# Bilateral Fixed-Points and Algebraic Properties of Viability Kernels and Capture Basins of Sets

**Jean-Pierre Aubin & Francine Catté**

*Université Paris-Dauphine*
F-75775 Paris cx (16)
e-mail: J.P.Aubin@wanadoo.fr, Catte@ceremade.dauphine.fr

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## Abstract

*Many concepts of viability theory such as viability or invariance kernels and capture or absorption basins under discrete multivalued systems, differential inclusions and dynamical games share algebraic properties that provide simple — yet powerful — characterizations as either largest or smallest fixed points or* unique minimax *(or bilateral fixed-point) of adequate maps defined on pairs of subsets. Further, important algorithms such as the Saint-Pierre viability kernel algorithm for computing viability kernels under discrete system and the Cardaliaguet algorithm for characterizing "discriminating kernels" under dynamical games are algebraic in nature. The Matheron Theorem as well as the Galois transform find applications in the field of control and dynamical games allowing us to clarify concepts and simplify proofs.*

**keywords** viability kernel, capture basin, discriminating kernel, Matheron Theorem, Saint-Pierre viability kernel algorithm, Cardaliaguet discirminating kernel algorithm, openings, closings, Galois transform.

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## 1 Introduction

Let us consider a differential inclusion $x' \in F(x)$ (summarizing the dynamics of a control system) and two subsets $C$ and $K$ of a finite dimensional vector space $X$ such that $C \subset K$. Here, $K$ is regarded as a constrained subset in which the solution must evolve until possibly reaching the subset $C$ regarded as a target. We recall the following definitions:

1. *The subset* $\mathrm{Viab}_F(K, C)$ *of initial states* $x_0 \in K$ *such that at least one solution* $x(\cdot)$ *to differential inclusion* $x' \in F(x)$ *starting at* $x_0$ *is viable in* $K$ *for all* $t \geq 0$ *or viable in* $K$

*until it reaches $C$ in finite time is called the* viability kernel of $K$ with target $C$ *under* $F$. *When the target* $C := \emptyset$ *is empty, we say that* $\text{Viab}(K) = \text{Viab}(K, \emptyset)$ *is the* viability kernel $\text{Viab}(K)$ *of* $K$.

2. *The subset* $\text{Capt}_F(K, C)$ *of initial states* $x_0 \in K$ *such that* $C$ *is reached in finite time before possibly leaving* $K$ *by at least one solution* $x(\cdot)$ *to differential inclusion* $x' \in F(x)$ *starting at* $x_0$ *is called the* viable-capture basin *of* $C$ *in* $K$ *and* $\text{Capt}_F(C) := \text{Capt}_F(X, C)$ *is said to be the* capture basin *of* $C$.

The subset $\text{Env}_F(C) := \text{Capt}_{-F}(C)$ is known under various names such as invariance envelope or accessibility map or controlled map of $C$ (See [57, Quincampoix] for properties of invariance envelopes under Lipschitz maps and [6, Aubin] for Marchaud maps). Henri Poincaré introduced the concept of shadow (in French, ombre) of $K$, which if the set of initial points of $K$ from which (all) solutions leave $K$ in finite time. It is thus equal to the complement $K \backslash \text{Viab}_F(K)$ of the viability kernel of $K$, which has been introduced in the context of differential inclusions in [1, Aubin]. The concept of viability kernel with a target by a Lipschitz set-valued map has been introduced and studied in [60, Quincampoix & Veliov] and the viable-capture basin in [10, Aubin].

Many applications of these concepts have been obtained (see [2, 3, Aubin], [34, Cardaliaguet, Quincampoix & Saint-Pierre] for a presentation of some of them). Applied to subsets such as epigraphs of extended real-valued functions or graphs of maps, they provide interesting existence and uniqueness results of adequately generalized solutions to systems of Hamilton-Jacobi inclusions (see [18, Aubin & Frankowska],[10, 7, Aubin] and, in mathematical finance, [56, Pujal], [23, Aubin, Pujal & Saint-Pierre]).

Various characterizations of the viability kernel and the capture basins finding their origin in the papers of Frankowska on Hamilton-Jacobi equations (see for instance [42, 43, Frankowska]) have been proposed in [10, Aubin] in terms of viability and backward invariance properties and thus, thanks to the Viability and Invariance Theorems, in terms of tangential or normal conditions.

Actually, fixed point properties of the two maps $K \mapsto \text{Viab}_F(K, C)$ and $C \mapsto \text{Viab}_F(K, C)$ popped up here and there, in the study of such and such application, until it was observed that the viability kernel $\text{Viab}_F(K, C)$ of $K$ with target $C$ is the unique "minimax (bilateral fixed-point)" $D$ between $C$ and $K$ of the map $(K, C) \mapsto \text{Viab}_F(K, C)$ in the sense that

$$C \subset D \subset K \ \& \ D = \text{Viab}(K, D) = \text{Viab}(D, C)$$

In other words, saying that a subset $K$ is viable outside $C$ if it coincides with its viability kernel $K$ with target $C$:

$$K = \text{Viab}_F(K, C)$$

and that a subset $C \subset K$ is isolated in $K$ by $F$ if it coincides with the viability kernel $K$ with target $C$:

$$\text{Viab}_F(K, C) \; = \; C$$

we can restate the above result by stating that the viability kernel $\text{Viab}_F(K, C)$ of $K$ with target $C$ is the unique subset $D$ viable outside $C$ and isolated in $K$:

$$\text{Viab}(K, C) \; = \; \text{Viab}(\text{Viab}(K, C), C) \; = \; \text{Viab}(K, \text{Viab}(K, C))$$

This is indeed the natural starting point for further characterizations of the above two fixed-point properties, for instance in terms of local viability of $K \backslash C$ and backward invariance of $C$ when viability kernels are concerned, as in [10, Aubin].

These algebraic properties are valid for other concepts that had been introduced in viability theory, such as invariance kernels $\text{Inv}(K, C)$ of $K$ with target $C$ and invariance-absorption basins $\text{Abs}(K, C)$ under differential inclusions as we shall see in the following pages.

They are valid not only for differential inclusions, but also for differential inclusions with memory and for general evolutionary systems $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ defined later in the paper, as well as for conditional and guaranteed viability kernels and capture basins in dynamical games (see [27, 28, 30, 31, Cardaliaguet]) and evolutionary games. They also hold true for impulse differential inclusions and hybrid systems, as it is shown in [19, Aubin & Haddad] (for more details on hybrid systems in the framework of viability theory, see [5, 9, 8, Aubin], [20, 21, Aubin & Haddad], [22, Aubin, Lygeros, Quincampoix, Sastry & Seube], [35, 36, Cruck], [62, Saint-Pierre]), qualitative viability kernel in [12, Aubin & Dordan], that are not presented in this paper.

Two words about this algebraic framework, that is also closely related to algebraic properties of mathematical morphology (see [52, Matheron], [63, 64, Serra], [45, Giardina & Dougherty], [55, Schmitt & Mattioli], [50, Heijmans] for instance) and max-plus algebras (see [24, Bacelli, Cohen, Olsder & Quadrat] and [44, Gaubert]). Indeed, they rely only on the following two properties of maps $(K, C) \mapsto \mathcal{A}(K, C)$:

$$\begin{cases} i) & C \; \subset \; \mathcal{A}(K, C) \; \subset K \text{ (extensivity/antiextensivity)} \\ ii) & (K, C) \mapsto \mathcal{A}(K, C) \text{ is increasing} \end{cases}$$

shared by the above viability kernels and capture basins. This has already been noticed first in [54, Mattioli, Doyen & Najman] and then, taken up in chapter 7 of [4, Aubin] for the partial maps $K \mapsto \text{Viab}(K)$ before the systematic study of viability kernels $\text{Viab}(K, C)$ with targets and capture basins $\text{Capt}(K, C)$ of targets viable in a constrained subset that involves functions defined on pairs of subsets.

This "minimax (bilateral fixed-point)" characterization, when valid, underlies some properties of [7, 11, Aubin], [56, Pujal], [23, Aubin, Pujal & Saint-Pierre]. In the same way, the Galois transform of the invariance kernel map $K \mapsto \mathrm{Inv}_F(K, C)$ provides the invariance envelope.

Not all maps $(K, C) \mapsto \mathcal{A}(K, C)$ defined on the family of subsets do enjoy this minimax property as the viability kernel map does, for instance. In this case, Matheron's theorem allows us to associate with $\mathcal{A}$ maps $K \mapsto \mathcal{A}^\sharp(K, C)$ and $C \mapsto \mathcal{A}^\flat(K, C)$ that have fixed points. They find applications here by providing alternate characterizations of the viability kernel under discrete systems or discriminating kernels under dynamical games. They project a new light on the viability kernel and capture basin algorithms introduced in [61, Saint-Pierre] and the Cardaliaguet algorithm for constructing discriminating kernels under dynamical games in [27, 28, 30, 31, Cardaliaguet] and [34, Cardaliaguet, Quincampoix & Saint-Pierre].

Although the abstract algebraic results we present below are valid on any lattice, we present them only on sub-lattices of the family of subsets of a given set $X$ ordered with the usual inclusion.

It usually takes a long time to discover the sources of a river — and to decide which is the real source. By now, one could "presume" that a source to uniqueness theorems of solutions to Hamilton-Jacobi-Bellman equations as the ones of [40, 41, 42, 43, Frankowska]) based on backward viability and invariance of epigraphs lies with these simple algebraic results that we now describe, and it took that time to isolate this source, indeed.

## 2 Minimax (bilateral fixed-points) of an opening/closing map

This section is devoted to the existence and uniqueness of maps $(K, C) \mapsto \mathcal{A}(K, C)$ that are pre-openings with respect to $K$ and opening with respect to $C$.

### 2.1 Openings and closings

**Definition 2.1** *Let us consider a family $\mathcal{D} \subset \mathcal{P}(X) := 2^X$ of subsets of a space $X$ stable under the operations of union and intersection.*

*We associate with any pair $(K, C) \in \mathcal{D} \times \mathcal{D}$ the family*

$$\mathcal{D}(K, C) \; := \; \{D \in \mathcal{D} \mid C \; \subset \; D \; \subset \; K\}$$

*Let us consider a map $\mathcal{A} : \mathcal{D} \mapsto \mathcal{D}$, We shall say that*

1. *the map $K \mapsto \mathcal{A}(K, C)$ is a* pre-opening *if it is* increasing *(in the sense that whenever $K_1 \subset K_2$, then $\mathcal{A}(K_1, C) \subset \mathcal{A}(K_2, C)$) and* antiextensive *(in the sense that for any pair $(K, C) \in \mathcal{D}$, $\mathcal{A}(K, C) \subset K$),*

2. *the map $C \mapsto \mathcal{A}(K, C)$ is a* pre-closing *if it is* increasing *(in the sense that whenever*

$C_1 \subset C_2$, *then* $\mathcal{A}(K, C_1) \subset \mathcal{A}(K, C_2)$*)* *and* extensive *(in the sense that for any pair* $(K, C) \in \mathcal{D}$, $C \subset \mathcal{A}(K, C)$*),*

*We recall that the map* $D \mapsto \mathcal{A}(D, C)$ *is an* opening *if it is a pre-opening and an* idempotent *map in the sense that*

$$\forall\, (K, C) \in \mathcal{D}^2, \;\; \mathcal{A}(\mathcal{A}(K, C), C) \; = \; \mathcal{A}(K, C)$$

*and that the map* $D \mapsto \mathcal{A}(K, D)$ *is a* closing *if it is a pre-closing and an* idempotent *map in the sense that*

$$\forall\, (K, C) \in \mathcal{D}^2, \;\; \mathcal{A}(K, \mathcal{A}(K, C)) \; = \; \mathcal{A}(K, C)$$

We observe that

- $\mathcal{D}(K, \emptyset)$ is the family of subset $D \in \mathcal{D}$ contained in $K$,
- $\mathcal{D}(X, C)$ is the family of subset $D \in \mathcal{D}$ containing $C$.

As we see, these properties involve only the order relations on the family $\mathcal{P}(X)$ of subsets of $X$, or on any sub-family $\mathcal{D} \subset \mathcal{P}(X)$ stable under the union ("max-closed") or intersection operations ("min-closed"). *Although they can be presented for arbitrary lattices or semi-lattices, we present them in the familiar context of families of subsets.*

We next associate with $\mathcal{A}$ the two following family of subsets:

1. the family $\mathcal{K}_{\mathcal{A}}(K, C)$ defined by

$$\mathcal{K}_{\mathcal{A}}(K, C) \; := \; \{D \in \mathcal{D}(K, C) \mid D \; \subset \; \mathcal{A}(D, C)\}$$

2. the family $\mathcal{I}_{\mathcal{A}}(K, C)$ defined by

$$\mathcal{I}_{\mathcal{A}}(K, C) \; := \; \{D \in \mathcal{D}(K, C) \mid \mathcal{A}(K, D) \; \subset \; D\}$$

We check at once the following claim:

**Lemma 2.2** *If the map* $D \mapsto \mathcal{A}(D, C)$ *is antiextensive, then* $\mathcal{K}_{\mathcal{A}}(K, C)$ *is the set of fixed-points of the map* $D \mapsto \mathcal{A}(D, C)$ *between* $C$ *and* $K$*:*

$$\mathcal{K}_{\mathcal{A}}(K, C) \; := \; \{D \in \mathcal{D}(K, C) \mid D \; = \; \mathcal{A}(D, C)\}$$

*Symmetrically, if the map* $D \mapsto \mathcal{A}(K, D)$ *is extensive, then* $\mathcal{I}_{\mathcal{A}}(K, C)$ *is the set of fixed-points of the map* $D \mapsto \mathcal{A}(K, D)$ *between* $C$ *and* $K$*:*

$$\mathcal{I}_{\mathcal{A}}(K, C) \; := \; \{D \in \mathcal{D}(K, C) \mid \mathcal{A}(K, D) \; = \; D\}$$

*Furthermore,*

1. *if the map* $D \mapsto \mathcal{A}(D,C)$ *is a pre-opening, then*

$$\forall\, D \in \mathcal{K}_{\mathcal{A}}(K,C),\ \ D \subset \mathcal{A}(K,C)$$

2. *if the map* $D \mapsto \mathcal{A}(K,D)$ *is a pre-closing, then for any*

$$\forall\, D \in \mathcal{I}_{\mathcal{A}}(K,C),\ \ \mathcal{A}(K,C) \subset D$$

*Consequently, if the map* $(K,C) \mapsto \mathcal{A}(K,C)$ *is a pre-opening with respect to* $K$ *and a pre-closing with respect to* $C$*, then* $C$ *belongs to* $\mathcal{K}_{\mathcal{A}}(K,C)$*,* $K$ *belongs to* $\mathcal{I}_{\mathcal{A}}(K,C)$ *and*

$$\forall\, D \in \mathcal{D}(K,C),\ \ C \subset \bigcup_{D \in \mathcal{K}_{\mathcal{A}}(K,C)} D \subset \mathcal{A}(K,C) \subset \bigcap_{D \in \mathcal{I}_{\mathcal{A}}(K,C)} D \subset K \qquad (1)$$

