On complexity and approximability of the Labeled maximum/perfect matching problems

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Abstract. In this paper, we deal with both the complexity and the approximability of the labeled perfect matching problem in bipartite graphs. Given a simple graph $G = (V, E)$ with n vertices with a color (or label) function $L : E \to \{c_1, \ldots, c_q\}$, the labeled maximum matching problem consists in finding a maximum matching on G that uses a minimum or a maximum number of colors.

Keywords: labeled matching; colored matching; bipartite graphs; **NP**-complete; approximate algorithms.

1 Introduction

The maximum matching problem is one of the most known combinatorial optimization problem and arises in several applications such as images analysis, artificial intelligence or scheduling. A matching M on a graph $G = (V, E)$ on n vertices is a subset of edges that are pairwise non adjacent; M is said maximum if the size of the matching is maximum among the matchings of G and perfect if it covers the vertex set of G (that is $|M| = \frac{n}{2}$). In the labeled maximum matching problem (Labeled MM in short), we are given a simple graph $G = (V, E)$ on $|V| = n$ vertices with a color (or label) function $L : E \to \{c_1, \ldots, c_q\}$ on the edge set of G. For $i = 1, \ldots, q$, we denote by $L^{-1}(\{c_i\}) \subseteq E$ the set of edges of color c_i. The goal of Labeled $Min\ MM$ (resp., $Max\ MM$) is to find a maximum matching on G using a minimum (resp., a maximum) number of colors. An equivalent formulation of Labeled $Min\ MM$ could be the following: if $G[\mathcal{C}]$ and m^* denote the subgraph induced by the edges of colors from $\mathcal{C} \subseteq \{c_1, \ldots, c_q\}$ and the size of the maximum matching of G respectively, then Labeled $Min\ MM$ aims at finding a subset $\mathcal{C}$ of minimum size such that $G[\mathcal{C}]$ contains a matching of size m^*. The restriction of Labeled MM to the case where each color occurs at most r times in $I = (G, L)$ (i.e., $|L^{-1}(\{c_i\})| \le r$ for $i = 1, \ldots, q$) will be denoted by Labeled MM_r. For the particular case where we deal with perfect matchings instead of maximum matchings, the labeled maximum matching problem is called the labeled perfect matching problem and denoted by Labeled PM.

The Labeled $Min\ MM$ problem has some relationship with the timetable problem, since a solution may be seen as a matching between classes and teachers that satisfies additional restrictions (for instance, a color corresponds to a school where we assume that a professor may teach in several schools). An inspector would like to assess all teachers during one lecture of each one of them and it would be desirable that (s)he visits not twice the same class. Hence the lectures to be attended would form a maximum matching. For convenience the inspector would like these lectures to take place in the smallest possible number of schools. Then clearly the inspector has to construct a maximum matching meeting a minimum number of colors in the graph associated with the lectures.

2 Previous related works and generalization

Labeled problems have been mainly studied, from a complexity and an approximability point of view, when Π is polynomial, [7–9, 12, 19, 23, 24]. For example, the first labeled problem introduced in the literature is the LABELED minimum spanning tree problem, which has several applications in communication network design. This problem is **NP**-hard and many complexity and approximability results have been proposed in [7, 9, 12, 19, 23, 24]. On the other hand, the LABELED maximum spanning tree problem has been shown polynomial in [7]. More recently, the LABELED path and the LABELED cycle problems have been studied in [8]; in particular, the authors proved that the LABELED minimum path problem is **NP**-hard and provided some exact and approximation algorithms. Note that the **NP**-completeness also appears in [11] since the LABELED minimum path problem is a special case of the red-blue set cover problem. To our knowledge, the LABELED minimum (or maximum) matching problem has not been studied yet in the literature. However, the restricted perfect matching problem [17] is very closed to the LABELED perfect matching. This latter problem aims at determining, given a graph $G = (V, E)$, a partition $E_1, \ldots, E_k$ of E and k positive integers $r_1, \ldots, r_k$, whether there exists a perfect matching M on G satisfying for all $j = 1, \ldots, k$ the restriction $|M \cap E_j| \leq r_j$. The restricted perfect matching problem is proved to be **NP**-complete in [17], even if (i) $|E_j| \leq 2$, (ii) $r_j = 1$, and (iii) G is a bipartite graph. On the other hand, it is shown in [25] that the restricted perfect matching problem is polynomial when G is a complete bipartite graph and $k = 2$; some others results of this problem can be found in [13]. A perfect matching M only verifying condition (ii) (that is to say $|M \cap E_i| \leq 1$) is called good in [10]. Thus, we deduce that the LABELED maximum perfect matching problem is **NP**-hard in bipartite graph since the value of an optimal solution $opt(I) = \frac{n}{2}$ iff G contains a good matching.

