Uniform controllability of a transport equation in zero diffusion-dispersion limit *

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Abstract

In this paper, we consider the controllability of a transport equation perturbed by small diffusion and dispersion terms. We prove that for a sufficiently large time, the cost of the null-controllability tends to zero exponentially as the perturbation vanishes. For small times, on the contrary, we prove that this cost grows exponentially.

1 Introduction

In this paper, we consider some null-controllability problems for a transport equation perturbed by small diffusion and dispersion effects. Precisely, the system under view is the following

$$\begin{cases} y_t - My_x + \delta y_{xxx} - \epsilon y_{xx} = 0 & \text{in } Q := (0,T) \times (0,1), \\ y_{|x=0} = v(t), \quad y_{|x=1} = 0, \quad y_{x|x=1} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,1), \end{cases} \tag{1}$$

where $T > 0$, $\delta > 0$, ϵ (typically non-negative) and M are real numbers. In (1), we have denoted y_0 the initial condition and v the control. Our main problem is the following. Consider the unperturbed transport equation :

$$y_t - My_x = 0 \text{ in } Q.$$

This equation is null-controllable from the boundary provided that $T > 1/|M|$. The question which arises is to determine whether it is possible for such times to control (1) at a uniform cost as ϵ and δ tend to 0. On the other hand, it is to expect that for times $T < 1/|M|$, the cost of null-controllability will dramatically increase. These problems have already been treated in the case of vanishing viscosity (see Coron-Guerrero [4] and Guerrero-Lebeau [9]), and in the case of vanishing dispersion (see Glass-Guerrero [8]). Several statements proved below are new even in the case of pure dispersion ($\epsilon = 0$), see Theorems 1 and 4.

The motivation for studying the dissipation-dispersion mechanism arises from continuous dynamics. In particular, in nonlinear elastodynamics, these terms can model viscosity and capillarity effects. These are particularly important in the theory of nonclassical shock waves (see in particular the book of LeFloch [11]). Nonclassical shock waves are shock waves for conservations laws with non-convex flux, which are selected through perturbative terms such as the ones of (1); in that case they can differ from the classical shock waves selected by vanishing viscosity. Hence, although the system which we consider here is linear, one can see the results described below as a first attempt to control nonlinear conservation laws in a dissipative-dispersive limit. Such a study in the purely dissipative limit has been handled by the authors in the case of the Burgers equation in [7].

We establish the following results.

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Theorem 1. *There exists $C_0 > 0$ such that the following holds. For any $M > 0$, any $T \geq C_0/M$, there are positive constants c and C (depending on T) such that for any $(\delta, \epsilon) \in (0,1]^2$, there exist $v \in L^2(0,T)$ driving $y_0 \in L^2(0,1)$ to 0 and which can be estimated as follows:*

$$\|v\|_{L^2} \leq \frac{C}{\sqrt{\delta M}} \exp\left\{-\frac{cM}{\max\{(M\delta)^{1/2}, \epsilon\}}\right\} \|y_0\|_{L^2}. \tag{2}$$

Remark 1. *Theorem 1 improves the one in [8]. In fact, for $\epsilon = 0$ a similar result was proved in [8] but with the help of three controls, one on each boundary condition.*

The meaning of a solution of system (1) will be given in Section 2 (see Definition 1 and Proposition 1). In a context of more regular solutions (which are easier to define via a lifting of boundary conditions), one can state the following.

Corollary 1. *In the above framework, we can obtain a more regular control v in $H^1(0,T)$ with the following estimate*

$$\|v\|_{H^1} \leq \frac{C(\delta, M)}{\sqrt{M}} \exp\left\{-\frac{cM}{\max\{(M\delta)^{1/2}, \epsilon\}}\right\} \|y_0\|_{L^2}, \tag{3}$$

where $C(\delta, M)$ behaves at most polynomially in δ^{-1} and in M (that is to say, $|C(\delta, M)| \leq K(\delta^{-1} + M)^r$ for some $K > 0$ and some $r \in \mathbb{N}$).

Let us recall that Theorem 1 is valid for the heat equation ($\delta = 0$) with Dirichlet boundary conditions regardless of the sign of M, see [4]. Hence it is natural to wonder if it is still valid when $\delta > 0$ is small with respect to ϵ. An answer is given in the next result.

Theorem 2. *Let $0 < \gamma \leq 1$. Then there exists C_0 (depending on γ), such that for any $M < 0$, any $T \geq C_0/|M|$, there are positive constants c and C (depending on T and γ) such that for any $(\delta, \epsilon) \in (0,1]^2$ satisfying*

$$\epsilon^2 \geq \gamma \delta |M|, \tag{4}$$

one can find a control driving y_0 to 0 and which can be estimated as follows:

$$\|v\|_{L^2} \leq \frac{C}{\sqrt{\delta |M|}} \exp\left\{-\frac{c|M|}{\epsilon}\right\} \|y_0\|_{L^2}. \tag{5}$$

In the next result we consider the case of negative ϵ. It is somewhat surprising that the dispersive term can overpower a small dissipation term with the wrong sign.

Theorem 3. *Suppose $\delta \in (0,1]$ and ϵ is* **negative** *but satisfies $-\epsilon < \kappa\delta$ (for some fixed $\kappa < 3/2$):*

- *the Cauchy problem for equation (1) is well-posed,*
- *if moreover one has $M > 0$ and $-\epsilon \leq \frac{3}{4}\sqrt{\delta M}$, then the conclusion of Theorem 1 still holds.*

Corollary 2. *In the framework of Theorem 3, it is possible to design a more regular control v in $H^1(0,T)$ with estimate (3) fulfilled.*

Now we consider the case where $T < 1/|M|$. In this case, the transport equation ($\delta = \epsilon = 0$) is no longer controllable. One should hence expect the cost of null-controllability to blow up as $\delta, \epsilon \to 0$. This is shown in the next result.

Theorem 4. *Consider $M \neq 0$ and $T > 0$ such that*

$$T < \frac{1}{|M|}. \tag{6}$$

Then there are some constants $c > 0$ and $\ell \in \mathbb{N}$ (independent of $\epsilon \in [0,1]$ and $\delta \in (0,1]$) and initial states $y_0 \in L^2(0,1)$ such that any control $v \in L^2(0,T)$ driving y_0 to 0 is estimated from below as follows

$$\|v\|_{L^2} \geq c\delta^\ell \exp\left\{\frac{c}{\max\{\delta^{1/2}, \epsilon\}}\right\} \|y_0\|_{L^2}. \tag{7}$$

Remark 2. *As far as we know, this result is new even in the case* $\epsilon = 0$.

Remark 3. *Let us recall that in a bounded domain, one can transform equation* (1) *in a linear KdV equation without diffusion term through the following transformation. We set*

$$z = \exp(-\alpha x) y \text{ with } \alpha = \frac{\epsilon}{3\delta}. \tag{8}$$

Then z *satisfies*

$$z_t + \delta z_{xxx} - \left(\frac{\epsilon^2}{3\delta} + M\right) z_x - \frac{\epsilon}{3\delta}\left(M + \frac{2\epsilon^2}{9\delta}\right) z = 0. \tag{9}$$

So the zero-controllability of (1) *for fixed* δ *and* ϵ *follows from Rosier's result [13] on the controllability of the linear KdV equation. The exact controllability with two Dirichlet controls also follows from this remark and the results from [8].*

However, the estimate of the cost (2) *cannot be obtained by this transformation and our method for the linear KdV equation from [8]. In particular, the above transformation gives bad estimates in the regime when the diffusion dominates, that is, when* $\epsilon^2 \gg \delta$. *It seems natural that difficulties appear in the above transformation when the third order term is very small.*

2 Cauchy problem

Let us briefly discuss the Cauchy problem for equation (1). For recent references concerning the initial boundary value problem for the Korteweg-de Vries equation, let us cite [2, 3, 10] and references therein.

First, we introduce the adjoint system

$$\begin{cases} -w_t - \delta w_{xxx} - \epsilon w_{xx} + M w_x = f & \text{in } (0,T)\times(0,1), \\ w_{|x=0} = w_{|x=1} = w_{x|x=0} = 0 & \text{in } (0,T), \\ w_{|t=T} = w_0 & \text{in } (0,1). \end{cases} \tag{10}$$

The solutions of system (1) are to be understood in the sense of transposition:

Definition 1. *Given* $T > 0$, $y_0 \in H^{-1}(0,1)$ *and* $v \in L^2(0,T)$, *we call* y *a solution of* (1), *a function* $y \in L^2((0,T)\times(0,1)) \cap C^0([0,T]; H^{-1}(0,1))$ *satisfying for all* $f \in L^2((0,T)\times(0,1)) + L^1(0,T; H^1_0(0,1))$,

$$\int_0^T \int_0^1 y\, f\, dx\, dt = \langle y_0, w|_{t=0}\rangle_{H^{-1}(0,1)\times H^1_0(0,1)} + \delta \int_0^T v\, w_{xx|x=0}\, dt, \tag{11}$$

where w *is the solution of* (10) *associated to* f *with* $w_0 = 0$.

Of course, any regular solution of (1) is a solution in the above sense, as easily shown by integration by parts.

Proposition 1. *For* $M \in \mathbb{R}$, $\delta \in (0,1]$, *and either* $\epsilon \in [0,1)$ *or* $\epsilon < 0$ *and* $-\epsilon < \kappa\delta$ *(for some fixed* $\kappa < \frac{3}{2}$*),* $T > 0$, $y_0 \in H^{-1}(0,1)$ *and* $v \in L^2(0,T)$, *there exists a unique solution of transposition of* (1). *Moreover, there exists* $C > 0$ *independent of* ϵ *and* δ *such that*

$$\|y\|_{L^2((0,T)\times(0,1))\cap C^0([0,T];H^{-1}(0,1))} \le \frac{C}{\delta^3}\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1}(0,1)}\big). \tag{12}$$

Proof of Proposition 1. It suffices to demonstrate that for $f \in L^2((0,T)\times(0,1)) + L^1(0,T; H^1_0(0,1))$, we have $w \in C^0(0,T; H^1_0(0,1))$ and $w_{xx|x=0} \in L^2(0,T)$, together with an estimate on this quantities in terms of f. For further purposes, we will consider w_0 not necessarily 0, although this is not needed to establish Proposition 1. Let us remark that the existence of solutions of (10) for regular data follows for instance from the transformation (8)-(9).

To do this, we perform several different energy inequalities. As we will see, this is valid in both cases when $\epsilon \in [0,1)$ and when $\epsilon < 0$ but $-\epsilon < \frac{3}{2}\delta$. Consider a regular solution w of (10). The general case follows from a regularization procedure.

First inequality. We multiply (10) by w: this yields

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 |w|^2\,dx + \frac{\delta}{2}w_{x|x=1}^2 + \epsilon\int_0^1 |w_x|^2\,dx = \int_0^1 wf\,dx. \tag{13}$$

Second inequality. We multiply (10) by $(1-x)w$: this yields

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 (1-x)|w|^2\,dx + \frac{3\delta}{2}\int_0^1 |w_x|^2\,dx + \epsilon\int_0^1 (1-x)|w_x|^2\,dx = \int_0^1 (1-x)wf\,dx. \tag{14}$$

When adding (13) and (14) we arrive at

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 (2-x)|w|^2\,dx + \Big(\frac{3\delta}{2}+\epsilon\Big)\int_0^1 |w_x|^2\,dx + \frac{\delta}{2}w_{x|x=1}^2 \\ \leq C\left\{\begin{array}{l} \|\sqrt{2-x}w\|_{L^2}\|\sqrt{2-x}f\|_{L^2}, \text{ if } f\in L^1(0,T;L^2(0,1)), \\ \|w\|_{H_0^1}\|f\|_{H^{-1}}, \text{ if } f\in L^2(0,T;H^{-1}(0,1)). \end{array}\right.$$

Hence we get an a priori estimate on w in $L^\infty([0,T];L^2(0,1))\cap L^2([0,T];H^1(0,1))$ in terms of f and w_0:

$$\|w\|_{L^\infty([0,T];L^2(0,1))} + \sqrt{\frac{3}{2}\delta+\epsilon}\,\|w\|_{L^2([0,T];H^1(0,1))} \leq C\|w_0\|_{L^2(0,1)} \\ + C\left\{\begin{array}{l} \|f\|_{L^1(0,T;L^2(0,1))}, \text{ if } f\in L^1(0,T;L^2(0,1)), \\ \frac{1}{\delta^{1/2}}\|f\|_{L^2(0,T;H^{-1}(0,1))}, \text{ if } f\in L^2(0,T;H^{-1}(0,1)). \end{array}\right. \tag{15}$$

Note that, due to the fact that $\kappa<3/2$, (15) provides an estimate of w in $L^2([0,T];H^1(0,1))$ in terms of w_0 and f, with a coefficient which grows as $1/\delta$.