We thus derive the following obvious but important statement:

**Proposition 2.3** *Let us consider a map* $\mathcal{A} : \mathcal{D}^2 \mapsto \mathcal{D}$*.*

1. *If the map* $D \mapsto \mathcal{A}(D,C)$ *is a pre-opening and if* $\mathcal{A}(K,C)$ *belongs to the family* $\mathcal{K}_{\mathcal{A}}(K,C)$*, then* $\mathcal{A}(K,C)$ *is the* largest *fixed-point of the map* $D \mapsto \mathcal{A}(D,C)$ *between* $C$ *and* $K$*. Furthermore, the map* $D \mapsto \mathcal{A}(D,C)$ *is an opening.*

2. *If the map* $D \mapsto \mathcal{A}(K,D)$ *is a pre-closing and if* $\mathcal{A}(K,C)$ *belongs to the family* $\mathcal{I}_{\mathcal{A}}(K,C)$*, then* $\mathcal{A}(K,C)$ *is the* smallest *fixed-point of the map* $D \mapsto \mathcal{A}(K,D)$ *between* $C$ *and* $K$*. Furthermore, the map* $D \mapsto \mathcal{A}(D,C)$ *is a closing.*

3. *If the map* $(K,C) \mapsto \mathcal{A}(K,C)$ *is a pre-opening with respect to* $K$ *and a pre-closing with respect to* $C$*, then any* minimax $D \in \mathcal{D}(K,C)$ *of the map* $\mathcal{A}$ *between* $C$ *and* $K$ *in the sense that:*

$$\mathcal{A}(D,C) = D = \mathcal{A}(K,D)$$

   *is* unique *and is equal to* $\mathcal{A}(K,C)$*.*

   *Consequently, if* $\mathcal{A}(K,C)$ *belongs to the intersection* $\mathcal{K}_{\mathcal{A}}(K,C) \cap \mathcal{I}_{\mathcal{A}}(K,C)$*, then* $\mathcal{A}(K,C)$ *is the* unique minimax *of the map* $\mathcal{A}$ *between* $C$ *and* $K$*:*

$$\mathcal{A}(\mathcal{A}(K,C),C) = \mathcal{A}(K,C) = \mathcal{A}(K,\mathcal{A}(K,C))$$

   *In this case,* $\mathcal{A}$ *is an opening with respect to* $K$ *and a closing with respect to* $C$*.*

**Proof** — The two first statements are obvious consequence of Lemma 2.2.

Therefore any minimax $D \in \mathcal{K}_{\mathcal{A}}(K,C) \cap \mathcal{I}_{\mathcal{A}}(K,C)$ is necessarily equal to $\mathcal{A}(K,C)$ of $K$ and $C$, because since $D$ belongs to $\mathcal{I}_{\mathcal{A}}(K,C)$, $\mathcal{A}(K,C) \subset \mathcal{A}(K,D) \subset D$ and since $D$ belongs to $\mathcal{K}_{\mathcal{A}}(K,C)$, then $D \subset \mathcal{A}(D,C) \subset \mathcal{A}(K,C)$. Hence $\mathcal{A}(K,C) = D$. □

The above Proposition 7.1 provides the uniqueness of the minimax of the map $\mathcal{A}$ that is a pre-opening with respect to $K$ and a pre-closing with respect to $C$. The existence of this minimax must be checked in each specific example by proving the two following inclusions

$$\begin{cases} i) & \mathcal{A}(K,C) \subset \mathcal{A}(\mathcal{A}(K,C),C) \\ ii) & \mathcal{A}(\mathcal{A}(K,C),C) \subset \mathcal{A}(K,C) \end{cases}$$

## 2.2 Galois Transform

Let us assume that $\emptyset \in \mathcal{D}$ and that the map $C \mapsto \mathcal{A}(K,C)$ is a dilation in the sense that for any family $I$ of indexes, we have

$$\mathcal{A}\left(K, \bigcup_{i \in I} C_i\right) = \bigcup_{i \in I} \mathcal{A}(K, C_i)$$

Therefore, for any $C \in \mathcal{D}(K,\emptyset)$, the family

$$D \in \mathcal{D}(K,\emptyset) \quad \text{such that} \;\; \mathcal{A}(K,D) \subset C$$

is max-closed (stable under the union) and has a largest element because, $\mathcal{A}$ being a dilation with respect to $C$,

$$\mathcal{A}\left(K, \bigcup_{\mathcal{A}(K,D) \subset C} D\right) = \bigcup_{\mathcal{A}(K,D) \subset C} \mathcal{A}(K,D) \subset C$$

In the same way, if $X \in \mathcal{D}$ and if the map $K \mapsto \mathcal{A}(K,C)$ is an erosion in the sense that

$$\mathcal{A}\left(\bigcap_{i \in I} K_i, C\right) = \bigcap_{i \in I} \mathcal{A}(K_i, C)$$

then for any $K \in \mathcal{D}(X,C)$, the family

$$D \in \mathcal{D}(X,C) \quad \text{such that} \;\; K \subset \mathcal{A}(D,C)$$

is min-closed (stable under the intersection) and has a smallest element because, $\mathcal{A}$ being an erosion with respect to $K$,

$$K \subset \mathcal{A}\left(K, \bigcap_{K \subset \mathcal{A}(D,C)} D\right) = \bigcap_{K \subset \mathcal{A}(D,C)} \mathcal{A}(D,C)$$

**Definition 2.4** *Let us assume that $\mathcal{A}$ is a dilation with respect to $C$ and that $\emptyset$ belongs to $\mathcal{D}$. Then we shall say that the map $C \mapsto \mathcal{A}^*(K,C)$ defined by*

$$\forall\, K \in \mathcal{D},\ \ \forall\, C \in \mathcal{D}(K,\emptyset),\ \ \mathcal{A}^*(K,C) \ := \bigcup_{\{D\in\mathcal{D}(K,\emptyset)\ \ \text{such that}\ \mathcal{A}(K,D)\subset C\}} D$$

*is called the* upper Galois transform *(or the* upper residual*) of $\mathcal{A}$ with respect to $C$.*

*If $\mathcal{A}$ is an erosion with respect to $K$ and is $X$ belongs to $\mathcal{D}$, then we shall say that the map $\mathcal{A}_* : K \mapsto \mathcal{A}_*(K,C)$ defined by*

$$\forall C \in \mathcal{D},\ \forall\, K \in \mathcal{D}(X,C),\ \ \mathcal{A}_*(K,C) \ := \bigcap_{D\in\mathcal{D}(X,K)\ \ \text{such that}\ K\subset\mathcal{A}(D,C)} D$$

*is called the* lower Galois transform *(or the* lower residual*) of $\mathcal{A}$ with respect to $K$.*

**Proposition 2.5** *Let us assume that $\mathcal{A}$ is a dilation and that $\emptyset$ belongs to $\mathcal{D}$.*

1. *The map $\mathcal{A}^*$ is increasing with respect to $C$ and the inclusions*

   $$\mathcal{A}(K,\mathcal{A}^*(K,C)) \ \subset\ C \ \ \&\ \ C \ \subset\ \mathcal{A}^*(K,\mathcal{A}(K,C))$$

   *hold true and actually, $C \mapsto \mathcal{A}^*(K,C)$ is the unique increasing map $\Phi$ satisfying*

   $$\mathcal{A}(K,\Phi(C)) \ \subset\ C \ \ \&\ \ C \ \subset\ \Phi(\mathcal{A}(K,C))$$

2. *If $\mathcal{A}$ is increasing with respect to $C$, then $\mathcal{A}^*$ is an erosion with respect to $C$, and its lower Galois transform with respect to $C$ coincides with $\mathcal{A}$:*

   $$(\mathcal{A}^*)_*(K,C) \ =\ \mathcal{A}(K,C)$$

3. *If $\mathcal{A}$ is extensive with respect to $C$, then $\mathcal{A}^*(K,C) \subset C$,*

4. *If $\mathcal{A}$ is increasing and idempotent with respect to $C$, then*

   $$\mathcal{A}^*(K,C) \ = \bigcup_{\{D\in\mathcal{I}_{\mathcal{A}}(K,\emptyset)\ \ \text{such that}\ \ D\subset C\}} D$$

   *is the largest element $D \in \mathcal{I}_{\mathcal{A}}(C,\emptyset)$ contained in $C$ and thus, $\mathcal{A}(K,\mathcal{A}^*(K,C)) = \mathcal{A}^*(K,C)$.*

*Let us assume that $\mathcal{A}$ is an erosion and that $X$ belongs to $\mathcal{D}$.*

1. *The map $\mathcal{A}^*$ is increasing with respect to $K$ and the inclusions*

$$K \subset \mathcal{A}(\mathcal{A}_*(K,C),C) \;\&\; \mathcal{A}_*(\mathcal{A}(K,C),C) \subset K$$

*hold true; Actually, $K \mapsto \mathcal{A}_*(K,C)$ is the unique increasing map $\Psi$ satisfying*

$$K \subset \mathcal{A}(K,\Psi(K)) \;\&\; \Psi(\mathcal{A}(K,C)) \subset K$$

2. *If $\mathcal{A}$ is increasing with respect to $K$, then $\mathcal{A}^*$ is a dilation with respect to $K$, and its upper Galois transform with respect to $K$ coincides with $\mathcal{A}$:*

$$(\mathcal{A}_*)^*(K,C) = \mathcal{A}(K,C)$$

3. *If $\mathcal{A}$ is idempotent with respect to $K$, then*

$$\mathcal{A}_*(K,C) = \bigcap_{D \in \mathcal{K}_{\mathcal{A}}(X,C) \text{ such that } K \subset D} D$$

*is the smallest element $D \in \mathcal{K}_{\mathcal{A}}(X,K)$ containing $K$ and thus, $\mathcal{A}(\mathcal{A}_*(K,C),C) = \mathcal{A}^*(K,C)$.*

**Proof** — The map $\mathcal{A}^*$ is increasing with respect to $C$ because the family of subsets $D$ such that $\mathcal{A}(K,D) \subset C_1$ is contained in the family of subsets $D$ such that $\mathcal{A}(K,D) \subset C_2$ whenever $C_1 \subset C_2$. Inclusion $\mathcal{A}(K,\mathcal{A}^*(K,C)) \subset C$ holds true by construction and replacing $C$ by $\mathcal{A}(K,C)$ in the definition of the upper Galois transform, we obtain the second inclusion $C \subset \mathcal{A}^*(K,\mathcal{A}(K,C))$.

Let $\Phi$ satisfy

$$\mathcal{A}(K,\Phi(C)) \subset C \;\&\; C \subset \Phi(\mathcal{A}(K,C))$$

Whenever $\mathcal{A}(K,D) \subset C$, then $D \subset \Phi(\mathcal{A}(K,D)) \subset \Phi(C)$ since $\Phi$ in increasing, and thus, $\mathcal{A}^*(K,C) \subset \Phi(C)$. Furthermore, inclusion $\mathcal{A}(K,\Phi(C)) \subset C$ implies that $\Phi(C) \subset \mathcal{A}^*(K,C)$.

Let us check now that $\mathcal{A}^*$ is an erosion with respect to $C$. Indeed, since $\mathcal{A}^*$ is increasing with respect to $C$, then $\mathcal{A}^*(K,\bigcap_{i\in I} C_i) \subset \bigcap_{i\in I} \mathcal{A}^*(K,C_i) =: D^*$. On the other hand, since for every $i \in I$, $D^* \subset \mathcal{A}^*(K,C_i) \subset C_i$, we infer that $D^* \subset \mathcal{A}^*(K,\bigcap_{i\in I} C_i)$.

Since $\mathcal{A}^*$ is an erosion with respect to $C$, we can take its lower Galois transform defined by

$$(\mathcal{A}^*)_*(K,C) := \bigcap_{D \mid C \subset \mathcal{A}^*(K,D)} D$$

Since $C \subset \mathcal{A}^*(K,\mathcal{A}(K,C))$, we infer that $(\mathcal{A}^*)_*(K,C) \subset \mathcal{A}(K,C)$ by taking $D := \mathcal{A}(K,C)$. On the other hand, for any $D$ satisfying $C \subset \mathcal{A}^*(K,D)$, we infer that $\mathcal{A}(K,C) \subset \mathcal{A}(K,\mathcal{A}^*(K,D))$

since $\mathcal{A}$ is increasing with respect to $C$ and we know that $\mathcal{A}(K, \mathcal{A}^*(K, D)) \subset D$. Hence $\mathcal{A}(K, C) \subset D$ for any $D$ satisfying $C \subset \mathcal{A}^*(K, D)$, and thus, $\mathcal{A}(K, C) \subset \mathcal{A}^*(K, \mathcal{A}(K, C))$, so that $\mathcal{A}(K, C) = \mathcal{A}^*(K, \mathcal{A}(K, C))$.

