can consider the problem of the existence and of the characterization of equivalent probability measures on (Ω, F, P) that make the discounted price processes $\tilde{S}$ in the primitive market and $\tilde{Z}$ in the full market martingales. We will see in section 6 that these probability measures are of great use for our problem. When we only observe the stock price process S, we already know, since Harrison and

Kreps [1979] that the fair price of any contingent claim, whose final payoff is in the form $h(S_T)$ for some function h, belongs to the following interval

$$\left[\inf_{Q \in \mathcal{M}_S} E^Q\left[\beta_T h(S_T)\right], \sup_{Q \in \mathcal{M}_S} E^Q\left[\beta_T h(S_T)\right]\right]$$

where $\mathcal{M}_S$ denotes the set of all equivalent probability measures that make the process $\tilde{S}$ a martingale.

3.1. In the primitive market

With the notations of section 2, we still consider the probability space (Ω, F, P) equipped with the filtration $(F_t)_{t \in \mathcal{T}}$.

Definition 3.1. *A probability measure Q defined on (Ω, F, P) is an S-equivalent martingale probability measure for $(F_t)_{t \in \mathcal{T}}$ if it satisfies :*

1. *The probability measures P and Q are equivalent.*
2. *The process $\tilde{S}$ is a Q-martingale for $(F_t)_{t \in \mathcal{T}}$.*

Using Itô's lemma and the fact that the dividend yield process is uniformly bounded, notice that an S-equivalent martingale probability measure is in fact an equivalent probability measure that makes the discounted "gain" process

$$G = \left\{ \frac{S_t}{S_t^0} + \int_0^t \exp\left(\int_0^s -r_u du\right) \delta_s S_s ds; t \in \mathcal{T} \right\}$$

a martingale; and we know that the existence of such a probability measure is essentially equivalent to the assumption of no arbitrage. Under such a probability measure, the expected return of the stock is equal to the (short term) interest rate minus the dividend yield.

Let $M_d^2(\mathcal{T})$ denote the set of R^d-valued processes $\Phi = \{\Phi_t; t \in \mathcal{T}\}$ in $L_d^2(\mathcal{T})$ such that

$$E\left[\mathcal{E}_T(\Phi)\right] \triangleq E\left[\exp\left\{\int_0^T (\Phi_s)^* \, dW_s - 1/2 \int_0^T \| \Phi_s \|^2 \, ds\right\}\right] = 1.$$

Notice that $M_d^2(\mathcal{T})$ corresponds to the set of processes $\{\Phi_t; t \in \mathcal{T}\}$ in $L_d^2(\mathcal{T})$ such that the exponential local martingale

$$\mathcal{E}(\Phi) = \left\{\exp\left\{\int_0^t (\Phi_s)^* \, dW_s - 1/2 \int_0^t \| \Phi_s \|^2 \, ds\right\}; t \in \mathcal{T}\right\}$$

is a true martingale: as a matter of fact, $\mathcal{E}(\Phi)$ being a nonnegative local martingale, we can use Fatou's lemma and get that this process is a supermartingale; therefore, it is a martingale if and only if its expected value is a constant for all t in $\mathcal{T}$.

We introduce the following set

$$K^\sigma \triangleq \left\{\nu \in L_d^2(\mathcal{T}) \text{ such that } \forall t,\ \sigma_t \nu_t = 0 \text{ and } -\theta^\nu \triangleq -(\theta + \nu) \in M_d^2(\mathcal{T})\right\}.$$

Notice that K^σ is never empty because the null process $n = \{n_t; t \in \mathcal{T}\}$ defined by $n_t = 0$ for all t always belongs to K^σ. Indeed, the first two conditions: n in $L_d^2(\mathcal{T})$ and $\sigma_t n_t = 0$ for all t are trivially satisfied and as θ is assumed to be uniformly bounded, the process $\mathcal{E}(-\theta)$ is a martingale -see, for instance, the Novikov condition in Karatzas and Shreve [1988] p. 199.

We shall now characterize all S-equivalent martingale probability measures:

Lemma 3.2. *Let Q be a probability measure defined on (Ω, F, P). The following are equivalent*

:

1. *The probability measure Q is an S-equivalent martingale probability measure for $(F_t)_{t\in\mathcal{T}}$.*

2. *The probability measure Q is such that $dQ/dP = \mathcal{E}_T(-\theta^\nu)$ for some ν in K^σ.*

Proof 1) Let us first show that Q is an equivalent probability measure if and only if it is such that $dQ/dP = \mathcal{E}_T(\rho)$ for some process $\rho = \{\rho_t; t \in \mathcal{T}\}$ in $M_d^2(\mathcal{T})$.

One implication is immediate: as ρ belongs to $L_d^2(\mathcal{T})$, the random variable $\mathcal{E}_T(\rho)$ is well defined. As it is nonnegative, it can be expressed as a measure density respectively to P, i.e., we can define on (Ω, F) a measure denoted by P^ρ given by $dP^\rho/dP = \mathcal{E}_T(\rho)$ and as $E[\mathcal{E}_T(\rho)] = 1$ -because ρ belongs to $M_d^2(\mathcal{T})$- this measure P^ρ is a probability measure. As M^ρ is positive, the probability measure P^ρ is equivalent to P.

For the reverse implication, we shall denote dQ/dP by M_T and consider the process $M = \{M_t; t \in \mathcal{T}\}$ given by $M_t \triangleq E[M_T \mid F_t]$ for all t in $\mathcal{T}$. The process $\{M_t; t \in \mathcal{T}\}$ is well defined because $M_T \in L^1(\Omega, F, P)$; it is in an obvious way a continuous $(F_t)_{t\in\mathcal{T}}$-martingale and $(F_t)_{t\in\mathcal{T}}$ is the P-augmentation of the filtration generated by W so, by the fundamental martingales representation theorem (see Karatzas-Shreve [1988], p.170), M can be written as a stochastic integral with respect to W: there exists a process $\{\gamma_t; t \in \mathcal{T}\}$ in $L_d^2(\mathcal{T})$ such that

$$M_t = E[M_T] + \int_0^t (\gamma_s)^* \, dW_s \quad 0 \le t \le T$$

so $M_t = 1 + \int_0^t (\gamma_s)^* \, dW_s$. As $M_T > 0$, the process M satisfies $M_t > 0$ for all t in $\mathcal{T}$, so we can apply Itô's lemma and obtain for all t in $\mathcal{T}$

$$\ln M_t = \ln M_0 + \int_0^t (\gamma_s / M_s)^* \, dW_s - 1/2 \int_0^t \| \gamma_s / M_s \|^2 \, ds$$

or $M_t = \exp \left\{ \int_0^t (\rho_s)^* \, dW_s - 1/2 \int_0^t \| \rho_s \|^2 \, ds \right\} = \mathcal{E}_t(\rho)$ for the d-dimensional process $\rho = \{\rho_t; t \in \mathcal{T}\}$ in $L_d^2(\mathcal{T})$, defined by $\rho_t \triangleq \frac{\gamma_t}{M_t}$ for all t in $\mathcal{T}$. As $E[M_T] = 1 = E[\mathcal{E}_T(\rho)]$, the process ρ belongs to $M_d^2(\mathcal{T})$ and this completes the proof.

So we can index the set of equivalent probability measures by $M_d^2(\mathcal{T})$ and for each process $\rho = \{\rho_t; t \in \mathcal{T}\}$ in $M_d^2(\mathcal{T})$ denote by P^ρ the equivalent probability measure such that $dP^\rho / dP = \mathcal{E}_T(\rho)$.