Higher order estimates. Now let us denote

$$P_1 := -\delta\partial_{xxx}^3 - \epsilon\partial_{xx}^2 + M\partial_x. \tag{16}$$

Let us also introduce

$$\tilde{H}_0^2(0,1) := \{u\in H^2(0,1)\cap H_0^1(0,1),\ u_{x|x=0}=0\}. \tag{17}$$

If $f\in L^1(0,T;(H^3\cap\tilde{H}_0^2)(0,1))$ or $f\in L^2(0,T;\tilde{H}_0^2(0,1))$, then we can perform the above estimates on $P_1 w$ (which indeed fulfills the same homogeneous boundary conditions as above). This yields a priori estimates on $P_1 w$ in $L^\infty([0,T];L^2(0,1))\cap L^2([0,T];H^1(0,1))$ in terms of $P_1 f$ and $P_1 w_0$:

$$\|P_1 w\|_{L^\infty([0,T];L^2(0,1))} + \sqrt{\frac{3}{2}\delta+\epsilon}\,\|P_1 w\|_{L^2([0,T];H^1(0,1))} \leq C\|P_1 w_0\|_{L^2(0,1)} \\ + C\left\{\begin{array}{l} \|P_1 f\|_{L^1(0,T;L^2(0,1))}, \text{ if } f\in L^1(0,T;(H^3\cap\tilde{H}_0^2)(0,1)), \\ \frac{1}{\delta^{1/2}}\|P_1 f\|_{L^2(0,T;H^{-1}(0,1))}, \text{ if } f\in L^2(0,T;\tilde{H}_0^2(0,1)). \end{array}\right. \tag{18}$$

Now we infer estimates on w (with polynomial growth in terms of δ) in the following way. We write

$$(\delta w_x + \epsilon w)_{xx} = -P_1 w + M w_x. \tag{19}$$

Suppose that $f\in L^2(0,T;\tilde{H}_0^2(0,1))$. Using (15), (17) and (18), we deduce

$$\|\delta w_x + \epsilon w\|_{L^2([0,T];H^2(0,1))} + \|\delta w_x + \epsilon w\|_{L^\infty([0,T];H^1(0,1))} \leq \frac{C}{\delta^{1/2}}\|w_0\|_{H^3(0,1)} + \frac{C}{\delta}\|f\|_{L^2(0,T;\tilde{H}_0^2(0,1))}. \tag{20}$$

(We use $\|h\|_{H^2(0,1)}\leq C(\|h\|_{L^2(0,1)}+\|h_{xx}\|_{L^2(0,1)})$ which holds in dimension 1.) Using again (15) to estimate ϵw in $L^2(0,T;H^1(0,1))\cap L^\infty(0,T;L^2(0,1))$, we reach

$$\|\delta w\|_{L^2([0,T];H^2(0,1))} + \|\delta w\|_{L^\infty([0,T];H^1(0,1))} \leq \frac{C}{\delta^{1/2}}\|w_0\|_{H^3(0,1)} + \frac{C}{\delta}\|f\|_{L^2(0,T;\tilde{H}_0^2(0,1))}.$$

Injecting in (20) we obtain

$$\|\delta w\|_{L^2([0,T];H^3(0,1))} + \|\delta w\|_{L^\infty([0,T];H^2(0,1))} \leq \frac{C}{\delta^{3/2}} \|w_0\|_{H^3(0,1)} + \frac{C}{\delta^2} \|f\|_{L^2(0,T;\tilde{H}_0^2(0,1))}.$$

Using (18) and (19) again, we easily get

$$\|w\|_{L^2([0,T];H^4(0,1))} + \|w\|_{L^\infty([0,T];H^3(0,1))} \leq \frac{C}{\delta^{7/2}} \|w_0\|_{H^3(0,1)} + \frac{C}{\delta^4} \|f\|_{L^2(0,T;\tilde{H}_0^2(0,1))}. \tag{21}$$

The same can be done in the same way in the case where $f \in L^1(0,T;(H^3 \cap \tilde{H}_0^2)(0,1))$ (see (15)-(18)).

Interpolation argument. Now an interpolation argument (see [1]) between (15) and (21) proves that if $f \in L^2(0,T;L^2(0,1)) + L^1(0,T;H_0^1(0,1))$ and $w_0 \in H_0^1$, then $w \in L^2(0,T;H^2(0,1)) \cap L^\infty(0,T;H_0^1(0,1))$ with the following estimate

$$\|w\|_{L^2(0,T;H^2(0,1)) \cap L^\infty(0,T;H_0^1(0,1))} \leq \frac{C}{\delta^2} \Big(\|f\|_{L^2(0,T;L^2(0,1)) + L^1(0,T;H_0^1(0,1))} + \|w_0\|_{H_0^1(0,1)} \Big). \tag{22}$$

Additional trace estimate. As in [8], we introduce $\rho \in C^3([0,1];\mathbb{R})$ satisfying

$$\rho_{|[0,1/2]} = 1 \quad \text{and} \quad \rho_{|[3/4,1]} = 0.$$

Now considering $f \in L^2(0,T;L^2(0,1)) + L^1(0,T;H_0^1(0,1))$ and $w_0 \in H_0^1$ as above, we multiply (10) by ρw_{xx} and integrate in x. This yields

$$\frac{\delta}{2} |w_{xx|x=0}|^2 = -\frac{\delta}{2} \int_0^1 \rho_x |w_{xx}|^2 \, dx - \int_0^1 w_t \rho_x w_x - \frac{1}{2} \int_0^1 (\rho w_x^2)_t + \epsilon \int_0^1 \rho |w_{xx}|^2 \, dx - M \int_0^1 \rho w_x w_{xx} \, dx + \int_0^1 f \rho w_{xx} \, dx.$$

Together with estimate (22), this implies that $w_{xx|x=0} \in L^2(0,T)$, with the estimate

$$\|w_{xx|x=0}\|_{L^2(0,T)} \leq \frac{C}{\delta^{5/2}} \Big(\|f\|_{L^2(0,T;L^2(0,1)) + L^1(0,T;H_0^1(0,1))} + \|w_0\|_{H_0^1(0,1)} \Big). \tag{23}$$

3 Proof of Theorem 1

3.1 Carleman inequality

Let us consider the following backwards (in time) problem, which is the adjoint system associated to (1):

$$\begin{cases} -\varphi_t - \delta \varphi_{xxx} - \epsilon \varphi_{xx} + M \varphi_x = 0 & \text{in } (0,T) \times (0,1), \\ \varphi_{|x=0} = \varphi_{|x=1} = \varphi_{x|x=0} = 0 & \text{in } (0,T), \\ \varphi_{|t=T} = \varphi_T & \text{in } (0,1). \end{cases} \tag{24}$$

The objective of this paragraph is to state a Carleman inequality for the solutions of this system. In order to state this estimate, let us set

$$\alpha(t,x) = \frac{\beta(x)}{t^\mu (T-t)^\mu}, \quad (t,x) \in Q, \tag{25}$$

for some $\mu \in [1/2, 1]$. Here, β is a strictly positive, strictly increasing and concave polynomial of degree 2. Weight functions of this kind were first introduced by A. V. Fursikov and O. Yu. Imanuvilov; we refer to [6] for a systematic use of them.

Observe that the function α satisfies

$$C \leq T^{2\mu} \alpha, \ C_0 \alpha \leq \alpha_x \leq C_1 \alpha, \ C_0 \alpha \leq -\alpha_{xx} \leq C_1 \alpha \ \text{ in } (0,T) \times [0,1], \tag{26}$$

and

$$|\alpha_t| + |\alpha_{xt}| + |\alpha_{xxt}| \le CT\alpha^{(\mu+1)/\mu}, \quad |\alpha_{tt}| \le CT^2\alpha^{(\mu+2)/\mu} \text{ in } (0,T)\times[0,1], \tag{27}$$

where C, C_0 and C_1 are positive constants independent of T.

We have:

Proposition 2. *There exists a positive constant C independent of T, $\delta > 0$, $\epsilon >\geq 0$ and $M \in \mathbb{R}$ such that, for any $\varphi_T \in L^2(0,1)$, we have*

$$s \iint_Q \alpha e^{-2s\alpha}\Big(\delta^2|\varphi_{xx}|^2 + (\delta^2 s^2\alpha^2 + \epsilon^2)|\varphi_x|^2 + (\delta^2 s^4\alpha^4 + \epsilon^2 s^2\alpha^2)|\varphi|^2\Big)\,dx\,dt$$
$$\le C\delta \int_0^T (\delta s\alpha_{|x=0} + \epsilon)e^{-2s\alpha_{|x=0}}|\varphi_{xx|x=0}|^2\,dt, \tag{28}$$

for any $s \ge CT^\mu(T^\mu + (1+T^\mu M^\mu)/(\delta^{1-\mu}\epsilon^{2\mu-1}))$, where φ is the solution of (24).

Remark 4. *Note that in the dispersive regime (that is, when $\delta \gtrsim \epsilon^2$), one could deduce Proposition 2 from the Carleman estimate of [8], by putting the diffusion term $\epsilon\varphi_{xx}$ on the right hand side. In passing, when this term is put in the right hand side, the sign of ϵ does no longer matter. See Proposition 6 for a precise statement with a negative ϵ.*

Since the proof of Proposition 2 is very technical, we postpone it to an appendix, at the end of the paper.

3.2 Exponential dissipation result

It follows from (13) that the solution of the adjoint system (24) satisfies

$$\int_0^1 |\varphi(t_1,x)|^2\,dx \le K(t_1,t_2)\int_0^1 |\varphi(t_2,x)|^2\,dx, \quad 0 \le t_1 \le t_2 \le T, \tag{29}$$

with $K(t_1,t_2) = 1$. In this paragraph we will prove that, whenever the time passed $t_2 - t_1$ is larger than $1/M$, the constant K can be dramatically improved: typically, it behaves like

$$\exp\left\{-\frac{C}{\max\{\delta^{1/2},\epsilon\}}\right\}.$$

The precise result is stated in the next proposition:

Proposition 3. *Let $T > 0$ and $\delta > 0$, $\epsilon \ge 0$. Let $0 \le t_1 < t_2 \le T$ such that $t_2 - t_1 \ge 1/M$. We have the following decay properties for the solution of (24):*

- *If $\epsilon^2 \ge 3\delta(M - 1/(t_2 - t_1))$, then*

$$K \le \exp\left\{-\frac{(M(t_2-t_1)-1)^2}{(1+\sqrt{2})^2\epsilon(t_2-t_1)}\right\}. \tag{30}$$

- *If $\epsilon^2 \le 3\delta(M - 1/(t_2 - t_1))$ then*

$$K \le \exp\left\{-\frac{2(M(t_2-t_1)-1)^{3/2}}{3\sqrt{3}(1+\sqrt{2})^2\delta^{1/2}(t_2-t_1)^{1/2}}\right\}. \tag{31}$$

Proof of Proposition 3. This is inspired by [5]. Let us multiply (24) by $\exp\{r(M(T-t)-x)\}\varphi$, where r is a positive constant which will be chosen below. Then, integrating in $(0,1)$ and integrating by parts with respect to x, we deduce

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 \exp\{r(M(T-t)-x)\}|\varphi|^2dx + \frac{\delta}{2}\int_0^1 \exp\{r(M(T-t)-x)\}\partial_x|\varphi_x|^2dx$$
$$-\delta r\int_0^1 \exp\{r(M(T-t)-x)\}\varphi_{xx}\,\varphi\,dx + \epsilon\int_0^1 \exp\{r(M(T-t)-x)\}|\varphi_x|^2dx$$
$$-\frac{\epsilon r}{2}\int_0^1 \exp\{r(M(T-t)-x)\}\partial_x|\varphi|^2dx = 0.$$

Integrating again by parts, we obtain

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_0^1 \exp\{r(M(T-t)-x)\}|\varphi|^2 dx + \frac{\delta}{2}\exp\{r(M(T-t)-1)\}|\varphi_{x|x=1}|^2 \\
&+\frac{3\delta r}{2}\int_0^1 \exp\{r(M(T-t)-x)\}|\varphi_x|^2 dx + \epsilon\int_0^1 \exp\{r(M(T-t)-x)\}|\varphi_x|^2 dx \\
&-\frac{1}{2}\int_0^1 \exp\{r(M(T-t)-x)\}(\delta r^3+\epsilon r^2)|\varphi|^2 dx = 0,
\end{aligned}
\tag{32}
$$

which implies

$$
-\frac{d}{dt}\left(\exp\{-(\delta r^3+\epsilon r^2)(T-t)\}\int_0^1 \exp\{r(M(T-t)-x)\}|\varphi(t,x)|^2\,dx\right) \le 0, \tag{33}
$$

for $t \in (0,T)$. Integrating between t_1 and t_2, we get

$$
\int_0^1 |\varphi(t_1,x)|^2 dx \le K\int_0^1 |\varphi(t_2,x)|^2 dx, \tag{34}
$$

with

$$
K = \exp\{\delta(t_2-t_1)r^3 + \epsilon(t_2-t_1)r^2 + (1-M(t_2-t_1))r\}. \tag{35}
$$

Now, we choose $r > 0$ to minimize K, that is to say, we take

$$
r = \frac{A_1}{A_3 + (A_3^2 + 3A_1A_2)^{1/2}}, \tag{36}
$$

with

$$
A_1 = M(t_2-t_1) - 1, \quad A_2 = \delta(t_2-t_1), \quad A_3 = \epsilon(t_2-t_1).
$$

With this choice of r, we have that K (given by (35)) coincides with

$$
K = \exp\left\{-\frac{A_1^2}{3}\frac{A_3 + 2(A_3^2+3A_1A_2)^{1/2}}{(A_3 + (A_3^2+3A_1A_2)^{1/2})^2}\right\}.
$$

First case: $3A_1A_2 \le A_3^2$.