Finally, assume that $\mathcal{A}$ is increasing and idempotent with respect to $C$ and prove that $\mathcal{A}(K, C)$ is the largest element $D \in \mathcal{I}_{\mathcal{A}}(K, \emptyset)$. First, we observe that inclusion

$$D^* \ := \bigcup_{D \in \mathcal{I}_{\mathcal{A}}(K,\emptyset) \ | \ D \subset C} D \ = \bigcup_{D \in \mathcal{I}_{\mathcal{A}}(K,\emptyset) \ | \ \mathcal{A}(K,D) \subset C} D \ \subset \ \mathcal{A}^*(K, C)$$

holds always true. On the other hand, if $\mathcal{A}$ is idempotent with respect to $C$, we deduce from inclusions

$$\mathcal{A}(K, \mathcal{A}(K, \mathcal{A}^*(K, C))) \ = \ \mathcal{A}(K, \mathcal{A}^*(K, C)) \ \subset \ D$$

that

$$\mathcal{A}(K, \mathcal{A}^*(K, C)) \ \subset \ \mathcal{A}^*(K, C)$$

and thus, that $\mathcal{A}(K, \mathcal{A}^*(K, C)) \subset D^*$. $\square$

We deduce the following consequence:

**Theorem 2.6** *If $\emptyset \in \mathcal{D}$ and if $C \mapsto \mathcal{A}(K, C)$ is a both a closing and a dilation, then its upper Galois transform $C \mapsto \mathcal{A}^*(K, C)$ is both an opening and an erosion associating with any $C$ the largest subset $D \in \mathcal{I}_{\mathcal{A}}(K, \emptyset)$ contained in $C$.*

*Symmetrically, if $X \in \mathcal{D}$ and if $K \mapsto \mathcal{A}(K, C)$ is a both an opening and an erosion, then its lower Galois transform $K \mapsto \mathcal{A}_*(K, C)$ is both a closing and a dilation associating with any $K$ the smallest subset $D \in \mathcal{K}_{\mathcal{A}}(X, C)$ containing $K$.*

**Remark** — We observe that

1. $\mathcal{A}(K, \emptyset)$ is the smallest subset $D \in \mathcal{I}_{\mathcal{A}}(K, \emptyset)$ contained in $K$,
2. $\mathcal{A}^*(K, K)$ is the largest subset $D \in \mathcal{I}_{\mathcal{A}}(K, \emptyset)$ contained in $K$,

so that, for any $C \subset K$,

$$\mathcal{A}(K, \emptyset) \ \subset \ \mathcal{A}(K, C) \ \subset \ \mathcal{A}^*(K, K) \ \subset \ K$$

In the same way,

1. $\mathcal{A}(X, C)$ is the largest subset $D \in \mathcal{K}_{\mathcal{A}}(X, C)$ containing $K$,
2. $\mathcal{A}_*(C, C)$ is the smallest subset $D \in \mathcal{K}_{\mathcal{A}}(X, C)$ containing $C$,

so that, for any $C \subset K$,

$$C \ \subset \ \mathcal{A}_*(C, C) \ \subset \ \mathcal{A}(K, C) \ \subset \ \mathcal{A}(X, C) \ \square$$

# 3 Minimax Characterization of Viability Kernels and Capture Basins

We shall prove in this section the statement announced in the introduction that the viability kernel $\text{Viab}_F(K, C)$ of $K$ with target $C$ is the unique "minimax (bilateral fixed-point)" $D$ between $C$ and $K$ of the map $(K, C) \mapsto \text{Viab}_F(K, C)$ in the sense that

$$C \subset D \subset K \ \& \ D = \text{Viab}(K, D) = \text{Viab}(D, C)$$

## 3.1 Evolutionary Systems

Actually, such results depend only on few properties of the solution map $x \rightsquigarrow \mathcal{S}_F(x) \subset \mathcal{C}(0, \infty; X)$ associating with any $x$ the set $\mathcal{S}_F(x)$ of solutions to differential inclusions $x' \in F(x)$ starting at $x$, that are shared by solution maps of

1. control problems with memory (see the contributions of [47, 48, 49, Haddad] and [21, Aubin & Haddad], some of them being presented in [2, Aubin]) — before known under the name of functional control problems, the new fashion calling them as "path dependent control systems"
2. parabolic type partial differential inclusions (see the contributions of [65, 66, 67, Shi Shuzhong], some of them being presented in [2, Aubin]) — also known as distributed control systems
3. "mutational equations" governing the evolution in metric spaces, including "morphological equations" governing the evolution of sets (see [4, Aubin] for instance).

Such solution maps define evolutionary system:

**Definition 3.1** *We regard $\mathcal{C}(0, \infty; X)$ as the space of evolutions. An* evolutionary system *is a set-valued map $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ satisfying*

1. *the* initial property*: $x(0) = x$,*
2. *the* translation property*: Let $x(\cdot) \in \mathcal{S}(x)$. Then for all $T \geq 0$, the function $y(\cdot)$ defined by $y(t) := x(t + T)$ is a solution $y(\cdot) \in \mathcal{S}(x(T))$ starting at $x(T)$,*
3. *the* concatenation property*: Let $x(\cdot) \in \mathcal{S}(x)$ and $T \geq 0$. Then for every $y(\cdot) \in \mathcal{S}(x(T))$, the function $z(\cdot)$ defined by*

$$z(t) := \begin{cases} x(t) & \text{if } t \in [0, T] \\ y(t - T) & \text{if } t \geq T \end{cases}$$

*belongs to $\mathcal{S}(x)$.*

### 3.2 Viability Kernels and Capture Basins

**Definition 3.2** *We say that an evolution $x(\cdot)$ is viable in $K$ if for any $t \geq 0$, $x(t) \in K$ and we shall denote by $\mathcal{S}^K(x)$ the set of evolutions of the evolutionary system $\mathcal{S}$ starting from $x \in K$ viable in $K$.*

*The* viability kernel

$$\text{Viab}(K, C) \;:=\; \text{Viab}_{\mathcal{S}}(K, C)$$

*of $K$ with target $C$ under the evolutionary system $\mathcal{S}$ is the subset of initial states $x \in K$ from which starts* at least *one evolution viable in $K$ forever or until it reaches the target $C$ in finite time.*

*The* viable-capture basin

$$\text{Capt}(K, C) \;:=\; \text{Capt}_{\mathcal{S}}(K, C)$$

*of $C$ viable in $K$ under the evolutionary system $\mathcal{S}$ is the subset of initial states $x \in K$ from which starts* at least *one evolution viable in $K$ until it reaches the target $C$ in finite time.*

*When the target $C = \emptyset$ is empty, we say that*

$$\text{Viab}(K) \;:=\; \text{Viab}_{\mathcal{S}}(K) \;:=\; \text{Viab}_{\mathcal{S}}(K, \emptyset)$$

*is the* viability kernel *of $K$ under the evolutionary system.*

*We shall say that*

1. *a subset $K$ is* viable outside a target *$C \subset K$ under the evolutionary system $\mathcal{S}$ if $K := \text{Viab}(K, C)$, i.e., if from any $x \in K$ starts* at least *one evolution viable in $K$ forever or until it reaches the target $C$ in finite time,*

2. *a subset $K$* captures the target *$C \subset K$ in $K$ under the evolutionary system $\mathcal{S}$ if $K := \text{Capt}(K, C)$, i.e., from any $x \in K$ starts* at least *one evolution viable in $K$ until it reaches the target $C$ in finite time,*

3. *that $C$ is* isolated *in $K$ if $C = \text{Viab}(K, C)$,*

4. *that $K$ is a* repeller *if $\text{Viab}(K) = \emptyset$, i.e., if the empty set is isolated in $K$.*

Naturally,

$$C \;\subset\; \text{Capt}(K, C) \;\subset\; \text{Viab}(K, C) \;\subset K$$

## 3.3 Minimax Characterization of Viability Kernels

Let us consider a target $C \subset K$ and the family $\mathcal{D}(K,C)$ of subsets $D$ satisfying $C \subset D \subset K$.

We associate with these subsets the two maps $\mathrm{Viab}(K,\cdot) : \mathcal{D}(K,C) \mapsto \mathcal{D}(K,C)$ and $\mathrm{Viab}(\cdot,C) : \mathcal{D}(K,C) \mapsto \mathcal{D}(K,C)$ associating with any subset $D \in \mathcal{D}(K,C)$ the viability kernels $\mathrm{Viab}(K,D)$ and $\mathrm{Viab}(D,C)$.

The family $\mathcal{K}_{\mathrm{Viab}}(K,C)$ is then the family of subsets $D \in \mathcal{D}(K,C)$ viable outside $C$ and the family $\mathcal{I}_{\mathrm{Viab}}(K,C)$ is then the family of subsets $D \in \mathcal{D}(K,C)$ isolated in $K$.

**Theorem 3.3** *The viability kernel* $\mathrm{Viab}(K,C)$ *of a subset* $K$ *with target* $C \subset K$ *is*

1. *the* largest *subset* $D \in \mathcal{D}(K,C)$ *viable outside the target* $C$*,*
2. *the* smallest *subset* $D \in \mathcal{D}(K,C)$ *isolated in* $K$*,*
3. *the* unique *minimax (bilateral fixed-point)* $D \in \mathcal{D}(K,C)$ *in the sense that*

$$D \;=\; \mathrm{Viab}(K,D) \;=\; \mathrm{Viab}(D,C)$$

*It is antiextensive with respect to* $K$ *and extensive with respect to* $C$*:*

$$C \;\subset\; \mathrm{Viab}(K,C) \;\subset\; K$$

*increasing in the sense that*

$$\text{If } C_1 \subset C_2 \;\&\; K_1 \subset K_2, \;\; \text{then } \mathrm{Viab}(K_1,C_1) \;\subset\; \mathrm{Viab}(K_2,C_2)$$

*is an opening with respect to* $K$*, a closing with respect to* $C$ *and the map* $C \mapsto \mathrm{Viab}(K,C)$ *is a* dilation *in the sense that*

$$\mathrm{Viab}\left(K, \bigcup_{i\in I} C_i\right) \;=\; \bigcup_{i\in I} \mathrm{Viab}(K,C_i)$$

*The upper Galois transform* $C \mapsto \mathrm{Viab}^*(K,C)$ *of* $C \mapsto \mathrm{Viab}(K,C)$ *is an opening and an erosion associating with* $C \subset K$ *the largest subset* $\mathrm{Viab}^*(K,C)$ *contained in* $C$ *and isolated in* $K$ *and* $\mathrm{Viab}^*(K,K)$ *is the largest subset of* $K$ *isolated in* $K$*.*

*The same properties hold true for the viable-capture map* $\mathrm{Capt}(K,C)$ *of a target* $C$ *viable in* $K$*.*

**Proof** — The viability kernel map is obviously a pre-opening with respect to $K$ and a pre-closing with respect to $C$. Therefore, thanks to Proposition 2.3, it is sufficient to prove that $\mathrm{Viab}(K,C)$ belongs to the intersection $\mathcal{K}_{\mathrm{Viab}}(K,C) \cap \mathcal{I}_{\mathrm{Viab}}(K,C)$, i.e., that it is viable in outside $C$ and isolated in $K$.

1. The translation property implies that the viability kernel $\mathrm{Viab}(K,C)$ is viable outside $C$:

$$\mathrm{Viab}(K,C) \subset \mathrm{Viab}(\mathrm{Viab}(K,C),C)$$

Take $x_0 \in \mathrm{Viab}(K,C)$ and prove that there exists a solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at $x_0$ viable in $\mathrm{Viab}(K,C)$ until it possibly reaches $C$. Indeed, there exists a solution $x(\cdot) \in \mathcal{S}(x_0)$ viable in $K$ until some time $T \geq 0$ either finite when it reaches $C$ or infinite. Then for all $t \in [0,T[$, the function $y(\cdot)$ defined by $y(\tau) := x(t+\tau)$ is a solution $y(\cdot) \in \mathcal{S}(x(t))$ starting at $x(t)$ and viable in $K$ until it reaches $C$ at time $T-t$. Hence $x(t)$ does belong to $\mathrm{Viab}(K,C)$ for every $t \in [0,T[$.

2. The concatenation property implies that the viability kernel $\mathrm{Viab}(K,C)$ is isolated in $K$:

$$\mathrm{Viab}(K,\mathrm{Viab}(K,C)) \subset \mathrm{Viab}(K,C)$$

Let $x$ belongs to $\mathrm{Viab}(K,\mathrm{Viab}(K,C))$. There exists at least one evolution $x(\cdot) \in \mathcal{S}(x)$ would either remain in $K$ or reach the viability kernel $\mathrm{Viab}(K,C)$ in finite time. In this case, it can be concatenated with an evolution either remaining in $\mathrm{Viab}(K,C) \subset K$ or reaching the target $C$ in finite time. This implies that $x \in \mathrm{Viab}(K,C)$.

The properties of the upper Galois transform $C \mapsto \mathrm{Viab}^*(K,C)$ follow from Theorem 2.6.

The same proof shows that viable-capture basins always enjoy these properties. □

Let us introduce the following notations. We associate with a subset $D$, a time $t \geq 0$ and a function $x(\cdot)$ the subset

$$J_D(t;x(\cdot)) := \begin{cases} x(0) & \text{if } x(t) \in D \\ \emptyset & \text{if } x(t) \notin D \end{cases}$$

and

$$I_D(t;x(\cdot)) := \bigcap_{s\in[0,t]} J(s,x(\cdot))$$

We next associate with a pair $(K,C)$ of subset where $C \subset K$ the subset

$$L_{(K,C)}(t;x(\cdot)) := I_K(t;x(\cdot)) \cap J_C(t;x(\cdot))$$

**Lemma 3.4** *The capture basin of $C$ viable in $K$ under the evolutionary system $\mathcal{S}$ is equal to*

$$\mathrm{Capt}(K,C) = \bigcup_{x\in K} \bigcup_{x(\cdot)\in\mathcal{S}(x)} \bigcup_{t\geq 0} L_{(K,C)}(t;x(\cdot)) \tag{2}$$

*and thus, the unique fixed set of the system*

$$\left\{\begin{array}{rl} D & = \bigcup_{x\in K}\bigcup_{x(\cdot)\in\mathcal{S}(x)}\bigcup_{t\geq 0} L_{(D,C)}(t;x(\cdot)) \\ & = \bigcup_{x\in K}\bigcup_{x(\cdot)\in\mathcal{S}(x)}\bigcup_{t\geq 0} L_{(K,D)}(t;x(\cdot)) \end{array}\right. \tag{3}$$

**Proof** — To say that $x$ belongs to the capture basin $\text{Capt}(K,C)$ of the target $C$ viable in $K$ amounts to saying that there exist at least one solution $x(\cdot)\in\mathcal{S}(x)$ and $t^*\geq 0$ such that

$$\left\{\begin{array}{lll} i) & \forall\, s\in[0,t^*], & x(s)\ \in K \\ ii) & & x(t^*)\ \in\ C \end{array}\right.$$

i.e., if $x = x(0)$ belongs to $L_{(K,C)}(t^*,x(\cdot))$. In other words, $x$ belongs to $\text{Capt}(K,C)$ if and only if

$$x\ \in\ \bigcup_{x(\cdot)\in\mathcal{S}(x)}\bigcup_{t\geq 0} L_{(K,C)}(t,x(\cdot))$$

Therefore formula (2) is proved. Using it, Theorem 3.3 can be reformulated as stating that the capture basin is the unique subset $D$ satisfying (3). □

# 4 Minimax Characterization of Invariance Kernels and Absorption Basins

## 4.1 Invariance Kernels and Absorption Basins

**Definition 4.1** *The* invariance kernel

$$\text{Inv}(K,C)\ :=\ \text{Inv}_{\mathcal{S}}(K,C)$$

*of $K$ with target $C$ under the evolutionary system $\mathcal{S}$ is the subset of initial states $x\in K$ such that* all *evolutions starting from $x$ are viable in $K$ forever or until they reach the target $C$ in finite time.*

*The* invariant-absorption basin

$$\text{Abs}(K,C)\ :=\ \text{Abs}_{\mathcal{S}}(K,C)$$

*of $C$ invariant in $K$ under the evolutionary system $\mathcal{S}$ is the subset of initial states $x\in K$ such that* all *evolutions starting from $x$ are viable in $K$ until they reach the target $C$ in finite time.*

*When the target $C=\emptyset$ is empty, we say that*

$$\text{Inv}(K)\ :=\ \text{Inv}_{\mathcal{S}}(K)\ :=\ \text{Inv}_{\mathcal{S}}(K,\emptyset)$$

*is the* invariance kernel *of $K$ under the evolutionary system.*

*We shall say that*

1. *a subset $K$ is* invariant outside a target $C \subset K$ *under the evolutionary system $\mathcal{S}$ if $K := \mathrm{Inv}(K, C)$, i.e., if from any $x \in K$ such that* all *evolutions starting from $x$ are viable in $K$ forever or until they reach the target $C$ in finite time,*

2. *a subset $K$* absorbs the target $C \subset K$ *in $K$ under the evolutionary system $\mathcal{S}$ if $K := \mathrm{Capt}(K, C)$, i.e., from any $x \in K$, such that* all *evolutions starting from $x$ are viable in $K$ until they reach the target $C$ in finite time,*

3. *that $C$ is* separated *in $K$ if $C = \mathrm{Inv}(K, C)$,*

Naturally,

$$C \subset \mathrm{Abs}(K, C) \subset \mathrm{Inv}(K, C) \subset K$$

## 4.2 Minimax Characterization

We associate with these subsets the two maps $\mathrm{Inv}(K, \cdot) : \mathcal{D}(K, C) \mapsto \mathcal{D}(K, C)$ and $\mathrm{Inv}(\cdot, C) : \mathcal{D}(K, C) \mapsto \mathcal{D}(K, C)$ associating with any subset $D \in \mathcal{D}(K, C)$ the invariance kernels $\mathrm{Inv}(K, D)$ and $\mathrm{Inv}(D, C)$.