Most of the labelled problems can be embedded in the following framework. Let Π be a **NPO** problem accepting simple graphs $G = (V, E)$ as instances, edge-subsets $E' \subseteq E$ verifying a given polynomial-time decidable property $Pred$ as solutions, and the solutions cardinality as objective function; the labeled problem associated to Π, denoted by LABELED Π, seeks, given an instance $I = (G, L)$ where $G = (V, E)$ is a simple graph and L is a mapping from E to $\{c_1, \ldots, c_q\}$, in finding a subset E' verifying $Pred$ that optimizes the size of the set $L(E') = \{L(e) : e \in E'\}$. Note that two versions of LABELED Π may be considered according to the optimization goal: LABELED $Min\ \Pi$ that consists of minimizing $|L(E')|$ and LABELED $Max\ \Pi$ that consists of maximizing $|L(E')|$. Roughly speaking, the mapping L corresponds to assigning a color (or a label) to each edge and the goal of LABELED $Min\ \Pi$ (resp., $Max\ \Pi$) is to find an edge subset using the fewest (resp., the most) number of colors. If a given **NPO** problem Π is **NP**-hard, then the associated labeled problem LABELED Π is clearly **NP**-hard (consider a distinct color per edge). For instance, the LABELED Longest path problem or the LABELED maximum induced matching problem are both **NP**-hard. Moreover, if the decision problem associated to Π is **NP**-complete, (the decision problem aims at deciding if a graph G contains an edge subset verifying $Pred$), then LABELED $Min\ \Pi$ can not be approximated within performance ratio better than $2-\varepsilon$ for all $\varepsilon > 0$ unless **P**=**NP**, even if the graph is complete. Indeed, if we color the edges from $G = (V, E)$ with a single color and then we complete the graph, adding a new color per edge, then it is **NP**-complete to decide between $opt(I) = 1$ and $opt(I) \geq 2$, where $opt(I)$ is the value of an optimal solution. Notably, it is the case of the LABELED traveling salesman problem (LABELED TSP in short) or the LABELED minimum partition problem into paths of length k for any $k \geq 2$. Note that the problem consisting in deciding whether $opt(I) = n$ in colored complete graphs K_n for LABELED $Max\ TSP$, has been studied. For instance in [1, 14, 16], some conditions (mainly using probabilistic methods) were mentioned for a colored complete graph K_n to contain a hamiltonian cycle using n colors.

In this paper, we go into the investigation of the complexity and the approximability of labeled matching problems in bipartite graphs. More precisely, we deal with 2 extreme classes of 2-regular or $\frac{n}{2}$-regular bipartite graphs. For both cases, we obtain hardness results. For these graphs, observe that a maximum matching is a perfect matching; thus, in these graphs LABELED MM and LABELED PM are the same problem. In section 3, we analyze both the complexity and the approximability of the LABELED minimum perfect matching problem and the LABELED maximum perfect matching problem in 2-regular bipartite graphs. Finally, section 4 focuses on the case of complete bipartite graphs $K_{n,n}$.

Now, we introduce some terminology and notations that will be used in the paper. We denote by $opt(I)$ and $apx(I)$ the value of an optimal and an approximate solution, respectively. We say that an algorithm $\mathcal{A}$ is an ε-approximation of LABELED $Min\ PM$ with $\varepsilon \geq 1$ (resp., $Max\ PM$ with $\varepsilon \leq 1$) if $apx(I) \leq \varepsilon \times opt(I)$ (resp., $apx(I) \geq \varepsilon \times opt(I)$) for any instance $I = (G, L)$, for more details see for instance [4].

