Inviscid limit for the 2-D stationary Euler system with arbitrary force in simply connected domains

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Abstract. In this paper, we study the convergence in the vanishing viscosity limit of the stationary incompressible Navier-Stokes equation towards the stationary Euler equation, in the presence of an arbitrary force term. This requires that the fluid is authorized to pass through some open part of the boundary.

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1 Introduction

We investigate the steady Navier-Stokes equations in a two dimensional bounded simply connected domain Ω with a smooth boundary $\partial\Omega$, with an external force term F. The system reads

$$\begin{array}{lll} v \cdot \nabla v - \nu \Delta v + \nabla p = F & \text{in} & \Omega, \\ \operatorname{div} v = 0 & \text{in} & \Omega, \end{array} \tag{1.1}$$

where $v = (v^1, v^2)$ represents the velocity of the fluid, p – its pressure and ν is the constant positive viscous coefficient.

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More precisely, we will be interested in the inviscid limit towards the Euler equation. Let us recall that for what concerns the stationary Euler system with homogeneous boundary conditions ($\vec{n}$ – the unit outward normal vector to $\partial\Omega$) and a force term:

$$
\begin{array}{llr}
v \cdot \nabla v + \nabla p = F & \text{in} & \Omega, \\
\operatorname{div} v = 0 & \text{in} & \Omega, \\
v \cdot \vec{n} = 0 & \text{on} & \partial\Omega,
\end{array}
\tag{1.2}
$$

there is no solution in general. Typically, integrating (1.2) on $\partial\Omega$ in the direction of the tangent ($\vec{\tau}$ – the tangent vector to $\partial\Omega$), we would get

$$
\int_{\partial\Omega} F \cdot \vec{\tau}\, d\sigma = 0,
\tag{1.3}
$$

which is not satisfied by any F (this is Kelvin's law for the stationary Euler equation). Moreover, even if we restrict ourselves to F satisfying (1.3), there might be no solution to (1.2). For instance, consider F satisfying (1.3) and $\operatorname{rot} F > 0$ on $\partial\Omega$. Writing (1.2) in the vorticity form:

$$
v \cdot \nabla \alpha = \operatorname{rot} F \qquad \text{in} \quad \Omega,
$$

where the vorticity is denoted

$$
\alpha = \operatorname{rot} v = \partial_{x_1} v^2 - \partial_{x_2} v^1.
$$

We see by using characteristics that one cannot define α completely on the boundary. On the other hand it was shown by Coron [3] that if one authorizes the fluid to pass through $\partial\Omega$ on an arbitrarily small part Γ of the boundary, then the Euler system has a solution for any F (in the case of a simply connected domain; see [5] in the general case).

Hence we investigate the Navier-Stokes equation with boundary conditions that will be authorized to be non-homogeneous on an arbitrary part of the boundary. Let Γ be an arbitrary nonempty open part of $\partial\Omega$, which will represent the zone where we are authorized to put non-homogeneous conditions. We will consider equation (1.1) supplied with the following boundary condition on $\partial\Omega \setminus \Gamma$:

$$
\begin{array}{lll}
v \cdot \vec{n} = 0 & \text{on} & \partial\Omega \setminus \Gamma, \\
\dfrac{\partial \alpha}{\partial \vec{n}} = 0 & \text{on} & \partial\Omega \setminus \Gamma.
\end{array}
\tag{1.4}
$$

Hence the question we raise is the following: given Ω and Γ, for any F, can we find a solution of (1.1)-(1.4) for all suitably small ν?

Let us mention that this problem is connected to control theory. It is indeed known that for finite-dimensional control systems, the stabilizability property involves the existence of stationary solutions for small right hand side (see Brockett [1]). A similar phenomenon is observed for the boundary stabilizability of the incompressible Euler equation and raises the question for the Navier-Stokes equation. See [3, 4, 6] for more details.

The solutions to (1.1),(1.4) will be found as solving the following system:

$$
\begin{array}{ll}
v\cdot\nabla\alpha-\nu\Delta\alpha=\operatorname{rot}F & \text{in}\quad\Omega,\\
\operatorname{rot}v=\alpha & \text{in}\quad\Omega,\\
\operatorname{div}v=0 & \text{in}\quad\Omega,\\
v\cdot\vec{n}=d & \text{on}\quad\partial\Omega,\\
\alpha=\alpha_{in} & \text{on}\quad\Gamma_{in},\\
\frac{\partial\alpha}{\partial\vec{n}}=0 & \text{on}\quad\partial\Omega\setminus\Gamma_{in}.
\end{array}
\tag{1.5}
$$

Let us emphasize that thanks to the simple connectedness of the domain Ω the system $(1.5)_{1,2,3}$ is completely equivalent to the original equation (1.1). A comment which is required here concerns the choice of the boundary condition $(1.5)_6$. Since we are required to examine the inviscid limit of solutions to (1.5) this relation should disappear at the limit. Additionally, inhomogeneous data $(1.5)_{4,5}$ are required to be preserved, to control the well posedness of the limit Euler system. Somehow we may look at (1.5) as a regularization of the system (1.2), thus we can find an analogy to considerations for the continuity equation in the theory of weak solutions to the compressible Navier-Stokes equations [9].

In the above equation, one will choose d suitably; in particular one requires

$$
\operatorname{supp}d\subset\Gamma\qquad\text{and}\qquad\int_{\partial\Omega}d\,d\sigma=0.
\tag{1.6}
$$

in order that $(1.4)_4$ is satisfied. Also we introduce

$$
\Gamma_{in}=\Big\{x\in\Gamma\;\Big|\;d(x)<0\Big\}\qquad\text{and}\qquad\Gamma_{out}=\Big\{x\in\Gamma\;\Big|\;d(x)>0\Big\}.
\tag{1.7}
$$

Our first aim is the following result.

Theorem 1 *For any (Ω,Γ), there exists $\overline{d}\in C^\infty(\Gamma)$ such that the following holds. Let $F\in H^1(\Omega;\mathbb{R}^2)$, and $\alpha_{in}\in H^{3/2}(\Gamma_{in};\mathbb{R})$, where Γ_{in} is defined in (1.7) with $d=\overline{d}$. Then there exists $\overline{l}$ such that for $0<\nu\le 1$, for any $l>\overline{l}$ there exists at least one weak solution to the system (1.5) with*

$$
d(x)=l\overline{d}(x),
\tag{1.8}
$$

such that $v\in C^a(\overline{\Omega})$ with $a<\frac{1}{2}$ and

$$
\|rot\,v\|_{L_2(\Omega)}+\|v\|_{C^a(\overline{\Omega})}\le C(DATA,l),
\tag{1.9}
$$

where the r.h.s. of (1.9) is independent from ν.

The main difficulty is to obtain the estimate (1.9), giving relatively high regularity of solutions with no dependence from the viscosity coefficient ν. The solutions will be constructed as perturbations of a given potential flow, which is related to the function $\overline{d}$ defined on Γ. Such flow will be constructed in the next section. Our technique is based on the energy approach, however, used in a non-standard way. For chosen d in (1.8) the solution to (1.5) is unique, which is a consequence of the application of the Banach fixed point theorem. However d is constructed in the proof, so this feature is not emphasized in the statement of the theorem. The obtained bound (1.9) is a motivation for the next result.