The family $\mathcal{K}_{\mathrm{Inv}}(K, C)$ is then the family of subsets $D \in \mathcal{D}(K, C)$ invariant outside $C$ and the family $\mathcal{I}_{\mathrm{Inv}}(K, C)$ is then the family of subsets $D \in \mathcal{D}(K, C)$ separated in $K$.

**Theorem 4.2** *The invariance kernel $\mathrm{Inv}(K, C)$ of a subset $K$ with target $C \subset K$ is*

1. *the* largest *subset $D \in \mathcal{D}(K, C)$ invariant outside the target $C$,*

2. *the* smallest *subset $D \in \mathcal{D}(K, C)$ separated in $K$,*

3. *the* unique *minimax (bilateral fixed-point) $D \in \mathcal{D}(K, C)$ in the sense that*

$$D = \mathrm{Inv}(K, D) = \mathrm{Inv}(D, C)$$

*It is antiextensive with respect to $K$ and extensive with respect to $C$:*

$$C \subset \mathrm{Inv}(K, C) \subset K$$

*increasing in the sense that*

$$\text{If } C_1 \subset C_2 \;\&\; K_1 \subset K_2, \;\; \text{then } \mathrm{Inv}(K_1, C_1) \subset \mathrm{Inv}(K_2, C_2)$$

*is an opening with respect to $K$, a closing with respect to $C$ and the map $C \mapsto \text{Inv}(K,C)$ is an* erosion *in the sense that*

$$\text{Inv}\left(\bigcap_{i \in I} K_i, C\right) \;=\; \bigcap_{i \in I} \text{Inv}(K_i, C)$$

*The lower Galois transform $K \mapsto \text{Inv}_*(K,C)$ of $K \mapsto \text{Inv}(K,C)$ is a closing and a dilation associating with $K$ the smallest subset $\text{Inv}_*(K,C)$ containing $K$ and invariant outside $C$, and $\text{Inv}_*(C,C)$ is the smallest invariant subset containing $C$. The same properties hold true for the invariant-absorption map $\text{Abs}(K,C)$ of a target $C$ invariant in $K$.*

We recognize the definition of invariance envelopes in the last statement of the above theorem:

**Definition 4.3** *We shall say that the smallest invariant subset containing $C$, equal to $\text{Inv}_*(C,C)$, is the* invariance envelope of *$C$ and that the smallest subset of $K$ containing $C$ invariant outside $C$, equal to $\text{Inv}_*(K,C)$, is the* invariance envelope of $K$ outside $C$.

**Proof** — The invariance kernel map is obviously a pre-opening with respect to $K$ and a pre-closing with respect to $C$. Therefore, thanks to Proposition 2.3, it is sufficient to prove that $\text{Inv}(K,C)$ belongs to the intersection $\mathcal{K}_{\text{Inv}}(K,C) \cap \mathcal{I}_{\text{Inv}}(K,C)$, i.e., that it is invariant outside $C$ and separated in $K$.

1. The invariance kernel $\text{Inv}(K,C)$ is invariant outside $C$:

   $$\text{Inv}(K,C) \;\subset\; \text{Inv}(\text{Inv}(K,C),C)$$

   Take $x_0 \in \text{Inv}(K,C)$ and prove all solutions $x(\cdot) \in \mathcal{S}(x_0)$ starting at $x_0$ are viable in $\text{Inv}(K,C)$ until they possibly reach $C$. Indeed, take any solution $x(\cdot) \in \mathcal{S}(x_0)$, which is viable in $K$ until some time $T \geq 0$ either finite when it reaches $C$ or infinite. Then for all $t \in [0,T[$, consider any solution $y(\cdot) \in \mathcal{S}(x(t))$ starting at $x(t)$. Then the solution $z(\cdot)$ defined by

   $$z(\tau) \;:=\; \begin{cases} x(\tau) & \text{if} \quad \tau \in [0,t] \\ y(\tau - t) & \text{if} \quad \tau \geq t \end{cases}$$

   is also a solution of $\mathcal{S}$ starting at $x_0$ at time 0, and thus viable in $K$ until it possibly reaches the target $C$ at time $T-t$ since $x_0$ belongs to the invariance kernel $\text{Inv}(K,C)$ of $K$ with target $C$. Therefore, for every $s \in [0,T]$, $y(s) = z(t+s) \in K$ and thus, $x(t)$ does belong to $\text{Inv}(K,C)$ for every $t \in [0,T[$.

2. The invariance kernel $\mathrm{Inv}(K,C)$ is separated from $K$:

$$\mathrm{Inv}(K, \mathrm{Inv}(K,C)) \subset \mathrm{Inv}(K,C)$$

Let $x$ belongs to $\mathrm{Inv}(K, \mathrm{Inv}(K,C))$. All evolutions $x(\cdot) \in \mathcal{S}(x)$ would either remain in $K$ or reach the invariance kernel $\mathrm{Inv}(K,C)$ in finite time $T$ at some $x(T) \in \mathrm{Inv}(K,C)$. Therefore, the translation $y(\tau) := x(\tau + T)$ is an evolution starting from $x(T)$, and thus, either remains in $\mathrm{Inv}(K,C) \subset K$ or reaches the target $C$ in finite time $S \geq 0$. This implies that for all $t \geq T$, $x(t) = y(t-T)$ is viable in $K$ for ever or reach the target $C$ at time $T+S$, i.e., that $x_0$ belongs to $\mathrm{Inv}(K,C)$.

The properties of the lower Galois transform $K \mapsto \mathrm{Inv}_*(K,C)$ follow from Theorem 2.6.

The same proof shows that invariant-absorption basins also enjoy these properties. □

**Lemma 4.4** *The Absorption basin of $C$ invariant in $K$ under the evolutionary system $\mathcal{S}$ is equal to*

$$\mathrm{Abs}(K,C) = \bigcup_{x \in K} \bigcap_{x(\cdot) \in \mathcal{S}(x)} \bigcup_{t \geq 0} L_{(K,C)}(t; x(\cdot)) \tag{4}$$

*and thus, the unique fixed set of the system*

$$\begin{cases} D & = \displaystyle\bigcup_{x \in K} \bigcap_{x(\cdot) \in \mathcal{S}(x)} \bigcup_{t \geq 0} L_{(D,C)}(t; x(\cdot)) \\ & = \displaystyle\bigcup_{x \in K} \bigcap_{x(\cdot) \in \mathcal{S}(x)} \bigcup_{t \geq 0} L_{(K,D)}(t; x(\cdot)) \end{cases} \tag{5}$$

**Proof** — By definition, $x$ belongs to the absorption basin $\mathrm{Abs}(K,C)$ of the target $C$ invariant in $K$, for all solutions $x(\cdot) \in \mathcal{S}(x)$, there exist $t^* \geq 0$ such that

$$\begin{cases} i) & \forall\, s \in [0, t^*], \quad x(s) \in K \\ ii) & \qquad\qquad\quad\;\; x(t^*) \in C \end{cases}$$

i.e., if $x = x(0)$ belongs to $L_{(K,C)}(t^*, x(\cdot))$. In other words, $x$ belongs to $\mathrm{Abs}(K,C)$ if and only if

$$x \in \bigcap_{x(\cdot) \in \mathcal{S}(x)} \bigcup_{t \geq 0} L_{(K,C)}(t, x(\cdot))$$

Therefore formula (4) is proved. Using it, Theorem 4.2 can be reformulated as stating that the absorption basin is the unique subset $D$ satisfying (5). □

# 5 Construction of Openings and Closings

When the map $\mathcal{A}$ is a pre-opening with respect to $K$, but not necessarily an opening, or/and a pre-closing with respect to $C$, but not necessarily a closing, Lemma 2.2 suggests to associate with the map $\mathcal{A}$ the maps $\mathcal{A}^\sharp$ and $\mathcal{A}^\flat$ defined by

$$\mathcal{A}^\sharp(K,C) \;:=\; \bigcup_{D\in\mathcal{K}_\mathcal{A}(K,C)} D \;\&\; \mathcal{A}^\flat(K,C) \;:=\; \bigcap_{D\in\mathcal{I}_\mathcal{A}(K,C)} D$$

## 5.1 Definition of Associated Openings and Closings

We observe that if $\mathcal{A}$ is a pre-opening with respect to $K$, then

$$C \;\subset\; \mathcal{A}^\sharp(K,C) \;\subset\; \mathcal{A}(K,C) \;\subset\; K$$

and thus, that

$$\forall\, D \in \mathcal{K}_\mathcal{A}(K,C),\;\; \mathcal{A}(D,C) = \mathcal{A}^\flat(D,C) \;=\; D$$

Symmetrically, if $\mathcal{A}$ is a pre-closing with respect to $C$, then

$$C \;\subset\; \mathcal{A}(K,C) \;\subset\; \mathcal{A}^\flat(K,C) \;\subset\; K$$

and

$$\forall\, D \in \mathcal{I}_\mathcal{A}(K,C),\;\; \mathcal{A}(D,C) = \mathcal{A}^\sharp(D,C) \;=\; D$$

Consequently, if the map $(K,C) \mapsto \mathcal{A}(K,C)$ is a pre-opening with respect to $K$ and a pre-closing with respect to $C$, then inequalities (1) can be written in the form

$$\forall\,(K,C) \;\in\; \mathcal{D}^2,\;\; C \;\subset\; \mathcal{A}^\sharp(K,C) \;\subset\; \mathcal{A}(K,C) \;\subset\; \mathcal{A}^\flat(K,C) \;\subset\; K \tag{6}$$

Therefore, Lemma 2.2 states that *if $\mathcal{A}$ is a pre-opening with respect to $K$ and if $\mathcal{A}(K,C)$ belongs to $\mathcal{K}_\mathcal{A}(K,C)$, then $\mathcal{A}(K,C) = \mathcal{A}^\sharp(K,C)$ and that if $\mathcal{A}$ is a pre-closing with respect to $C$ and if $\mathcal{A}(K,C)$ belongs to $\mathcal{I}_\mathcal{A}(K,C)$, then $\mathcal{A}(K,C) = \mathcal{A}^\flat(K,C)$.*

## 5.2 The Matheron Theorem

We begin by observing that the supremum $\mathcal{A}^\sharp(K,C)$ belongs to $\mathcal{K}_\mathcal{A}(K,C)$ and is actually the "maximum" and that $\mathcal{A}^\flat(K,C)$ belongs to $\mathcal{I}_\mathcal{A}(K,C)$ and is the minimum:

**Proposition 5.1** *Assume that the family $\mathcal{D}$ is max-closed and that the map $D \mapsto \mathcal{A}(D,C)$ is a pre-opening. Then the family $\mathcal{K}_\mathcal{A}(K,C)$ is max-closed (stable for the union) and $\mathcal{A}^\sharp(K,C) \in \mathcal{K}_\mathcal{A}(K,C)$ is the largest fixed-point of the map $D \mapsto \mathcal{A}(D,C)$ in $\mathcal{D}(K,C)$:*

$$\mathcal{A}(\mathcal{A}^\sharp(K,C),C) \;=\; \mathcal{A}^\sharp(K,C)$$

*Symmetrically, if the family $\mathcal{D}$ is min-closed (stable for the intersection) and if the map $D \mapsto \mathcal{A}(K,D)$ is a pre-closing, then the family $\mathcal{I}_{\mathcal{A}}(K,C)$ is min-closed and $\mathcal{A}^\flat(K,C) \in \mathcal{I}_{\mathcal{A}}(K,C)$ is the smallest fixed-point of the map $D \mapsto \mathcal{A}(K,D)$ in $\mathcal{D}(K,C)$:*

$$\mathcal{A}(K, \mathcal{A}^\flat(K,C)) \;=\; \mathcal{A}^\flat(K,C)$$

**Proof** — We reproduce the proof that can be found for instance in [55, Mattioli & Schmitt], Theorem III.3.3, p.46. Consider a family of subsets $D_i \in \mathcal{K}_{\mathcal{A}}(K,C)$. Therefore $D_i \subset \mathcal{A}(D_i,C) \subset \mathcal{A}\left(\bigcup_{i\in I} D_i, C\right)$ since $\mathcal{A}$ is antiextensive and increasing. Consequently,

$$\bigcup_{i\in I} D_i \;\subset\; \mathcal{A}\left(\bigcup_{i\in I} D_i, C\right)$$

so that $\bigcup_{i\in I} D_i$ belongs to $\mathcal{K}_{\mathcal{A}}(K,C)$. This proves that the family $\mathcal{K}_{\mathcal{A}}(K,C)$ is stable for the union. Hence its supremum $\mathcal{A}^\sharp(K,C)$ belongs to $\mathcal{K}_{\mathcal{A}}(K,C)$ and thus, satisfies

$$\mathcal{A}(\mathcal{A}^\sharp(K,C),C) \;=\; \mathcal{A}^\sharp(K,C)$$

If $\mathcal{A}(K,C)$ belongs to $\mathcal{K}_{\mathcal{A}}(K,C)$, then

$$\mathcal{A}(K,C) \;\subset\; \mathcal{A}(\mathcal{A}(K,C),C) \;\subset\; \mathcal{A}^\sharp(K,C) \;\subset\; \mathcal{A}(K,C) \;\square$$

Matheron's theorem (see [52, Matheron']) states in essence that the map $D \mapsto \mathcal{A}^\sharp(D,C)$ is an opening and that the map $D \mapsto \mathcal{A}^\flat(K,D)$ is a closing:

**Theorem 5.2 (Matheron)** *Assume that the family $\mathcal{D}$ is max-closed and that the map $D \mapsto \mathcal{A}(D,C)$ is a pre-opening and $D \mapsto \mathcal{A}^\sharp(K,D)$ is extensive whenever $D \mapsto \mathcal{A}(K,D)$ is extensive[1]. Then the map $D \mapsto \mathcal{A}^\sharp(D,C)$ is the* unique *opening onto the family $\mathcal{K}_{\mathcal{A}}(K,C)$:*

$$\mathcal{A}^\sharp(\mathcal{A}^\sharp(K,C),C) \;=\; \mathcal{A}^\sharp(K,C)$$

*and the families*

$$\mathcal{K}_{\mathcal{A}}(K,C) \;=\; \mathcal{K}_{\mathcal{A}^\sharp}(K,C)$$

*coincide.*

*Similarly, if the family $\mathcal{D}$ is min-closed and if the map $D \mapsto \mathcal{A}(K,D)$ is a pre-closing, then the map $D \mapsto \mathcal{A}^\flat(K,D)$ is the unique closing onto the family $\mathcal{I}_{\mathcal{A}}(K,C)$:*

$$\mathcal{A}^\flat(K, \mathcal{A}^\flat(K,C)) \;=\; \mathcal{A}^\flat(K,C)$$

*and the families*

$$\mathcal{I}_{\mathcal{A}}(K,C) \;=\; \mathcal{I}_{\mathcal{A}^\flat}(K,C)$$

*coincide.*

[1] However, $D \mapsto \mathcal{A}^\sharp(K,D)$ is not necessarily increasing whenever $D \mapsto \mathcal{A}(K,D)$ is increasing, although this happens for some specific examples.