3 The 2-regular bipartite case

In this section, we deal with a particular class of graphs that consist in a collection of pairwise disjoints cycles of even length; note that such graphs are 2-regular bipartite graphs.

Theorem 1. *LABELED $Min\ PM_r$ is **APX**-complete in 2-regular bipartite graphs for any $r \geq 2$.*

Proof. Observe that any solution of LABELED $Min\ PM_r$ is an r-approximation. The rest of the proof will be done via an approximation preserving reduction from the minimum balanced satisfiability problem with clauses of size at most r, MIN BALANCED r-SAT for short. An instance $I = (\mathcal{C}, X)$ of MIN BALANCED r-SAT consists of a collection $\mathcal{C} = (C_1, \ldots, C_m)$ of clauses over the set $X = \{x_1, \ldots, x_n\}$ of boolean variables, such that each clause C_j has at most r literals and each variable appears positively as many times as negatively; let B_i denote this number for any $i = 1, \ldots, n$. The goal is to find a truth assignment f satisfying a minimum number of clauses. MIN BALANCED 2-SAT where $2 \leq B_i \leq 3$ has been shown **APX**-complete by the way of an L-reduction from MAX BALANCED 2-SAT where $B_i = 3$, [6, 18].

We only prove the case $r = 2$. Let $I = (\mathcal{C}, X)$ be an instance of MIN BALANCED 2-SAT on m clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$ and n variables $X = \{x_1, \ldots, x_n\}$ such that each variable x_i has either 2 occurrences positive and 2 occurrences negative, or 3 occurrences positive and 3 occurrences negative. We build the instance $I' = (H, L)$ of LABELED $Min\ PM_2$ where H is a collection of pairwise disjoints cycles $\{H(x_1), \ldots, H(x_n)\}$ and L colors edges of H with colors $c_1, \ldots, c_j, \ldots, c_m$, by applying the following process:

- For each variable x_i, create $2B_i$-long cycle $H(x_i) = \{e_{i,1}, \ldots, e_{i,k}, \ldots, e_{i,2B_i}\}$.
- Color the edges of $H(x_i)$ as follows: if x_i appears positively in clauses $C_{j_1}, \ldots, C_{j_{B_i}}$ and negatively in clauses $C_{j'_1}, \ldots, C_{j'_{B_i}}$, then set $L(e_{i,2k}) = c_{j_k}$ and $L(e_{i,2k-1}) = c_{j'_k}$ for $k = 1, \ldots, B_i$.

Clearly, H is made of n disjoint cycles and is painted with m colors. Moreover, each color appears at most twice.

Let f^* be an optimal truth assignment on I satisfying m^* clauses and consider the perfect matching $M = \cup_{i=1}^{n} M_i$ where $M_i = \{e_{i,2k} | k = 1, \ldots, B_i\}$ if $f(x_i) = true$, $M_i = \{e_{i,2k-1} | k = 1, \ldots, B_i\}$ otherwise; M uses exactly m^* colors and thus:

$$opt(I') \leq m^* \tag{1}$$

Conversely, let M' be a perfect matching on H using $apx(I') = m'$ colors; if one sets $f'(x_i) = true$ if $e_{i,2} \in M'$, $f'(x_i) = false$ otherwise, we can easily observe that the truth assignment f' satisfies m' clauses.

$$apx(I) = m' \tag{2}$$

Hence, using inequalities (1) and (2) the result follows.

Trivially, the problem becomes obvious when each color is used exactly once. We now show that we have a 2-approximation in 2-regular bipartite graphs, showing that the restriction of LABELED $Min\ PM$ to 2-regular bipartite graphs is as hard as approximate as MINSAT.