We first use that K written in the form $\exp\{-y/z\}$ $(y,z>0)$ is an increasing function of z and we get

$$
K \le \exp\left\{-\frac{A_1^2}{3(1+\sqrt{2})^2}\frac{A_3 + 2(A_3^2+3A_1A_2)^{1/2}}{A_3^2}\right\}.
$$

Now, taking into account that $(A_3^2+3A_1A_2)^{1/2} \ge A_3$, we obtain

$$
K \le \exp\left\{-\frac{A_1^2}{(1+\sqrt{2})^2 A_3}\right\}.
$$

Second case: $A_3^2 \le 3A_1A_2$.

Similarly as in the previous case, we first have that

$$
K \le \exp\left\{-\frac{A_1^2}{9(1+\sqrt{2})^2}\frac{A_3 + 2(A_3^2+3A_1A_2)^{1/2}}{A_1A_2}\right\}.
$$

Finally, we take into account that $(A_3^2+3A_1A_2)^{1/2} \ge (3A_1A_2)^{1/2}$ and $A_3 \ge 0$ and we obtain

$$
K \le \exp\left\{-\frac{2A_1^{3/2}}{3\sqrt{3}(1+\sqrt{2})^2 A_2^{1/2}}\right\}.
$$

This establishes (30)-(31).

3.3 Proof of Theorem 1

1. Let us first deduce an observability inequality from the Carleman inequality (28). To do this we must distinguish two regimes: the "dispersive regime" where $M\delta \gtrsim \epsilon^2$ and the diffusive regime where $M\delta \lesssim \epsilon^2$. We consider φ a regular solution of (24) and use Proposition 2 for a time $T_1 = 1/M$. Denote $Q_1 := [0, T_1] \times [0, 1]$ and $\tilde{Q}_1 := [T_1/3, 2T_1/3] \times [0, 1]$.

- *First regime:* $M\delta \geq \epsilon^2$. We fix $\mu = 1/2$. We consider s fulfilling the assumptions of Proposition 2:

$$s = CT_1^{1/2}\Big(T_1^{1/2} + \frac{1 + T_1^{1/2}M^{1/2}}{\delta^{1/2}}\Big) = \frac{1}{M} + \frac{1}{(M\delta)^{1/2}}.$$

  Observe in particular that $T_1/s \lesssim 1$. From (28), we infer

$$\delta^2 s^5 \iint_{Q_1} \alpha^5 e^{-2s\alpha}|\varphi|^2 dx\, dt \leq C \int_0^{T_1} (\delta^2 s\alpha(t,0) + \epsilon\delta)e^{-2s\alpha(t,0)}|\varphi_{xx}(t,0)|^2 dt,$$

  for some $C > 0$ independent of δ, ϵ and M. From the definition of α (see (25)), this yields

$$\frac{\delta^2 s^5}{T_1^5} e^{-C_2 s/T_1} \iint_{\tilde{Q}_1} |\varphi|^2 dx\, dt \leq C_3(\frac{\delta^2 s}{T_1} + \epsilon\delta)e^{-C_3 s/T_1} \int_0^{T_1} |\varphi_{xx}(t,0)|^2 dt, \tag{37}$$

  for some C_2, $C_3 > 0$. Then, we use here the following energy inequality

$$\int_0^1 |\varphi(t_1,x)|^2\, dx \leq \int_0^1 |\varphi(t_2,x)|^2\, dx \quad 0 \leq t_1 \leq t_2 \leq T_1.$$

  In particular, using $T_1/s \lesssim 1$, this allows us to deduce the following observability inequality from (37):

$$\int_0^1 |\varphi(0,x)|^2\, dx \leq C^* \int_0^{T_1} |\varphi_{xx|x=0}|^2\, dt, \tag{38}$$

  with

$$C^* = C_4(1 + \frac{\epsilon}{\delta}) \exp\left\{\frac{C_4 M^{1/2}}{\delta^{1/2}}\right\},$$

  for some $C_4 > 0$ independent of M, $\epsilon < 1$ and $\delta < 1$. Now, we use $\epsilon \leq \sqrt{\delta M}$ and deduce an estimate on the observability constant

$$C^* \leq C_5 \exp\left\{\frac{C_5 M^{1/2}}{\delta^{1/2}}\right\}. \tag{39}$$

- *Second regime:* $M\delta \lesssim \epsilon^2$. We choose $\mu = 1$. Here we deduce from (28) that for s as in Proposition 2 (observe now that $T_1^2/s \lesssim 1$):

$$\epsilon^2 s^3 \iint_{Q_1} \alpha^3 e^{-2s\alpha}|\varphi|^2 dx\, dt \leq C \int_0^{T_1} (\delta^2 s\alpha(t,0) + \epsilon\delta)e^{-2s\alpha(t,0)}|\varphi_{xx}(t,0)|^2 dt,$$

  for some $C > 0$ independent of δ, ϵ and M. From the definition of α (see (25)), this yields

$$\frac{\epsilon^2 s^3}{T_1^6} e^{-C_6 s/T_1^2} \iint_{\tilde{Q}_1} |\varphi|^2 dx\, dt \leq C_7(\frac{\delta^2 s}{T_1^2} + \epsilon\delta)e^{-C_7 s/T_1^2} \int_0^{T_1} |\varphi_{xx}(t,0)|^2 dt,$$

  for some C_6, $C_7 > 0$. Proceeding as previously we obtain (38) with C^* estimated by

$$C^* \leq C_8\Big(\frac{\delta^2}{\epsilon^2} + \frac{\delta}{\epsilon}\Big) \exp\left\{\frac{C_8 M}{\epsilon}\right\}. \tag{40}$$

2. Now, given $T \geq C_0/M$ (with $C_0 > 2$ to be chosen large enough later), we use the above observability inequality (38) between times $T - 1/M$ and T; we deduce

$$\|\varphi(T-1/M)\|_{L^2}^2 \leq C^* \int_{T-1/M}^{T} |\varphi_{xx|x=0}|^2 \, dt.$$

During the time interval $[0, T-1/M]$, we use Proposition 3 to compare $\|\varphi(T-1/M)\|_{L^2}^2$ to $\|\varphi(0)\|_{L^2}^2$ (that is, we take $t_1 = 0$ and $t_2 = T - 1/M$). We finally obtain

$$\int_0^1 |\varphi(0,x)|^2 \, dx \leq KC^* \int_{T-1/M}^{T} |\varphi_{xx|x=0}|^2 \, dt,$$

with $K(0, T-1/M)$ and C^* given by (29)-(31) and (39) in the first regime and by (29)-(30) and (40) in the second one. It is then clear that by taking C_0 large enough (independently of the parameters δ, ϵ, M), we can bound the observability constant in the following way:

$$C_{\text{obs}} \leq \begin{cases} C_9 \exp\left\{\dfrac{-c_1 M^{1/2}}{\delta^{1/2}}\right\} & \text{in the first regime,} \\ C_9\Big(\dfrac{\delta^2}{\epsilon^2} + \dfrac{\delta}{\epsilon}\Big) \exp\left\{\dfrac{-c_1 M}{\epsilon}\right\} & \text{in the second regime.} \end{cases}$$

Now, from these observability inequalities for the solutions of (24), it is classical to prove that for any $y_0 \in L^2(0,1)$, there exists a control $v \in L^2(0,T)$ such that the solution $y \in L^2(0,T;H^1(0,1))$ of (1) satisfies $y(T,x) = 0$ for $x \in (0,1)$ with v_1 estimated by

$$\|v_1\|_{L^2(0,T)}^2 \leq \frac{C_{\text{obs}}}{\delta^2} \|y_0\|_{L^2(0,1)}^2.$$

Let us emphasize that the δ factor comes from writing the duality relation between (1) and (24) and applying the standard H.U.M. procedure (see [12]). Then, one can estimate the factor C_{obs}/δ^2 in the following way:

$$\frac{C_{\text{obs}}}{\delta^2} \leq \begin{cases} \dfrac{C_{10}}{\delta M} \exp\left\{\dfrac{-c_2 M^{1/2}}{\delta^{1/2}}\right\} & \text{in the first regime,} \\ C_{10}\Big(\dfrac{1}{\delta M} + \dfrac{1}{\delta\epsilon}\Big) \exp\left\{\dfrac{-c_2 M}{\epsilon}\right\} & \text{in the second regime,} \end{cases}$$

and hence one obtain the form (2) in both cases (slightly reducing c_2). This concludes the proof of Theorem 1.

3.4 Proof of Corollary 1

In this proof, $C(\delta, M)$ will denote a generic positive constant depending on δ^{-1} and M at most polynomially.

The construction consists of two steps: first, we let the control be zero in (1), and prove that this regularizes the state of the system. Next, we find a convenient control for more regular initial data.

First step. We consider some $t^* \in (0,T)$. Let us prove that setting the control to zero yields a state $y(t^*)$ in $H^3(0,1)$.

We recall that we already have that $y \in L^2((0,T)\times(0,1)) \cap C^0([0,T];H^{-1}(0,1))$ and

$$\|y\|_{L^2((0,T)\times(0,1))\cap C^0([0,T];H^{-1}(0,1))} \leq C(\delta, M)\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1}(0,1)}\big), \tag{41}$$

(see (12)). We also remark that changing x into $1-x$ and t into $T-t$ transforms system (42) into system (10), so that we can use (22) and (21) in the sequel.

Let $\eta_1 \in C^1([0,t^*/2];\mathbb{R})$ be a non-negative function such that

$$\begin{cases} \eta_1(t) = 0 & \text{if } t \in [0, t^*/4], \\ \eta_1(t) = 1 & \text{if } t \in [t^*/3, t^*/2]. \end{cases}$$

Let $y_1 := \eta_1(t)y$; it satisfies

$$\begin{cases} y_{1t} + \delta y_{1xxx} - \epsilon y_{1xx} - M y_{1x} = \eta_{1t} y & \text{in } (0, t^*/2) \times (0, 1), \\ y_{1|x=0} = 0, \quad y_{1|x=1} = 0, \quad y_{1x|x=1} = 0 & \text{in } (0, t^*/2), \\ y_{1|t=0} = 0 & \text{in } (0, 1). \end{cases} \tag{42}$$

Thanks to (41), the right-hand side of (42) belongs to $L^2((0, t^*/2) \times (0, 1))$ and so inequality (22) gives that $y_1 \in L^2(0, t^*/2; H^2(0, 1))$ and

$$\|y_1\|_{L^2(0,t^*/2;H^2(0,1))} \leq C(\delta, M)\|y\|_{L^2((0,T)\times(0,1))} \leq C(\delta, M)\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1}(0,1)}\big). \tag{43}$$

Let now consider $\eta_2 \in C^1([0, t^*])$ a non-negative function such that

$$\begin{cases} \eta_2(t) = 0 & \text{if } t \in [0, t^*/3], \\ \eta_2(t) = 1 & \text{if } t \in [t^*/2, t^*]. \end{cases}$$

Let $y_2 := \eta_2(t)y$; it satisfies system (42) with $\eta_{2t} y$ in the right hand side, which, thanks to (43) belongs to $L^2(0, t^*; \check{H}^2_0(0, 1))$, where

$$\check{H}^2_0(0, 1) := \{u \in H^2(0, 1) \cap H^1_0(0, 1),\ u_{x|x=1} = 0\}. \tag{44}$$

Using estimate (21) we deduce that $y_2 \in L^2(0, t^*; H^4(0, 1)) \cap C^0([0, t^*]; H^3(0, 1))$ and

$$\|y_2\|_{L^2(0,t^*;H^4(0,1))\cap L^\infty(0,t^*;H^3(0,1))} \leq C(\delta, M)\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1}(0,1)}\big), \tag{45}$$

which concludes the desired proof.