**Proof** — We reproduce the proof that can be found for instance in [52, Matheron] and [55, Mattioli & Schmitt], Theorem III.3.3, p.46. The map $K \mapsto \mathcal{A}^\sharp(K,C) := \bigcup_{D \in \mathcal{K}_\mathcal{A}(K,C)} D \subset K$ is antiextensive. Since $\mathcal{K}_\mathcal{A}(K_1,C) \subset \mathcal{K}_\mathcal{A}(K_2,C)$ whenever $K_1 \subset K_2$, the map $K \mapsto \mathcal{A}^\sharp(K,C)$ is increasing.

If $D$ belongs to $\mathcal{K}_\mathcal{A}(K,C)$, then $D \subset \mathcal{A}^\sharp(D,C)$ and thus belongs to $\mathcal{K}_{\mathcal{A}^\sharp}(K,C)$. Taking $D := \mathcal{A}^\sharp(K,C)$ that belongs to $\mathcal{K}_{\mathcal{A}^\sharp}(K,C)$, we infer that

$$\mathcal{A}^\sharp(K,C) \subset \mathcal{A}^\sharp(\mathcal{A}^\sharp(K,C),C)$$

By Proposition 2.3, $\mathcal{A}^\sharp(K,C)$ is the largest fixed point $D \in \mathcal{K}_{\mathcal{A}^\sharp}(K,C)$ of the map $K \mapsto \mathcal{A}^\sharp(K,C)$. Hence this map is an opening.

Conversely, if $D$ belongs to $\mathcal{K}_{\mathcal{A}^\sharp}(K,C)$, then

$$D \ \subset \ \mathcal{A}^\sharp(D,C) \ := \ \mathcal{A}(D,C)$$

so that $D \subset \mathcal{A}(D,C)$ and thus belongs to $\mathcal{K}_\mathcal{A}(K,C)$. Consequently, the families $\mathcal{A}$ and $\mathcal{A}^\sharp$ coincide:

$$\mathcal{K}_\mathcal{A}(K,C) \ = \ \mathcal{K}_{\mathcal{A}^\sharp}(K,C)$$

Let us consider now a surjective opening $\Pi : \mathcal{D}(K,C) \mapsto \mathcal{K}_\mathcal{A}(K,C)$. Since $\Pi(K)$ belongs to $\mathcal{K}_\mathcal{A}(K,C) = \mathcal{K}^\sharp_\mathcal{A}(K,C)$, then $\Pi(K) \subset \mathcal{A}^\sharp(\Pi(K),C)$. Since $\mathcal{A}^\sharp(\cdot,C)$ is increasing and $\Pi$ is antiextensive, we deduce that $\mathcal{A}^\sharp(\Pi(K),C) \subset \mathcal{A}^\sharp(K,C)$. We finally check that $\mathcal{A}^\sharp(K,C) \subset \Pi(K)$ because, for every $D \in \mathcal{K}_{\mathcal{A}^\sharp}(K,C)$,

$$D \ \subset \ \mathcal{A}^\sharp(D,C) \ = \ \mathcal{A}^\sharp(\Pi(D),C) \ \subset \ \Pi(D) \ \subset \ \Pi(K)$$

since $\Pi(D) = D \subset \Pi(K)$ whenever $D \in \mathcal{K}_\mathcal{A}(K,C)$ and $\Pi(D) \subset \Pi(K)$. Therefore $\Pi(K) = \mathcal{A}^\sharp(K,C)$. □

### 5.3 The Opening and Closing Algorithms

One can define the Opening and Closing Algorithms for computing the opening $\mathcal{A}^\sharp$ and closing $\mathcal{A}^\flat$ associated with $\mathcal{A}$.

For the closing $\mathcal{A}^\flat$, we define recursively the Closing Algorithm by a sequence of subsets $C_n$ by $C_0 := C$ and

$$\forall\, n \geq 0, \ \ C_{n+1} \ := \ \mathcal{A}(K,C_n)$$

Then, if $\mathcal{A}$ is increasing with respect to $K$,

$$\bigcup_{n \geq 0} C_n \ \subset \ \mathcal{A}^\flat(K,C)$$

Indeed, let any $D$ belong to $\mathcal{I}_{\mathcal{A}}(K,C)$. Hence $C_0 := C \subset D$, and thus $\mathcal{A}(K,D) \subset D$ and $C_1 := \mathcal{A}(K,C_0) \subset \mathcal{A}(K,D)$ since $\mathcal{A}$ is increasing with respect to $C$, so that $C_1 \subset D$. Assume now that $C_n \subset D$ and check that $C_{n+1} \subset D$. We infer that $\mathcal{A}(K,D) \subset D$ and $C_{n+1} := \mathcal{A}(K,C_n) \subset \mathcal{A}(K,D)$, so that

$$\forall\, D \in \mathcal{I}_{\mathcal{A}}(K,C),\ \ \bigcup_{n\geq 0} C_n \ \subset\ D$$

In the same way, the Opening Algorithm for computing the closing $\mathcal{A}^\sharp$ associated with $\mathcal{A}$ is defined recursively by $K_0 := K$ and

$$\forall\, n \geq 0,\ \ K_{n+1} \ :=\ \mathcal{A}(K_n, C)$$

Then, if $\mathcal{A}$ is increasing with respect to $K$,

$$\mathcal{A}^\sharp(K,C) \ \subset\ \bigcap_{n\geq 0} K_n$$

Indeed, let any $D$ belong to $\mathcal{K}_{\mathcal{A}}(K,C)$. Hence $D \subset K_0 := K$, and thus $D \subset \mathcal{A}(D,C)$ and $\mathcal{A}(D,C) \subset \mathcal{A}(K_0,C) =: K_1$ since $\mathcal{A}$ is increasing with respect to $K$. Assume now that $D \subset K_n$ and check that $D \subset K_{n+1}$. We infer that $D \subset \mathcal{A}(D,C)$ and $\mathcal{A}(D,C) \subset \mathcal{A}(K_n,C) =: K_{n+1}$, so that

$$\forall\, D \in \mathcal{K}_{\mathcal{A}}(K,C),\ \ D \subset \bigcap_{n\geq 0} K_n$$

# 6 Viability Kernels and Capture Basins Algorithms for Discrete Systems

Let us consider a set-valued map $\Phi : X \rightsquigarrow X$ from a metric space $X$ to itself, governing the evolution $\vec{x} : n \mapsto x_n$ defined by

$$\forall\, n \geq 0,\ \ x_{n+1} \ \in\ \Phi(x_n)$$

We denote by $\mathcal{S}_\Phi(x)$ the set of evolutions $\vec{x}$ of solutions to the above discrete system starting at $x_0$. Replacing the space $\mathcal{C}(0,\infty;X)$ of continuous time-dependent functions by the space $X^{\mathbf{N}}$ of discrete-time dependent function (sequences) and making the necessary adjustments in definitions, we can still regard $\mathcal{S}_\Phi$ as an evolutionary system from $X$ to $X^{\mathbf{N}}$. We thus define the viability kernels $\mathrm{Viab}_{\mathcal{S}_\Phi}(K,C)$ and the invariance kernels $\mathrm{Inv}_{\mathcal{S}_\Phi}(K,C)$ and derive from Theorems 3.3 and 4.2 the following

**Theorem 6.1** *The viability kernel* $\mathrm{Viab}_{\mathcal{S}_\Phi}(K,C)$ *of a subset* $K$ *with target* $C \subset K$ *is the* unique *minimax (bilateral fixed-point)* $D \in \mathcal{D}(K,C)$ *in the sense that*

$$D \;=\; \mathrm{Viab}_{\mathcal{S}(\oplus)}(K,D) \;=\; \mathrm{Viab}_{\mathcal{S}(\oplus)}(D,C)$$

*and the invariance kernel* $\mathrm{Inv}_{\mathcal{S}(\oplus)}(K,C)$ *of a subset* $K$ *with target* $C \subset K$ *is the* unique *minimax (bilateral fixed-point)* $D \in \mathcal{D}(K,C)$ *in the sense that*

$$D \;=\; \mathrm{Inv}_{\mathcal{S}(\oplus)}(K,D) \;=\; \mathrm{Inv}_{\mathcal{S}(\oplus)}(D,C)$$

## 6.1 An Alternate Characterization of Viability and Invariance Kernels

We shall prove that the viability and invariance kernels under a discrete systems are fixed points of openings associated with maps $\mathcal{A}$ that we shall define.

We set $\Phi^{-1}(K) := \{x \mid \Phi(x) \cap K \neq \emptyset\}$ and $\Phi^{\ominus 1}(K) := \{x \mid \Phi(x) \subset K\}$ and we introduce the maps $\mathcal{A}_{\Phi^{-1}} : \mathcal{D} \times \mathcal{D} \mapsto \mathcal{D}$ and $\mathcal{A}_{\Phi^{\ominus 1}}$ defined on the family $\mathcal{D}$ of all subsets of $X$ by

$$\begin{cases} i) & \mathcal{A}_{\Phi^{-1}}(K,C) \;:=\; C \cup (K \cap \Phi^{-1}(K)) \\ ii) & \mathcal{A}_{\Phi^{\ominus 1}}(K,C) \;:=\; C \cup (K \cap \Phi^{\ominus 1}(K)) \end{cases}$$

They are pre-openings with respect to $K$ and a pre-closings with respect to $C$.

We then can go one step farther and characterize these maps as openings and closings of these associated maps : For that purpose, we provide a characterization of the discrete viability and invariance kernels:

**Theorem 6.2** *The viability kernel* $\mathrm{Viab}_{\mathcal{S}_\Phi}(K,C)$ *of* $K$ *with target* $C$ *under* $\Phi$ *is the largest subset* $D$ *of the family* $\mathcal{K}_{\mathcal{A}_{\Phi^{-1}}}(K,C)$ *of subsets* $D \in \mathcal{D}(K,C)$ *such that* $D \subset C \cup (D \cap \Phi^{-1}(D))$.

*In the same way, the invariance kernel* $\mathrm{Inv}_{\mathcal{S}_\Phi}(K,C)$ *of* $K$ *with target* $C$ *under* $\Phi$ *is the largest subset* $D$ *of the family* $\mathcal{K}_{\mathcal{A}_{\Phi^{\ominus 1}}}(K,C)$ *of subsets* $D \in \mathcal{D}(K,C)$ *such that* $D \subset C \cup (D \cap \Phi^{\ominus 1}(D))$.

**Proof —** Indeed, to say that $D \subset C \cup (D \cap \Phi^{-1}(D))$ means that $D \backslash C \subset \Phi^{-1}(D)$, i.e., that for any $x \in D$, either $x$ belongs to $C$ or else, that there exists $y \in \Phi(x) \cap D$. Consequently, for any $x_0 \in D$, either $x_0 \in C$ and $x_0$ belongs to the viability kernel $\mathrm{Viab}_{\mathcal{S}_\Phi}(K,C)$ of $K$ with target $C$ or else, there exists $x_1 \in \Phi(x_0) \cap D$, so that we can continue by induction constructing a solution $\vec{x} \in \mathcal{S}_\Phi(x_0)$ starting from $x_0 \in D$ viable in $D$ — and thus, in $K$ — either until some finite $N$ when $x_N \in C$ or forever. Hence $D \subset \mathrm{Viab}_{\mathcal{S}_\Phi}(D,C) \subset \mathrm{Viab}_{\mathcal{S}_\Phi}(K,C)$.

Conversely, if $D \subset \mathrm{Viab}_{\mathcal{S}_\Phi}(D,C)$, there exists $\vec{x} \in \mathcal{S}_\Phi(x_0)$ starting from $x_0$ and viable in $D$ outside $C$. Therefore, starting from any $x_0 \in D$, either $x_0 \in C$ or else, there exists $x_1 \in \Phi(x_0)$ that belongs to $D$, i.e., $x_0 \in \Phi^{-1}(x_1) \subset \Phi^{-1}(D)$. Hence $D \subset C \cup (D \cap \Phi^{-1}(D))$.
$\square$

The above Characterization Theorem 6.2 and Matheron's Theorem 5.2 imply the following result:

**Theorem 6.3** *Let $\Phi : X \mapsto X$ be a set-valued map from a set $X$ to itself.*

1. *The map $K \mapsto \mathrm{Viab}_{\mathcal{S}_\Phi}(K, C)$ is the opening associated with the map $\mathcal{A}_{\Phi^{-1}} : K \mapsto C \cup (K \cap \Phi^{-1}(K))$,*
2. *The map $K \mapsto \mathrm{Inv}_{\mathcal{S}_\Phi}(K, C)$ is the opening associated with the map $\mathcal{A}_{\Phi^{\ominus 1}} : K \mapsto C \cup (K \cap \Phi^{\ominus 1}(K))$.*

**Proof** — Indeed, Theorem 6.2 states that $\mathrm{Viab}_{\mathcal{S}_\Phi}(K, C)$ is the largest of subsets $D \in \mathcal{K}_{\mathcal{A}_{\Phi^{-1}}}(K, C)$, and thus, Theorem 5.2 implies that

$$\mathrm{Viab}_{\mathcal{S}_\Phi}(K, C) \ := \ \mathcal{A}^{\sharp}_{\Phi^{-1}}(K, C)$$

The proof for the invariance kernel is identical. □

## 6.2 The Viability Kernel Algorithm

Since the discrete viability kernel map $K \mapsto \mathrm{Viab}_{\mathcal{S}_\Phi}(K, C)$ is the opening associated with the pre-opening $K \mapsto C \cup (K \cap \Phi^{-1}(K))$, we can define the Opening Algorithm which, here, is the Viability Algorithm designed by Patrick Saint-Pierre in [61, Saint-Pierre] :

Starting with $K_0 := K$, we define recursively the subsets $K_n$ by

$$\forall\, n \geq 0, \ \ K_{n+1} \ := \ C \cup (K_n \cap \Phi^{-1}(K_n))$$

We already know that

$$\mathrm{Viab}_{\mathcal{S}_\Phi}(K, C) \ \subset \ \bigcap_{n \geq 0} K_n$$

Adequate topological assumptions imply the equality:

**Proposition 6.4** *Let us assume that $K$ and $C \subset K$ are closed and that the graphical restriction*[2] *$\Phi|^{K}_{|K}$ of $\Phi$ to $K \times K$ is upper semicompact*[3]*. Then the subsets $K_n$ are closed and*

$$\mathrm{Viab}_{\mathcal{S}_\Phi}(K, C) \ = \ \bigcap_{n \geq 0} K_n$$

[2] defined by $\mathrm{Graph}(\Phi|^{K}_{|K}) := \mathrm{Graph}(\Phi)$.