Theorem 2. *There exists an approximation preserving reduction from* LABELED *$Min\ PM$ in 2-regular bipartite graphs to* MINSAT *of expansion $c(\varepsilon) = \varepsilon$.*

Proof. The result comes from the reciprocal of the previous transformation. Let $I = (G, L)$ be an instance of LABELED *Min PM* where $G = (V, E)$ is a collection $\{H_1, \ldots, H_n\}$ of disjoint cycles of even length and $L(E) = \{c_1, \ldots, c_m\}$ defines the label set, we describe every cycle H_i as the union of two matchings M_i and $\overline{M_i}$. We construct an instance $I' = (\mathcal{C}, X)$ of MINSAT where $\mathcal{C} = \{C_1, \ldots, C_m\}$ is a set of m clauses and $X = \{x_1, \ldots, x_n\}$ is a set of n variables, as follows. The clause set $\mathcal{C}$ is in one to one correspondence with the color set $L(E)$ and the variable set X is in one to one correspondence with the connected components of G; a literal x_i (resp., $\overline{x_i}$) appears in C_j iff $c_j \in L(M_i)$ (resp., $c_j \in L(\overline{M_i})$). We easily deduce that any truth assignment f on I' that satisfies k clauses can be converted into a perfect matching M_f on I that uses k colors.

Using the 2-approximation of MINSAT [20] and the Theorem 2, we deduce:

Corollary 1. LABELED *Min PM in 2-regular bipartite graphs is 2-approximable.*

Dealing with LABELED *Max* PM_r, the result of [17] shows that computing a *good matching* is **NP**-hard even if the graph is bipartite and each color appears at most twice; a good matching M is a perfect matching using $|M|$ colors. Thus, we deduce from this result that LABELED *Max* PM_r is **NP**-hard for any $r \geq 2$. We strengthen this result using a reduction from MAX BALANCED 2-SAT.

Theorem 3. LABELED *Max* PM_r *is* ***APX****-complete in 2-regular bipartite graphs for any* $r \geq 2$ *.*

In the same way, there exists an approximation preserving reduction from LABELED *Max PM* in 2-regular bipartite graphs to MAXSAT of expansion $c(\varepsilon) = \varepsilon$. Thus, using the approximate result for MAXSAT [3], we obtain

Corollary 2. LABELED *Max PM in 2-regular bipartite graphs is 0.7846-approximable.*

4 The complete bipartite case

When considering complete bipartite graphs, we obtain several results:

Theorem 4. LABELED *Min* PM_r *is* ***APX****-complete in bipartite complete graphs* $K_{n,n}$ *for any* $r \geq 6$.

Proof. We give an approximation preserving L-reduction (cf. Papadimitriou & Yannakakis [21]) from the restriction MINSC_3 of the set cover problem, MINSC for short. Given a family $\mathcal{S} = \{S_1, \ldots, S_{n_0}\}$ of subsets and a ground set $X = \{x_1, \ldots, x_{m_0}\}$ (we assume that $\cup_{i=1}^{n_0} S_i = X$), a set cover of X is a sub-family $\mathcal{S}' = \{S_{f(1)}, \ldots, S_{f(p)}\} \subseteq \mathcal{S}$ such that $\cup_{i=1}^{p} S_{f(i)} = X$; MINSC is the problem of determining a minimum-size set cover $\mathcal{S}^* = \{S_{f^*(1)}, \ldots, S_{f^*(q)}\}$ of X. Its restriction MINSC_3 to instances where each set is of size at most 3 and each element x_j appears in at most 3 and at least 2 different sets has been proved **APX**-complete in [21].

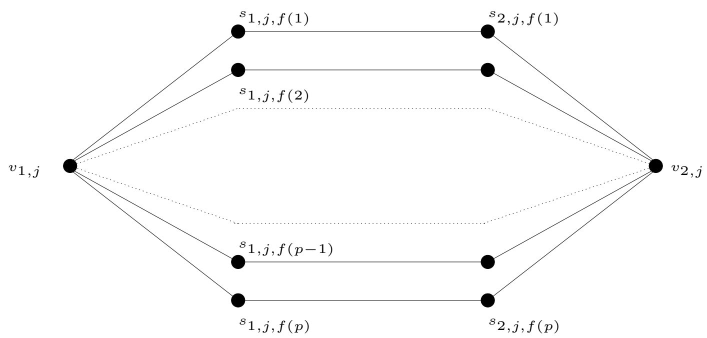

Fig. 1. The gadget $H(x_j)$.