Our second aim is to analyze the inviscid limit of solutions given by Theorem 1.

Theorem 2 *Let the assumptions of Theorem 1 be fulfilled and let v^ν denote the solution to (1.5) with the viscosity ν. Then there exists a function $v^E \in C^{1/2-\delta}(\overline{\Omega})$ with $\delta > 0$ such that for a subsequence $\nu_k \to 0^+$ as $k \to \infty$*

$$(1.10) \qquad v^{\nu_k} \to v^E \quad in \quad C^{1/2-\delta}(\overline{\Omega}) \quad and \quad v^{\nu_k} \rightharpoonup v^E \quad in \quad H^1(\Omega)$$

for $k \to \infty$ and v^E fulfills the Euler system

$$(1.11) \qquad \begin{array}{lll} v^E \cdot \nabla v^E + \nabla p^E = F & in & \Omega, \\ div\, v^E = 0 & in & \Omega, \\ v^E \cdot \vec{n} = d & on & \partial\Omega, \\ rot\, v^E = \alpha_{in} & on & \Gamma_{in}, \end{array}$$

where p^E is a constructible pressure.

The main element in the proof of Theorem 2 is the a priori estimate (1.9). In the evolutionary case the basic bound follows from the energy estimate, however in the stationary case we lose this possibility. The information given by Theorem 1 is not sufficient. We are required to find more sophisticated estimates for higher derivatives of solutions in terms of dependence of ν. It will allow us to control the dependence from the boundary condition α_{in}, which could be omitted in straightforward considerations. The meaning of the solution to (1.11) will be defined later by (4.10). The obtained regularity and weak formulation will allow us to control the dependence from the boundary datum α_{in}.

There are no general results concerning the inviscid limit of the stationary Navier-Stokes system towards the stationary Euler system, the only known results have been proved in [7], [8], but for the unforced system in a special type of domains.

The same as for Theorem 1 fixing d we are able to obtain the uniqueness of the limit, so (1.10) holds for arbitrary sequence, not only for a subsequence. Additionally in that case (1.11) admits unique solutions and it is again the consequence of the chosen fixed point approach in the proof of estimate (1.9). We omit these considerations in proofs, since this property holds for our particular d and is not proven for general datum.

The structure of the paper is the following. In Section 2, we introduce the function $\overline{d}$ announced in Theorem 1; in Section 3, we establish Theorem 1; in Section 4, we establish Theorem 2; finally, Section 5 is an Appendix where we have put technical yet central results.

Throughout the paper we use the standard notation. Letter C denotes a generic constant independent from ν and λ, $DATA$ depends on norms of data and known quantities and it is independent from ν and λ, too.

2 A proposition

In this section we introduce a function θ on which the construction depends. This is given in the next proposition;

Proposition 2.1 *Let Ω be a bounded smooth simply connected domain in $\mathbb{R}^2$, and let Γ a nonempty open part of $\partial\Omega$. Then there exists $\theta \in C^\infty(\overline{\Omega};\mathbb{R})$ such that*

$$\Delta\theta = 0 \ \text{in } \Omega, \qquad |\nabla\theta| \geq k > 0 \ \text{in } \overline{\Omega}, \tag{2.1}$$

$$\nabla\theta \cdot \vec{n} = 0 \ \text{on } \partial\Omega \setminus \Gamma, \tag{2.2}$$

$$\forall t, \ \left\{ x \in \overline{\Omega} \ \middle| \ \theta(x) \leq t \right\} \ \text{is a piecewise smooth domain.} \tag{2.3}$$

Remark 2.1 *For the rest of the paper, we will fix*

$$\overline{d} := \nabla\theta \cdot \vec{n} \ \text{on } \partial\Omega,$$

and as we will consider boundary conditions $(1.5)_4$ of the type $d = l\overline{d}$, we will systematically have according to the definition (1.7):

$$\begin{aligned} \Gamma_{in} &= \left\{x \in \partial\Omega : \nabla\theta(x)\cdot\vec{n}(x) < 0\right\} = \left\{x \in \partial\Omega : d(x) < 0\right\}, \\ \Gamma_{out} &= \left\{x \in \partial\Omega : \nabla\theta(x)\cdot\vec{n}(x) > 0\right\} = \left\{x \in \partial\Omega : d(x) > 0\right\}. \end{aligned} \tag{2.4}$$

Shrinking Γ if necessary, we will suppose from now on that $\Gamma = \Gamma_{in} \cup \Gamma_{out}$.

Proof. Such a proposition without condition (2.3) was established in [2]. Here we proceed as follows. Consider in $\mathbb{R}^2$ the square $[0,1]^2$. Extend it inside the strip $\mathbb{R}\times[0,1]$ into a smooth bounded contractile domain U. Now it follows from Riemann's mapping theorem that U and Ω are conformally equivalent. Moreover it is a standard result that the corresponding mapping is C^∞ up to the boundary, as follows from the smoothness of Ω and U (see for instance [10]).

Now a conformal map of a simply connected domain is defined up to the conformal group of the disc, which is the following 3-parameter group:

$$G = \left\{ g(z) = e^{i\phi}\frac{a+z}{1+\overline{a}z}, \ a \in \mathbb{C}, \ |a| < 1, \ \phi \in \mathbb{R} \right\}.$$

Hence one can extract a unique conformal mapping φ between U and Ω by fixing the image of three points of the boundary of U in the boundary of Ω. We do as described in Figure 1. Precisely, shrinking Γ if necessary, we can suppose that it is connected. Call $\{\tilde{A}, \tilde{B}\}$ its boundary inside $\partial\Omega$. Denote $A := (1, 0)$, $B = (1, 1)$ and $C = (0, 1/2)$. We choose the conformal map φ from U to Ω so that it sends A to $\tilde{A}$, B to $\tilde{B}$, and C inside Γ.

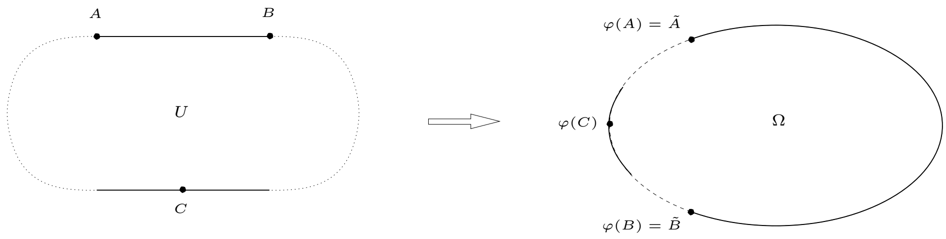

Figure 1: Conformal map

We recall that on simply connected domains, there is an equivalence between holomorphic functions and gradients of harmonic functions via the following rule:

$$\psi = \psi^1 + i\psi^2 \text{ is holomorphic in } \Omega \iff (\psi^1, -\psi^2) \text{ is the gradient of a harmonic function.} \tag{2.5}$$

Now we consider the application θ obtained by transporting on Ω the harmonic map $(x_1, x_2) \mapsto x_1$ defined on U through φ, $\theta(\varphi(x)) = x$ on U. Now for this θ, properties (2.1) and (2.2) come from the conformality and the fact that $(1, 0)$ is tangent to the part of the boundary ∂U given by $[A, B]$. Finally, (2.3) directly comes from the fact that

$$\left\{ x = (x_1, x_2) \in U \;\middle|\; x_1 \leq t \right\} \text{ is a piecewise smooth domain.}$$

Proposition 2.1 is proved.