[3] A set-valued map $\Phi : X \rightsquigarrow Y$ is said to be *upper semicompact* at $x$ if for every sequence $x_n \in \mathrm{Dom}(\Phi)$ converging to $x$ and for every sequence $y_n \in \Phi(x_n)$, there exists a subsequence $y_{n_p}$ converging to some $y \in \Phi(x)$.

**Proof** — Let $x$ belongs to $K_\infty := \bigcap_{n\geq 0} K_n$. If $x \in K_\infty \backslash C$, then for every $n \geq 0$, $x \in \Phi^{-1}(K_n)$, and thus, there exists $y_n \in \Phi(x) \in K_n$. Since the images $\Phi(x) \cap K$ are compact, a subsequence (again denoted by) $y_n$ converges to some $y_* \in \Phi(x) \cap \bigcap_{n\geq 0} \overline{K_n}$. Hence, whenever the subsets $K_n$ are closed,

$$x \;\in\; K_\infty \cap \Phi^{-1}(y_*) \;\subset\; K_\infty \cap \Phi^{-1}(K_\infty)$$

and thus, $K_\infty$ is viable outside $C$ under $\Phi$, and consequently, contained in $\mathrm{Viab}_{\mathcal{S}_\Phi}(K, C)$.

To prove recursively that the subsets $K_n$ are closed, it is enough to prove that $K_{n+1} := K_n \cap \Phi^{-1}(K_n)$ is closed whenever $K_n$ is closed. Let $x_j \in K_{n+1}$ converge to $x$, that belongs to $K_n$. Furthermore, there exists $y_j \in K_n \cap \Phi(x_j)$. Since the graphical restriction $\Phi_{|K}^{|K}$ of $\Phi$ is assumed to be upper semicompact, a subsequence (again denoted by) $y_j$ converges to $y \in \Phi_{|K}^{|K}(x)$ and actually, to $\Phi(x) \in K_n$. This implies that $K_{n+1}$ is closed. □

For the invariance kernels, starting with $L_0 := K$, we define recursively the subsets $L_n$ by

$$\forall\, n \geq 0, \;\; L_{n+1} \;:=\; C \cup (L_n \cap \Phi^{\ominus 1}(L_n))$$

We already know that

$$\mathrm{Inv}_{\mathcal{S}_\Phi}(K, C) \;\subset\; \bigcap_{n\geq 0} L_n$$

**Proposition 6.5** *If we assume that $K$ and $C \subset K$ are closed and that $\Phi$ is lower semicontinuous, then the subsets $L_n$ are closed and*

$$\mathrm{Inv}_{\mathcal{S}_\Phi}(K, C) \;=\; \bigcap_{n\geq 0} L_n$$

**Proof** — Let $x$ belongs to $L_\infty := \bigcap_{n\geq 0} L_n$. If $x \in L_\infty \backslash C$, then for every $n \geq 0$, $x \in \Phi^{\ominus 1(}K_n)$, and thus, $\Phi(x_n) \subset K_n$. Hence

$$\Phi(x) \;\subset\; \bigcap_{n\geq 0} L_n \;=:\; L_\infty$$

Therefore,

$$x \;\in\; L_\infty \cap \Phi^{\ominus 1}(L_\infty)$$

and thus, $L_\infty$ is invariant outside $C$ under $\Phi$, and consequently, contained in $\mathrm{Inv}_{\mathcal{S}_\Phi}(K, C)$.

To prove recursively that the subsets $L_n$ are closed, it is enough to prove that $L_{n+1} := L_n \cap \Phi^{\ominus 1}(L_n)$ is closed whenever $L_n$ is closed. Let $x_j \in L_{n+1}$ converge to $x$, that belongs to

$L_n$. Since $\Phi$ is lower semicontinuous, for any $y \in \Phi(x)$, there exists $y_j \in \Phi(x_j)$ converging to $y$. Since $\Phi(x_j) \subset L_n$, we know that $y_j$ belongs to $L$, and consequently, that $y$ belongs to $L_n$. Hence $\Phi(x) \subset L_n$, so that $x$ belongs to $L_{n+1}$. □

## 6.3 An Alternate Characterization of Capture and Absorptions Basins

Let us consider now the capture basins $\mathrm{Capt}_{\mathcal{S}_\Phi}(K,C)$ and the absorption basins $\mathrm{Abs}_{\mathcal{S}_\Phi}(K,C)$ and provide another characterization involving the maps $\mathcal{B}_{\Phi^{-1}} : \mathcal{D}\times\mathcal{D} \mapsto \mathcal{D}$ and $\mathcal{B}_{\Phi^{\ominus 1}}$ defined by

$$\begin{cases} i) & \mathcal{B}_{\Phi^{-1}}(K,C) \ := \ K \cap (C \cup \Phi^{-1}(C)) \\ ii) & \mathcal{B}_{\Phi^{\ominus 1}}(K,C) \ := \ K \cap (C \cup \Phi^{\ominus 1}(C)) \end{cases}$$

They are pre-openings with respect to $D$ and a pre-closings with respect to $C$.

**Theorem 6.6** *The capture basin* $\mathrm{Capt}_{\mathcal{S}_\Phi}(K,C)$ *of* $K$ *of* $C$ *viable in* $K$ *under* $\Phi$ *is the smallest subset* $D$ *of the family* $\mathcal{I}_{\mathcal{B}_{\Phi^{-1}}}(K,C)$ *of subsets* $D \in \mathcal{D}(K,C)$ *such that* $K\cap(D\cup\Phi^{-1}(D)) \subset D$.

*In the same way, the absorption basin* $\mathrm{Abs}_{\mathcal{S}_\Phi}(K,C)$ *of* $K$ *with target* $C$ *under* $\Phi$ *is the smallest subset* $D$ *of the family* $\mathcal{I}_{\mathcal{B}_{\Phi^{\ominus 1}}}(K,C)$ *of subsets* $D \in \mathcal{D}(K,C)$ *such that* $K \cap (D \cup \Phi^{\ominus 1}(D)) \subset D$.

**Proof —** The capture basin $\mathrm{Capt}_{\mathcal{S}_\Phi}(K,C)$ belongs to the family $\mathcal{I}_{\mathcal{B}_{\Phi^{-1}}}(K,C)$. We have to check that whenever we take $x \in K\backslash\mathrm{Capt}_{\mathcal{S}_\Phi}(K,C)$ satisfying

$$x \ \in \ \Phi^{-1}(\mathrm{Capt}_{\mathcal{S}_\Phi}(K,C))$$

then such an $x$ belongs to the capture basin. Indeed, such an element satisfies $\Phi(x) \cap \mathrm{Capt}_{\mathcal{S}_\Phi}(K,C) \neq \emptyset$. Then $x$ captures $\mathrm{Capt}_{\mathcal{S}_\Phi}(K,C)$ in one step and thus, the target $C$ in a finite number of steps. Hence

$$K \cap (\mathrm{Capt}_{\mathcal{S}_\Phi}(K,C) \cup \Phi^{-1}(\mathrm{Capt}_{\mathcal{S}_\Phi}(K,C))) \ \subset \ \mathrm{Capt}_{\mathcal{S}_\Phi}(K,C)$$

Let us prove now that the capture basin is the smallest of the subsets $D \in \mathcal{I}_{\mathcal{B}_{\Phi^{-1}}}(K,C)$. Otherwise, there would exist some $D_0 \in \mathcal{I}_{\mathcal{B}_{\Phi^{-1}}}(K,C)$ and some

$$x_0 \ \in \ \mathrm{Capt}_{\mathcal{S}_\Phi}(K,C)\backslash D$$

Taking the complement, we see that $K\backslash D_0 \subset \Phi^{\ominus 1}(X\backslash D_0)$. Since $x_0 \in K\backslash D_0$, we infer that $\Phi(x_0) \subset X\backslash D_0 \subset X\backslash C$, so that no solution $\vec{x} \in \mathcal{S}(x_0)$ can ever reach the target $C$ before leaving $K$. □

The above Characterization Theorem 6.2 and Matheron's Theorem 5.2 imply the following result:

**Theorem 6.7** *Let $\Phi : X \mapsto X$ be a set-valued map from a set $X$ to itself.*

1. *The map $C \mapsto \mathrm{Capt}_{\mathcal{S}_\Phi}(K,C)$ is the closing associated with the map $C \mapsto K \cap (C \cup \Phi^{-1}(C))$,*

2. *The map $C \mapsto \mathrm{Abs}_{\mathcal{S}_\Phi}(K,C)$ is the closing associated with the map $C \mapsto K \cap (C \cup \Phi^{\ominus 1}(C))$,*

**Proof** — Indeed, Theorem 6.6 states that $\mathrm{Capt}_{\mathcal{S}_\Phi}(K,C)$ is the smallest of subsets $D \in \mathcal{I}_{\mathcal{B}_{\Phi^{-1}}}(K,C)$, and thus, Theorem 5.2 implies that

$$\mathrm{Capt}_{\mathcal{S}_\Phi}(K,C) \ := \ \mathcal{B}^\flat_{\Phi^{-1}}(K,C)$$

The proof for the absorption basin is identical. □

## 6.4 The Capture Basin Algorithm

Since the discrete capture basin map $C \mapsto \mathrm{Capt}_{\mathcal{S}_\Phi}(K,C)$ is the closing associated with the pre-closing $C \mapsto K \cap (C \cup \Phi^{-1}(C))$, we can define the Closing Algorithm which, here, is the Capture Basin Algorithm designed by Patrick Saint-Pierre and Dominique Pujal :

Starting with $C_0 := C$, we define recursively the subsets $C_n$ by

$$\forall\, n \geq 0, \ \ C_{n+1} \ := \ K \cap (C_n \cup \Phi^{-1}(C_n))$$

We already know that

$$\mathrm{Capt}_{\mathcal{S}_\Phi}(K,C) \ \subset \ \bigcup_{n \geq 0} C_n$$

**Proposition 6.8** *The discrete viable-capture basin*

$$\mathrm{Capt}_{\mathcal{S}_\Phi}(K,C) \ = \ \bigcup_{n \geq 0} C_n \ = \ \mathcal{B}^\flat_{\Phi^{-1}}(K,C)$$

*is the union of these subsets $C_j$.*

**Proof** — Let us prove first that $C_j$ *is the subset of elements $x \in K$ from which starts a solution $\{x_n\}_{n \geq 0}$ to $x_{n+1} \in \Phi(x_n)$ such that $x_n \in K$ for $n = 0, \ldots, j-1$ and $x_j \in C$.*

This is obvious for $j = 1$. By induction, assume that it is true for $j$ and let us prove that it is true for $j+1$: Let $x$ belong to $C_{j+1} := K \cap \Phi^{-1}(C_j)$. There exist $y \in \Phi(x) \cap C_j$, and thus, a sequence $y_{n+1} \in \Phi(y_n)$ such that $y_0 = y$, $y_n \in K$ for $j = 0, \ldots, j-1$ and $y_j \in C$. Setting $x_0 := x$ and $x_n := y_{n-1}$ for $n \geq 1$, we see that $x_0 := x$ belongs to $K$, that $x_1 := y \in \Phi(x) = \Phi(x_0)$ belongs to $K$, that $x_n \in \Phi(x_{n-1})$ belongs to $K$ for $n \leq j$ and that $x_{j+1} := y_j \in \Phi(x_j)$ belongs to $C$.

Therefore, if $x$ belongs to one of the subsets $C_j$, a solution $\{x_n\}_{n\geq 0}$ to the discrete problem $x_{n+1} \in \Phi(x_n)$ is viable in $K$ until it reaches $C$ in $j$ steps. □

For the absorption basins, Starting with $B_0 := C$, we define recursively the subsets $B_n$ by

$$\forall\, n \geq 0,\;\; B_{n+1} \;:=\; K \cap (B_n \cup \Phi^{\ominus 1}(B_n))$$

We already know that

$$\mathrm{Abs}_{\mathcal{S}_\Phi}(K,C) \;\subset\; \bigcup_{n\geq 0} B_n$$

**Proposition 6.9** *If we assume that $K$ and $C \subset K$ are closed and that $\Phi$ is lower semicontinuous, then the subsets $B_n$ are closed and*

$$\mathrm{Abs}_{\mathcal{S}_\Phi}(K,C) \;=\; \bigcup_{n\geq 0} B_n$$

**Proof** — Let $x$ belongs to $B_\infty := \bigcup_{n\geq 0} B_n$. If $x \in B_\infty \backslash C$, then for every $n \geq 0$, $x \in \Phi^{\ominus 1(}K_n)$, and thus, $\Phi(x_n) \subset K_n$. Hence

$$\Phi(x) \;\subset\; \bigcup_{n\geq 0} B_n \;=:\; B_\infty$$

Hence,

$$x \;\in\; B_\infty \cap \Phi^{\ominus 1}(B_\infty)$$

and thus, $B_\infty$ is invariant outside $C$ under $\Phi$, and consequently, contained in $\mathrm{Abs}_{\mathcal{S}_\Phi}(K,C)$.

To prove recursively that the subsets $B_n$ are closed, it is enough to prove that $B_{n+1} := B_n \cup \Phi^{\ominus 1}(B_n)$ is closed whenever $B_n$ is closed. Let $x_j \in B_{n+1}$ converge to $x$, that belongs to $B_n$. Since $\Phi$ is lower semicontinuous, for any $y \in \Phi(x)$, there exists $y_j \in \Phi(x_j)$ converging to $y$. Since $\Phi(x_j) \subset B_n$, we know that $y_j$ belongs to $B_n$, and consequently, that $y$ belongs to $B_n$. Hence $\Phi(x) \subset B_n$, so that $x$ belongs to $B_{n+1}$. □

# 7 Conditional Maps

Let us consider an evolutionary game described by a map $(x,v) \in X \times \mathcal{V} \mapsto \mathcal{S}_v(x) \in \mathcal{C}(0,\infty;X)$, where, for each $v \in \mathcal{V}$, $x \mapsto \mathcal{S}_v(x)$ is an evolutionary system.