2) Let us now show the lemma: by Girsanov's theorem (see e.g. Karatzas-Shreve [1988], p.191), for all process ρ in $M_d^2(\mathcal{T})$, the d-dimensional process $W^{P^\rho} = \left\{ W_t^{P^\rho}; t \in \mathcal{T} \right\}$ defined for all t in $\mathcal{T}$ by

$$W_t^{P^\rho} \triangleq W_t - \int_0^t (\rho_s) \, ds$$

is a P^ρ- Brownian motion for $(F_t)_{t \in \mathcal{T}}$. As $d\tilde{S}_t = \tilde{S}_t \left[(b_t - r_t) \, dt + \sigma_t dW_t \right]$, we have for all process ρ in $M_d^2(\mathcal{T})$, $d\tilde{S}_t = \tilde{S}_t \left[(b_t - r_t + \sigma_t \rho_t) \, dt + \sigma_t dW_t^{P^\rho} \right]$; defining the process $\{\nu_t; t \in \mathcal{T}\}$ by $\nu_t \triangleq -(\rho_t + \theta_t)$ for all t in $\mathcal{T}$, we have $d\tilde{S}_t = \tilde{S}_t \left[-\sigma_t \nu_t dt + \sigma_t dW_t^{P^\rho} \right]$ and $\tilde{S}$ is a P^ρ-martingale for $(F_t)_{t \in \mathcal{T}}$ if and only if $\sigma_t \nu_t = 0$ for all t. This ends the proof of the lemma.$\square$

We shall denote by $\mathcal{M}_S$ the set of S-equivalent martingale probability measures for $(F_t)_{t\in\mathcal{T}}$. As we have seen, the null process n always belongs to K^σ so $\mathcal{M}_S$ is never reduced to the empty set and there always exists at least one $S-$equivalent martingale probability measure denoted by P^0 and given by $dP^0/dP = \mathcal{E}_T(-\theta)$; it is the so-called minimal martingale-measure of Föllmer-Schweitzer [1991].

There exists a unique martingale probability measure if and only if we have $K^\sigma = \{0\}$ which is the case if and only if $d = 1$. In that case, the unique S-equivalent martingale probability measure is the one given by Föllmer and Schweitzer [1991]. More generally, $\mathcal{M}_S$ can be considered as indexed by K^σ: for each ν in K^σ, we shall denote by P^ν the corresponding martingale measure, i.e., such that

$$dP^\nu/dP = \exp\left\{\int_0^T -(\theta_s^\nu)^* \, dW_s - 1/2\int_0^T \parallel \theta_s \parallel^2 + \parallel \nu_s \parallel^2 ds\right\}$$

and by $M^\nu \triangleq \mathcal{E}[-\theta^\nu]$ the corresponding process and then we have

$$\mathcal{M}_S = \{P^\nu; \nu \in K^\sigma\} \neq \emptyset.$$

Notice that for each $S-$equivalent martingale probability measure P^ν, we have $d\tilde{S}_t = \tilde{S}_t\left[\sigma_t dW_t^{P^\nu}\right]$.

3.2. In the full market

We are interested in what we shall call $Z-$equivalent martingale probability measures, i.e., equivalent probability measures Q that make the full process

$$\tilde{Z} \triangleq \left(\tilde{S}, C^1/S^0, ..., C^{d-1}/S^0\right)$$

a Q-martingale for $(F_t)_{t\in\mathcal{T}}$. Notice that any $Z-$equivalent martingale probability measure is in an obvious way an S-equivalent martingale probability measure. Following exactly the same approach as in the preceding section for $d = 1$, we show the following result:

Lemma 3.3. *There exists a unique equivalent probability measure $\bar{P}$ defined on (Ω, F, P) that makes the full process $\tilde{Z}$ a martingale for $(F_t)_{t\in\mathcal{T}}$. It is given by*

$$d\bar{P}/dP = \mathcal{E}_T\left(-\bar{\theta}\right) = \exp\left\{-\int_0^T \left(\bar{\theta}_s\right)^* dW_s - 1/2\int_0^T \parallel \bar{\theta}_s \parallel^2 ds\right\}.$$

Proof Analogous to the previous one in the primitive market for $d = 1$. As a matter of fact, replacing σ with $\bar{\sigma}$ and θ with $\bar{\theta}$, if we let

$$K^{\bar{\sigma}} \triangleq \left\{\nu \in L_d^2\left(\mathcal{T}\right) \text{ such that } \bar{\sigma}_t \nu_t = 0 \text{ for all } t \text{ and } -\left(\bar{\theta} + \nu\right) \in M_d^2\left(\mathcal{T}\right)\right\},$$

then $K^{\bar{\sigma}} = \{0\}$, because $\bar{\sigma}$ admits an inverse. So there exists a unique equivalent martingale measure which is in the form given above.□

We then have $d\tilde{Z}_t = diag\tilde{Z}_t\left[\bar{\sigma}_t dW_t^{\bar{P}}\right]$ where $\left\{W_t^{\bar{P}}; t \in \mathcal{T}\right\}$ is the $\bar{P}-$Brownian motion for $(F_t)_{t\in\mathcal{T}}$ defined by $W_t^{\bar{P}} \triangleq W_t + \int_0^t \bar{\theta}_s ds$ for all t in $\mathcal{T}$. We shall in the remainder of the paper denote the martingale process $\left\{E\left[\frac{d\bar{P}}{dP} \mid F_t\right]; t \in \mathcal{T}\right\}$ by $\bar{M} = \left\{\bar{M}_t; t \in \mathcal{T}\right\}$. As $\bar{P}$ belongs to $\mathcal{M}_S$, it can be written in the form $P^{\bar{\nu}}$ for $\bar{\nu} = \bar{\theta} - \theta$ satisfying

$$\sigma_t \bar{\nu}_t = \sigma_t\left(\bar{\theta}_t - \theta_t\right) = (1\ 0\ ...0)\left(\bar{b}_t - r_t 1_d\right) - (b_t - r_t) = 0.$$

Notice also that $\bar{P}$ not only depends on the productive asset price process but also on the financial assets price processes.

4. Trading strategies

Let us now consider an economic agent, who invests in the full market.

4.1. Wealth process and admissible strategies

We shall denote respectively by π_t^S and $\pi_t^{C^i}$ the amounts that the agent invests at time t in the stock and in the ith contingent claim respectively, by c_t the rate at which he withdraws funds for consumption and by $X_t^{\pi^S,\left(\pi^{C^i}\right)_i,c}$ the corresponding wealth of this agent at time t. We allow here any π_t^S or $\pi_t^{C^i}$ to become negative, which amounts to allowing the agent to sell short any risky asset. Similarly, the amount of money $\pi_t^{S^0} = X_t^{\pi^S,\left(\pi^{C^1},\dots,\pi^{C^{d-1}}\right),c} - \pi_t^S - \sum_{i=1}^{d-1} \pi_t^{C^i}$ invested in the bond at time t may also become negative, which is to be interpreted as borrowing at the interest rate r_t. More precisely:

Definition 4.1. *A trading strategy or a portfolio process*

$$\pi = \left\{ \left(\pi_t^S, \pi_t^{C^1}, \dots, \pi_t^{C^{d-1}} \right)^*; t \in \mathcal{T} \right\}$$

is an element of $L_d^2(\mathcal{T})$.

Definition 4.2. *A consumption strategy or a consumption rate process*

$$c = \{c_t; t \in \mathcal{T}\}$$

is a nonnegative, progressively measurable, real-valued process that satisfies $\int_0^T c_t dt < \infty$ *a.s.* P.