3 Proof of Theorem 1

In this section we prove Theorem 1. The proof is divided in two parts. First, we restrict our attention to sufficiently small data, but the viscosity coefficient is not restricted, i.e. it can be arbitrarily small. Next, we consider the general case by using a homogeneity argument.

Let us consider the case of small data. We look for solutions in a vicinity of the potential flow $\nabla\theta$ constructed by Proposition 2.1.

Let the solution to (1.5) be considered in the following form

$$v = \nabla\theta + u \quad \text{and we put} \quad v \cdot \vec{n} = \nabla\theta \cdot \vec{n} =: \overline{d} \text{ on } \partial\Omega. \tag{3.1}$$

Then the system (1.5) in terms of u takes the form

$$
(3.2) \qquad \begin{array}{lll}
\nabla\theta \cdot \nabla\alpha - \nu\Delta\alpha = \operatorname{rot} F - u \cdot \nabla\alpha & \text{in} & \Omega, \\
\operatorname{rot} u = \alpha & \text{in} & \Omega, \\
\operatorname{div} u = 0 & \text{in} & \Omega, \\
u \cdot \vec{n} = 0 & \text{on} & \partial\Omega, \\
\alpha = \alpha_{in} & \text{on} & \Gamma_{in}, \\
\frac{\partial\alpha}{\partial\vec{n}} = 0 & \text{on} & \partial\Omega \setminus \Gamma_{in}.
\end{array}
$$

Our technique requires a special parameterization of the domain Ω. Thanks to Proposition 2.1 we are allowed to use the properties of the function θ. The set Ω is parameterized by a set $D \subset \mathbb{R}^2$ in that way

$$Y_t = \{x \in \Omega : \theta(x) = t\}$$

and Y_t is parameterized by a coordinate $s \in \mathbb{R}$. Thus, we find a diffeomorphism

$$D \ni (s,t) \leftrightarrow x \in \Omega.$$

Precisely, one chooses D to be U such as described in the proof of Proposition 2.1. Additionally, we introduce

$$X_t = \bigcup_{t'<t} Y_{t'} = \{x \in \Omega : \theta(x) < t\}$$

for $t \in (t_{min}, t_{max})$ and $X_{t_{max}} = \Omega$ and $X_{t_{min}} = \emptyset$.

Let us remark that ∂X_t can be divided in several parts: the part Y_t, and the part $\partial X_t \cap \partial\Omega$, which can itself be divided into $\partial X_t \cap \Gamma_{in}$, $\partial X_t \cap (\partial\Omega \setminus \Gamma)$ and $\partial X_t \cap \Gamma_{out}$ (which is not empty for large t). Clearly, this decomposition is trivial when transported in U.

The proof of existence of solutions to (3.2) will rely on the Banach fixed point theorem. First, we show the a priori estimate, describing the function spaces for the solutions.

A priori estimate. To start our estimation we are required to find an extension of the boundary vorticity. We easily find $\tilde{\alpha} \in H^2(\Omega)$ such that

$$(3.3) \quad \|\tilde{\alpha}\|_{H^2(\Omega)} \le C\|\alpha_{in}\|_{H^{3/2}(\Gamma_{in})}, \qquad \tilde{\alpha}|_{\Gamma_{in}} = \alpha_{in} \quad \text{and} \quad \tilde{\alpha}|_{\Gamma_{out}} = 0.$$

Additionally, we choose $\tilde{\alpha}$ as a harmonic function

$$(3.4) \qquad \Delta\tilde{\alpha} = 0 \quad \text{in} \quad \Omega.$$

Multiplying $(3.2)_1$ by $(\alpha - \tilde{\alpha})$ and integrating over X_t we get

$$(3.5)$$
$$\int_{X_t} [\nabla\theta \cdot \nabla\alpha(\alpha - \tilde{\alpha}) - \nu\Delta\alpha(\alpha - \tilde{\alpha})]dx = \int_{X_t} [\operatorname{rot} F(\alpha - \tilde{\alpha}) + u\nabla\alpha(\alpha - \tilde{\alpha})]dx.$$

Let us consider the first term in the l.h.s. of (3.5). It reads

$$\frac{1}{2}\int_{X_t} \nabla\theta\cdot\nabla\alpha^2 dx - \int_{X_t}\nabla\theta\cdot\nabla\alpha\tilde{\alpha}dx = \frac{1}{2}\int_{Y_t}|\nabla\theta|\alpha^2 d\sigma + \int_{X_t}\nabla\theta\cdot\nabla\tilde{\alpha}\alpha dx - \int_{Y_t}|\nabla\theta|\alpha\tilde{\alpha}d\sigma - \frac{1}{2}\int_{\partial X_t\cap\Gamma_{in}}\nabla\theta\cdot\vec{n}\alpha_{in}^2 d\sigma + \frac{1}{2}\int_{\partial X_t\cap\Gamma_{out}}\nabla\theta\cdot\vec{n}\alpha^2 d\sigma,$$

where we used the fact $\nabla\theta\cdot\vec{n} = |\nabla\theta|$ at Y_t and that $u\cdot\vec{n} = 0$ on $\partial\Omega\setminus\Gamma$. Now we remark that by (2.4) the integral over $\partial X_t\cap\Gamma_{out}$ is non-negative; hence we can forget this term and obtain a lower bound for the first term in the l.h.s. of (3.5).

The second term takes the form

$$-\nu\int_{X_t}\Delta\alpha(\alpha-\tilde{\alpha})dx = \nu\int_{X_t}|\nabla\alpha|^2 dx - \nu\int_{X_t}\nabla\alpha\cdot\nabla\tilde{\alpha}dx - \nu\int_{Y_t}\frac{\partial\alpha}{\partial n}\alpha d\sigma + \nu\int_{Y_t}\frac{\partial\alpha}{\partial n}\tilde{\alpha}d\sigma.$$

The remaining boundary terms vanish by $(3.2)_{5,6}$ and (3.3).

The last term of the r.h.s. of (3.5) is considered in the form

$$\int_{X_t} u\cdot\nabla\alpha\alpha\, dx - \int_{X_t} u\cdot\nabla\alpha\tilde{\alpha}\, dx = \frac{1}{2}\int_{Y_t}\vec{n}\cdot u\alpha^2\, d\sigma + \int_{X_t} u\cdot\nabla\tilde{\alpha}\alpha\, dx - \int_{Y_t}\vec{n}\cdot u\alpha\tilde{\alpha}\, d\sigma.$$

Here the remaining boundary terms vanish thanks to (3.3) and $(3.2)_4$.

Applying the standard Schwarz inequality we infer that for some constant C independent of ν,

$$\left|\nu\int_{Y_t}\frac{\partial\alpha}{\partial n}\alpha d\sigma\right| \le \frac{k}{4}\sup_t\int_{Y_t}\alpha^2\, d\sigma + C\sup_t \nu^2\int_{Y_t}|\nabla\alpha|^2\, d\sigma.$$

In the above estimate, the constant k is given by (2.1).