In this case, we define the conditional viability kernel as

$$\bigcap_{v\in\mathcal{V}} \mathrm{Viab}_v(K,C)$$

and the guaranteed viability kernel as

$$\bigcup_{v\in\mathcal{V}} \mathrm{Inv}_v(K,C)$$

One can check that only the map $C \mapsto \bigcap_{v\in\mathcal{V}} \mathrm{Viab}_v(K,C)$ is a closing. The map $K \mapsto \bigcap_{v\in\mathcal{V}} \mathrm{Viab}_v(K,C)$ is only a pre-opening, but not necessarily an opening.

However, some of the properties of the conditional and guaranteed viability kernels that we shall review are algebraic in nature.

Let us consider a family of maps $(K,C) \mapsto \mathcal{A}_v(K,C)$ where $v \in \mathcal{V}$ ranges over a subset $\mathcal{V}$. We associate with $\mathcal{A}$ the conditional map $\mathcal{A}_{\mathcal{V}}$ defined by

$$\mathcal{A}_{\mathcal{V}}(K,C) \;:=\; \bigcap_{v\in\mathcal{V}} \mathcal{A}_v(K,C)$$

the guaranteed map $\mathcal{A}^{\mathcal{V}}$ defined by

$$\mathcal{A}^{\mathcal{V}}(K,C) \;:=\; \bigcup_{v\in\mathcal{V}} \mathcal{A}_v(K,C)$$

**Lemma 7.1** *If for any $v \in \mathcal{V}$,*

1. *the maps $K \mapsto \mathcal{A}_v(K,C)$ are pre-openings, so are the maps $K \mapsto \mathcal{A}_{\mathcal{V}}(K,C)$ and $K \mapsto \mathcal{A}^{\mathcal{V}}(K,C)$,*
2. *the maps $C \mapsto \mathcal{A}_v(K,C)$ are pre-closings, so are the maps $C \mapsto \mathcal{A}_{\mathcal{V}}(K,C)$ and $K \mapsto \mathcal{A}^{\mathcal{V}}(K,C)$.*

We begin by providing sufficient conditions for the existence of a (unique) minimax (bilateral fixed-point) of the guaranteed map $\mathcal{A}^{\mathcal{V}}$ and the conditional map $\mathcal{A}_{\mathcal{V}}$:

**Proposition 7.2** *Let us consider a family of maps $\{\mathcal{A}_v\}_{v\in\mathcal{V}}$ that are pre-openings with respect to $K$ and pre-closings with respect to $C$.*

1. *If a subset $D \in \mathcal{D}(K,C)$ satisfies*

$$\begin{cases} i) & \forall\, v \in \mathcal{V},\;\; \mathcal{A}_v(K,D) \;\subset\; D \\ ii) & \exists\, v_0 \in \mathcal{V} \;\; \text{such that}\; D \;\subset\; \mathcal{A}_{v_0}(D,C) \end{cases} \tag{7}$$

*then $D$ is equal to $\mathcal{A}^{\mathcal{V}}(K,C)$ and is the* unique minimax (bilateral fixed-point) *of the map $\mathcal{A}^{\mathcal{V}}$:*

$$\mathcal{A}^{\mathcal{V}}(\mathcal{A}^{\mathcal{V}}(K,C),C) \;=\; \mathcal{A}^{\mathcal{V}}(K,C) \;=\; \mathcal{A}^{\mathcal{V}}(K,\mathcal{A}^{\mathcal{V}}(K,C))$$

2. *If a subset* $D \in \mathcal{D}(K,C)$ *satisfies*

$$\left\{ \begin{array}{ll} i) & \forall\, v \in \mathcal{V},\;\; D \;\subset\; \mathcal{A}_v(D,C) \\ ii) & \exists\, v_0 \in \mathcal{V} \;\; \text{such that}\, \mathcal{A}_{v_0}(K,D) \;\subset\; D \end{array} \right. \tag{8}$$

*then* $D$ *is equal to* $\mathcal{A}_{\mathcal{V}}(K,C)$ *and is the* unique minimax (bilateral fixed-point) *of the conditional map* $\mathcal{A}_{\mathcal{V}}$*:*

$$\mathcal{A}_{\mathcal{V}}(\mathcal{A}_{\mathcal{V}}(K,C),C) \;=\; \mathcal{A}_{\mathcal{V}}(K,C) \;=\; \mathcal{A}_{\mathcal{V}}(K,\mathcal{A}_{\mathcal{V}}(K,C))$$

**Proof** — By Lemma 7.1, $\mathcal{A}_{\mathcal{V}}$ is a pre-opening with respect to $K$ and a pre-closing with respect to $C$.

Condition (7)i) implies that $\mathcal{A}^{\mathcal{V}}(K,D) \subset D$, and thus, that $D$ belongs to $\mathcal{I}_{\mathcal{A}^{\mathcal{V}}}(K,C)$. Condition (7)ii) implies that $D \;\subset\; \mathcal{A}^{v_0}(K,D) \;\subset\; \mathcal{A}^{\mathcal{V}}(K,D)$, and thus that $D$ belongs to $\mathcal{K}_{\mathcal{A}^{\mathcal{V}}}(K,C)$. Then $D \in \mathcal{D}(K,C)$ is a minimax (bilateral fixed-point) of $\mathcal{A}^{\mathcal{V}}$, and thus is equal to $\mathcal{A}^{\mathcal{V}}(K,C)$ by Proposition 2.3.

The proof of the second statement is the same one. □

Hence $\mathcal{A}^{\mathcal{V}}(K,C)$ is the unique candidate for being a of minimax (bilateral fixed-point) of $\mathcal{A}^{\mathcal{V}}$ and $\mathcal{A}_{\mathcal{V}}(K,C)$ is the unique candidate for being a of minimax (bilateral fixed-point) of $\mathcal{A}_{\mathcal{V}}$. Unfortunately, there are examples of maps $\mathcal{A}^{\mathcal{V}}$ that do not belong to $\mathcal{I}_{\mathcal{A}_{\mathcal{V}}}(K,C)$, even though the maps $\mathcal{A}_{\mathcal{V}}$ are openings whenever the maps $\mathcal{A}_v$ are openings for all $v \in \mathcal{V}$ and examples of maps $\mathcal{A}_{\mathcal{V}}$ that do not belong to $\mathcal{K}_{\mathcal{A}_{\mathcal{V}}}(K,C)$, even though the maps $\mathcal{A}_{\mathcal{V}}$ are closings whenever the maps $\mathcal{A}_v$ are closing for all $v \in \mathcal{V}$:

**Lemma 7.3** *Let us assume that for any* $v \in \mathcal{V}$, $K \mapsto \mathcal{A}_v(K,C)$ *is an opening. Then* $K \mapsto \mathcal{A}^{\mathcal{V}}(K,C)$ *is also an opening, so that*

$$\mathcal{A}^{\mathcal{V}}(\mathcal{A}^{\mathcal{V}}(K,C),C) \;=\; \mathcal{A}^{\mathcal{V}}(K,C)$$

*and*

$$\mathcal{A}^{\mathcal{V}}(K,C) \;=\; \bigcup_{D \in \mathcal{K}_{\mathcal{A}_{\mathcal{V}}}(K,C)} D$$

*is the largest fixed point of* $D \mapsto \mathcal{A}^{\mathcal{V}}(D,C)$ *between* $C$ *and* $K$*.*

*If for any* $v \in \mathcal{V}$, $C \mapsto \mathcal{A}_v(K,C)$ *is a closing, then* $C \mapsto \mathcal{A}_{\mathcal{V}}(K,C)$ *is also a closing, so that*

$$\mathcal{A}_{\mathcal{V}}(K,\mathcal{A}_{\mathcal{V}}(K,C)) \;=\; \mathcal{A}_{\mathcal{V}}(K,C)$$

*and*

$$\mathcal{A}_{\mathcal{V}}(K,C) \;=\; \bigcap_{D \in \mathcal{I}_{\mathcal{A}_{\mathcal{V}}}(K,C)} D$$

*is the smallest fixed point of* $D \mapsto \mathcal{A}_{\mathcal{V}}(K,D)$ *between* $C$ *and* $K$*.*

**Proof** — Indeed, since for any $v \in \mathcal{V}$, the maps $D \mapsto \mathcal{A}_v(D,C)$ are increasing, since $\mathcal{A}_v(K,C) \subset \mathcal{A}^{\mathcal{V}}(K,C)$, since $\mathcal{A}_v(\mathcal{A}_v(K,C),C) = \mathcal{A}_v(K,C)$ and since $\mathcal{A}_v(D,C) \subset D$, we infer that

$$\begin{cases} \mathcal{A}^{\mathcal{V}}(\mathcal{A}^{\mathcal{V}}(K,C),C) \;:=\; \bigcup_{v\in\mathcal{V}} \mathcal{A}_v(\mathcal{A}_{\mathcal{V}}(K,C),C) \\ \subset\; \bigcup_{v\in\mathcal{V}} \mathcal{A}_v(K,C) \;=\; \mathcal{A}^{\mathcal{V}}(K,C) \;=\; \bigcup_{v\in\mathcal{V}} \mathcal{A}_v(\mathcal{A}_v(K,C),C) \\ \subset\; \bigcup_{v\in\mathcal{V}} \mathcal{A}_v(\mathcal{A}_{\mathcal{V}}(K,C),C) \end{cases}$$

so that $\mathcal{A}^{\mathcal{V}}(K,C)$ is the largest fixed point of $D \mapsto \mathcal{A}^{\mathcal{V}}(D,C)$ between $C$ and $K$.

In the same way, since for any $v \in \mathcal{V}$, the maps $D \mapsto \mathcal{A}_v(K,D)$ are increasing, since $\mathcal{A}_{\mathcal{V}}(K,C) \subset \mathcal{A}_v(K,C)$, since $\mathcal{A}_v(K,\mathcal{A}_v(K,C)) = \mathcal{A}_v(K,C)$ and since $D \subset \mathcal{A}_v(K,D)$, we infer that

$$\begin{cases} \mathcal{A}_{\mathcal{V}}(K,\mathcal{A}_{\mathcal{V}}(K,C)) \;\subset\; \mathcal{A}_{\mathcal{V}}(K,\mathcal{A}_v(K,C)) \\ :=\; \bigcap_{v\in\mathcal{V}} \mathcal{A}_v(K,\mathcal{A}_v(K,C)) \;=\; \bigcap_{v\in\mathcal{V}} \mathcal{A}_v(K,C) \;=\; \mathcal{A}_{\mathcal{V}}(K,C) \\ \subset\; \bigcap_{v\in\mathcal{V}} \mathcal{A}_v(K,\mathcal{A}_{\mathcal{V}}(K,C)) \;=:\; \mathcal{A}_{\mathcal{V}}(K,\mathcal{A}_{\mathcal{V}}(K,C)) \end{cases}$$

Therefore, $\mathcal{A}_{\mathcal{V}}(K,C)$ is the smallest fixed point of $D \mapsto \mathcal{A}_{\mathcal{V}}(K,D)$ between $C$ and $K$. □

The question arises whether we can associate with such a map $\mathcal{A}^{\mathcal{V}}$ a canonical opening, which in the when $\mathcal{A}_v(K,C) := \text{Inv}\ _v(K,C)$ case is what is called the discriminating kernel.

They follow from adaptations of the Matheron's theorem.

Indeed, applying the Matheron's Theorem 5.2 to a conditional map $\mathcal{A}_{\mathcal{V}} := \bigcap_{v\in\mathcal{V}} \mathcal{A}_v$, we obtain the following consequence:

**Theorem 7.4** *Let us assume that $\mathcal{D}$ is stable for the union and the intersection. If for any $v \in \mathcal{V}$, the maps $\mathcal{A}_v$ are pre-openings with respect to $K$ and extensive with respect to $C$, and if we assume furthermore that*

1. *the map $\mathcal{A}^{\sharp}_{\mathcal{V}}$ is increasing with respect to $C$,*
2. *the map $\mathcal{A}^{\sharp}_{\mathcal{V}}$ is idempotent with respect to $C$:*

$$\forall\ (K,C) \in \mathcal{D}^2,\ \ \mathcal{A}^{\sharp}_{\mathcal{V}}(K,\mathcal{A}^{\sharp}_{\mathcal{V}}(K,C)) \;\subset\; \mathcal{A}^{\sharp}_{\mathcal{V}}(K,C)$$

*then $\mathcal{A}^{\sharp}_{\mathcal{V}}(K,C)$ is the* unique minimax (bilateral fixed-point) *$D \in \mathcal{D}(K,C)$ of $\mathcal{A}^{\sharp}_{\mathcal{V}}$ between $C$ and $K$:*

$$\mathcal{A}^{\sharp}_{\mathcal{V}}(\mathcal{A}^{\sharp}_{\mathcal{V}}(K,C),C) \;=\; \mathcal{A}^{\sharp}_{\mathcal{V}}(K,C) \;=\; \mathcal{A}^{\sharp}_{\mathcal{V}}(K,\mathcal{A}^{\sharp}_{\mathcal{V}}(K,C))$$

# 8 Discriminating Kernels under Differential Games

## 8.1 Definition of Evolutionary Games

An evolutionary game is a family of evolutionary systems parametrized by a given family $\widetilde{\mathcal{V}}$ of feedbacks $\widetilde{v} : x \in X \mapsto \widetilde{v}(x) \in \mathcal{V}$.

**Definition 8.1** *Let $\mathcal{V}$ be a metric space and $\widetilde{\mathcal{V}}$ be a family of* feedbacks $\widetilde{v} : x \in X \mapsto \widetilde{v}(x) \in \mathcal{V}$. *Then an evolutionary game is a set-valued map*

$$(x, \widetilde{v}) \in X \times \widetilde{\mathcal{V}} \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x) \in \mathcal{C}(0, \infty; X)$$

*such that, for any $\widetilde{v} \in \widetilde{\mathcal{V}}$, $x \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ is an evolutionary system.*

*It is said to be upper semicompact if, for every $\widetilde{v}$, $x \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ is upper semicompact, and lower semicontinuous if the map $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ is lower semicontinuous.*

Usual differential games provide examples of evolutionary games.

**Definition 8.2** *A* dynamical game $(P, Q, F)$ *is defined by*

- *a "set-valued feedback map" $P : X \rightsquigarrow \mathcal{U}$*
- *a "perturbation" set-valued map $Q : X \rightsquigarrow \mathcal{V}$*
- *a set-valued map $F : X \times \mathcal{U} \times \mathcal{V} \rightsquigarrow X$ describing the dynamics of the dynamical game:*

$$\begin{cases} i) & x'(t) \in F(x(t), u(t), v(t))) \\ ii) & u(t) \in P(x(t)) \\ iii) & v(t) \in Q(x(t)) \end{cases}$$

Naturally, a dynamical game generates the evolutionary game $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ where $\mathcal{S}_{\widetilde{v}}(x)$ is the set of solutions to the differential inclusion

$$x'(t) \in F(x(t), P(x(t)), \widetilde{v}(x(t)))$$

starting at $x$.