Assuming that the trading-consumption strategy is self financing, i.e., that at each time t, sales and dividends must finance purchases and consumption, we obtain, with the above interpretations and definitions, for each t in $\mathcal{T}$, the following equation for the wealth of the agent

$$\begin{aligned} dX_t^{\pi,c} &= \frac{\pi_t^{S^0}}{S_t^0} dS_t^0 + \frac{\pi_t^S}{S_t} dS_t + \sum_{i=1}^{d-1} \frac{\pi_t^{C^i}}{C_t^i} dC_t^i \\ & \quad -c_t dt + \frac{\pi_t^S}{S_t} \delta_t S_t dt. \end{aligned} \tag{4.1}$$

where the terms on the right-hand side of the equation account respectively for capital gains or losses from the productive asset held, capital gains or losses from financial assets held, the decrease in wealth due to consumption and the increase in wealth due to dividends paid by the productive asset. It is easy to see that, with the assumptions made on the trading and consumption strategies, all quantities are well defined. Using what has been done in the preceding section, the dynamics of the wealth process can be rewritten

$$\begin{aligned} dX_t^{\pi,c} &= \left[X_t^{\pi,c} - \pi_t^S - \sum_{i=1}^{d-1} \pi_t^{C^i} \right] r_t dt + \pi_t^S \left[(b_t - \delta_t) dt + \sigma_t dW_t \right] \\ & \quad + \sum_{i=1}^{d-1} \pi_t^{C^i} \left[a_t^i dt + \mu_t^i dW_t \right] - c_t dt + \frac{\pi_t^S}{S_t} \delta_t S_t dt \\ &= \left[r_t X_t^{\pi,c} - c_t \right] dt + (\pi_t)^* \left[\bar{b}_t - r_t 1_d \right] dt + (\pi_t)^* \bar{\sigma}_t dW_t \\ &= \left[r_t X_t^{\pi,c} - c_t \right] dt + (\pi_t)^* \bar{\sigma}_t dW_t^{\bar{P}}. \end{aligned}$$

where, as above, $W_t^{\bar{P}} \triangleq W_t + \int_0^t \bar{\theta}_s ds$ is a $\bar{P}$-Brownian motion. The unique solution of this equation with initial wealth $X_0^{\pi,c} = x \geq 0$ is denoted by $\{X_t^{x;\pi,c}; t \in \mathcal{T}\}$ and is easily seen to be given for all t in $\mathcal{T}$ by

$$\beta_t X_t^{x;\pi,c} = x - \int_0^t \beta_s c_s ds + \int_0^t \beta_s (\pi_s)^* \bar{\sigma}_s dW_s^{\bar{P}}. \tag{4.2}$$

We shall now single out those pairs (π, c) for which the investor avoids negative wealth by defining admissible strategies:

Definition 4.3. *A pair (π, c) of portfolio and consumption rate processes is called admissible for the initial capital $x \geq 0$ if the unique corresponding wealth process $\{X_t^{x;\pi,c}; t \in \mathcal{T}\}$ given by equation (4.2) above satisfies*

$$X_t^{x;\pi,c} \geq 0 \quad \textit{for all } t \in \mathcal{T}. \tag{4.3}$$

The class of such pairs is denoted by $\mathbf{A}(x)$. Notice that we don't need the assumption that π satisfies $E\left[\int_0^T \|\pi_s\|^2 ds\right] < \infty$, which is often found in the literature and implies that the process

$$M \triangleq \left\{\beta_t X_t^{x;\pi,c} + \int_0^t \beta_s c_s ds; t \in \mathcal{T}\right\}$$

consisting of current discounted wealth plus total discounted consumption is a $\bar{P}$-martingale. One may note the requirement in (4.3) that wealth is always nonnegative which makes budget feasibility somewhat more restrictive than the usual notion. We impose a no-bankruptcy condition not only at terminal time but at each time t in $\mathcal{T}$, i.e., $X_t \geq 0$ for all t in $\mathcal{T}$, which amounts to saying that at each time t, the investor must be able to cover his debts -see e.g. Karatzas-Lehoczky-Shreve [1987] or Duffie [1996] where the same assumption is made. It is technically useful as it enables us to apply Fatou's lemma in equation (4.2) and get that the above mentioned process

$$M \triangleq \left\{ \beta_t X_t^{x;\pi,c} + \int_0^t \beta_s c_s ds; t \in \mathcal{T} \right\} \tag{4.4}$$

consisting of current discounted wealth plus total discounted consumption is a $\bar{P}$-supermartingale. We then get the inequality

$$E^{\bar{P}} \left[\beta_t X_t^{x;\pi,c} + \int_0^t \beta_s c_s ds \right] \leq x \tag{4.5}$$

which can be interpreted as a budget constraint: the expected total value of current wealth and consumption-to-date, both deflated to $t = 0$, does not exceed the initial capital.

If we consider an economic agent who only invests in the primitive market, all definitions and interpretations remain the same, provided we adapt them in a natural way to the primitive market: more precisely, a trading strategy is an element $\pi = \left\{ \left(\pi_t^S \right)^* ; t \in \mathcal{T} \right\}$ of $L_1^2 (\mathcal{T})$; a consumption strategy is a nonnegative, progressively measurable, real-valued process $c = \{c_t; t \in \mathcal{T}\}$ that satisfies $\int_0^T c_t dt < \infty \quad a.s.\ P$. The wealth process corresponding to a trading-consumption strategy (π, c) is given for all t in $\mathcal{T}$ by

$$\beta_t X_t^{x;\pi,c} = x - \int_0^t \beta_s c_s ds + \int_0^t \beta_s (\pi_s)^* \sigma_s dW_s^{P^0}. \tag{4.6}$$

Finally, a pair (π, c) of trading and consumption strategies is called admissible for the initial capital $x \geq 0$ if the unique corresponding wealth process $\{X_t^{x;\pi,c}; t \in \mathcal{T}\}$ satisfies $X_t^{x;\pi,c} \geq 0$ for all $t \in \mathcal{T}$.

Notice that both our full market model and our primitive market model exclude arbitrage opportunities; an arbitrage opportunity is an admissible plan that yields through some combination of buying and selling a positive gain in some circumstances without a countervailing threat of loss in other circumstances or equivalently in our setting, a trading strategy that achieves with zero initial capital an amount of terminal wealth which is almost surely nonnegative and positive with positive probability; so here, an arbitrage opportunity consists in a pair (π, c) of portfolio and consumption rate processes such that (π, c) is in $A(0)$ and such that the corresponding wealth process with initial capital $x = 0$ is almost surely nonnull at terminal time. In both cases (full and primitive markets), the existence of at least one equivalent martingale probability measure (for $\tilde{Z}$ in the first case and for $\tilde{S}$ in the second case) rules out such opportunities: indeed, in both cases, as we have seen with equation (4.5), the wealth at initial time is greater than the expected value of the discounted wealth at terminal time: the discount process being positive and the wealth process being nonnegative, it is then impossible, starting from the initial capital $x = 0$ to reach a nonnull wealth at terminal time.

4.2. Achievable consumption and wealth processes

For every given real number $x \geq 0$, denote by $\mathbf{C}(x)$ the class of consumption rate processes c which satisfy $E^{\bar{P}}\left[\int_0^T \beta_s c_s ds\right] \leq x$ and by $\mathbf{L}(x)$ the class of nonnegative, F-measurable random variables B which satisfy $E^{\bar{P}}[\beta_T B] \leq x$.