We conclude

$$\frac{k}{4}\sup_t\int_{Y_t}\alpha^2 d\sigma + \nu\int_\Omega|\nabla\alpha|^2 dx \le \|u\|_{L_\infty(\Omega)}\sup_t\int_{Y_t}\alpha^2 d\sigma + C\sup_t\nu^2\int_{Y_t}|\nabla\alpha|^2 d\sigma + C\sup_t\int_{Y_t}\tilde{\alpha}^2 d\sigma + \int_\Omega|\nabla\tilde{\alpha}|^2 dx + \int_\Omega(\operatorname{rot} F)^2 dx + C\int_{\Gamma_{in}}\alpha_{in}^2 d\sigma. \tag{3.6}$$

The main difficulty is the term containing the trace of the gradient of α at Y_t – the second term in the r.h.s. of (3.6). To estimate it we have to apply the trace theorem for the Besov space $B^{1/2}_{2,1}(\Omega)$. This critical case still controls the trace since Y_t is a smooth submanifold, i.e. in our case for a sufficiently regular function f there holds

$$\|f|_{Y_t}\|_{L_2(Y_t)} \leq C\|f\|_{B^{1/2}_{2,1}(\Omega)} \qquad \text{for all} \quad t \in (t_{min}, t_{\max}). \tag{3.7}$$

The definition of this space is quite complex, however it can be represented as an interpolation space

$$B^{1/2}_{2,1}(\Omega) = (H^1(\Omega), L_2(\Omega))_{1/2,1}$$

– see [11]; this guarantees the estimate

$$\|f\|_{B^{1/2}_{2,1}(\Omega)} \leq C\|f\|^{1/2}_{H^1(\Omega)}\|f\|^{1/2}_{L_2(\Omega)}. \tag{3.8}$$

Thus for, (3.7) and (3.8) we easily conclude that the boundary term can be estimated as follows

$$\nu^2 \int_{Y_t} |\nabla\alpha|^2 d\sigma \leq C\nu^2\|\nabla\alpha\|_{L_2(\Omega)}(\|\nabla^2\alpha\|_{L_2(\Omega)} + \|\nabla\alpha\|_{L_2(\Omega)}). \tag{3.9}$$

To make use of the above inequality there is a need to control the second gradient of α in terms of ν. We will need the following result, which we prove in the Appendix.

Theorem 3 *Let $G \in L_2(\Omega)$, $\theta \in W^1_\infty(\Omega)$, and consider λ – a parameter of the localization defined by (5.2) – be sufficiently small, then solutions to*

$$\begin{array}{lcr} \nabla\theta\cdot\nabla\beta - \nu\Delta\beta = G & in & \Omega, \\ \beta = \beta_{in} & on & \Gamma_{in}, \\ \frac{\partial\beta}{\partial\vec{n}} = 0 & on & \partial\Omega\setminus\Gamma_{in} \end{array} \tag{3.10}$$

fulfill the following bound (for $0 < \lambda < \lambda_0(\partial\Omega)$):

$$\begin{aligned} \nu\|\nabla^2\beta\|_{L_2(\Omega)} \leq C\,\big[&(\nu\lambda^{-1}+\lambda)\|\nabla\beta\|_{L_2(\Omega)} + (\nu\lambda^{-2}+1)\|\beta\|_{L_2(\Omega)} \\ &+\|G\|_{L_2(\Omega)} + \|\beta_{in}\|_{H^{3/2}(\Gamma_{in})}\big]. \end{aligned} \tag{3.11}$$

Applying Theorem 3 to estimate $\nu\|\nabla^2\alpha\|_{L_2(\Omega)}$ (and using $(3.2)_1$) we conclude

$$\begin{aligned} \nu^2\int_{Y_t}|\nabla\alpha|^2 d\sigma \leq C\nu\|\nabla\alpha\|_{L_2(\Omega)}\Big(&\|\nabla\alpha\|_{L_2(\Omega)}(\nu\lambda^{-1}+\lambda+\nu+\|u\|_{L_\infty(\Omega)}) \\ &+\|\alpha\|_{L_2(\Omega)}(\nu\lambda^{-2}+1) + \|\mathrm{rot}\,F\|_{L_2(\Omega)} + \|\alpha_{in}\|_{H^{3/2}(\Gamma_{in})}\Big). \end{aligned} \tag{3.12}$$

Now, we restrict ourselves to the case

$$\nu \leq \lambda^2 \leq 1, \text{ and, hence, } \nu\lambda^{-1} + \lambda + \nu \leq 3\lambda \text{ and } \nu\lambda^{-2} + 1 \leq 2. \tag{3.13}$$

This can be obtained for any $\lambda > 0$ by choosing ν sufficiently small.

Using $\nu\|\nabla\alpha\|_{L^2(\Omega)}\|\alpha\|_{L^2(\Omega)} \leq \nu^2\lambda^{-1}\|\nabla\alpha\|^2_{L^2(\Omega)} + \lambda\|\alpha\|^2_{L^2(\Omega)}$ and taking (3.13) into account we deduce

$$\nu^2 \int_{Y_t} |\nabla\alpha|^2 d\sigma \leq C\nu(3\lambda + \|u\|_{L_\infty(\Omega)})\|\nabla\alpha\|^2_{L_2(\Omega)} + C\lambda\|\alpha\|^2_{L_2(\Omega)} + DATA. \tag{3.14}$$

Provided sufficiently smallness of λ and $\|u\|_{L_\infty(\Omega)} \leq \epsilon_0$ in terms of the domain only (and in particular independently of ν which has to be less than a constant, say $\nu_0(\Omega, \Gamma)$, which also depends on the domain only), we obtain

$$\frac{k}{4}\sup_t \int_{Y_t} \alpha^2 d\sigma + \nu\int_\Omega |\nabla\alpha|^2 dx \leq \frac{k}{8}\sup_t \int_{Y_t} \alpha^2 d\sigma + \frac{1}{2}\nu\int_\Omega |\nabla\alpha|^2 dx + DATA. \tag{3.15}$$

So we get the main a priori bound

$$\sup_t \int_{Y_t} \alpha^2 d\sigma + \nu\int_\Omega |\nabla\alpha|^2 dx \leq DATA. \tag{3.16}$$

The quantities in $DATA$ are independent from ν and are sufficiently small. To close the estimation we are required to show that inequality (3.16) implies $\|u\|_{L_\infty(\Omega)} \leq \epsilon_0$, but

$$\begin{array}{lll} \operatorname{rot} u = \alpha & \text{in} & \Omega, \\ \operatorname{div} u = 0 & \text{in} & \Omega, \\ \vec{n}\cdot u = 0 & \text{on} & \partial\Omega. \end{array} \tag{3.17}$$

We want to show that solutions to (3.17) belong to $C^a(\overline{\Omega})$ for $a < \frac{1}{2}$. By $(3.17)_2$ we can introduce a stream function ϕ such that $u = \nabla^\perp\phi$ and

$$\begin{array}{lll} \Delta\phi = \alpha & \text{in} & \Omega, \\ \phi = 0 & \text{in} & \partial\Omega. \end{array} \tag{3.18}$$