Second step. Hence, considering $y(t^*)$ as our new initial condition, we can consider that $y_0 \in (H^3 \cap \check{H}^2_0)(0, 1)$. We now reduce Corollary 1 to an internal regularity property for system (1). In order to do this, we extend y_0 into $\tilde{y}_0 \in (H^3 \cap \check{H}^2_0)(-1, 1)$ (in a continuous manner) and we consider system (1) in $[-1, 1]$ rather than in $[0, 1]$:

$$\begin{cases} \tilde{y}_t + \delta \tilde{y}_{xxx} - \epsilon \tilde{y}_{xx} - M \tilde{y}_x = 0 & \text{in } \tilde{Q} := (0, T) \times (-1, 1), \\ \tilde{y}_{|x=-1} = \tilde{v}(t), \quad \tilde{y}_{|x=1} = 0, \quad \tilde{y}_{x|x=1} = 0 & \text{in } (0, T), \\ \tilde{y}_{|t=0} = \tilde{y}_0 & \text{in } (-1, 1). \end{cases} \tag{46}$$

Due to Theorem 1, there exists a control $\tilde{v} \in L^2(0, T)$ driving $\tilde{y}_0$ to 0 at time T. Now for system (46), we establish in the Appendix the following internal regularity result.

Proposition 4. *Let $(\delta, \epsilon) \in (0, 1]^2$ (or $\delta > 0$ and $-\epsilon < \kappa\delta$ with $\kappa < 3/2$), $M \in \mathbb{R}$. Consider $\tilde{y}$ a solution of (46) for some $\tilde{v} \in L^2(0, T)$ and $\tilde{y}_0 \in (H^3 \cap \check{H}^2_0)(-1, 1)$. Then $\tilde{y}_{|[-\frac{1}{2},1]} \in L^2(0, T; H^4(-\frac{1}{2}, 1)) \cap H^1(0, T; H^1(-\frac{1}{2}, 1))$, with the estimate*

$$\|\tilde{y}_{|[-\frac{1}{2},1]}\|_{L^2(H^4)\cap H^1(H^1)} \leq C(\delta, M)\Big[\|\tilde{y}_0\|_{H^3(-1,1)} + \|\tilde{v}\|_{L^2(0,T)}\Big], \tag{47}$$

for some constant $C(\delta, M)$ depending at most polynomially in δ^{-1} and $|M|$.

Now it is clear that $y = \tilde{y}_{|[0,1]}$ fulfills the requirements of Corollary 1, since $\tilde{y}$ has a trace at $x = 0$ belonging to $H^1(0, T)$ and satisfying estimate (47).

3.5 Proof of Theorem 2

The only part of the proof of Theorem 1 which needs to be changed is the dissipation inequality. For that, we start from (32), but here, due to the sign of M, we consider $r < 0$. Provided that

$$r \geq -\frac{2\epsilon}{3\delta}, \tag{48}$$

the integral concerning φ_x has a positive coefficient. It follows that (34) is still valid in this situtation, with here K given by

$$K = \exp\{\delta(t_2 - t_1)r^3 + \epsilon(t_2 - t_1)r^2 + (-1 - M(t_2 - t_1))r\}, \tag{49}$$

instead of (35). Now we consider

$$r^* = -\frac{2\gamma(|M| - \frac{1}{t_2 - t_1})}{3\epsilon}. \tag{50}$$

Observe that, since

$$\epsilon^2 \geq \gamma\delta\big(|M| - \frac{1}{t_2 - t_1}\big),$$

(see condition (4)), r^* indeed satisfies (48). Now injecting r^* in (49) and neglecting the first term in (49) yields

$$K \leq \exp\Big\{ -\frac{(t_2 - t_1)}{\epsilon}\Big(-\frac{4}{9}\gamma^2 + \frac{2}{3}\gamma\Big)\big[|M| - \frac{1}{t_2 - t_1}\big]^2\Big\},$$

which, recalling that $\gamma \leq 1$, proves the exponential decay property. Now the rest of the proof follows the lines of Theorem 1.

4 Proof of Theorem 3

First, let us recall that the Cauchy problem for such ϵ was investigated in Section 2. That the required Carleman inequality holds in the context of this result (viz. when ϵ is negative but small) was explained in Remark 4; see Proposition 6 for a precise statement. Also, we need to extend the validity of the dissipation estimate (Proposition 3) to our context.

4.1 Dissipation and Carleman estimates

The dissipation result which we use here is the following.

Proposition 5. *Consider $T > 0$, $\epsilon < 0$ and $\delta > 0$ satisfying $-\epsilon < 3\delta/2$. Fix $c_0 > 0$. If*

$$-\epsilon \leq \frac{3}{2\sqrt{2}}\sqrt{\frac{M\delta c_0}{1 + c_0}}, \tag{51}$$

then for every $0 \leq t_1 < t_2 \leq T$ such that $t_2 - t_1 \geq (1 + c_0)/M$, the solutions of (24) satisfy the decay property (29) *with the constant K estimated by:*

$$K \leq \exp\left\{ -\left(\frac{c_0}{2(1 + c_0)}\right)^{3/2} \frac{M^{3/2}(t_2 - t_1)}{\delta^{1/2}} \right\}. \tag{52}$$

Proof of Proposition 5. The computations that led to (32) are still valid. But due to the negative sign of ϵ, estimate (33) could possibly no longer occur. However, if we choose r properly and prove that

$$r \geq -\frac{2\epsilon}{3\delta}, \tag{53}$$

then the sum of the two terms of the second line of (32) is non-negative, so that (33) holds.

Hence it remains to choose r satisfying (53), in order to make our constant K given in (35) satisfy (52). First, we remark that due to the sign of ϵ, we have:

$$K \leq \exp\{\delta(t_2 - t_1)r^3 + (1 - M(t_2 - t_1))r\}.$$

We choose

$$r = \sqrt{\frac{c_0 M}{2(1 + c_0)\delta}}.$$

In particular, (53) is a direct consequence of (51). Due to $M(t_2 - t_1) \geq 1 + c_0$, we have

$$M(t_2 - t_1) - 1 \geq \frac{c_0}{1 + c_0} M(t_2 - t_1),$$

hence we deduce easily

$$K \leq \exp\Big\{ - \frac{c_0}{2(1 + c_0)} M(t_2 - t_1) r \Big\},$$

which yields (52).

The Carleman estimate that we use here is the following

Proposition 6. *Consider $T > 0$, $M > 0$, $\delta > 0$ and $\epsilon < 0$ satisfying $-\epsilon < 3\delta/2$. There exists a positive constant C independent of the previous quantities such that, for any $\varphi_0 \in L^2(0,1)$, we have*

$$\iint_Q \alpha e^{-2s\alpha} \Big(|\varphi_{xx}|^2 + s^2\alpha^2 |\varphi_x|^2 + s^4 \alpha^4 |\varphi|^2 \Big)\, dx\, dt \leq C \int_0^T \alpha_{|x=0} e^{-2s\alpha_{|x=0}} |\varphi_{xx|x=0}|^2\, dt, \tag{54}$$

for any $s \geq C(T + (T/\delta)^{1/2} + TM^{1/2}/\delta^{1/2})$, where φ is the solution of (24).

Proof of Proposition 6. We modify the proof of Proposition 2, by taking $\mu = 1/2$ and placing the $\epsilon \varphi_{xx}$ term in (24) on the right-hand side. This yields

$$\begin{aligned} s\delta^2 \iint_Q & \alpha e^{-2s\alpha} (|\varphi_{xx}|^2 + s^2 \alpha^2 |\varphi_x|^2 + s^4 \alpha^4 |\varphi|^2)\, dx\, dt \\ & \leq C \Big(s\delta^2 \int_0^T \alpha_{|x=0} e^{-2s\alpha_{|x=0}} |\varphi_{xx|x=0}|^2\, dt + \epsilon^2 \iint_Q e^{-2s\alpha} |\varphi_{xx}|^2\, dx\, dt \Big), \end{aligned} \tag{55}$$

for any $s \geq C(T + (T/\delta)^{1/2} + TM^{1/2}/\delta^{1/2})$. (Essentially, this is the Carleman estimate from [8, Proposition 4], with an additional right-hand side and a slightly more general weight function.)

Now to absorb the last term on the right-hand side by the first term on the left-hand side, it suffices to take

$$s \geq \frac{\epsilon^2 T}{\delta^2}. \tag{56}$$

But since $3\delta/2 > -\epsilon$, we see that (56) does not provide an additional condition.

4.2 Conclusion: proof of Theorem 3

We take $c_0 = 1$ in Proposition 5 (in that case, (51) is satisfied by assumption of Theorem 3). Hence (29) and (52) are valid for the solutions of (24).

Now from the Carleman inequality (54) applied for a time $T_1 = 1/M$, we deduce the observability inequality (38) as in Paragraph 3.3; here the observability constant C^* is estimated by

$$C^* \leq \exp\left\{ C \Big[1 + \frac{1}{\delta^{1/2}} \left(M^{1/2} + \frac{1}{T_1^{1/2}} \right) \Big] \right\}. \tag{57}$$

We proceed as in the proof of Theorem 1 and get an observability constant which is the product of C^* and K. If we take a sufficiently long amount of time for the dissipation, we see that we can absorb the constant from (57) by the constant from (52) (used during the time interval $[0, T - 1/M]$). This yields an observability constant of the form:

$$C_{\text{obs}} \leq C \exp\Big(- c \frac{M^{1/2}}{\delta^{1/2}} \Big). \tag{58}$$

Recall that the constant giving the cost of the control is $C_{\text{obs}}^{1/2}/\delta$. This yields the conclusion.

4.3 Proof of Corollary 2

One can reproduce the proof of Corollary 1. Concerning the first step (using a null control regularizes the data), inequality (45) is still true, since we did not use the sign of ϵ (apart from the fact that $3/2\delta + \epsilon > (3/2 - \kappa)\delta$). Concerning the second step (the interior regularity result), we can see in Section 7 that we only use $\frac{3}{2}\delta + \epsilon > 0$.

5 Proof of Theorem 4

We first consider the case where $M > 0$. At the end, we will describe the modificiations needed in the case where $M < 0$.

We introduce $R > 0$ such that

$$0 < 2R < 1 - MT. \tag{59}$$

We introduce $\hat{\varphi}_T \in C_0^\infty(0,1)$ such that

$$\begin{cases} \operatorname{Supp}(\hat{\varphi}_T) \subset (R, 2R), \\ \hat{\varphi}_T \geq 0, \\ \displaystyle\int_0^1 \hat{\varphi}_T^2\, dx = 1. \end{cases} \tag{60}$$

We consider the corresponding solution $\hat{\varphi}$ of (24). Now the proof is twofold. First we show that the mass of $\hat{\varphi}$ is essentially conserved in the sense that

$$\int_0^1 |\hat{\varphi}(0,x)|^2\, dx \geq c > 0, \tag{61}$$

for some constant $c > 0$. Next, we prove that $\hat{\varphi}_{xx|x=0}$ decays exponentially as $\epsilon, \delta \to 0^+$.

First step. We introduce $\theta(t,x)$ as the solution of

$$\begin{cases} \theta_t - M\theta_x = 0 \text{ in } (0,T) \times (0,1), \\ \theta_{|t=T} = \hat{\varphi}_T \text{ in } (0,1). \end{cases} \tag{62}$$

Due to (59) and (60), we have for all $t \in [0,T]$

$$\operatorname{Supp} \theta(t,\cdot) \subset (0,1). \tag{63}$$

Now, using

$$\int_0^T \int_0^1 \theta(-\hat{\varphi}_t - \delta\hat{\varphi}_{xxx} - \epsilon\hat{\varphi}_{xx} + M\hat{\varphi}_x)\, dt\, dx = 0,$$

we easily get

$$\int_0^1 (\theta(0,x)\hat{\varphi}(0,x) - \theta(T,x)\hat{\varphi}(T,x))\, dx + \int_0^T \int_0^1 (\delta\hat{\varphi}\theta_{xxx} - \epsilon\hat{\varphi}\theta_{xx})\, dt\, dx = 0. \tag{64}$$

Now we have a uniform $L^2(0,1)$ estimate of $\hat{\varphi}$ independently of δ and ϵ, see (15). It follows that for ϵ and δ suitably small (depending only on T and R), we have

$$\int_0^1 \theta(0,x)\hat{\varphi}(0,x)\, dx \geq \frac{1}{2}\int_0^1 \theta(T,x)\hat{\varphi}(T,x)\, dx = \frac{1}{2}\|\hat{\varphi}_T\|^2_{L^2(0,1)} = \frac{1}{2},$$

which implies (61).