We have just seen with equation (4.5) that $(\pi, c) \in A(x)$ implies that $c \in C(x)$ and $X_T \in L(x)$. We shall now study to which extent the "opposite implications" are true, i.e., for every c in $C(x)$, does there exist a trading strategy π such that $(\pi, c) \in A(x)$; for every B in $L(x)$, does there exist (π, c) in $A(x)$ such that $X_T^{x;\pi,c} = B$ and for every pair (c, B) in $C(x) \times L(x)$ satisfying

$$E^{\bar{P}}\left[\beta_T B + \int_0^t \beta_s c_s ds\right] \leq x,$$

does there exist a trading strategy π such that $(\pi, c) \in A(x)$ and $X_T^{x;\pi,c} = B$.

Achievable consumption processes

Given an initial wealth $x > 0$, we want to know which consumption processes an investor can achieve and we shall give a positive answer to the first question just raised.

Proposition 4.4. *1. For every c in $C(x)$, there exists a portfolio process π such that (π, c) belongs to $A(x)$.*

2. For every c in $\mathbf{D}(x) \triangleq \left\{c \in C(x); E^{\bar{P}}\left[\int_0^T \beta_s c_s ds\right] = x\right\}$, the preceding π is unique, the corresponding wealth process satisfies $X_T^{x;\pi,c} = 0$ and the process M given in (4.4) is a $\bar{P}$-martingale.

Proof See Karatzas [1989]: the proof uses a representation result and is analogous to the proof of the completeness of the full market, given in theorem 4.8 below.□

Achievable terminal wealth and completeness issues

We shall see now that the primitive market in the case $d = 1$ as well as the full market enable agents to hedge against all risk.

Definition 4.5. *A contingent claim is a financial instrument consisting of a payment B at maturity, where B is a nonnegative, F_T-measurable random variable satisfying $E[B^\mu] < \infty$ for some $\mu > 1$.*

We shall denote any contingent claim by its payment B. Using the boundedness of the processes $\bar{\theta}$, θ and r as well as Hölder's inequality, it is not hard to see that any contingent claim B satisfies $E^{\bar{P}}\left[B\beta_T\right] < \infty$ as well as $E^{P^0}\left[B\beta_T\right] < \infty$.

Definition 4.6. *The market is complete if, for all contingent claim B, there exist a trading strategy π and an initial capital $x \geq 0$ such that $(\pi, 0)$ is in $A(x)$ and the terminal value of the corresponding wealth process is equal to B, i.e., $X_T^{x;\pi,0} = B$.*

We say that the full market -resp. the primitive market- is complete if the conditions of the definition are satisfied for a trading strategy in the form $\pi = \left(\pi^S, \pi^{C_1}, ..., \pi^{C_{d-1}}\right)$ -resp. in the form $\pi = \left(\pi^S\right)$- and for a wealth process satisfying equation (4.2) -resp. equation (4.6).

We first state a representation result, which is an easy corollary of the fundamental martingales representation theorem:

Lemma 4.7. *Let $Y = \{Y_t; t \in \mathcal{T}\}$ be a $\bar{P}-$martingale for $(F_t)_{t \in \mathcal{T}}$. Then there exists a d-dimensional process Φ in $L_d^2(\mathcal{T})$ such that*

$$Y_t = Y_0 + \int_0^t (\Phi_s)^* \, dW_s^{\bar{P}} \qquad 0 \leq t \leq \mathcal{T}.$$

Proof Apply the martingales representation theorem (see Karatzas and Shreve [1988]) to the process $\bar{M}_t Y_t$, which is a continuous P-martingale for $(F_t)_{t \in \mathcal{T}}$, where $(F_t)_{t \in \mathcal{T}}$ is the P-augmentation of the filtration generated by W and the lemma is obtained through the use of Itô's lemma (see lemma 8.4 in Karatzas-Lehoczy-Shreve [1990] for a detailed proof).□

We can now prove the following

Theorem 4.8. *1. The primitive market is complete if and only if $d = 1$.*

2. The full market is complete.

Proof 2. For each contingent claim B, we consider the process X given for all t in $\mathcal{T}$ by $X_t = \frac{1}{\beta_t} E^{\bar{P}} [\beta_T B \mid F_t]$. The process $\beta X = \{\beta_t X_t; t \in \mathcal{T}\}$ is in a trivial way a $\bar{P}-$martingale for $(F_t)_{t\in\mathcal{T}}$. Using the lemma, we can write βX in the form

$$\beta_t X_t = E^{\bar{P}} [\beta_T B] + \int_0^t (\Phi_s)^* \, dW_s^{\bar{P}}, \quad 0 \leq t \leq T$$

for some d-dimensional process Φ in $L_d^2(\mathcal{T})$. Defining the process π by

$$\pi_t = (1/\beta_t) \left(\bar{\sigma}_t^{-1}\right)^* \Phi_t, \quad 0 \leq t \leq T$$

we get that π is a portfolio process and that

$$\beta_t X_t = E^{\bar{P}} [\beta_T B] + \int_0^t \beta_s (\pi_s)^* \, \bar{\sigma}_s dW_s^{\bar{P}},$$

which shows that X is the wealth process corresponding to the trading strategy $(\pi, 0)$ with initial value $E^{\bar{P}} [\beta_T B]$, i.e., $X = X^{E^{\bar{P}}[\beta_T B];\pi,0}$ -see equation (4.2). The terminal value satisfies $X_T = B$ and as X is nonnegative, the trading strategy $(\pi, 0)$ is admissible.

1. The proof is analogous to the proof of 2. and can be found for instance in Musiela-Rutkowski [1997], p.250. □

Following exactly the same approach and using the same lemma, we can characterize the levels of wealth attainable by an initial capital $x \geq 0$. For every real number $x \geq 0$, we have

Proposition 4.9. *Given an initial wealth $x \geq 0$,*

1. *For every B in $L(x)$, there exists a pair (π, c) in $A(x)$ such that the corresponding wealth process $X^{x;\pi,c}$ satisfies $X_T^{x;\pi,c} = B$ almost surely.*

2. *For any B in $\mathbf{M}(x) \triangleq \left\{B \in L(x); E^{\bar{P}}[\beta_T B] = x\right\}$, the pair (π, c) in $A(x)$ above is unique and $c \equiv 0$; moreover, the corresponding wealth process is given by $\beta_t X_t^{x;\pi,0} = E^{\bar{P}}[\beta_T B \mid F_t]$.*

Proof See Karatzas [1989].

Achievable pairs of terminal wealth and consumption processes

Let $\mathbf{A}$ denote the set of pairs (c, X) where c is an adapted nonnegative consumption rate process and X is a nonnegative F_T-measurable random variable describing terminal wealth; we want to know which pairs (c, X) in A an investor can achieve starting with an initial capital $x > 0$ and following an admissible strategy.

Proposition 4.10. *If a pair (c, X) in A is such that $E^{\bar{P}}\left[\int_0^T \beta_s c_s ds + \beta_T X\right] = x$, then there exists a trading strategy π such that (π, c) belongs to $A(x)$ and $X_T^{x;\pi,c} = X$.*

Proof We consider the following quantities

$$x_1 \triangleq E^{\bar{P}}\left[\int_0^T \beta_t c_t dt\right] \text{ and } x_2 \triangleq x - x_1 = E^{\bar{P}}[\beta_T X],$$

for which it is easy to see that we have $c \in D(x_1)$ and $X \in M(x_2)$.

As c belongs to $D(x_1)$, according to proposition 4.4, there exists a unique trading strategy π_1 such that (π_1, c) is in $A(x_1)$ and the corresponding wealth process satisfies $X_T^{x_1;\pi_1,c} = 0$.