## 8.2 Conditional Viability Kernels and Capture Basins

**Definition 8.3** *Let us consider an evolutionary game $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ and two subsets $K$ and $C \subset K$. We shall say that the subset*

$$\bigcap_{\widetilde{v}} \mathrm{Viab}_{\widetilde{v}}(K, C)$$

*is the* conditional viability kernel with a target under the evolutionary game $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ *and that the subset*

$$\bigcap_{\widetilde{v}} \text{Capt}_{\widetilde{v}}(K, C)$$

*is the* conditional viable-capture basin of a target under the evolutionary game $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$.

*We shall say that $K$ is conditionally viable outside $C$ if*

$$K \subset \bigcap_{\widetilde{v}} \text{Viab}_{\widetilde{v}}(K, C)$$

*i.e., if for any state $x_0 \in K$,* for all feedbacks $\widetilde{v}$ *played by Victor,* there exists at least *one evolution $x(\cdot) \in \mathcal{S}_{\widetilde{v}}(x_0)$ viable in $K$ forever or until it reaches $C$.*

*We shall say that $K$* traps $C \subset K$ *in [27, 31, Cardaliaguet] under the evolutionary game $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ if*

$$K \subset \bigcap_{\widetilde{v}} \text{Capt}_{\widetilde{v}}(K, C)$$

*i.e., if for any state $x_0 \in K$,* for all feedbacks $\widetilde{v}$ *played by Victor,* there exists at least *one evolution $x(\cdot) \in \mathcal{S}_{\widetilde{v}}(x_0)$ viable in $K$ until it reaches $C$ in finite time.*

## 8.3 Fixed-Point Characterization

Proposition 7.2 implies

**Proposition 8.4** *If a subset $D \in \mathcal{D}(K, C)$ satisfies*

$$\begin{cases} i) & \forall\, \widetilde{v} \in \widetilde{\mathcal{V}}, \;\; D \;\subset\; \text{Viab}_{\widetilde{v}}(K, D) \\ ii) & \exists\, \widetilde{v}_0 \in \widetilde{\mathcal{V}} \;\; \textit{such that}\, \text{Viab}_{\widetilde{v}_0}(D, C) \;\subset\; D \end{cases} \tag{9}$$

*then $D$ is equal to $\bigcap_{\widetilde{v}} \text{Viab}_{\widetilde{v}}(K, C)$ and is the* unique minimax (bilateral fixed-point) *of $\bigcap_{\widetilde{v}} \text{Viab}_{\widetilde{v}}(\cdot, \cdot)$:*

$$\bigcap_{\widetilde{v}} \text{Viab}_{\widetilde{v}}\left(\bigcap_{\widetilde{v}} \text{Viab}_{\widetilde{v}}(K, C), C\right) \;=\; \bigcap_{\widetilde{v}} \text{Viab}_{\widetilde{v}}(K, C) \;=\; \bigcap_{\widetilde{v}} \text{Viab}_{\widetilde{v}}\left(K, \bigcap_{\widetilde{v}} \text{Viab}_{\widetilde{v}}(K, C)\right)$$

We deduce from Theorem 3.3 and Lemma 7.3 the following consequence:

**Theorem 8.5** *The map* $C \in \mathcal{D}(K,C) \mapsto \bigcap_{\widetilde{v}} \mathrm{Viab}_{\widetilde{v}}(K,C)$ *is a closing so that the conditional viability kernel* $\bigcap_{\widetilde{v}} \mathrm{Viab}_{\widetilde{v}}(K,C)$ *of a subset* $K$ *with target* $C \subset K$ *is the* smallest *subset* $D \in \mathcal{D}(K,C)$ *satisfying:*

$$\bigcap_{\widetilde{v}} \mathrm{Viab}_{\widetilde{v}}(K,C) \;=\; \bigcap_{\widetilde{v}} \mathrm{Viab}_{\widetilde{v}}\left(K, \bigcap_{\widetilde{v}} \mathrm{Viab}_{\widetilde{v}}(K,C)\right)$$

*It is antiextensive with respect to* $K$ *and extensive with respect to* $C$*:*

$$C \;\subset\; \bigcap_{\widetilde{v}} \mathrm{Viab}_{\widetilde{v}}(K,C) \;\subset\; K$$

*The same properties hold true for the conditional viable-capture map* $\bigcap_{\widetilde{v}} \mathrm{Capt}_{\widetilde{v}}(K,C)$ *of a target* $C$ *viable in* $K$ *in a conditional way.*

**Unfortunately**, the conditional viability kernel $\bigcap_{\widetilde{v}} \mathrm{Viab}_{\widetilde{v}}(K,C)$ of $K$ outside the target $C \subset K$ is not itself necessarily conditionally viable outside the target $C$, i.e., it is not a fixed point of the map $K \mapsto \bigcap_{\widetilde{v}} \mathrm{Viab}_{\widetilde{v}}(K,C)$. In other words, this map is a pre-opening, but not necessarily an opening.

Pierre Cardaliaguet introduced in [27, 31, Cardaliaguet] the opening associated with this map, that he called the discriminating kernel $\mathrm{Disc}(K,C)$ of $K$ with target $C$:

**Definition 8.6** *Let us consider an evolutionary game* $(x,\widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ *and two subsets* $K$ *and* $C \subset K$*.*

*We shall say that the largest closed subset* $D \in \mathcal{D}(K,C)$ *conditionally viable in* $K$ *outside* $C$ *is the* discriminating kernel of $K$ with target $C$ under the evolutionary game $(x,\widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$, *denoted by* $\mathrm{Disc}(K,C)$*, and that the largest closed subset* $D \in \mathcal{D}(K,C)$ *trapping* $C$ *is the* trapping basin *under the evolutionary game* $(x,\widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$*, denoted by* $\mathrm{Trap}(K,C)$*.*

Matheron's Theorem 5.2 implies that the map $K \mapsto \mathrm{Disc}(K,C)$ is the unique opening onto the family of subsets $D \in \mathcal{D}(K,C)$ conditionally viable in $K$ outside $C$ associated with the map $K \in \mathcal{D}(K,C) \mapsto \bigcap_{\widetilde{v}} \mathrm{Viab}_{\widetilde{v}}(K,C)$ and that $\mathrm{Disc}(K,C)$ is the largest fixed point of this map:

$$\mathrm{Disc}(\mathrm{Disc}(K,C),C) \;=\; \mathrm{Disc}(K,C)$$

We shall prove

**Proposition 8.7** *Let us consider an evolutionary game* $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$*.*
*Then*

$$\begin{cases} i) & C \mapsto \text{Disc}(K, C) \text{ is increasing} \\ ii) & \text{Disc}(K, \text{Disc}(K, C)) \; = \; \text{Disc}(K, C) \end{cases} \tag{10}$$

**Proof** — Let $C_1 \subset C_2 \subset K$ and $x \in \text{Disc}(K, C_1)$. Hence, for any feedback $\widetilde{v} \in \widetilde{\mathcal{V}}$, there exists at least one evolution $x(\cdot) \in \mathcal{S}_{\widetilde{v}}(x)$ viable in $\text{Disc}(K, C_1)$ forever or until it reaches $C_1$ — and thus, $C_2$ — in finite time. Hence $\text{Disc}(K, C_1)$ is conditionally viable outside $C_2$, so that $\text{Disc}(K, C_1) \subset \text{Disc}(K, C_2)$.

For proving that $\text{Disc}(K, \text{Disc}(K, C))$ traps $C$, take any $x_0 \in \text{Disc}(K, \text{Disc}(K, C))$. For any $\widetilde{v}$, there exists an evolution $x(\cdot) \in \mathcal{S}_{\widetilde{v}}(x_0)$ viable in $\text{Disc}(K, \text{Disc}(K, C))$ forever or else, until it possibly reaches the subset $\text{Disc}(K, C)$ of $\text{Disc}(K, \text{Disc}(K, C))$ at some *finite time* $T > 0$ at $x(T) \in \text{Disc}(K, C)$. In this case, we concatenate $x(\cdot)$ with one solution $y(\cdot) \in \mathcal{S}_{\widetilde{v}}(x(T))$ starting from $x(T)$ viable in $\text{Disc}(K, C)$, and thus, in $\text{Disc}(K, \text{Disc}(K, C))$, until it possibly reaches $C$. Hence, for any $\widetilde{v}$, $\text{Disc}(K, \text{Disc}(K, C))$ captures $C$, i.e., $\text{Disc}(K, \text{Disc}(K, C))$ traps $C$ by the evolutionary game. □

Theorem 7.4 and Proposition 8.7 imply

**Theorem 8.8** *The discriminating kernel* $\text{Disc}(K, C)$ *of a subset* $K$ *with target* $C \subset K$ *is*

1. *the* largest *subset* $D \in \mathcal{D}(K, C)$ *conditionally viable outside the target* $C$*,*
2. *the* smallest *subset* $D \in \mathcal{D}(K, C)$ *satisfying* $\text{Disc}(K, D) = D$*,*
3. *the* unique *minimax (bilateral fixed-point)* $D \in \mathcal{D}(K, C)$ *in the sense that*

$$D \; = \; \text{Disc}(K, D) \; = \; \text{Disc}(D, C)$$

*It is antiextensive with respect to* $K$ *and extensive with respect to* $C$*:*

$$C \; \subset \; \text{Disc}(K, C) \; \subset \; K$$

*is an opening with respect to* $K$*, a closing with respect to* $C$ *The same properties hold true for the trapping basin map* $\text{Trap}(K, C)$ *of a target* $C$ *viable in* $K$*.*

## 8.4 The Cardaliaguet Theorem

Pierre Cardaliaguet has related the concepts of discriminating kernels and conditional viability kernels: Since $K \mapsto \text{Disc}(K, C)$ is the opening associated to the map $K \mapsto \bigcap_{\widetilde{v}} \text{Viab}_{\widetilde{v}}(K, C)$, we know that setting

$$K_0 \ := \ K \ \& \ \forall\, i \geq 0, \ \ K_i \ := \ \bigcap_{\widetilde{v} \in \widetilde{\mathcal{V}}} \mathrm{Viab}_{\widetilde{v}}(K_{i-1}, C)$$

then

$$\mathrm{Disc}(K,C) \ \subset \ \bigcap_{i=1}^{\infty} K_i$$

Under topological assumptions, we obtain the convergence of the above "Cardaliaguet algorithm (see [27, Cardaliaguet]):

**Theorem 8.9 (Cardaliaguet)** *Let us assume that the evolutionary game* $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ *is upper semicompact. Then*

$$\mathrm{Disc}(K,C) \ = \ \bigcap_{i=1}^{\infty} K_i$$

**Proof** — We have to check that the intersection $K_\infty := \bigcap_{i \geq 0} K_i$ of the closed subsets $K_i$ is conditionally viable outside $C$, i.e., that for each $\widetilde{v}$, $K_\infty$ captures $C$ under $\mathcal{S}_{\widetilde{v}}$. Indeed, for a fixed $\widetilde{v}$,

$$C \ \subset \ K_{i+1} \ \subset \ \mathrm{Viab}_{\widetilde{v}}(K_i, C) \ \subset \ K_i$$

So, $K_\infty$ is the intersection of the decreasing sequence of the viability kernels $\mathrm{Viab}_{\widetilde{v}}(K_i, C)$ viable under the set-valued map $\mathcal{S}_{\widetilde{v}}$. Since $\mathcal{S}_{\widetilde{v}}$ is assumed to be upper semicompact, the stability theorem of viability kernels implies that $K_\infty$ itself is viable outside $C$ under this map $\mathcal{S}_{\widetilde{v}}$. Hence, $K_\infty$ is viable outside $C$ under every set-valued maps $\mathcal{S}_{\widetilde{v}}$, and thus, is contained in the discriminating kernel $\mathrm{Disc}(K, C)$. □

## 8.5 Guaranteed Viability Kernels and Capture Basins

**Definition 8.10** *We shall say that the subset*

$$\bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}(K,C)$$

*is the* guaranteed viability kernel with a target under the evolutionary game $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ *and that the subset*

$$\bigcup_{\widetilde{v}} \mathrm{Abs}_{\widetilde{v}}(K,C)$$

*is the* guaranteed viable-capture basin of a target under the evolutionary game $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$.

*We shall say that $K$ is viable outside $C$ in a guaranteed way if*

$$K \ \subset \ \bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}(K,C)$$

*i.e., if for any state $x_0 \in K$,* there exists a feedback *$\widetilde{v}$ played by Victor such that* all evolutions *$x(\cdot) \in \mathcal{S}_{\widetilde{v}}(x_0)$ are viable in $K$ forever or until they reach $C$.*

*We shall say that $K$* captures *$C$* in a guaranteed way *or* catches *$C$ in $K$ under the evolutionary game $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$*

$$K \subset \bigcup_{\widetilde{v}} \mathrm{Abs}_{\widetilde{v}}(K, C)$$

*i.e., if for any state $x_0 \in K$, there exists a feedback $\widetilde{v}$ played by Victor such that all evolutions $x(\cdot) \in \mathcal{S}_{\widetilde{v}}(x_0)$ reach $C$ (in finite time) before leaving $K$.*

Naturally, this definition depends upon the choice of the set $\widetilde{\mathcal{V}}$ of feedbacks defining the evolutionary game.

## 8.6 Fixed-Point Characterization

Proposition 7.2 implies

**Proposition 8.11** *If a subset $D \in \mathcal{D}(K, C)$ satisfies*

$$\begin{cases} i) & \forall\, \widetilde{v} \in \widetilde{\mathcal{V}},\ \mathrm{Inv}_{\widetilde{v}}(K, D) \subset D \\ ii) & \exists\, \widetilde{v}_0 \in \widetilde{\mathcal{V}} \ \text{such that } D \subset \mathrm{Inv}_{\widetilde{v}_0}(D, C) \end{cases} \tag{11}$$

*then $D$ is equal to $\bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}(K, C)$ and is the* unique minimax (bilateral fixed-point) *of $\bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}(\cdot, \cdot)$:*

$$\bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}\left(\bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}(K, C), C\right) = \bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}(K, C) = \bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}\left(K, \bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}(K, C)\right)$$

We deduce from Theorem 3.3 and Lemma 7.3 the following consequence:

**Theorem 8.12** *The map $C \in \mathcal{D}(K, C) \mapsto \bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}(K, C)$ is an opening so that the guaranteed viability kernel $\bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}(K, C)$ of a subset $K$ with target $C \subset K$ enjoys the guaranteed viability property*

$$\bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}\left(K, \bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}(K, C)\right) = \bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}(K, C)$$

*and is the* largest *subset $D \in \mathcal{D}(K, C)$ satisfying:*

$$D \subset \bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}(D, C)$$

*It satisfies*

$$C \subset \bigcup_{\widetilde{v}} \mathrm{Inv}_{\widetilde{v}}(K, C) \subset K$$

*The same properties hold true for the guaranteed viable-capture map* $\bigcup_{\widetilde{v}} \mathrm{Abs}_{\widetilde{v}}(K, C)$ *of a target* $C$ *viable in* $K$ *in a guaranteed way.*

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