Second step. First, we prove the following estimate

$$\int_0^{R/4} |\hat{\varphi}(t,x)|^2\, dx \leq K \int_0^1 |\hat{\varphi}_T|^2\, dx, \tag{65}$$

with

$$K = C\exp\Big(-\frac{R^2}{500\max((\delta TR)^{1/2},\epsilon T)}\Big). \tag{66}$$

For that, we proceed as in Proposition 3. Introduce $\psi \in C^\infty(\mathbb{R})$ satisfying

$$\begin{cases} \psi = 0 \text{ in } [R,+\infty), \\ \psi = 1 \text{ in } (-\infty, R/2], \\ \psi' \leq 0. \end{cases} \tag{67}$$

We multiply (24) by $\psi(x-M(T-t))\exp\{r(M(T-t)-x)\}\hat{\varphi}$, for some $r \geq 0$. In Proposition 3 we multiplied (24) with $\exp\{r(M(T-t)-x)\}\hat{\varphi}$, which led to (33). Here there are additional terms due to the presence of $\psi(x-M(T-t))$; we put them in the right-hand side. This yields

$$\begin{aligned} -\frac{d}{dt}\Big(\exp\{-(\delta r^3+\epsilon r^2)(T-t)\}\int_0^1 \exp\{r(M(T-t)-x)\}\psi(x-M(T-t))|\hat{\varphi}(t,x)|^2\,dx\Big) \\ \leq C\int_{R/2+M(T-t)}^{R+M(T-t)} (\|\psi'\|_\infty + \|\psi''\|_\infty + \|\psi'''\|_\infty)\exp\{r(M(T-t)-x)\}|\hat{\varphi}(t,x)|^2\,dx, \end{aligned} \tag{68}$$

where C depends on r in a polynomial way. Using (15), we easily obtain

$$\begin{aligned} -\frac{d}{dt}\Big(\exp\{-(\delta r^3+\epsilon r^2)(T-t)\}\int_0^1 \exp\{r(M(T-t)-x)\}\psi(x-M(T-t))|\hat{\varphi}(t,x)|^2\,dx\Big) \\ \leq C\exp\{-rR/2\}\int_0^1 |\hat{\varphi}_T|^2\,dx. \end{aligned} \tag{69}$$

We integrate over $[t,T]$ to infer that for all $t \in [0,T]$

$$\int_0^1 \exp\{r(M(T-t)-x)\}\psi(x-M(T-t))|\hat{\varphi}(t,x)|^2\,dx \leq C\exp\{\delta r^3(T-t)+\epsilon r^2(T-t)-rR/2\}\int_0^1 |\hat{\varphi}_T|^2\,dx. \tag{70}$$

We estimate from below the left-hand side by

$$\exp\{r(M(T-t)-\frac{R}{4})\}\int_0^{R/4} |\hat{\varphi}(t,x)|^2\,dx, \tag{71}$$

where we have used that $\psi(x-M(T-t)) = 1$ for $(t,x) \in [0,T]\times[0,R/4]$. Finally, we get

$$\begin{aligned} \int_0^{R/4} |\hat{\varphi}(t,x)|^2\,dx &\leq C\exp\{\delta r^3(T-t)+\epsilon r^2(T-t)-r[M(T-t)+R/4]\}\int_0^1 |\hat{\varphi}_T|^2\,dx \\ &\leq C\exp\{\delta r^3 T+\epsilon r^2 T-rR/4\}\int_0^1 |\hat{\varphi}_T|^2\,dx. \end{aligned} \tag{72}$$

Now we choose r as follows

$$r = \frac{R}{5}\min\Big(\frac{1}{(R\delta T)^{1/2}},\frac{1}{\epsilon T}\Big). \tag{73}$$

Using, $r \leq 1/(R\delta T)^{1/2}$, this yields (65)-(66) with C at most polynomial in $1/\delta$.

Now from (65)-(66), we are going to deduce an estimate of the type

$$\|\hat{\varphi}\|^2_{L^2(0,T;H^3(0,R/16))} \leq C(\delta)\exp\Big(-\frac{R^2}{500\max((\delta TR)^{1/2},\epsilon T)}\Big)\int_0^1 |\hat{\varphi}_T|^2\,dx, \tag{74}$$

for some constant $C(\delta)$ whose growth in $1/\delta$ is at most polynomial.

To do so, first, we consider equation (24) in $[0,R/4]$ and multiply by $(R/4-x)^3\hat\varphi$ and get as for (14):

$$-\frac{1}{2}\frac{d}{dt}\int_0^{R/4}(R/4-x)^3|\hat\varphi|^2\,dx+\frac{9\delta}{2}\int_0^{R/4}(R/4-x)^2|\hat\varphi_x|^2\,dx-3\delta\int_0^{R/4}|\hat\varphi|^2\,dx$$
$$+\frac{3M}{2}\int_0^{R/4}(R/4-x)^2|\hat\varphi|^2\,dx+\epsilon\int_0^{R/4}(R/4-x)^3|\hat\varphi_x|^2\,dx-3\epsilon\int_0^{R/4}(R/4-x)|\hat\varphi_x|^2\,dx=0.$$

This yields (using (60))

$$\|(R/4-x)^{3/2}\hat\varphi\|_{L^\infty(0,T;L^2(0,R/4))}+\|(R/4-x)\hat\varphi\|_{L^2(0,T;H^1(0,R/4))}\leq C(\frac{\epsilon}{\delta}+1)\|\hat\varphi\|_{L^2(0,T;L^2(0,R/4))}. \quad (75)$$

From now, we consider the problem in $[0,R/8]$. We use Proposition 4; since we consider the adjoint problem in $[0,R/8]$ rather than the direct one in $[-1,1]$, the same result holds replacing $[-1/2,1]$ by $[0,R/16]$ for instance. Hence we deduce

$$\|\hat\varphi\|_{L^2(0,T;H^4(0,R/16))}\leq C(\delta)\|\hat\varphi\|_{L^2(0,T;H^1(0,R/8))}. \quad (76)$$

Then (74) follows from (65), (66), (75) and (76). Finally we get

$$\|\hat\varphi_{xx|x=0}\|_{L^2(0,T)}\leq C(\delta)\exp\Big(-\frac{R^2}{1000\max((\delta TR)^{1/2},\epsilon T)}\Big)\|\hat\varphi_T\|_{L^2(0,1)}. \quad (77)$$

With (61), this proves Theorem 4, when $M>0$.

When $M<0$, we define R essentially as in (59)

$$0<4R<1-|M|T. \quad (78)$$

and replace the condition on the support in (60) by

$$\mathrm{Supp}\,(\hat\varphi_T)\subset(1-2R,1-R).$$

Then the step 1 in the above analysis is easily adapted in this situation. Concerning the second step, we redefine $\psi\in C^\infty(\mathbb{R})$ by

$$\begin{cases}\psi=0 \text{ in } [1-2R,+\infty),\\ \psi=1 \text{ in } (-\infty,1-3R],\\ \psi'\leq 0.\end{cases} \quad (79)$$

Then the goal is again to establish (65)-(66). The same computation as previously gives (68) where the limits of the time-integral in the right-hand side have to be replaced with $1-3R+M(T-t)$ and $1-2R+M(T-t)$. Next, (69) still holds with a different coefficient

$$-\frac{d}{dt}\Big(\exp\{-(\delta r^3+\epsilon r^2)(T-t)\}\int_0^1\exp\{r(M(T-t)-x)\}\psi(x-M(T-t))|\hat\varphi(t,x)|^2\,dx\Big)$$
$$\leq C\exp\{r(3R-1)\}\int_0^1|\hat\varphi_T|^2\,dx.$$

As previously we integrate in time; here with the new definition of ψ we again have $\psi(x-M(T-t))=1$ for $(t,x)\in[0,T]\times[0,R/4]$, and we get

$$\int_0^{R/4}|\hat\varphi(t,x)|^2\,dx\leq C\exp\{\delta r^3(T-t)+\epsilon r^2(T-t)-r[M(T-t)+1-13R/4]\}\int_0^1|\hat\varphi_T|^2\,dx.$$

Then one may conclude as previously (note in particular that due to (78), (72) is valid), which ends the proof of Theorem 4.

6 Appendix 1: Proof of the Carleman inequality

Let $\psi := e^{-s\alpha}\varphi$, where α is given by (25) and φ fulfills system (24). We deduce that

$$L_1\psi + L_2\psi = L_3\psi,$$

with

$$L_1\psi = \delta\psi_{xxx} + \psi_t + 3\delta s^2\alpha_x^2\psi_x + 2\epsilon s\alpha_x\psi_x - M\psi_x, \tag{80}$$

$$L_2\psi = \delta s^3\alpha_x^3\psi + \epsilon s^2\alpha_x^2\psi + 3\delta s\alpha_x\psi_{xx} + \epsilon\psi_{xx} + s\alpha_t\psi + 3\delta s\alpha_{xx}\psi_x - sM\alpha_x\psi \tag{81}$$

and

$$L_3\psi = -\delta s\alpha_{xxx}\psi - 3\delta s^2\alpha_x\alpha_{xx}\psi - \epsilon s\alpha_{xx}\psi. \tag{82}$$

Then, we have

$$\|L_1\psi\|^2_{L^2(Q)} + \|L_2\psi\|^2_{L^2(Q)} + 2\iint_Q L_1\psi\, L_2\psi\, dx\, dt = \|L_3\psi\|^2_{L^2(Q)}. \tag{83}$$

In the following lines, we will compute the double product term. For the sake of simplicity, let us denote by $(L_i\psi)_j$ $(1 \le i \le 2,\ 1 \le j \le 7)$ the j-th term in the expression of $L_i\psi$. To identify the signs of the following integrals, we recall that $\alpha > 0$, $\alpha_x > 0$ and $\alpha_{xx} < 0$.

• First, integrating by parts with respect to x, we have

$$\begin{aligned}((L_1\psi)_1, (L_2\psi)_1)_{L^2(Q)} &= -\delta^2\frac{s^3}{2}\iint_Q \alpha_x^3\partial_x|\psi_x|^2\,dx\,dt + 3\delta^2 s^3\iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,dx\,dt \\ &+3\delta^2 s^3\iint_Q \alpha_x\alpha_{xx}^2\partial_x|\psi|^2\,dx\,dt = \delta^2\frac{9s^3}{2}\iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,dx\,dt - \delta^2\frac{s^3}{2}\int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,dt \\ &-3\delta^2 s^3\iint_Q \alpha_{xx}^3|\psi|^2\,dx\,dt \ge \delta^2\frac{9s^3}{2}\iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,dx\,dt - \delta^2\frac{s^3}{2}\int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,dt \\ &- C\delta^2 T^{4\mu}s^3\iint_Q \alpha^5|\psi|^2\,dx\,dt.\end{aligned} \tag{84}$$

Here, we have used that $\psi_{|x=0,1} = \psi_{x|x=0} = 0$, (26) and the fact that $\alpha_{xxx} = 0$.