The estimate (3.16) guarantees that for any $2 \leq p < \infty$ we have

$$\alpha \in L_p(t_{min}, t_{max}; L_2(Y_t)).$$

Trivially we have $u = \nabla^\perp\phi \in H^1(\Omega)$. To obtain the Hölder continuity of u we examine a model problem in the plane

$$\Delta\psi = \beta \quad \text{in } \mathbb{R}^2, \qquad \text{where } \beta \in L_p(\mathbb{R}; L_2(\mathbb{R})), \tag{3.19}$$

then using the Fourier transform

$$\nabla^2\psi = \mathcal{F}_x^{-1}\left[\frac{\xi\otimes\xi}{|\xi|^2}\right]\beta, \quad \text{hence} \quad \nabla^2\psi \in L_p(\mathbb{R}; L_2(\mathbb{R})), \text{ too.}$$

Then the embedding theorem implies

$$\|\nabla\psi\|_{C^a(\mathbb{R}^2)} \le C(\|\beta\|_{L_p(\mathbb{R};L_2(\mathbb{R}))} + \|\nabla\psi\|_{H^1(\mathbb{R}^2)}) \quad \text{for} \;\; a < \frac{1}{2} - \frac{1}{p}. \tag{3.20}$$

To apply the above result to (3.18) we just solve the equation (3.19) with α extended by zero, using the localization techniques. Then by (3.20), we can remove inhomogeneity from the r.h.s. of (3.18), getting a standard problem in the Hölder spaces for which we are able to show

$$u \in C^a(\overline{\Omega}) \qquad \text{with} \qquad a < \frac{1}{2} - \frac{1}{p}.$$

Finally we get

$$\|u\|_{L_\infty(\Omega)} + < u >_{C^a(\overline{\Omega})} \le C_p DATA \le \epsilon_0 \tag{3.21}$$

as $DATA$ are sufficiently small and $< \cdot >_{C^a(\overline{\Omega})}$ denotes the main seminorm in the Hölder space $C^a(\overline{\Omega})$. The estimate (3.21) enables us to proceed the next step in our proof.

Existence. Let us define the set

$$\Xi = \left\{ u \in C(\overline{\Omega};\mathbb{R}^2) : \vec{n}\cdot u = 0 \quad \text{and} \quad \|u\|_{L_\infty(\Omega)} \le \epsilon_0 \right\}.$$

Next, we define a map $K : \Xi \to C(\overline{\Omega};\mathbb{R}^2)$ such that

$$K(\bar{u}) = u,$$

where u is the solution to the following problem

$$\begin{array}{lll} \operatorname{rot} u = \alpha & \text{in} & \Omega, \\ \operatorname{div} u = 0 & \text{in} & \Omega, \\ \vec{n}\cdot u = 0 & \text{on} & \partial\Omega \end{array} \tag{3.22}$$

and α is given as the solution to the problem below

$$\begin{array}{lll} \nabla\theta\cdot\nabla\alpha - \nu\Delta\alpha = \operatorname{rot} F - \bar{u}\cdot\nabla\alpha & \text{in} & \Omega, \\ \alpha = \alpha_{in} & \text{on} & \Gamma_{in}, \\ \frac{\partial\alpha}{\partial\vec{n}} = 0 & \text{on} & \partial\Omega\setminus\Gamma_{in}. \end{array} \tag{3.23}$$

We see that a fixed point of the map K defines a solution to system (3.2).

Lemma 3.1 *Let $\bar{u} \in \Xi$ and*

$$\|rot F\|_{L_2(\Omega)} + \|\alpha_{in}\|_{H^{3/2}(\Gamma_{in})} \le \epsilon_1, \tag{3.24}$$

then $u \in \Xi$, i.e. $K : \Xi \to \Xi$. Moreover the map K is a contraction on the set Ξ. Subsequently, there exists unique fixed point of K belonging to Ξ.

Proof. To obtain $u \in \Xi$ it is enough to follow the steps of the a priori bound. To prove that K is the contraction we note that

$$\|K(\bar{u}_1) - K(\bar{u}_2)\|_{L_\infty(\Omega)} \leq \frac{1}{2}\|\bar{u}_1 - \bar{u}_2\|_{L_\infty(\Omega)}, \tag{3.25}$$

and

$$\begin{array}{lll} \mathrm{rot}\,(u_1 - u_2) = \alpha_1 - \alpha_2 & \text{in} & \Omega, \\ \mathrm{div}\,(u_1 - u_2) = 0 & \text{in} & \Omega, \\ \vec{n} \cdot (u_1 - u_2) = 0 & \text{on} & \partial\Omega, \end{array} \tag{3.26}$$

where $u_1 = K(\bar{u}_1)$, $u_2 = K(\bar{u}_2)$ and α_1, α_2 are the vorticities satisfying (3.23) for $\overline{u} = \bar{u}_1$ and $\overline{u} = \bar{u}_2$, respectively. The difference $(\alpha_1 - \alpha_2)$ fulfills the equations

$$\begin{array}{lll} \nabla\theta \cdot \nabla(\alpha_1 - \alpha_2) - \nu\Delta(\alpha_1 - \alpha_2) & & \\ \qquad = \bar{u}_1 \cdot \nabla(\alpha_1 - \alpha_2) + (\bar{u}_1 - \bar{u}_2)\nabla\alpha_2 & \text{in} & \Omega, \\ \alpha_1 - \alpha_2 = 0 & \text{on} & \Gamma_{in}, \\ \frac{\partial}{\partial \vec{n}}(\alpha_1 - \alpha_2) = 0 & \text{on} & \partial\Omega \setminus \Gamma_{in}. \end{array} \tag{3.27}$$

Again we repeat the steps (3.3)-(3.21) and get (3.25), provided that the data and ϵ_0 are sufficiently small. Inequality (3.25) implies the existence of unique fixed point $K(u) = u$, which yields a solution of the system (1.5).

However the above considerations concerned only the small data problem: at this step we have proven the following.

Proposition 3.1 *There exist $\epsilon_0 = \epsilon_0(\Omega, \Gamma)$ and $\nu_0 = \nu_0(\Omega, \Gamma)$ such that for any $\alpha_{in} \in H^{3/2}(\Gamma_{in})$, for any $F \in H^1(\Omega)$ satisfying*

$$\|\alpha_{in}\|_{H^{3/2}(\Gamma_{in})} + \|F\|_{H^1(\Omega)} \leq \epsilon_0,$$

for any $\nu \in (0, \nu_0)$, there exists a solution to the system (1.5) *with $d = \overline{d} = \nabla\theta \cdot \vec{n}$ on $\partial\Omega$, and with the estimate* (1.9) *satisfied independently from ν.*

Now let us justify the passage from Proposition 3.1 to Theorem 1. This is a homogeneity argument: if (u, p) satisfies

$$(u \cdot \nabla)u - \nu\Delta u + \nabla p = F,$$

then $u^\rho := \rho u$ and $p^\rho := \rho^2 p$ satisfy

$$(u^\rho \cdot \nabla)u^\rho - \nu\rho\Delta u^\rho + \nabla p^\rho = \rho^2 F,$$

Now given $\alpha_{in} \in H^{3/2}(\Gamma_{in})$ and $F \in H^1(\Omega)$ which do not necessarily satisfy a smallness assumption, we choose $\rho \leq 1$ small enough in order that

$$\|\rho\alpha_{in}\|_{H^{3/2}(\Gamma_{in})} + \|\rho^2 F\|_{H^1(\Omega)} \leq \epsilon_0. \tag{3.28}$$

For this data $(\rho\alpha_{in}, \rho^2 F)$ one can employ Proposition 3.1, and deduce a solution for any $\nu \le \nu_0$. Let us call $\tilde{u}$ this solution. It satisfies in particular $\tilde{u} \cdot \vec{n} = \overline{d}$ on $\partial\Omega$.