As X belongs to $M(x_2)$, according to proposition 4.9, there exists a unique pair (π_2, c_2) in $A(x_2)$ such that $X_T^{x_2;\pi_2,c_2} = X$ and it satisfies $c_2 \equiv 0$.

We then consider the strategy π given by $\pi \triangleq \pi_1 + \pi_2$ and it is easy to check that (π, c) belongs to $A(x)$ and that

$$X_T^{x;\pi,c} = X_T^{x_1;\pi_1,c} + X_T^{x_2;\pi_2,c_2} = X$$

which completes the proof.

We can sketch the proof of a direct approach, that leads to the same result using the martingales representation theorem: consider the martingale process

$$\left\{ M_t \triangleq E^{\bar{P}} \left[\int_0^T \beta_s c_s ds + \beta_T X \mid F_t \right]; t \in \mathcal{T} \right\}.$$

Using the martingale representation theorem, M_t can be written in the form

$$M_t = E^{\bar{P}} \left[\int_0^T \beta_s c_s ds + \beta_T X \right] + \int_0^t \beta_s \pi_s^* \bar{\sigma}_s dW_s^{\bar{P}}$$

for some portfolio process π. Then, according to equation (4.2), $X_t \triangleq M_t - \int_0^t \beta_s c_s ds = \beta_t X_t^{x;\pi,c}$ and $X_T^{x;\pi,c} = X$.$\Box$

We are now in a position to answer the last question raised at the beginning of the section:

Corollary 4.11. *For any pair (c, X) in A, there exist an initial wealth $x > 0$ and a trading strategy π such that (π, c) belongs to $A(x)$ and $X_T^{x;\pi,c} = X$.*

Proof Immediate using the proof of the preceding theorem and considering $x \triangleq E^{\bar{P}}\left[\int_0^T \beta_s c_s ds + \beta_T X\right]$.□

5. Optimal demand in the full market

We still consider an economic agent, who invests in the so-called full market. We assume that his preferences are represented by a utility function for consumption and terminal wealth. The problem is the following: how should this agent choose at every time his portfolio and his consumption rate processes from among admissible pairs in order to obtain a maximum expected utility from both consumption over the time-interval $\mathcal{T}$ and terminal wealth.

More precisely, the agent has a utility function $U : A \to R$ given by

$$U(c, X) = E\left[\int_0^T u(t, c_t)\, dt + V(X)\right]$$

where

- $V : R_+ \to R$ is strictly increasing and concave;
- $u : \mathcal{T} \times R_+ \to R$ is continuous and, for each t in $\mathcal{T}$, $u(t, \cdot) : R_+ \to R$ is strictly increasing and concave;
- V is strictly concave or, for each t in $\mathcal{T}$, $u(t, \cdot)$ is strictly concave.

We assume that the agent is endowed with an initial capital $x > 0$ and that there is no exogenous endowment during the trading period $\mathcal{T}$. We now have the problem for each initial wealth x,

$$\sup_{(\pi,c)\in A(x)} U\left(c, X_T^{x;\pi,c}\right).$$

Using proposition 4.10 and the strict monotonicity of either or both of V and $\{u(t,\cdot); t \in \mathcal{T}\}$, we get that the agent's optimization problem is equivalent to

$$\sup_{(c,X)\in A} U(c,X)$$

$$\text{subject to } E^{\bar{P}}\left[\int_0^T \beta_t c_t dt + \beta_T X\right] \le x.$$

Proposition 5.1. *A pair (c^*, X^*) in A is optimal for the agent if and only if*

$$E\left[\int_0^T \beta_t M_t c_t^* dt + \beta_T M_T X^*\right] = x$$

and there is a constant $\gamma^* > 0$ such that (c^*, X^*) solves

$$\sup_{(c,X)\in A} E\left[\int_0^T u(t,c_t) - \gamma^* \beta_t M_t c_t dt + V(X) - \gamma^* \beta_T M_T X\right]. \tag{5.1}$$

Proof By the Saddle Point Theorem (see Duffie [1996], p.276) and the strict monotonicity of U, $(c^*, X^*) \in A$ solves our problem if and only if there is a Lagrange multiplier $\gamma > 0$ such that (c^*, X^*) solves the unconstrained problem

$$\sup_{(c,X)\in A} U(c,X) - \gamma E^{\bar{P}}\left[\int_0^T \beta_t c_t dt + \beta_T X - x\right]$$

with the complementary slackness condition

$$E^{\bar{P}}\left[\int_0^T \beta_t c_t dt + \beta_T X\right] = x.$$

Then, by Fubini's theorem, the fact that c is an adapted process, the law of iterated expectations and the fact that the process $\left\{\bar{M}_t; t \in \mathcal{T}\right\}$ is a martingale, we get that

$$\begin{aligned} E^{\bar{P}}\left[\int_0^T \beta_t c_t dt + \beta_T X\right] &= E\left[M_T\left(\int_0^T \beta_t c_t dt + \beta_T X\right)\right] \\ &= E\left[\int_0^T M_T \beta_t c_t dt + M_T \beta_T X\right] \\ &= E\left[\int_0^T E_t\left[M_T\right] \beta_t c_t dt + M_T \beta_T X\right] \\ &= E\left[\int_0^T M_t \beta_t c_t dt + M_T \beta_T X\right] \end{aligned}$$

which completes the proof.□

We can be a little more systematic about the properties of U and V in order to characterize optimal pairs. A strictly concave increasing function $F : R^+ \to R$ that is C^1 on $(0, \infty)$ satisfies Inada conditions if $\inf_x F'(x) = 0$ and $\sup_x F'(x) = +\infty$. If F satisfies these Inada conditions, then the inverse I_F of F' is well defined as a strictly decreasing continuous function on $(0, \infty)$ whose image is $(0, \infty)$.

Assumption A: *The function V is C^1 on $(0, \infty)$, strictly concave and satisfies Inada conditions. For all $t \in \mathcal{T}, u(t, \cdot)$ is C^1 on $(0, \infty)$, strictly concave and satisfies Inada conditions.*

Under *Assumption A*, we shall denote by $u_c(t, \cdot)$ the derivative of $u(t, \cdot)$ and by $I_u(t, \cdot)$ the inverse function of $u_c(t, \cdot)$. Then

Proposition 5.2. *Under Assumption A, a pair (c^*, X^*) in A is optimal for the agent if and* *only if there exists a constant* $\gamma^* > 0$ *such that*

$$\beta_t \bar{M}_t = \gamma^* u_c(t, c_t^*) \qquad 0 \le t \le T \quad a.s.\ P \tag{5.2}$$

$$\beta_T \bar{M}_T = \gamma^* V'(X^*) \tag{5.3}$$

$$E\left[\int_0^T \beta_t \bar{M}_t c_t^* dt + \beta_T \bar{M}_T X^*\right] = x. \tag{5.4}$$

Proof As conditions (5.2) and (5.3) are equivalent to

$$c_t^* = I_u\left(t, \gamma^* \beta_t \bar{M}_t\right) \ \forall t \in \mathcal{T} \text{ and } X^* = I_V\left(\gamma^* \beta_T \bar{M}_T\right), \tag{5.5}$$

we only need to check that the solution of the optimization problem (5.1) is given by (c^*, X^*) like in (5.5). We easily get from elementary calculus that for all t in $\mathcal{T}$,

$$\min_{c \ge 0}\left[cy - u(t, c)\right] = y I_u(t, y) - u(t, I_u(t, y)) \qquad \text{for all } y \text{ in } (0, \infty)$$

and that for all y in $(0, \infty)$, the minimum is uniquely attained at $I_u(t, y)$, so that

$$u(t, I_u(t, y)) \ge u(t, c) + y\left[I_u(t, y) - c\right] \qquad \text{for all } c \ge 0 \text{ and all } y \text{ in } (0, \infty),$$

the inequality being strict for $c \ne I_u(t, y)$. Then

$$u(t, c_t^*) \ge u(t, c_t) + \gamma^* \beta_t \bar{M}_t \left[I_u\left(t, \gamma^* \beta_t \bar{M}_t\right) - c_t\right] \text{ and}$$

$$E\left[\int_0^T u(t,c_t^*) - \gamma^* \beta_t \bar{M}_t c_t^* dt\right] \geq E\left[\int_0^T u(t,c_t) - \gamma^* \beta_t \bar{M}_t c_t dt\right]$$

the inequality being strict for $c \neq c^*$, which proves our proposition.□

Now that we have characterized the optimal pairs $(c^*, X^*) \in A$, we can turn to multi-agent equilibrium considerations.