For the second term, we do very similar computations and we obtain

$$\begin{aligned}((L_1\psi)_1, (L_2\psi)_2)_{L^2(Q)} &= -\epsilon\delta\frac{s^2}{2}\iint_Q \alpha_x^2\partial_x|\psi_x|^2\,dx\,dt + 2\epsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_x|^2\,dx\,dt \\ &\ge 3\epsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_x|^2\,dx\,dt - \epsilon\delta\frac{s^2}{2}\int_0^T \alpha^2_{x|x=1}|\psi_{x|x=1}|^2\,dt.\end{aligned} \tag{85}$$

For the third term, we integrate by parts with respect to the x variable and we obtain

$$\begin{aligned}((L_1\psi)_1, (L_2\psi)_3)_{L^2(Q)} &\ge -\delta^2\frac{3s}{2}\iint_Q \alpha_{xx}|\psi_{xx}|^2\,dx\,dt + \delta^2\frac{3s}{2}\int_0^T \alpha_{x|x=1}|\psi_{xx|x=1}|^2\,dt \\ &- C\delta^2 s\int_0^T \alpha_{|x=0}|\psi_{xx|x=0}|^2\,dt.\end{aligned} \tag{86}$$

We consider now the fourth term of $L_2\psi$ and we readily get

$$((L_1\psi)_1, (L_2\psi)_4)_{L^2(Q)} = \frac{\epsilon\delta}{2}\left(\int_0^T |\psi_{xx|x=1}|^2 dt - \int_0^T |\psi_{xx|x=0}|^2 dt\right) \ge -\frac{\epsilon\delta}{2}\int_0^T |\psi_{xx|x=0}|^2 dt. \tag{87}$$

The next term gives

$$
\begin{aligned}
((L_1\psi)_1,(L_2\psi)_5)_{L^2(Q)} &= -\delta\frac{s}{2}\iint_Q \alpha_t\partial_x|\psi_x|^2\,dx\,dt + \delta\frac{s}{2}\iint_Q \alpha_{xxt}\partial_x|\psi|^2\,dx\,dt \\
&+ \delta s\iint_Q \alpha_{xt}|\psi_x|^2\,dx\,dt = \frac{3}{2}\delta s\iint_Q \alpha_{xt}|\psi_x|^2\,dx\,dt \\
&- \delta\frac{s}{2}\int_0^T \alpha_{t|x=1}|\psi_{x|x=1}|^2\,dt \\
&\geq -C\delta sT\left(\iint_Q \alpha^{(\mu+1)/\mu}|\psi_x|^2\,dx\,dt + \int_0^T \alpha_{|x=1}^{(\mu+1)/\mu}|\psi_{x|x=1}|^2\,dt\right),
\end{aligned}
\tag{88}
$$

thanks to $\alpha_{xxx}=0$ and (26)-(27).

Furthermore, since $\alpha_{xxx}=0$ and $\psi_{|x=0}=\psi_{x|x=0}=0$, we have

$$
\begin{aligned}
((L_1\psi)_1,(L_2\psi)_6)_{L^2(Q)} &= -3\delta^2 s\iint_Q \alpha_{xx}|\psi_{xx}|^2\,dx\,dt + 3\delta^2 s\int_0^T \alpha_{xx|x=1}\psi_{xx|x=1}\psi_{x|x=1}\,dt \\
&\geq -3\delta^2 s\iint_Q \alpha_{xx}|\psi_{xx}|^2\,dx\,dt - \delta^2\frac{s}{2}\int_0^T \alpha_{x|x=1}|\psi_{xx|x=1}|^2\,dt. \\
&- C\delta^2 sT^{4\mu}\int_0^T \alpha_{|x=1}^3|\psi_{x|x=1}|^2\,dt.
\end{aligned}
\tag{89}
$$

Observe that for the last integral of the first line of (89) we have used Cauchy-Schwarz inequality and the estimate $\alpha_{xx}^2/\alpha_x \leq CT^{4\mu}\alpha^3$ for some constant $C>0$. The last term in the second line of (89) yields a positive term when combined with the last term in the first line of (86) thanks to $\alpha_x>0$.

For the last term of $L_2\psi$, we have

$$
\begin{aligned}
((L_1\psi)_1,(L_2\psi)_7)_{L^2(Q)} &= \delta\frac{sM}{2}\iint_Q \alpha_x\partial_x|\psi_x|^2\,dx\,dt + \delta sM\iint_Q \alpha_{xx}\psi_{xx}\psi\,dx\,dt \\
&\geq -C\delta s|M|\Big(\int_0^T \alpha_{|x=1}|\psi_{x|x=1}|^2\,dt + \iint_Q \alpha|\psi_x|^2\,dx\,dt\Big).
\end{aligned}
\tag{90}
$$

All these computations ((84)-(90)) show that

$$
\begin{aligned}
&((L_1\psi)_1,L_2\psi)_{L^2(Q)} \geq \frac{9\delta^2 s^3}{2}\iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,dx\,dt - \frac{9\delta^2 s}{2}\iint_Q \alpha_{xx}|\psi_{xx}|^2\,dx\,dt \\
&-C\delta s|M|\iint_Q \alpha|\psi_x|^2 dx\,dt + 3\epsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_x|^2\,dx\,dt - \frac{\delta^2 s^3}{2}\int_0^T \alpha_{x|x=1}^3|\psi_{x|x=1}|^2\,dt \\
&-C\delta s|M|\int_0^T \alpha_{|x=1}|\psi_{x|x=1}|^2\,dt - \frac{\epsilon\delta s^2}{2}\int_0^T \alpha_{x|x=1}^2|\psi_{x|x=1}|^2\,dt - C\delta sT\iint_Q \alpha^{(\mu+1)/\mu}|\psi_x|^2\,dx\,dt \\
&-CT^{4\mu}\delta^2 s^3\iint_Q \alpha^5|\psi|^2\,dx\,dt - C\int_0^T (\delta^2 s\alpha_{|x=0} + \frac{\epsilon\delta}{2})|\psi_{xx|x=0}|^2\,dt \\
&-Cs\int_0^T (\delta T\alpha_{|x=1}^{(\mu+1)/\mu} + \delta^2 T^{4\mu}\alpha_{|x=1}^3)|\psi_{x|x=1}|^2\,dt.
\end{aligned}
\tag{91}
$$

- Concerning the second term of $L_1\psi$, we first integrate by parts with respect to t:

$$
((L_1\psi)_2,(L_2\psi)_1)_{L^2(Q)} = -\delta\frac{3s^3}{2}\iint_Q \alpha_x^2\alpha_{xt}|\psi|^2\,dx\,dt \geq -C\delta s^3 T\iint_Q \alpha^{(3\mu+1)/\mu}|\psi|^2\,dx\,dt. \tag{92}
$$

Similar computations give the following for the second term:

$$((L_1\psi)_2,(L_2\psi)_2)_{L^2(Q)} = -\epsilon s^2 \iint_Q \alpha_x\alpha_{xt}|\psi|^2\,dx\,dt \geq -C\epsilon s^2 T \iint_Q \alpha^{(2\mu+1)/\mu}|\psi|^2\,dx\,dt. \tag{93}$$

For the third one we use that $\psi_{x|t=0} = \psi_{x|t=T} = 0$ and (27) and we get

$$\begin{aligned}((L_1\psi)_2,(L_2\psi)_3)_{L^2(Q)} &= -\delta\frac{3s}{2}\iint_Q \alpha_x\partial_t|\psi_x|^2\,dx\,dt - 3\delta s\iint_Q \alpha_{xx}\psi_x\psi_t\,dx\,dt\\ &= \delta\frac{3s}{2}\iint_Q \alpha_{xt}|\psi_x|^2\,dx\,dt - 3\delta s\iint_Q \alpha_{xx}\psi_x\psi_t\,dx\,dt\\ &\geq -C\delta sT\iint_Q \alpha^{(\mu+1)/\mu}|\psi_x|^2\,dx\,dt - 3\delta s\iint_Q \alpha_{xx}\psi_x\psi_t\,dx\,dt.\end{aligned} \tag{94}$$

Then, we readily see that $((L_1\psi)_2,(L_2\psi)_4)_{L^2(Q)} = 0$.

Again using $\psi_{|t=0} = \psi_{|t=T} = 0$ and (27), we deduce

$$((L_1\psi)_2,(L_2\psi)_5)_{L^2(Q)} = -s\iint_Q \alpha_{tt}|\psi|^2\,dx\,dt \geq -CsT^2\iint_Q \alpha^{(\mu+2)/\mu}|\psi|^2\,dx\,dt. \tag{95}$$

Furthermore,

$$((L_1\psi)_2,(L_2\psi)_6)_{L^2(Q)} = 3\delta s\iint_Q \alpha_{xx}\psi_x\psi_t\,dx\,dt. \tag{96}$$

This term cancels with the last term in (94).

Finally, the last product of the second term of $L_1\psi$ provides

$$((L_1\psi)_2,(L_2\psi)_7)_{L^2(Q)} = \frac{sM}{2}\iint_Q \alpha_{xt}|\psi|^2\,dx\,dt \geq -Cs|M|T\iint_Q \alpha^{(\mu+1)/\mu}|\psi|^2\,dx\,dt. \tag{97}$$

Putting together all the computations concerning the second term of $L_1\psi$ ((92)-(97)), we obtain

$$\begin{aligned}((L_1\psi)_2,L_2\psi)_{L^2(Q)} \geq{}& -C\delta sT\iint_Q \alpha^{(\mu+1)/\mu}|\psi_x|^2\,dx\,dt - C\iint_Q \delta s^3T\alpha^{(3\mu+1)/\mu}|\psi|^2dx\,dt\\ &- C\iint_Q(\epsilon s^2T\alpha^{(2\mu+1)/\mu} + sT^2\alpha^{(\mu+2)/\mu} + s|M|T\alpha^{(\mu+1)/\mu})|\psi|^2\,dx\,dt.\end{aligned} \tag{98}$$

- We consider now the products concerning the third term of $L_1\psi$. First, we have

$$((L_1\psi)_3,(L_2\psi)_1)_{L^2(Q)} = -\delta^2\frac{15s^5}{2}\iint_Q \alpha_x^4\alpha_{xx}|\psi|^2\,dx\,dt \geq c\delta^2s^5\iint_Q \alpha^5|\psi|^2\,dx\,dt. \tag{99}$$

Secondly

$$((L_1\psi)_3,(L_2\psi)_2)_{L^2(Q)} = -6\epsilon\delta s^4\iint_Q \alpha_x^3\alpha_{xx}|\psi|^2\,dx\,dt \geq c\epsilon\delta s^4\iint_Q \alpha^4|\psi|^2\,dx\,dt. \tag{100}$$

Thirdly,

$$((L_1\psi)_3,(L_2\psi)_3)_{L^2(Q)} = -\delta^2\frac{27s^3}{2}\iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,dx\,dt + \delta^2\frac{9s^3}{2}\int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,dt. \tag{101}$$

For the fourth term, we have

$$((L_1\psi)_3,(L_2\psi)_4)_{L^2(Q)} = -3\epsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_x|^2\,dx\,dt + \epsilon\delta\frac{3s^2}{2}\int_0^T \alpha^2_{x|x=1}|\psi_{x|x=1}|^2\,dt. \tag{102}$$

Using (27), we obtain the following for the fifth term:

$$((L_1\psi)_3, (L_2\psi)_5)_{L^2(Q)} = -\delta\frac{3s^3}{2}\iint_Q (\alpha_x^2\alpha_t)_x|\psi|^2\,dx\,dt \geq -C\delta s^3 T\iint_Q \alpha^{(3\mu+1)/\mu}|\psi|^2\,dx\,dt. \tag{103}$$

Furthermore,

$$((L_1\psi)_3, (L_2\psi)_6)_{L^2(Q)} = 9\delta^2 s^3\iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,dx\,dt. \tag{104}$$

Finally,

$$((L_1\psi)_3, (L_2\psi)_7)_{L^2(Q)} = \frac{9}{2}\delta s^3 M\iint_Q \alpha_x^2\alpha_{xx}|\psi|^2\,dx\,dt \geq -C\delta s^3|M|\iint_Q \alpha^3|\psi|^2\,dx\,dt. \tag{105}$$

Consequently, we get the following for the third term of $L_1\psi$ ((99)-(105)):

$$\begin{aligned}((L_1\psi)_3, L_2\psi)_{L^2(Q)} &\geq C_0\iint_Q (\delta^2 s^5\alpha^5 + \epsilon\delta s^4\alpha^4)|\psi|^2\,dx\,dt - \frac{9}{2}\delta^2 s^3\iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,dx\,dt \\ &-3\epsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_x|^2\,dx\,dt + \frac{3s^2}{2}\int_0^T (3\delta^2 s\alpha_{x|x=1}^3 + \epsilon\delta\alpha_{x|x=1}^2)|\psi_{x|x=1}|^2 dt \\ &-C\delta s^3|M|\iint_Q \alpha^3|\psi|^2\,dx\,dt - C\delta s^3 T\iint_Q \alpha^{(3\mu+1)/\mu}|\psi|^2\,dx\,dt.\end{aligned} \tag{106}$$

• Now, we compute the fourth term. First, we have:

$$((L_1\psi)_4, (L_2\psi)_1)_{L^2(Q)} = -4\epsilon\delta s^4\iint_Q \alpha_x^3\alpha_{xx}|\psi|^2\,dx\,dt. \tag{107}$$

Similar computations give

$$((L_1\psi)_4, (L_2\psi)_2)_{L^2(Q)} = -3\epsilon^2 s^3\iint_Q \alpha_x^2\alpha_{xx}|\psi|^2\,dx\,dt. \tag{108}$$

For the third term, we get

$$((L_1\psi)_4, (L_2\psi)_3)_{L^2(Q)} = -6\epsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_x|^2\,dx\,dt + 3\epsilon\delta s^2\int_0^T \alpha_{x|x=1}^2|\psi_{x|x=1}|^2 dt. \tag{109}$$

Then,

$$((L_1\psi)_4, (L_2\psi)_4)_{L^2(Q)} = -\epsilon^2 s\iint_Q \alpha_{xx}|\psi_x|^2\,dx\,dt + \epsilon^2 s\int_0^T \alpha_{x|x=1}|\psi_{x|x=1}|^2 dt. \tag{110}$$

The fifth term gives

$$((L_1\psi)_4, (L_2\psi)_5)_{L^2(Q)} = -\epsilon s^2\iint_Q (\alpha_x\alpha_t)_x|\psi|^2\,dx\,dt \geq -C\epsilon T s^2\iint_Q \alpha^{(2\mu+1)/\mu}|\psi|^2\,dx\,dt. \tag{111}$$

Direct computations for the sixth term provides

$$((L_1\psi)_4, (L_2\psi)_6)_{L^2(Q)} = 6\epsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_x|^2\,dx\,dt. \tag{112}$$

This term cancels with the first integral in the right hand side of (109).