Now we go back to the original data (α_{in}, F), that is, we use the above homogeneity argument with coefficient $1/\rho$. We find a solution u of

$$
\begin{aligned}
(u \cdot \nabla)u - \frac{\nu}{\rho}\Delta u + \nabla p = F \text{ in } \Omega, \\
u \cdot \vec{n} = \frac{1}{\rho}\overline{d} \text{ on } \partial\Omega, \\
\operatorname{rot} u = \alpha_{in} \text{ on } \Gamma_{in},
\end{aligned}
$$

for any $\nu \in (0, \nu_0]$. Hence, a solution of (1.1) for a wider range of ν, which can include $(0, 1]$, reducing ρ if necessary.

Now, since ρ is chosen as to satisfy (3.28) and can consequently be chosen arbitrarily small, the boundary condition $d = \frac{1}{\rho}\overline{d}$ becomes (1.8) for large enough l. That u satisfies (1.9) is a straightforward consequence of the fact that $\tilde{u}$ satisfies (1.9) (of course, the constant depends on l). This ends the proof of Theorem 1.

4 Proof of Theorem 2

A key problem in the passage to the limit is the control of dependence of the obtained inviscid solutions from the boundary data. The main difficulty is related to α_{in} on Γ_{in}, since the vorticity is uniformly bounded in a space which does not control the trace (we may take the L_2-space). That is the reason we shall choose a special class of the test functions.

For fixed $\nu > 0$ the solution to the Navier-Stokes equations are regular. In particular there holds

$$
(4.1) \qquad \int_\Omega v^\nu \cdot \nabla\alpha^\nu \phi dx - \nu \int_\Omega \Delta\alpha^\nu \phi dx = \int_\Omega \operatorname{rot} F \, \phi \, dx
$$

for each $\phi \in C^\infty(\overline{\Omega}; \mathbb{R})$ such that $\phi|_{\Gamma_{out}} = 0$.

The choice of ϕ makes us possible to take into account the influence of the boundary vorticity α_{in} at Γ_{in} as well as to neglect information at Γ_{out}.

Then (4.1) reads

$$
\begin{aligned}
(4.2) \quad -\int_\Omega v^\nu \alpha^\nu \nabla\phi \, dx + \int_{\Gamma_{in}} \vec{n} \cdot v^\nu \alpha_{in} \phi \, d\sigma - \nu \int_{\Gamma_{in}} \frac{\partial \alpha^\nu}{\partial \vec{n}} \phi \, d\sigma \\
+ \nu \int_\Omega \nabla\alpha^\nu \nabla\phi \, dx = \int_\Omega \operatorname{rot} F \, \phi \, dx.
\end{aligned}
$$

The above equality follows from integration by parts and the form of the boundary terms is a consequence of the boundary conditions:

$$
(4.3) \qquad
\begin{array}{lll}
\vec{n} \cdot v^\nu = 0 & \text{on} & \partial\Omega \setminus (\Gamma_{in} \cup \Gamma_{out}), \\
\phi = 0 & \text{on} & \Gamma_{out}, \\
\frac{\partial \alpha^\nu}{\partial \vec{n}} = 0 & \text{on} & \partial\Omega \setminus \Gamma_{in}.
\end{array}
$$

The estimates proved in Theorem 1 guarantee for $a < \frac{1}{2}$ the following bound uniformly in ν:

$$\|v^\nu\|_{C^a(\overline{\Omega})} + \|v^\nu\|_{H^1(\Omega)} + \nu^{1/2}\|\nabla\alpha^\nu\|_{L_2(\Omega)} \le C. \tag{4.4}$$

So for any $\delta > 0$ we find a subsequence $\nu_k \to 0^+$ such that

$$v^{\nu_k} \to v^E \quad \text{in} \quad C^{1/2-\delta}(\overline{\Omega}) \quad \text{and} \quad v^{\nu_k} \rightharpoonup v^E \quad \text{in} \quad H^1(\Omega), \tag{4.5}$$

for $k \to \infty$ for some divergence-free function v^E.

To examine the limit in (4.2) for this subsequence we analyze the behavior of the third and fourth terms of the l.h.s. of (4.2). By (4.4) the last term vanishes as $\nu \to 0$, since

$$\nu\Big|\int_\Omega \nabla\alpha^\nu\nabla\phi dx\Big| \le \nu^{1/2}\big[\nu^{1/2}\|\nabla\alpha^\nu\|_{L_2(\Omega)}\big]\|\nabla\phi\|_{L_2(\Omega)} \le C\nu^{1/2} \to 0. \tag{4.6}$$

However the main difficulty is located in the third term. To find a good estimate we are required to control the normal derivative of α^ν. As in the proof of Theorem 1 – see considerations for (3.7)-(3.14) – we follow (ϕ is given and fixed)

$$\begin{aligned} \nu\Big|\int_{\Gamma_{in}} \frac{\partial\alpha^\nu}{\partial\vec{n}}\phi d\sigma\Big| &\le C\nu\Big\|\frac{\partial\alpha^\nu}{\partial\vec{n}}\Big\|_{L_2(\Gamma_{in})} \\ &\le C\nu\|\nabla\alpha^\nu\|^{1/2}_{L_2(\Omega)}(\|\nabla^2\alpha^\nu\|^{1/2}_{L_2(\Omega)} + \|\nabla\alpha^\nu\|^{1/2}_{L_2(\Omega)}) \\ &\le C\nu\|\nabla\alpha^\nu\|_{L_2(\Omega)} \\ &\quad +C\nu^{1/4}\Big[\nu^{1/4}\|\nabla\alpha^\nu\|^{1/2}_{L_2(\Omega)}\Big]\Big[\nu^{1/2}\|\nabla^2\alpha^\nu\|^{1/2}_{L_2(\Omega)}\Big] \\ &\le C(\nu^{1/2}I_\nu + I_\nu^{1/2}J^\nu), \end{aligned} \tag{4.7}$$

with $I_\nu := \nu^{1/2}\|\nabla\alpha^\nu\|_{L_2(\Omega)}$ and $J_\nu := \nu^{1/4}[\nu^{1/2}\|\nabla^2\alpha^\nu\|^{1/2}_{L_2(\Omega)}]$. Note that the first term in the r.h.s. of (4.7) satisfies $\nu^{1/2}I^\nu \to 0$ by the same arguments as for (4.6). To control the second derivatives, a modification of Theorem 3 is required.