6. Equilibrium and compatible state price densities

We have so far considered a single economic agent, trading in the full market. We shall now assume that our economy consists of a finite number n of agents, who all have utility functions $U_j : A \to R$ given by

$$U_j(c,X) = E\left[\int_0^T u_j(t,c_t)\,dt + V_j(X)\right] \quad \text{for } j = 1, ..., n$$

where u_j and V_j satisfy the same conditions as u and V at the beginning of section 5. Each agent j has an initial endowment x_j and tries to maximize his utility $U_j(c,X)$ from both consumption over the time-interval $\mathcal{T}$ and terminal wealth. So the optimal demand $(c_j)^*$ of each agent j in the consumption commodity as well as his optimal portfolio choice $(\pi_j)^*$ are determined by the optimization problem studied in the preceding section

$$\sup_{(\pi,c)\in A(x_j)} U_j\left(c, X_T^{x_j;\pi,c}\right).$$

Besides, the total supply in the economy at time t consists of one unit of the productive asset S_t and of the dividend paid by the stock D_t. In equilibrium, the aggregated optimal demands of the agents must equal the total supply available.

More precisely, an equilibrium consists in price processes $S^0, S, C^1, ..., C^{d-1}$ and trading-consumption choices $\left((\pi_j^*)^S, \left[(\pi_j^*)^{C^i}\right]_{i \leq d-1}; c_j^*\right)_{1 \leq j \leq n}$ which are optimal for the agents and such that for all t in $\mathcal{T}$, the following market clearing conditions hold almost surely:

$$\sum_{j=1}^{n} \left(c_j^*\right)_t = D_t$$

$$\sum_{j=1}^{n} (\pi_j^*)_t^S = S_t$$

$$\sum_{j=1}^{n} (\pi_j^*)_t^{C^i} = 0 \qquad 1 \leq i \leq (d-1)$$

$$\sum_{j=1}^{n} X_t^{\pi_j^*, c_j^*} = S_t$$

where the last relation follows from the equilibrium condition on the amount invested in the bond: $\sum_j (\pi_j^*)_t^{S^0} = \sum_j X_t^{\pi_j^*, c_j^*} - \sum_j (\pi_j^*)_t^S - \sum_j \sum_{i=1}^{d-1} (\pi_j^*)_t^{C^i} = 0.$

As described in the introduction, our problem consists in finding a fair price for contingent claims that only uses information on the productive asset's price process.

As implied by the next lemma and mentioned in the introduction, by only using the assumption of no arbitrage, our problem is solved for any contingent claim in the case $d = 1$ and for contingent claims B, which are redundant with respect to the primitive market in the case

$d > 1$.

Definition 6.1. *We say that a contingent claim B is redundant -with respect to the primitive market- if there exist a nonnegative initial capital x and an admissible trading-consumption strategy $(\pi, 0)$ in $A(x)$ such that the corresponding discounted wealth process is a P^0-martingale and has a terminal value equal to the discounted contingent claim, i.e., $X_T^{x;\pi,0} = B$.*

Notice that if the trading strategy π is in the form $\pi = \left(\pi^S, 0, ..., 0\right)$ then the corresponding wealth process is in the form

$$dX_t^{x;\pi,0} = [r_t X_t]\, dt + \left(\pi_t^S\right)^* \sigma_t dW_t^{P^0}$$

so that the condition $E\left[\int_0^T \left\|\pi_s^S\right\|^2 ds\right] < \infty$ ensures that $\beta X^{x;\pi,0}$ is a P^0-martingale.

Lemma 6.2. *Let B be a given contingent claim.*

1. *If $d = 1$, then the unique fair price for B is equal to $E^{P^0}[\beta_T B]$.*

2. *If $d > 1$ and if B is redundant then its unique fair price is also equal to $E^{P^0}[\beta_T B]$.*

Proof 1. As any contingent claim B belongs to $M\left(E^{P^0}[\beta_T B]\right)$, proposition 4.9 tells us that in the case $d = 1$, all contingent claims are redundant: for any contingent claim B, there exists a trading strategy π such that $(\pi, 0)$ is in $A\left(E^{P^0}[\beta_T B]\right)$, $\beta X^{E^{P^0}[\beta_T B];\pi,0}$ is a martingale and $X_T^{E^{P^0}[\beta_T B];\pi,0} = B$. Then, by absence of arbitrage opportunity, the price for B at time 0 is necessarily equal to $E^{P^0}[\beta_T B]$.

2. As $B = X_T^{x;\pi,0}$ for some x in $(0, \infty)$ and some pair $(\pi, 0)$ in $A(x)$, by absence of arbitrage opportunity, the contingent claim B must have a price equal to x, so we only need to compute

the value of x. As the corresponding discounted wealth process $\beta X^{x;\pi,0}$ is a P^0-martingale, we have

$$x = X_0^{x;\pi,0} = E^{P^0}\left[\beta_T X_T^{x;\pi,0}\right] = E^{P^0}\left[\beta_T B\right],$$

which is the result announced.□

Notice that P^0 only depends on the productive asset's price process so that in both cases the unique fair price for any contingent claim B is perfectly determined without any knowledge about the financial assets price processes.

Assume now that the contingent claim B is nonredundant (with respect to the primitive market); following the same approach as above in the case $d = 1$, its unique fair price is $E^{\bar{P}}[\beta_T B]$; the problem is that, as we have noticed at the end of section 3.2, we need to know all the additional purely financial assets price processes in order to compute this price, which is not supposed to be the case here. As the equivalent probability measure $\bar{P}$ belongs to the set $\mathcal{M}_S$ of all S-equivalent martingale measures, we know that this fair price lies in the interval consisting of the expected values of the discounted contingent claim with respect to all S-equivalent martingale measures. But this interval has been shown to be too large (Cvitanic-Pham-Touzi [1997]). Our purpose here is to find prices or equivalently S-equivalent martingale measures that are compatible with what we have called equilibrium and to restrict this way the fair pricing interval. In the remainder of the paper, a fair price will denote a price that is compatible with both equilibrium and the assumption of no arbitrage.

6.1. A necessary condition for equilibrium

We have seen in section 3 that as $\bar{P}$ belongs to $\mathcal{M}_S$, we can write it in the form $P^{\bar{\nu}}$ for some $\bar{\nu}$ in K^{σ}. We shall here suppose that there is an equilibrium and, in order to grab more information on $\bar{\nu}$, deduce necessary conditions that the process $\{M_t^{\bar{\nu}}; t \in \mathcal{T}\}$ must satisfy. We emphasize the fact that we are in this section only interested in necessary conditions for equilibrium.
We assume that the agents' utility functions satisfy Inada conditions or more precisely *Assumption A* of section 5.