At last,

$$((L_1\psi)_4,(L_2\psi)_7)_{L^2(Q)} = 2\epsilon M s^2 \iint_Q \alpha_x\alpha_{xx}|\psi|^2\,dx\,dt \geq -C\epsilon|M|s^2\iint_Q \alpha^2|\psi|^2 dx\,dt. \tag{113}$$

All these computations ((107)-(113)) gives

$$\begin{aligned}((L_1\psi)_4, L_2\psi)_{L^2(Q)} &\geq -\epsilon s^2 \iint_Q (4\delta s^2\alpha_x^2 + 3\epsilon s\alpha_x)\alpha_x\alpha_{xx}|\psi|^2 dx\,dt\\ &-\epsilon^2 s\iint_Q \alpha_{xx}|\psi_x|^2 dx\,dt + 3\epsilon\delta s^2\int_0^T \alpha^2_{x|x=1}|\psi_{x|x=1}|^2 dt + \epsilon^2 s\int_0^T \alpha_{x|x=1}|\psi_{x|x=1}|^2 dt\\ &-C\epsilon T s^2\iint_Q \alpha^{(2\mu+1)/\mu}|\psi|^2 dx\,dt - C\epsilon|M|s^2\iint_Q \alpha^2|\psi|^2 dx\,dt.\end{aligned} \tag{114}$$

• Concerning the fifth and last term of $L_1\psi$, we have:

$$((L_1\psi)_5,(L_2\psi)_1)_{L^2(Q)} = \frac{3}{2}\delta M s^3\iint_Q \alpha_x^2\alpha_{xx}|\psi|^2 dx\,dt \geq -C\delta|M|s^3\iint_Q \alpha^3|\psi|^2 dx\,dt. \tag{115}$$

Then,

$$((L_1\psi)_5,(L_2\psi)_2)_{L^2(Q)} = \epsilon M s^2\iint_Q \alpha_x\alpha_{xx}|\psi|^2 dx\,dt \geq -C\epsilon|M|s^2\iint_Q \alpha^2|\psi|^2 dx\,dt. \tag{116}$$

Now, we compute the third one:

$$\begin{aligned}((L_1\psi)_5,(L_2\psi)_3)_{L^2(Q)} &= \frac{3}{2}\delta M s\iint_Q \alpha_{xx}|\psi_x|^2\,dx\,dt - \frac{3}{2}\delta M s\int_0^T \alpha_{x|x=1}|\psi_{x|x=1}|^2\,dt\\ &\geq \frac{3}{2}\delta|M|s\iint_Q \alpha_{xx}|\psi_x|^2\,dx\,dt - C\delta|M|s\int_0^T \alpha_{|x=1}|\psi_{x|x=1}|^2\,dt.\end{aligned} \tag{117}$$

Then,

$$((L_1\psi)_5,(L_2\psi)_4)_{L^2(Q)} = -\frac{\epsilon M}{2}\int_0^T |\psi_{x|x=1}|^2\,dt \geq -C\epsilon|M|\int_0^T |\psi_{x|x=1}|^2 dt. \tag{118}$$

Additionally, integrating by parts again with respect to x, we have

$$((L_1\psi)_5,(L_2\psi)_5)_{L^2(Q)} = \frac{Ms}{2}\iint_Q \alpha_{xt}|\psi|^2\,dx\,dt \geq -C|M|Ts\iint_Q \alpha^{(\mu+1)/\mu}|\psi|^2\,dx\,dt. \tag{119}$$

Furthermore,

$$((L_1\psi)_5,(L_2\psi)_6)_{L^2(Q)} = -3\delta M s\iint_Q \alpha_{xx}|\psi_x|^2\,dx\,dt. \tag{120}$$

This term can be combined with the first integral in the second line of (117).

Finally,

$$((L_1\psi)_5,(L_2\psi)_7)_{L^2(Q)} = -\frac{M^2 s}{2}\iint_Q \alpha_{xx}|\psi|^2\,dx\,dt \geq 0. \tag{121}$$

All these computations for $(L_1\psi)_5$ ((115)-(119)) show that

$$\begin{aligned}((L_1\psi)_5, L_2\psi)_{L^2(Q)} &\geq -C|M|\Big(\delta s^3\iint_Q \alpha^3|\psi|^2 dx\,dt + \epsilon s^2\iint_Q \alpha^2|\psi|^2 dx\,dt\\ &+\epsilon\int_0^T |\psi_{x|x=1}|^2\,dt + \delta s\int_0^T \alpha_{|x=1}|\psi_{x|x=1}|^2\,dt + \delta s\iint_Q \alpha|\psi_x|^2 dx\,dt\\ &+Ts\iint_Q \alpha^{(\mu+1)/\mu}|\psi|^2\,dt\Big).\end{aligned} \tag{122}$$

Let us now gather all the product $(L_1\psi, L_2\psi)_{L^2(Q)}$ coming from (91), (98), (106), (114) and (122):

$$
\begin{aligned}
&\iint_Q (\delta^2 s^5\alpha^5 + \epsilon\delta s^4\alpha^4 + \epsilon^2 s^3\alpha^3)|\psi|^2 dx\,dt + \epsilon^2 s\iint_Q \alpha|\psi_x|^2 dx\,dt + \delta^2 s\iint_Q \alpha|\psi_{xx}|^2 dx\,dt\\
&+\int_0^T (\delta^2 s\alpha_{|x=1} + \epsilon\delta)|\psi_{xx|x=1}|^2 dt + \int_0^T (\delta^2 s^3\alpha^3_{|x=1} + \epsilon^2 s\alpha_{|x=1})|\psi_{x|x=1}|^2 dt\\
&\le C\Bigg(T^{4\mu}\delta^2 s^3\iint_Q \alpha^5|\psi|^2 dx\,dt + |M|\delta s^3\iint_Q \alpha^3|\psi|^2 dx\,dt + |M|\epsilon s^2\iint_Q \alpha^2|\psi|^2 dx\,dt\\
&+\delta s\iint_Q \big(T\alpha^{(\mu+1)/\mu} + |M|\alpha\big)|\psi_x|^2 dx\,dt + |M|\delta s\int_0^T \alpha_{|x=1}|\psi_{x|x=1}|^2 dt + |M|\epsilon\int_0^T |\psi_{x|x=1}|^2 dt\\
&+\iint_Q (\delta s^3 T\alpha^{(3\mu+1)/\mu} + \epsilon s^2 T\alpha^{(2\mu+1)/\mu} + sT^2\alpha^{(\mu+2)/\mu} + sT|M|\alpha^{(\mu+1)/\mu})|\psi|^2 dx\,dt\\
&+s\int_0^T (\delta^2 T^{4\mu}\alpha^3_{|x=1} + \delta T\alpha^{(\mu+1)/\mu}_{|x=1})|\psi_{x|x=1}|^2 dt + \int_0^T (\delta^2 s\alpha_{|x=0} + \frac{\epsilon\delta}{2})|\psi_{xx|x=0}|^2 dt\Bigg).
\end{aligned}
\tag{123}
$$

• Concerning the zero order terms, we see that the first term in the right hand side of (123) is absorbed by the two first terms in the first line of (123) by taking $s \ge CT^{2\mu}$. Let us now prove that all the terms in the fifth line of (123) can be eliminated with the help of the first integral of (123) with the choice

$$
s \ge \frac{T^\mu}{\delta^{1-\mu}\epsilon^{2\mu-1}} \text{ and } s \ge C\frac{T^{2\mu}|M|^\mu}{\delta^{1-\mu}\epsilon^{2\mu-1}}. \tag{124}
$$

Firstly we observe that the integral

$$
\delta^{2\theta}\epsilon^{2-2\theta}s^{2\theta+3}\iint_Q \alpha^{2\theta+3}|\psi|^2 dx\,dt \tag{125}
$$

is bounded by the first integral of (123) for all $\theta \in [0,1]$. In particular, for $\theta = 1/(2\mu)$ (recall that $\mu \in [1/2, 1]$) this gives

$$
\delta^{1/\mu}\epsilon^{(2\mu-1)/\mu}s^{(3\mu+1)/\mu}\iint_Q \alpha^{(3\mu+1)/\mu}|\psi|^2 dx\,dt. \tag{126}
$$

With a choice of s as the first one in (124), we readily see that the first term in the fifth line of (123) is absorbed by (126). Furthermore, for $\theta = (1-\mu)/(2\mu)$ in (125), we have the integral

$$
\delta^{(1-\mu)/\mu}\epsilon^{(3\mu-1)/\mu}s^{(2\mu+1)/\mu}\iint_Q \alpha^{(2\mu+1)/\mu}|\psi|^2 dx\,dt
$$

which absorbs the second term in the fifth line of (123) with the first choice in (124). The same can be done for the third term in the fifth line of (123) for $\theta = (1-\mu)/\mu$.

Analogously, one can use (125) to absorb the last term in the fifth line of (123) with $\theta = (1-\mu)/\mu$ and

$$
s \ge C\frac{T^{3\mu/2}|M|^{\mu/2}}{\delta^{1-\mu}\epsilon^{2\mu-1}}, \tag{127}
$$

which can be obtained as an interpolation of the two choices in (124).

We also use (125) to absorb the second (resp. third) integral in the right hand side of (123) with $\theta = 1/2\mu$ (resp. with $\theta = (1-\mu)/2\mu$) for a choice of s like the second choice in (124).

• On the other hand, the presence of the first and last terms in the first line of (123) provides the term

$$
\delta^2 s^3\iint_Q \alpha^3|\psi_x|^2\,dx\,dt. \tag{128}
$$

Analogously as before, we also obtain the following integral in the left hand side of (123):

$$
\delta^{2\theta}\epsilon^{2-2\theta}s^{2\theta+1}\iint_Q \alpha^{2\theta+1}|\psi_x|^2 dx\,dt, \tag{129}
$$

for all $\theta \in [0,1]$. Thus, taking $\theta = 1/(2\mu)$, this gives

$$\delta^{1/\mu}\epsilon^{(2\mu-1)/\mu}s^{(\mu+1)/\mu}\iint_Q \alpha^{(\mu+1)/\mu}|\psi_x|^2 dx\,dt \tag{130}$$

which serves to eliminate the integral

$$C\delta T s\iint_Q \alpha^{(\mu+1)/\mu}|\psi_x|^2\,dx\,dt$$

by using the first choice in (124).

Next to eliminate the integral

$$C\delta s|M|\iint_Q \alpha|\psi_x|^2\,dx\,dt$$

we also use $\theta = 1/(2\mu)$, but we use the second choice in (124).

• Concerning the traces, the first term in the last line of (123) is absorbed by the third term in the second line by taking

$$s \geq CT^{2\mu} \tag{131}$$

Now, from the two last terms in the second line of (123) we also have the following integral in the left hand side of (123):

$$\delta^{2\theta}\epsilon^{2-2\theta}s^{2\theta+1}\int_0^T \alpha_{|x=1}^{2\theta+1}|\psi_{x|x=1}|^2\,dt. \tag{132}$$

Making the choice $\theta = 1/(2\mu)$ and taking again s like in the first choice of (124), we absorb the second term in the last line of (123).