Given an instance $I_0 = (\mathcal{S}, X)$ of MinSC, its characteristic graph $G_{I_0} = (L_0, R_0; E_{I_0})$ is a bipartite graph with a left set $L_0 = \{l_1, \dots, l_{n_0}\}$ that represents the members of the family $\mathcal{S}$ and a right set $R_0 = \{r_1, \dots, r_{m_0}\}$ that represents the elements of the ground set X; the edge-set E_{I_0} of the characteristic graph is defined by $E_{I_0} = \{[l_i, r_j] : x_j \in S_i\}$. Note that G_{I_0} is of maximum degree 3 iff I_0 is an instance of MinSC$_3$. From I_0 an instance of MinSC$_3$, we construct the instance $I = (K_{n,n}, L)$ of Labeled *Min PM*$_6$. First, we start from a bipartite graph having m_0 connected components $H(x_j)$ and $n_0 + m_0$ colors $\{c_1, \dots, c_{n_0+m_0}\}$, described as follows:

- For each element $x_j \in X$, we build a gadget $H(x_j)$ that consists of a bipartite graph of $2(d_{G_{I_0}}(r_j)+1)$ vertices and $3d_{G_{I_0}}(r_j)$ edges, where $d_{G_{I_0}}(r_j)$ denotes the degree of vertex $r_j \in R_0$ in G_{I_0}. The graph $H(x_j)$ is illustrated in Figure 1.
- Assume that vertices $\{l_{f(1)}, \dots, l_{f(p)}\}$ are the neighbors of r_j in G_{I_0}, then color $H(x_j)$ as follows: for any $k = 1, \dots, p$, $L(v_{1,j}, s_{1,j,f(k)}) = L(v_{2,j}, s_{2,j,f(k)})$ $= c_{f(k)}$ and $L(s_{1,j,f(k)}, s_{2,j,f(k)}) = c_{n_0+j}$.
- We complete $H = \cup_{x_j \in X} H(x_j)$ into $K_{n,n}$, by adding a new color per edge.

Clearly, $K_{n,n}$ is complete bipartite and has $2n = 2\sum_{r_j \in R_0}(d_{G_{I_0}}(r_j) + 1) = 2|E_{I_0}| + 2m_0$ vertices. Moreover, each color is used at most 6 times.

Let $\mathcal{S}^*$ be an optimal set cover on I_0. From $\mathcal{S}^*$, we can easily construct a perfect matching M^* on I using exactly $|\mathcal{S}^*| + m_0$ colors (since we assume that each element appears in at least 2 sets) and thus:

$$opt_{\text{Labeled } Min\ PM_6}(I) \leq opt_{\text{MinSC}_3}(I_0) + m_0 \tag{3}$$

Conversely, we can show that any perfect matching M may be polynomially transformed into a perfect matching M" with value $apx(I)$, using the edges of H and verifying:

Property 1. $apx(I) \leq |L(M)|$

From such a matching, we may obtain a set cover $\mathcal{S}" = \{S_k | c_k \in L(M")\}$ on I_0 verifying:

$$|\mathcal{S}"| = apx(I) - m_0 \tag{4}$$

Using (3) and (4), we deduce $opt_{\text{LABELED } Min\ PM_6}(I) = opt_{\text{MINSC}_3}(I_0) + m_0$ and $|\mathcal{S}"| - opt_{\text{MINSC}_3}(I_0) \leq |L(M))| - opt_{\text{LABELED } Min\ PM_6}(I)$. Finally, since $opt_{\text{MINSC}_3}(I_0) \geq \frac{m_0}{3}$ the result follows.

Applying the same kind of proof from the vertex cover problem (MINVC in short) in cubic graphs [2], we obtain a stronger result.

Theorem 5. LABELED *Min PM$_r$ is **APX**-complete in bipartite complete graphs $K_{n,n}$ for any $r \geq 3$.*

Proof. Starting from a cubic graph $G = (V, E)$ instance of MINVC, we associate to each edge $e = [x, y] \in E$ a 4-long cycle $\{a_{1,e}, a_{2,e}, a_{3,e}, a_{4,e}\}$ together with a coloration L given by: $L(a_{1,e}) = c_x$, $L(a_{2,e}) = c_y$ and $L(a_{3,e}) = L(a_{4,e}) = c_e$. We complete this graph into a complete bipartite graph, adding a new color per edge.