As there is an equilibrium, each agent j must achieve an optimal consumption rate process c_j^* as well as an optimal terminal wealth X_j^*. According to section 5, this implies that for all $j = 1, ...n$, there exists a positive constant $\gamma_j^* > 0$ such that

$$\beta_t M_t^{\bar{\nu}} = \gamma_j^* \left(u_j\right)_c \left[t, \left(c_j^*\right)_t\right] \qquad 0 \leq t \leq T \quad a.s.\ P$$

or

$$\left(c_j^*\right)_t = I_{u_j}\left(t, \frac{1}{\gamma_j^*}\beta_t M_t^{\bar{\nu}}\right) \qquad 0 \leq t \leq T \quad a.s.\ P.$$

On the other hand, as there is an equilibrium, markets must clear. As we have seen at the beginning of this section, this implies that

$$\sum_{j=1}^{n} \left(c_j^*\right)_t = D_t.$$

So, with the notations introduced in the preceding sections, we get the following

Lemma 6.3. *A necessary condition for an equilibrium to be reached in our model is that there exist positive constants γ_j^*, $j = 1, ...n$, such that*

$$\sum_{j=1}^{n} I_{u_j}\left(t, \frac{1}{\gamma_j^*}\beta_t M_t^{\bar{\nu}}\right) = D_t.$$

Proof Immediate.□

6.2. Assuming that the utility functions are "regular"

We shall now assume that for each j, the utility function u_j satisfies certain regularity conditions, to wit,

$$I_{u_j} : \mathcal{T} \times R_+^* \to R_+$$

is of class $C^{1,2}$. We also assume that the coefficients of the primitive risky asset are such that for all t in $\mathcal{T}$, $b_t \neq r_t$.

We show that the compatibility with equilibrium enables us to price the contingent claims in a unique way, only using information on the primitive assets.

Let $\varphi(t,x) \triangleq \sum_{j=1}^{n} I_{u_j}\left(t, \frac{1}{\gamma_j^*}x\right)$. The regularity assumptions made on all utility functions imply that φ is of class $C^{1,2}$ on $R_+ \times R_+^*$ and enable us to apply Itô's lemma to the process $\Phi = \{\varphi(t, \beta_t M_t^{\bar{\nu}}); t \in \mathcal{T}\}$ and get that for all t in $\mathcal{T}$

$$d\Phi_t = a_t dt - \varphi_x\left(t, \beta_t M_t^{\bar{\nu}}\right)\beta_t M_t^{\bar{\nu}}\left(\theta_t^{\bar{\nu}}\right)^* dW_t$$

for some progressively measurable process $a = \{a_t; t \in \mathcal{T}\}$. Using lemma 6.3, if equilibrium is reached in our model, the dividend process $\{D_t; t \in \mathcal{T}\}$ must follow a diffusion process given by

$$dD_t = b_t^D dt + \sigma_t^D dW_t \qquad \text{for all } t \text{ in } \mathcal{T}$$

where

$$\sigma_t^D \triangleq -\varphi_x\left(t, \beta_t M_t^{\bar{\nu}}\right) \beta_t M_t^{\bar{\nu}} \left(\theta_t^{\bar{\nu}}\right)^* .$$

As for all j and for all t, $I_{u_j}(t, \cdot)$ is assumed to be strictly decreasing, $\varphi_x(t, \cdot)$ is negative; for all $t \in \mathcal{T}$, the random variable $\beta_t M_t^{\bar{\nu}}$ is positive; this implies that there exists a measurable positive process λ such that for all t in $\mathcal{T}$,

$$\theta_t^{\bar{\nu}} = \lambda_t \left(\sigma_t^D\right)^* \text{ or}$$

$$\theta_t + \bar{\nu}_t = \lambda_t (\sigma_t^D)^* \quad a.s.\ P.$$

Notice that this implies that, for all i in $\{1, ..., d\}$,

$$(\theta^{\bar{\nu}})_t^i = \lambda_t (\sigma^D)_t^i \quad a.s.\ P,$$

so that the coefficients $(\theta^{\bar{\nu}})_t^i$ and $(\sigma^D)_t^i$ are of the same sign.

The process $\bar{\nu}$ we are looking for satisfies

$$\sigma_t \bar{\nu}_t = 0 \text{ for all } t \text{ in } \mathcal{T}$$

in order to belong to K^σ. We must then have

$$\sigma_t \theta_t = \lambda_t \sigma_t (\sigma_t^D)^* . \tag{6.2}$$

For all t in $\mathcal{T}$, we have assumed that $b_t - r_t \neq 0$, so that $\sigma_t(\sigma_t^D)^* \neq 0$ and we get

$$\lambda_t = \frac{(b_t - r_t)}{\sigma_t(\sigma_t^D)^*}.$$

Consequently, $\bar{\nu}_t$ must be equal to

$$\hat{\nu}_t \triangleq \frac{(b_t - r_t)}{\sigma_t(\sigma_t^D)^*}(\sigma_t^D)^* - \theta_t$$

The only martingale measure to be compatible with equilibrium is then $P^{\hat{\nu}}$. The problem of fair pricing of nonredundant contingent claims is then reduced to taking the expected value with respect to $P^{\hat{\nu}}$, which only involves the productive asset and its dividends price processes: our problem is solved.

Notice that, according to relation (6.2), the condition we have imposed on b_t and r_t is equivalent to the condition that for all t, $\sigma_t(\sigma_t^D)^* \neq 0$ which amounts to saying that the price process S and its associated dividend process are in a way correlated, which seems reasonable.

We have then two ways to define a valuation operator. The first one is based on the arbitrage approach and leads to a unique martingale measure defined by

$$\bar{\nu}_t \stackrel{\triangle}{=} (\bar{\sigma}_t)^{-1}\left[\left(\bar{b}_t - r_t 1_d\right)\right] - \theta_t \quad a.s. \quad P, \quad 0 \leq t \leq T.$$

It only depends on the asset price processes and does not depend on the dividend process. The second one is based on the equilibrium approach and leads to

$$\hat{\nu}_t \triangleq \frac{(b_t - r_t)}{\sigma_t(\sigma_t^D)^*}(\sigma_t^D)^* - \theta_t.$$

It only depends on the productive asset price and dividend processes and does not involve the purely financial assets price processes. We have then the following :

Corollary 6.4. *At the equilibrium we must have $\bar{\nu} = \hat{\nu}$ or in other words*

$$(\bar{\sigma}_t)^{-1}\left[\left(\bar{b}_t - r_t 1_d\right)\right] = \frac{(b_t - r_t)}{\sigma_t(\sigma_t^D)^*}(\sigma_t^D)^* \quad a.s. \quad P, \quad 0 \le t \le T.$$

In particular, recall that the minimal martingale-measure also called Föllmer and Schweitzer [1991] martingale-measure does not take into account "orthogonal" risks and coincides with our martingale-measure P^0. We have then the following :

Corollary 6.5. *The Föllmer and Schweitzer [1991] martingale-measure is compatible with an equilibrium in the model $(S_0, S, (C^i), D)$ only if $\hat{\nu} = 0$ or*

$$\frac{(b_t - r_t)}{\sigma_t(\sigma_t^D)^*}(\sigma_t^D)^* = \frac{(b_t - r_t)}{\sigma_t \sigma_t^*}\sigma_t^* \quad a.s. \quad P, \quad 0 \le t \le T$$

or finally, if there exists a process λ such that $\sigma = \lambda \sigma^D$.