We have

Theorem 4 *Let $G \in L_2(\Omega)$, $\beta_{in} \in H^{3/2}(\Gamma_{in})$ and $V \in C^a(\overline{\Omega})$ with $a < \frac{1}{2}$; then for $0 < \nu \le 1$ there exists a unique solution to*

$$\begin{array}{lll} V\cdot\nabla\beta - \nu\Delta\beta = G & in & \Omega, \\ \beta = \beta_{in} & on & \Gamma_{in}, \\ \frac{\partial\beta}{\partial n} = 0 & on & \partial\Omega\setminus\Gamma_{in} \end{array} \tag{4.8}$$

such that $\beta \in H^2(\Omega)$. Additionally the following estimate is valid

$$\begin{aligned} \nu\|\nabla^2\beta\|_{L_2(\Omega)} \le C\,\big[&\|G\|_{L_2(\Omega)} + \|\beta_{in}\|_{H^{3/2}(\Gamma_{in})} \\ &+(\nu\lambda^{-1} + \lambda^a)\|\nabla\beta\|_{L_2(\Omega)} + \lambda^{a-1}\|\beta\|_{L_2(\Omega)}\big]. \end{aligned} \tag{4.9}$$

where $\lambda : 0 < \lambda \le \lambda_0$ is a localization parameter and the constant C is independent from λ and ν and depends only on the shape of $\partial\Omega$.

Applying (4.9) to J^ν from (4.7) we conclude

$$\begin{aligned} J^\nu &\le C\nu^{1/4}\Big[\|\mathrm{rot}\, F\|_{L_2(\Omega)} + \|\alpha_{in}\|_{H^{3/2}(\Gamma_{in})} \\ &\qquad +(\nu\lambda^{-1}+\lambda^a)\|\nabla\alpha^\nu\|_{L_2(\Omega)} + \lambda^{a-1}\|\alpha^\nu\|_{L_2(\Omega)}\Big]^{1/2} \\ &\le \kappa(\nu) + C[(\nu\lambda^{-1}+\lambda^a)\nu^{1/2}\|\nabla\alpha^\nu\|_{L_2(\Omega)} + \nu^{1/2}\lambda^{a-1}\|\alpha^\nu\|_{L_2(\Omega)}]^{1/2}, \end{aligned}$$

where $\kappa(\nu) \to 0$ as $\nu \to 0$.

Keeping in mind (4.4) we describe relations between ν and λ. It is allowed since the constant in (4.9) is independent of ν and λ. Taking

$$\nu \le \lambda^2 \le 2\nu \le 2,$$

we obtain

$$J^\nu \le \kappa(\nu) + C[\nu^{1/2} + \nu^{a/2} + \nu^{a/2}]^{1/2} \to 0.$$

So the limit of (4.2) reads

$$-\int_\Omega v^E\alpha^E\cdot\nabla\phi dx + \int_{\Gamma_{in}} \vec{n}\cdot v^E\alpha_{in}\phi d\sigma = \int_\Omega \mathrm{rot}\, F\phi dx \tag{4.10}$$

for $\phi \in C^\infty(\overline{\Omega})$ with $\phi|_{\Gamma_{out}} = 0$.

Thanks to the choice of the test functions we obtain a dependence from the boundary vorticity α_{in}. Additionally the above integral identity is a weak formulations of the Euler system (1.11). Theorem 2 is proved.

5 Appendix

Here we prove Theorems 3 and 4.

Proof of Theorem 3. By taken assumptions we are able to find an extension of β_{in} such that

$$\tilde\beta \in H^2(\Omega), \qquad \frac{\partial\tilde\beta}{\partial n}|_{\partial\Omega} = 0 \quad \text{and} \quad \tilde\beta|_{\Gamma_{in}} = \beta_{in}.$$

Then we are allowed to consider $\beta_{new} = \beta_{old} - \tilde\beta$, getting (3.10) with the homogeneous condition: $\beta|_{\Gamma_{in}} = 0$ and

$$G_{new} = G_{old} + \nu\Delta\tilde\beta + \nabla\theta\cdot\nabla\tilde\beta$$

with suitable estimate in the L_2-space.

Now we start with local estimates. Let us consider a partition of unity over Ω. We find smooth functions $\{\pi_k\}_{k\in I}$ such that

$$\pi_k : \Omega \to [0,1] \ \text{ and } \ \sum_{k\in I}\pi_k \equiv 1 \text{ on } \Omega.$$

Additionally, we divide the set of indexes I into two parts $I = \mathcal{N} \cup \mathcal{I}$, the sets are finite, in that way

$$\text{for } k \in \mathcal{N},\ \operatorname{supp} \pi_k \cap \partial\Omega \neq \emptyset \ \text{ and } \ \text{for } k \in \mathcal{I},\ \operatorname{supp} \pi_k \cap \partial\Omega = \emptyset; \tag{5.1}$$

moreover

$$\sup_k \operatorname{diam}(\operatorname{supp} \pi_k) \leq \lambda \ \text{ and } \ |\nabla \pi_k| \leq C/\lambda, \quad |\nabla^2 \pi_k| \leq C/\lambda^2 \tag{5.2}$$

and the Lebesgue cover number is denoted by N_0, which does not increase for all $0 < \lambda \leq \lambda_0$. It depends only on the regularity of the boundary $\partial\Omega$. In our considerations we are required to choose $\{\pi_k\}_{k \in I}$ with sufficiently small λ.

The interior estimate. Applying π_k with $k \in \mathcal{I}$ to system we obtain

$$\overline{\nabla\theta} \cdot \nabla(\pi_k \beta) - \nu \Delta(\pi_k \beta) = \pi G + R_1 \quad \text{in} \quad \mathbb{R}^2, \tag{5.3}$$

where

$$R_1 = 2\nu \nabla \pi_k \cdot \nabla \beta + \nu(\Delta \pi_k)\beta + (\overline{\nabla\theta} - \nabla\theta)\pi_k \nabla \beta + (\overline{\nabla\theta} - \nabla\theta)\beta \nabla \pi_k \tag{5.4}$$

with $\overline{\nabla\theta} = \nabla\theta(x_k)$ and $x_k \in \text{int supp } \pi_k$.

The symbol of the operator in the l.h.s. of (5.3) has the following form

$$i\overline{\nabla\theta} \cdot \xi + \nu|\xi|^2,$$

and in particular satisfies

$$|i\overline{\nabla\theta} \cdot \xi + \nu|\xi|^2| \geq \nu|\xi|^2. \tag{5.5}$$

A direct application of Parseval's identity and the definition of H^m together with (5.5) lead straightforwardly to the estimate

$$\nu \|\nabla^2(\pi_k \beta)\|_{L_2(O_k)} \leq C(\|\pi_k G\|_{L_2(O_k)} + \|R_1\|_{L_2(O_k)}), \tag{5.6}$$

where $O_k = \text{supp } \pi_k$ and by (5.2) and (5.4) we find

$$\|R_1\|_{L_2(O_k)} \leq C[\nu\lambda^{-1}\|\nabla\beta\|_{L_2(O_k)} + \nu\lambda^{-2}\|\beta\|_{L_2(O_k)} + \lambda\|\nabla\beta\|_{L_2(O_k)} + \|\beta\|_{L_2(O_k)}]. \tag{5.7}$$

where the constant in (5.7) is independent from ν and λ.