Unfortunately, we can not apply the proof of Theorem 2 since in this latter, on the one hand, we have some cycles of size 6 and, on the other hand, a color may occur in different gadgets. One open question concerns the complexity of LABELED *Min PM$_2$* in bipartite complete graphs. Moreover, from Theorem 4, we can also obtain a stronger inapproximability result concerning the general problem LABELED *Min PM*: one can not compute in polynomial-time an approximate solution of LABELED *Min PM* that uses less than $(1/2 - \varepsilon)\ln(opt_{\text{LABELED } Max\ PM}(I))$ colors in complete bipartite graphs.

Corollary 3. *For any $\varepsilon > 0$,* LABELED *Min PM is not $(\frac{1}{2} - \varepsilon) \times \ln(n)$ approximable in complete bipartite graphs $K_{n,n}$, unless **NP**$\subset$**DTIME**$(n^{loglogn})$.*

Proof. First, we apply the construction made in Theorem 4, except that $I_0 = (\mathcal{S}, X)$ is an instance of MINSC such that the number of elements m_0 is strictly larger than the number of sets n_0. From I_0, we construct n_0 instances $I'_1, \ldots, I'_{n_0}$ of LABELED *Min PM* where $I'_i = (H, L_i)$. The colors $L_i(E)$ are the same as $L(E)$, except that we replace colors $c_{n_0+1}, \ldots, c_{n_0+m_0}$ by c_i. Finally, as previously, we complete each instance I'_i into a complete bipartite graph $K_{n,n}$ by adding a new color by edge.

Let $\mathcal{S}^*$ be an optimal set cover on I_0 and assume that $S_i \in \mathcal{S}^*$, we consider the instance I_i of LABELED *Min PM*. From $\mathcal{S}^*$, we can easily construct a perfect matching M_i^* of I_i that uses exactly $|\mathcal{S}^*|$ colors. Conversely, let M_i be a perfect matching on I_i; by construction, the subset $\mathcal{S}' = \{S_k : c_k \in L(M_i)\}$ of $\mathcal{S}$ is a set cover of X using $|L(M_i)|$ sets. Finally, let A be an approximate algorithm for LABELED *Min PM*, we compute n_0 perfect matchings M_i, applying A on instances I_i. Thus, if we pick the matching that uses the minimum number of colors, then we can polynomially construct a set cover on I_0 of cardinality this number of colors.

Since $n_0 \leq m_0 - 1$, the size n of a perfect matching of $K_{n,n}$ verifies: $n = |E_{I_0}| + m_0 \leq n_0 \times m_0 + m_0 \leq m_0(m_0 - 1) + m_0 = m_0^2$. Hence, from any algorithm A solving LABELED *Min PM* within a performance ratio $\rho_{\text{A}}(I) \leq \frac{1}{2} \times \ln(n)$, we can deduce an algorithm for MINSC that guarantees the performance ratio $\frac{1}{2}\ln(n) \leq \frac{1}{2}\ln(m_0^2) = \ln(m_0)$. Since the negative result of [15] holds when $n_0 \leq m_0 - 1$, i.e., MINSC is not $(1 - \varepsilon) \times \ln(m_0)$ approximable for any $\varepsilon > 0$, unless $\mathbf{NP} \subset \mathbf{DTIME}(n^{loglogn})$, we obtain a contradiction.

On the other hand, dealing with LABELED *Max* PM_r in $K_{n,n}$, the result of [10] shows that the case $r = 2$ is polynomial, whereas it becomes **NP**-hard when $r = \Omega(n^2)$. Indeed, it is proved in [10] that, on the one hand, we can compute a good matching in $K_{n,n}$ within polynomial-time when each color appears at most twice and, on the other hand, there always exists a good matching in such a graph if $n \geq 3$. An interesting question is to decide the complexity and the approximability of LABELED *Max* PM_r when r is a constant greater than 2.