It means that the Föllmer and Schweitzer [1991] martingale-measure is compatible with an equilibrium only if all the stock price risk is explained by the current dividend risk and this is, in particular, the case when the stock price at date t is a regular function of the dividend at the same date, i.e. $S_t = f(t, D_t)$.

Let us now compare our result with the one concerning our problem obtained in Pham-Touzi [1996]. They consider a model in which there is a bond S^0, one risky productive asset, whose price process S is described by a stochastic volatility model, and one contingent claim completing the market. They fix the productive asset price process S and the contingent claim's one and this determines a unique equivalent martingale measure. Then they study the consistency of such a martingale measure with an equilibrium model; they answer the question: do there exist utility functions such that the price process S is an equilibrium price process? If the answer is yes, they say that the martingale measure is viable, and they show that this induces strong constraints on the coefficients of the price diffusion process. More specifically related to our problem, it is shown -see proposition 5.1- that in the positive dividend case, as long as the coefficients of the model respect the above mentioned constraints, the minimal martingale measure of Föllmer and Schweitzer is viable if and only if the dividend process is in the form $D_t = a(t) S_t + b(t)$ for some continuous functions $a(t) > 0$ and $b(t) \geq 0$ and if the utility functions are logarithmic ones which seems to be a very restrictive condition.

We have shown in section 6.2 that in a model which is not necessarily markovian anymore, if the dividend process is a diffusion process then there exists a unique admissible martingale measure. If we use the result of Pham-Touzi [1996], then we obtain that in a stochastic volatility setting, if the dividend process is in the form mentioned above, the minimal martingale measure being viable, it is necessarily the unique viable martingale measure. This is what we find if we adapt our result to their specific setting and if we let the function d be an affine function.

Besides, we have shown that there is also a unique admissible price in any other case.

In fact, in Jouini and Napp [2000], under some mild technical conditions, we prove that our necessary condition is also sufficient and that the Föllmer and Schweitzer martingale-measure is compatible with many other utility functions and for prices that are not affine functions of the dividends. This big difference with Pham and Touzi [1996] results is due to our non markovian setting.

7. Incomplete markets and many productive assets

If we assume now that there is many productive assets denoted by $S^1, \cdots, S^k$ with dividend processes $D^1, \cdots, D^k$, it is easy to check that the previous results still apply:

1. at the equilibrium we must have

$$(\bar{\sigma}_t)^{-1}\left[\left(\bar{b}_t - r_t 1_d\right)\right] = \frac{\left(b_t^h - r_t\right)}{\sigma_t^h (\sigma_t^D)^*}(\sigma_t^D)^* \quad a.s. \quad P, \quad 0 \leq t \leq T, \quad h = 1, \cdots, k.$$

   where $D = D^1 + \cdots + D^k$ and $\sigma_t^D = \sigma_t^{D^1} + \cdots + \sigma_t^{D^k}$.

2. the Föllmer and Schweitzer [1991] martingale-measure is compatible with an equilibrium in the model $(S_{0,}(S^h)_{h=1,\cdots,k}, (C^i)_{i=1,\cdots d-k}, (D^j)_{j=1,\cdots,k})$ only if

$$\frac{\left(b_t^h - r_t\right)}{\sigma_t^h (\sigma_t^D)^*}(\sigma_t^D)^* = \sigma_t^*(\sigma_t \sigma_t^*)^{-1}\left(b_t - r_t 1_k\right) \quad a.s. \quad P, \quad 0 \leq t \leq T, \quad h = 1, \cdots, k$$

   where $\sigma_t = \begin{pmatrix} \sigma_{t,1}^{S^1} & \cdots & \sigma_{t,d}^{S^1} \\ \vdots & & \vdots \\ \sigma_{t,1}^{S^k} & \cdots & \sigma_{t,d}^{S^k} \end{pmatrix}$ and $b_t = \begin{pmatrix} b_t^{S^1} \\ \vdots \\ b_t^{S^k} \end{pmatrix}$.

When the markets are incomplete, we can't assume that there is only one $Z-$equivalent martingale measure anymore. However, as in Lemma 6.2, all the $Z-$equivalent martingale measure coincide on the set of attainable assets (assets that are redundant in the full market).

If the gradient of the utility of agent j at an optimal consumption process c^j exists and has a representation similar to the representation given by equation (6.1), i.e.

$$\beta_t M_t^j = \gamma_j (u_j)_c \left[t, (c_j^*)_t\right] \qquad 0 \leq t \leq T \quad a.s. \ P$$

where M_t^j is a martingale, then it is easy to check that M_t^j is the density process of a $Z-$equivalent martingale measure. Furthermore under the regularity conditions of section 6.2, M_t^j is an Ito process and we have

$$dM_t^j = \zeta_j^t \sigma_c^j dW_t$$

where $\zeta_t^j = \frac{\gamma_j}{\beta_t}(u_j)_{cc}\left[t, (c_j^*)_t\right].$

Let us now consider the process $\hat{M}_t$ defined by

$$d\hat{M}_t = \left(\sum_{j=1}^{n}(\zeta_t^j)^{-1}\right)^{-1} \sum_{j=1}^{n}(\zeta_t^j)^{-1} dM_t^j$$

we have

$$\begin{aligned} d\hat{M}_t &= \left(\sum_{j=1}^{n}(\zeta_t^j)^{-1}\right)^{-1} \sum_{j=1}^{n}\sigma_c^j dW_t \\ &= \left(\sum_{j=1}^{n}(\zeta_t^j)^{-1}\right)^{-1} \sigma_D dW_t \\ &\triangleq \hat{\theta}_t^* dW_t \end{aligned} \tag{7.1}$$

and it is easy to check that $\hat{M}_t$ is the density process of a $Z-$equivalent martingale measure. Therefore, we must have

$$\bar{\sigma}_t \hat{\theta}_t = \left[\left(\bar{b}_t - r_t 1_d\right)\right]. \tag{7.2}$$

Moreover, since $\hat{M}_t$ is the density process of a S^h-equivalent martingale for any h in $\{1, \cdots, k\}$, we must have, using equation 7.1,

$$\hat{\theta}_t = \frac{\left(b_t^h - r_t\right)}{\sigma_t^h (\sigma_t^D)^*} (\sigma_t^D)^*$$

and in particular, for all pair h, ℓ in $\{1, \cdots, k\}$,

$$\frac{\left(b_t^h - r_t\right)}{\sigma_t^h (\sigma_t^D)^*} (\sigma_t^D)^* = \frac{\left(b^\ell - r_t\right)}{\sigma_t^\ell (\sigma_t^D)^*} (\sigma_t^D)^*, \quad a.s. \quad P, \quad 0 \leq t \leq T, \quad h, \ell \in \{1, \cdots, k\}\,. \tag{7.3}$$

Equation 7.2 can then be rewritten as follows

$$\frac{\left(b_t^h - r_t\right)}{\sigma_t^h (\sigma_t^D)^*} \bar{\sigma}_t \left(\sigma_t^D\right)^* = \bar{b}_t - r_t 1_d, \quad a.s. \quad P, \quad 0 \leq t \leq T, \quad h = 1, \cdots, k. \tag{7.4}$$

Equations 7.3 and 7.4 are then necessary conditions for $(S_{0,}(S^h), (C^i), (D^h))$ to be an equilibrium price system.

Note that these conditions are only necessary ones and as underlined by Duffie [1996], there is as yet no set of conditions that is sufficient for the existence of equilibrium in such an incomplete market framework except in trivial or simple parametric examples.

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