Next, we can also use (132) to absorb the second integral in the fourth line of (123) with $\theta = 1/(2\mu)$ using the second choice in (124).

Finally, we can use (132) to absorb the last integral in the fourth line of (123) with $\theta = (1-\mu)/(2\mu)$ using the second choice in (124).

With all this, we get

$$\begin{aligned}&\delta^2 s\iint_Q \alpha|\psi_{xx}|^2 dx\,dt + \iint_Q (\delta^2 s^3\alpha^3 + \epsilon^2 s\alpha)|\psi_x|^2 dx\,dt + \int_0^T (\delta^2 s^2\alpha_{|x=1}^3 + \epsilon^2 s\alpha_{|x=1})|\psi_{x|x=1}|^2 dt\\ &+ \iint_Q (\delta^2 s^5\alpha^5 + \epsilon^2 s^3\alpha^3)|\psi|^2\,dx\,dt \leq C\left(\int_0^T (\delta^2 s\alpha_{|x=0} + \epsilon\delta)|\psi_{xx|x=0}|^2\,dt + \|L_3\psi\|^2_{L^2(Q)}\right),\end{aligned} \tag{133}$$

for a choose of s like $s \geq CT^\mu(T^\mu + (1+T^\mu M^\mu)/(\delta^{1-\mu}\epsilon^{2\mu-1}))$. Now, from the expression of $L_3\psi$ (see (82)), we see that

$$\|L_3\psi\|^2_{L^2(Q)} \leq C\iint_Q (\delta^2(s^2T^{6\mu} + s^4T^{2\mu})\alpha^5 + \epsilon^2 s^2 T^{2\mu}\alpha^3)|\psi|^2\,dx\,dt$$

which can also be absorbed by the left hand side of (133) with $s \geq CT^{2\mu}$.

Finally, we come back to φ by using the definition of $\psi = e^{-s\alpha}\varphi$ and the properties on the weight function α given in (26). As a consequence, we deduce estimate (28).

7 Appendix 2: interior regularity estimates

An alternative proof can be found in [13, Section 2]. Let us consider $\tilde{y}$ a smooth solution of (46). The general case follows from a density argument. Our goal is to estimate $\tilde{y}_{|[-1/2,1]}$ in $L^2(0,T;H^4(-1/2,1))\cap H^1(0,T;H^1(-1/2,1))$ in terms of $\|\tilde{y}\|_{L^2(0,T;L^2(-1,1))} + \|\tilde{y}_0\|_{H^3(-1,1)}$. Once this is established, (47) follows from (12).

We introduce $\rho \in C^\infty([-1,1])$, such that

$$\begin{cases} 0 \le \rho \le 1 \\ \rho = 1 \text{ in } [-\frac{1}{2}, 1], \\ \rho(-1) = 0. \end{cases}$$

We introduce $\mathcal{R}_k(x) := \int_{-1}^x \rho^k$. The estimates are done in four steps. In what follows, C denotes a constant independent of $\tilde{y}$, whose growth in $1/\delta$ and M is at most polynomial, and which can change from one line to another. Consider $m \ge 5$.

Step 1. We differentiate (46) with respect to t, multiply it by $\mathcal{R}_{2m-2}\tilde{y}_t$ and integrate; we get

$$\frac{1}{2}\Big[\int_{-1}^1 \mathcal{R}_{2m-2}|\tilde{y}_t|^2\,dx\Big]_0^T + \frac{3}{2}\delta\int_0^T\int_{-1}^1 \rho^{2m-2}|\tilde{y}_{xt}|^2\,dt\,dx + \epsilon\int_0^T\int_{-1}^1 \mathcal{R}_{2m-2}|\tilde{y}_{xt}|^2\,dt\,dx$$
$$-\delta(m-1)\int_0^T\int_{-1}^1 (\rho_x\rho^{2m-3})_x|\tilde{y}_t|^2\,dt\,dx - \epsilon(m-1)\int_0^T\int_{-1}^1 \rho_x\rho^{2m-3}|\tilde{y}_t|^2\,dt\,dx = -\frac{M}{2}\int_0^T\int_{-1}^1 \rho^{2m-2}|\tilde{y}_t|^2\,dt\,dx.$$

It follows that for some $C > 0$,

$$\|\rho^{m-1}\tilde{y}_t\|_{L^2(0,T;H_0^1(-1,1))} \le C(\|\rho^{m-2}\tilde{y}_t\|_{L^2(0,T;L^2(-1,1))} + \|\tilde{y}_0\|_{H^3(-1,1)}). \tag{134}$$

Now we fix $\breve{y} := \rho^{m-1}\tilde{y}$; it satisfies the following equation:

$$\begin{cases} -P_1^*\breve{y} = \rho^{m-1}\tilde{y}_t + [\rho^{m-1}, P_1^*]\tilde{y}, \\ \breve{y}_{|x=-1} = \breve{y}_{|x=1} = \breve{y}_{x|x=1} = 0 \end{cases}$$

where

$$P_1^* = \delta\partial^3_{xxx} - \epsilon\partial^2_{xx} - M\partial_x.$$

It follows that

$$\|\breve{y}\|_{L^2(0,T;H^4(-1,1))} \le C\|\rho^{m-1}\tilde{y}_t + [\rho^{m-1}, P_1]\tilde{y}\|_{L^2(0,T;H^1(-1,1))},$$

which implies that

$$\|\breve{y}\|_{L^2(0,T;H^4(-1,1))} \le C\big(\|\rho^{m-1}\tilde{y}_t\|_{L^2(0,T;H^1(-1,1))} + \sum_{\alpha\le 3}\|\rho^{m-5+\alpha}\partial_x^\alpha\tilde{y}\|_{L^2(0,T;L^2(-1,1))}\big).$$

With (134), we deduce that

$$\|\rho^{m-1}\tilde{y}_t\|_{L^2(0,T;H^1(-1,1))} + \|\rho^{m-1}\tilde{y}\|_{L^2(0,T;H^4(-1,1))}$$
$$\le C\big(\sum_{\alpha\le 3}\|\rho^{m-5+\alpha}\partial_x^\alpha\tilde{y}\|_{L^2(0,T;L^2(-1,1))} + \|\tilde{y}_0\|_{H^3(-1,1)} + \|\rho^{m-2}\tilde{y}_t\|_{L^2(0,T;L^2(-1,1))}\big). \tag{135}$$

Step 2. We multiply (46) by $\rho^{2m-4}\tilde{y}_{xxx}$, integrate and integrate by parts; we get

$$\delta\int_0^T\int_{-1}^1 \rho^{2m-4}|\tilde{y}_{xxx}|^2\,dt\,dx = (2m-4)\int_0^T\int_{-1}^1 \rho^{2m-5}\rho_x\tilde{y}_{xx}\tilde{y}_t\,dt\,dx$$
$$+\int_0^T\int_{-1}^1 \rho^{2m-4}\tilde{y}_{xx}\tilde{y}_{xt}\,dt\,dx + \int_0^T\int_{-1}^1 \rho^{2m-4}\tilde{y}_{xxx}(\epsilon\tilde{y}_{xx} + M\tilde{y}_x)\,dt\,dx.$$

Now we estimate the first two terms in the right-hand side in the following way: for all $\gamma > 0$, there exists $C > 0$ such that

$$\Big|(2m-4)\int_0^T\int_{-1}^1 \rho^{2m-5}\rho_x\tilde{y}_{xx}\tilde{y}_t\,dt\,dx + \int_0^T\int_{-1}^1 \rho^{2m-4}\tilde{y}_{xx}\tilde{y}_{xt}\,dt\,dx\Big|$$
$$\le C\|\rho^{m-3}\tilde{y}_{xx}\|^2_{L^2(0,T;L^2(-1,1))} + \gamma\big(\|\rho^{m-1}\tilde{y}_t\|^2_{L^2(0,T;H^1(-1,1))} + \|\rho^{m-2}\tilde{y}_t\|^2_{L^2(0,T;L^2(-1,1))}\big).$$

This yields

$$\|\rho^{m-2}\tilde{y}_{xxx}\|^2_{L^2(0,T;L^2(-1,1))} \leq C\big(\|\rho^{m-3}\tilde{y}_{xx}\|^2_{L^2(0,T;L^2(-1,1))} + \|\rho^{m-2}\tilde{y}_x\|^2_{L^2(0,T;L^2(-1,1))}\big) + \gamma\big(\|\rho^{m-1}\tilde{y}_t\|^2_{L^2(0,T;H^1(-1,1))} + \|\rho^{m-2}\tilde{y}_t\|^2_{L^2(0,T;L^2(-1,1))}\big).$$

We use (46), and we absorb the last term of the previous inequality with the left hand side, taking γ suitably small in terms of δ (in a polynomial way); we get:

$$\|\rho^{m-2}\tilde{y}_t\|_{L^2(0,T;L^2(-1,1))} + \|\rho^{m-2}\tilde{y}_{xxx}\|_{L^2(0,T;L^2(-1,1))} \leq C\big(\|\rho^{m-3}\tilde{y}_{xx}\|_{L^2(0,T;L^2(-1,1))} + \|\rho^{m-2}\tilde{y}_x\|_{L^2(0,T;L^2(-1,1))}\big) + \gamma\|\rho^{m-1}\tilde{y}_t\|_{L^2(0,T;H^1(-1,1))}.$$

Step 3. We multiply (46) by $\rho^{2m-6}\tilde{y}_x$, integrate and integrate by parts; we get

$$\delta\int_0^T\int_{-1}^1 \rho^{2m-6}|\tilde{y}_{xx}|^2\,dt\,dx = \int_0^T\int_{-1}^1 \rho^{2m-6}\tilde{y}_x\tilde{y}_t\,dt\,dx - \delta(2m-6)\int_0^T\int_{-1}^1 \rho^{2m-7}\rho_x\tilde{y}_{xx}\tilde{y}_x\,dt\,dx - \epsilon\int_0^T\int_{-1}^1 \rho^{2m-6}\tilde{y}_{xx}\tilde{y}_x\,dt\,dx - M\int_0^T\int_{-1}^1 \rho^{2m-6}|\tilde{y}_x|^2\,dt\,dx.$$

It follows that for all $\gamma > 0$, there exists $C > 0$ such that

$$\delta\int_0^T\int_{-1}^1 \rho^{2m-6}|\tilde{y}_{xx}|^2\,dt\,dx \leq \frac{\delta}{2}\int_0^T\int_{-1}^1 \rho^{2m-6}|\tilde{y}_{xx}|^2\,dt\,dx + \gamma\int_0^T\int_{-1}^1 \rho^{2m-4}|\tilde{y}_t|^2\,dt\,dx + C\int_0^T\int_{-1}^1 \rho^{2m-8}|\tilde{y}_x|^2\,dt\,dx.$$

Step 4. We multiply (46) by $\mathcal{R}_{2m-8}\tilde{y}$, integrate and integrate by parts; we get

$$\frac{1}{2}\Big[\int_{-1}^1 \mathcal{R}_{2m-8}|\tilde{y}|^2\,dt\,dx\Big]_0^T + \frac{3}{2}\delta\int_0^T\int_{-1}^1 \rho^{2m-8}|\tilde{y}_x|^2\,dt\,dx + \epsilon\int_0^T\int_{-1}^1 \mathcal{R}_{2m-8}|\tilde{y}_x|^2\,dt\,dx$$
$$-\delta(m-4)\int_0^T\int_{-1}^1 (\rho_x\rho^{2m-9})_x|\tilde{y}|^2\,dt\,dx - \epsilon(m-4)\int_0^T\int_{-1}^1 \rho_x\rho^{2m-9}|\tilde{y}|^2\,dt\,dx = -\frac{M}{2}\int_0^T\int_{-1}^1 \rho^{2m-8}|\tilde{y}|^2\,dt\,dx.$$

We deduce

$$\|\rho^{m-4}\tilde{y}_x\|_{L^2(0,T;L^2(-1,1))} \leq C\big(\|\rho^{m-5}\tilde{y}\|_{L^2(0,T;L^2(-1,1))} + \|\tilde{y}_0\|_{L^2(-1,1)}\big).$$

Conclusion. Putting the above inequalities together, we obtain

$$\|\rho^{m-1}\tilde{y}_t\|_{L^2(0,T;H^1(-1,1))} + \|\rho^{m-1}\tilde{y}\|_{L^2(0,T;H^4(-1,1))} \leq C\big(\|\rho^{m-5}\tilde{y}\|_{L^2(0,T;L^2(-1,1))} + \|\tilde{y}_0\|_{H^3(-1,1)}\big), \quad (136)$$

which due to the choice of ρ yields the desired estimate (47).

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