4.1 Approximation algorithm for LABELED *Min* PM_r

Let us consider the greedy algorithm for LABELED *Min* PM_r in complete bipartite graphs that iteratively picks the color that induces the maximum-size matching in the current graph and delete the corresponding vertices. Formally, if $L(G')$ denotes the colors that are still available in the graph G' at a given iteration and if $G'[c]$ (resp., $G'[V']$) denotes the subgraph of G' that is induced by the edges of color c (resp., by the vertices V'), then the greedy algorithm consists of the following process:

`Greedy`

1 Set $\mathcal{C}' = \emptyset$, $V' = V$ and $G' = G$;
2 While $V' \neq \emptyset$ do
  2.1 For all $c \in L(G')$, compute a maximum matching M_c in $G'[c]$;
  2.2 Select a color c^* maximizing $|M_c|$;
  2.3 $\mathcal{C}' \leftarrow \mathcal{C}' \cup \{c^*\}$, $V' \leftarrow V' \setminus V(M_{c^*})$ and $G' = G[V']$;
3 output $\mathcal{C}'$;

Theorem 6. *`Greedy` is an $\frac{H_r + r}{2}$-approximation of* LABELED *Min* PM_r *in complete bipartite graphs where* H_r *is the r-th harmonic number* $H_r = \sum_{i=1}^{r} \frac{1}{i}$*, and this ratio is tight.*

Proof. Let $I = (G, L)$ be an instance of LABELED *Min* PM_r. We denote by $\mathcal{C}'_i$ for $i = 1, \ldots, r$ be the set of colors of the approximate solution which appears exactly i times in $\mathcal{C}'$ and by p_i its cardinality (thus, $\forall c \in \mathcal{C}'_i$ we have $|M_c| = i$ in

$G'[c]$); finally, let M_i denote the matching with colors $\mathcal{C}'_i$. If $apx(I) = |\mathcal{C}'|$, then we have:

$$apx(I) = \sum_{i=1}^{r} p_i \tag{5}$$

Let $\mathcal{C}^*$ be an optimal solution corresponding to the perfect matching M^* of size $opt(I) = |\mathcal{C}^*|$; we denote by E_i the set of edges of M^* that belong to $G[\cup_{k=1}^{i} V(M_k)]$, the subgraph induced by $\cup_{k=1}^{i} V(M_k)$ and we set $q_i = |E_i \setminus E_{i-1}|$ (where we assume that $E_0 = \emptyset$). For any $i = 1, \ldots, r-1$, we get:

$$opt(I) \geq \frac{1}{i} \sum_{k=1}^{i} q_k \tag{6}$$

Indeed, $\sum_{k=1}^{i} q_k = |E_i|$ and by construction, each color appears at most i times in $G[\cup_{k=1}^{i} V(M_k)]$.

We also have the following inequality for any $i = 1, \ldots, r-1$:

$$opt(I) \geq \frac{1}{r} \left(2 \sum_{k=1}^{i} k \times p_k - \sum_{k=1}^{i} q_k \right) \tag{7}$$

Since M^* is a perfect matching, the quantity $2\sum_{k=1}^{i} k \times p_k - \sum_{k=1}^{i} q_k$ counts the edges of M^* of which at least one endpoint belongs to $G[\cup_{k=1}^{i} V(M_k)]$. Because each color appears on at most r edges, the result follows.

Finally, since $\sum_{k=1}^{r} k \times p_k$ is the size of a perfect matching of G, the following inequality holds:

$$opt(I) \geq \frac{1}{r} \sum_{k=1}^{r} k \times p_k \tag{8}$$

Using equality (5) and adding inequalities (6) with coefficient $\alpha_i = \frac{1}{2(i+1)}$ for $i = 1, \ldots, r-1$, inequalities (7) with coefficient $\beta_i = \frac{r}{2i(i+1)}$ for $i = 1, \ldots, r-1$ and inequality (8), we obtain:

$$apx(I) \leq \left(\frac{H_r + r}{2} \right) opt(I) \tag{9}$$

The proof of the tightness is omitted.

We conjecture that Labeled *Min PM* is not $O(n^{\varepsilon})$-approximable in complete bipartite graphs. Thus, a challenge will be to give better approximate algorithms or to improve the lower bound.

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