The boundary estimate. Taking π_k, but with $k \in \mathcal{N}$ and applying it to the system (3.10) we obtain

$$\begin{array}{c} \overline{\nabla\theta} \cdot \nabla(\pi_k \beta) - \nu \Delta(\pi_k \beta) = \pi G + R_1 \quad \text{in} \quad \Omega, \\ \frac{\partial(\pi_k \beta)}{\partial \vec{n}} = 0 \qquad \text{and / or} \qquad (\pi_k \beta) = 0 \quad \text{on} \quad \partial\Omega \cap O_k, \end{array} \tag{5.8}$$

where R_1 is given by (5.4) and $\overline{\nabla\theta} = \nabla\theta(x_k)$, but with $x_k \in \operatorname{int}\operatorname{supp}\pi_k \cap \partial\Omega$. The boundary conditions depend on the localization of the support of taken π_k with respect to the localization of Γ_{in}.

The smoothness of the boundary allows us to transport this system onto the halfspace $\mathbb{R}^2_+$ with a local coordinate system (z_1, z_2). For each function π_k we consider a map $Z_k : \Omega \cap O_k \to \mathbb{R}^2_+$, then equations (5.8) reads

(5.9)
$$\begin{array}{ll}\overline{\nabla\theta} \cdot \nabla_z Z_k^{-1*}(\pi\beta) - \nu\Delta_z Z_k^{-1*}(\pi\beta) = Z_k^{-1*}(\pi G) + Z_k^{-1*}(R_1) + R_2 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial Z_k^{-1*}(\pi_k\beta)}{\partial z_2} = 0 \qquad \text{and / or} \qquad Z_k^{-1*}(\pi_k\beta) = 0 & \text{on } \mathbb{R}\times\{0\},\end{array}$$

where

$$R_2 = \overline{\nabla\theta}(\nabla_x - \nabla_z) Z_k^{-1*}(\pi_k\beta) + \nu(\Delta_x - \Delta_z) Z_k^{-1*}(\pi_k\beta), \tag{5.10}$$

and ∇_z denotes the gradient in $\mathbb{R}^2$ in the z-coordinates and ∇_x denotes the gradient in the x-coordinates transformed by Z_k.

The above problem reduces to a model problem with three possibilities. We have to consider the equation

$$\overline{\nabla\theta} \cdot \nabla\gamma - \nu\Delta\gamma = H \qquad \text{in} \quad \mathbb{R}^2_+, \tag{5.11}$$

with three types of boundary relations

$$\begin{array}{ll} (i) & \frac{\partial\gamma}{\partial\vec{n}} = 0 \text{ on } \mathbb{R}\times\{0\}; \\ (ii) & \gamma = 0 \text{ on } \mathbb{R}\times\{0\}; \\ (iii) & \frac{\partial\gamma}{\partial\vec{n}} = 0 \text{ for } z_1 < 0 \text{ and } \gamma = 0 \text{ for } z_1 \geq 0 \text{ on } \mathbb{R}\times\{0\}. \end{array} \tag{5.12}$$

We are required to obtain the following bound on the solutions to (3.10)

$$\nu\|\nabla^2\gamma\|_{L_2(\mathbb{R}^2_+)} \leq C\|H\|_{L_2(\mathbb{R}^2_+)}. \tag{5.13}$$

The first two cases follow from the standard approach. The case (i) used the method of symmetry to transform the system into the whole space, only. The case (ii) requires the standard energy estimate. The case (iii) is not straightforward, because of the structure of the boundary conditions. Here we have to specify the choice of the point x_k. If $\partial\Gamma_{in} \in \operatorname{int}\operatorname{supp}\pi_k$ then we choose x_k as the end of Γ_{in} – see (5.4). This choice implies that $\overline{\nabla\theta} \perp \vec{n}$ at $\partial\Gamma_{in}$ and this form of (5.11) allow us to apply the standard energy method to get the estimate (5.13). It is enough to test the equation by γ_{z_1}, $\gamma_{z_2z_2}$.

Thus the bound used for (5.11) delivered the following estimate for (5.9)

$$\begin{aligned}\nu\|\nabla^2 Z_k^{-1*}(\pi_k\beta)\|_{L_2(\mathbb{R}^2_+)} \leq C\Big[&\|Z_k^{-1*}(\pi_k G)\|_{L_2(\mathbb{R}^2_+)} \\ &+\|Z_k^{-1*}(R_1)\|_{L_2(\mathbb{R}^2_+)} + \|R_2\|_{L_2(\mathbb{R}^2_+)}\Big]\end{aligned} \tag{5.14}$$

with (where we use regularity of the boundary, i.e. regularity of maps Z_k)

$$\|R_2\| \leq C\Big(\lambda\|\nabla_z Z_k^{-1*}(\pi_k\beta)\|_{L_2(\mathbb{R}^2_+)} + \nu\lambda\|\nabla_z^2 Z_k^{-1*}(\pi_k\beta)\|_{L_2(\mathbb{R}^2_+)} + \nu\|\nabla_z Z_k^{-1*}(\pi_k\beta)\|_{L_2(\mathbb{R}^2_+)}\Big) \tag{5.15}$$

and the constants in (5.14) and (5.15) are independent from ν and λ. Now we first apply Poincaré's inequality to the last term in the r.h.s.: this allows to include it in the second one. And at this point we require to the parameter λ be so small that two last terms can be put on the l.h.s. of (5.14).

So for $k \in \mathcal{N}$ we have

$$\nu\|\nabla^2(\pi_k\beta)\|_{L_2(O_k)} \leq C(\lambda\|\nabla\beta\|_{L_2(O_k)} + \|\beta\|_{L_2(O_k)} + \|\pi_k G\|_{L_2(O_k)} + \|R_1\|_{L_2(O_k)}), \tag{5.16}$$

but now C in (5.16) depends on maps Z_k, so depends on the regularity of $\partial\Omega$.

Summing up (5.6) and (5.16), noting that

$$\nu^2\|\nabla^2\beta\|^2_{L_2(\Omega)} \leq \nu^2 N_0 \sum_k \|\nabla^k(\pi_k\beta)\|^2_{L_2(\Omega)}.$$

Since the cover number N_0 is independent from the smallness of the localization parameter λ, we get (3.11). In particular we can use it in considerations for the limit $\lambda \to 0$.

Theorem 3 has been proved.

Proof of Theorem 4. We present here only main difference between proofs of Theorems 3 and 4. Since for (4.8) the vector field $V \in C^a(\overline{\Omega})$, only, we look closer on the estimate of R_1 – see (5.4). Pointing out the difference we have

$$\|(\bar{V} - V)\pi_k\nabla\beta\|_{L_2(O_k)} + \|(\bar{V} - V)\beta\nabla\pi_k\|_{L_2(O_k)} \leq C\lambda^a\|\nabla\beta\|_{L_2(O_k)} + C\lambda^{a-1}\|\beta\|_{L_2(O_k)}. \tag{5.17}$$

The rest of the estimation is the same since the other assumptions are identical. This way we prove (4.9) and Theorem 4.

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